Proceedings of the Royal Society of Edinburgh, **146A**, 393–413, *2016*
DOI:10.1017/S0308210515000542

A general formula for the anisotropic outer Minkowski content of a set

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(MS received 24 June 2014; accepted 13 February 2015)

We generalize to the anisotropic case some classical and recent results on the $(n-1)$-Minkowski content of rectifiable sets in $\mathbb{R}^n$, and on the outer Minkowski content of subsets of $\mathbb{R}^n$. In particular, a general formula for the anisotropic outer Minkowski content is provided; it applies to a wide class of sets that are stable under finite unions.

Keywords: Minkowski content; outer Minkowski content; anisotropy; rectifiability

2010 *Mathematics subject classification:* Primary 28A75; 49Q15; 52A39

1. Introduction

The notion of Minkowski content was introduced by Minkowski in order to study an intrinsic definition of the k-dimensional area of a compact set. Precisely, if $S \subset \mathbb{R}^n$ is a closed set, the $(n-1)$-*dimensional Minkowski content* of S is defined by

$$\mathcal{M}^{n-1}(S) := \lim_{\varepsilon \to 0} \frac{|\{x \in \mathbb{R}^n\colon\ \operatorname{dist}(x,S) \leqslant \varepsilon\}|}{2\varepsilon} \tag{1.1}$$

whenever the limit on the right-hand side exists and is finite; here $|\cdot|$ denotes the Lebesgue measure in $\mathbb{R}^n$. Note that if B denotes the unit ball in $\mathbb{R}^n$, then $|\{x \in \mathbb{R}^n\colon\ \operatorname{dist}(x,E) \leqslant \varepsilon\}| = |E+\varepsilon B|$; in other words, we are thus looking for the limit of the volume of the tube around S divided by the thickness 2ε. The set $E+\varepsilon B$ is called the *parallel set* or *Minkowski enlargement* of E at distance ε. A natural question arises: is it true that the $(n-1)$-dimensional Minkowski content of S coincides with the $(n-1)$-dimensional Hausdorff measure of S? A first result in this direction can be found in [10], and the answer is affirmative if the set S is smooth enough in the sense of geometric measure theory, i.e. it has good *rectifiability* properties (see § 2 for details). The notion of the Minkowski content of sets recently played a fundamental role in the approximation and estimation of the mean density of $(n-1)$-dimensional random closed sets (see, for example, [5,18]), and thus has applications in statistics, stochastic geometry and image analysis. A computer graphics representation of lower dimensional sets in $\mathbb{R}^2$ is in any case provided in terms of pixels, which can offer only a two-dimensional box approximation of points in $\mathbb{R}^2$ (an interesting discussion on this is given in [14]). Therefore, the possibility of evaluating and estimating the surface measure of a set (the mean surface density for random sets) by the volume measure of the Minkowski enlargement of the involved set, which is much more robust and computable with respect to the $\mathcal{H}^{n-1}$-measure, could provide a solution to problems of this kind. Other examples of applications to statistical problems concerning non-parametric estimation of the boundary of deterministic sets can be found, for instance, in [3, 4].

More recently, in [2], motivated by problems arising in stochastic geometry, the notion of *outer Minkowski content* of a set was introduced:

$$\mathcal{SM}(E) := \lim_{\varepsilon\to 0} \frac{|\{x \in \mathbb{R}^n : \operatorname{dist}(x, E) \leqslant \varepsilon\} \setminus E|}{\varepsilon} = \lim_{\varepsilon\to 0} \frac{|E + \varepsilon B| - E|}{\varepsilon}; \tag{1.2}$$

in this case, we are looking for the limit of the volume of the part of the tube 'outside' E, now divided by the thickness ε. Ambrosio *et al*. [2] investigated the general conditions that ensure the existence of $\mathcal{SM}$; in particular, they prove that $\mathcal{SM}(E)$ coincides with the perimeter $\mathcal{P}(E)$ of E whenever E has finite perimeter and $\mathcal{M}^{n-1}(\partial E)$ equals the perimeter of E, where ∂E denotes the topological boundary of E (see theorem 3.1).

A more general formula for the outer Minkowski content of a set has been investigated in [16], and also holds when some points of E have *density 0* in the sense of geometric measure theory; roughly speaking, the points of density 0 of E, denoted by E^0 (see § 2 for a precise definition), form the part of E that looks like a $(n-1)$-dimensional manifold, or any of its subsets. In order to understand what happens in this case, take, for instance, the set E in $\mathbb{R}^2$ given by a square and a line segment outside the square: the line segment is the set of points of E of density 0. We expect that $\mathcal{SM}(E)$ takes into account the perimeter of the square plus twice the $\mathcal{H}^1$-measure of the added line segment. A rigorous formula that formalizes this intuition holds for a suitable class of sets that is stable under finite unions (see theorem 4.3), and such stability is a particularly relevant feature in connection with applications to the study of some stochastic processes. Precisely, the formula takes the form

$$\mathcal{SM}(E) = \mathcal{P}(E) + \mathcal{H}^{n-1}(\partial E \cap E^0). \tag{1.3}$$

Possible applications of this are the study of the evolution equations of the mean density of the surface measure of random sets evolving in time and modelling grain growth in recrystallization processes in materials science (see, for example, [17] and the references therein for a more exhaustive treatment). Note that isotropic growth may be modelled by the Minkowski enlargement of the involved crystals, and so the role played by the outer Minkowski content in the study of the surface measure of the crystallized region is evident.

It thus seems crucially important to try to extend the above results on the (outer) Minkowski content to the anisotropic case, in order to deal with the problem of anisotropic growth in recrystallization processes. Very recently, an anisotropic variant of the outer Minkowski content of a set was considered in [7] from a purely mathematical point of view, also motivated by the study of anisotropic perimeters arising from discrete perimeters (see [6]). So, by replacing B in (1.2) with a *convex body* $C \subset \mathbb{R}^n$, i.e. a compact and convex set with 0 in its interior, we may look at the limit

$$\mathcal{SM}_C(E) := \lim_{\varepsilon \to 0} \frac{|E + \varepsilon C| - |E|}{\varepsilon} \tag{1.4}$$

whenever it exists and is finite.

Note that $\mathcal{SM}(E)$ and $\mathcal{M}^{n-1}(\partial E)$ may suggest using statistical methods to estimate the $\mathcal{H}^{n-1}$-measure of the boundary of E (see, for example, [8]); similarly, their anisotropic generalizations $\mathcal{SM}_C(E)$ and $\mathcal{M}_C(\partial E)$, and related results proved in theorems 3.2, 3.4 and 4.4, may be applied analogously in order to estimate the measure of the anisotropic perimeter of E, or, eventually, the $\mathcal{H}^{n-1}$-measure of its boundary, whenever they coincide. Moreover, the study of $\mathcal{SM}_C(E)$ and $\mathcal{M}_C(\partial E)$ with $C = [-1, 1]^2$ may also be of interest in image analysis in order to compare or improve the existing results in boundary estimation by taking into account the fact that pixels are actually squares in digitized images. Finally, an anisotropic version of the outer Minkowski content may be of further applications in stochastic geometry concerning the so-called contact distribution function of random closed sets (see, for example, [12, 13]).

Surprisingly, much less is known about the anisotropic case than the isotropic case. In order to understand easily what happens in the anisotropic case, take a convex body C that is not a ball: there exists a direction $\nu \in \mathbb{R}^n$ for which $h_C(\nu)$ is not 1, where h_C is the so-called *support function* of C, defined as $h_C(v) := \sup_{x \in C} x \cdot v$ (for a very nice explanation of h_C we refer the interested reader to [11]); without loss of generality, we can assume $\nu = e_n$, where e_n is the last vector of the canonical basis of $\mathbb{R}^n$. Consider thus the half-space $E = \{x \in \mathbb{R}^n : x_n \geqslant 0\}$, which has ν as exterior normal, and measure locally (see what happens on a bounded set A that intersects ∂E) its anisotropic Minkowski content using (1.4): note that, since E has a flat boundary, it is very easy to compute the measure of the ε-tube around ∂E, and we obtain $\varepsilon h_C(\nu)\mathcal{H}^{n-1}(\partial E \cap A) + o(\varepsilon)$ for any A such that $\mathcal{H}^{n-1}(\partial E \cap \partial A) = 0$. This very simple computation suggests that for sufficiently smooth sets the anisotropic outer Minkowski content of E takes the form

$$\mathcal{SM}_C(E) = \int_{\partial E} h_C(\nu_E)\, \mathrm{d}\mathcal{H}^{n-1}, \tag{1.5}$$

where ν_E is the exterior unit normal at E: therefore, $\mathcal{SM}_C(E)$ is not, in general, the perimeter of E, unless $C = B$, when we have $h_C = 1$ identically; of course it may be that, for particular sets E and for a particular choice of C, $\mathcal{SM}_C(E)$ coincides with $\mathcal{H}^{n-1}(\partial E)$ (think of sets with flat boundaries and take a suitable C), but this is not the general case, as (1.5) shows. As a very simple example in $\mathbb{R}^2$, take $E = [0, L_1] \times [0, L_2] \subset \mathbb{R}^2$; if $C = [-1, 1]^2$, then

$$\mathcal{SM}_C(E) = 2(L_1 + L_2) = \mathcal{H}^1(\partial E),$$

whereas if $\tilde{C}$ is the rotation of C through $\pi/4$, then $\mathcal{SM}_C(E) = \sqrt{2}\mathcal{H}^1(\partial E)$. Chambolle *et al*. [7] prove that for sufficiently smooth sets (always in the sense of geometric measure theory) (1.5) holds.

In this paper, first of all we want to find an integral formula for the anisotropic Minkowski content of a set, i.e. the quantity, whenever it is well defined, given by

$$\mathcal{M}_C(S) := \lim_{\varepsilon\to 0} \frac{|S+\varepsilon C|}{2\varepsilon}.$$

Note that if S is a boundary, taking $S = \partial E$, the anisotropic Minkowski content of S should be equal to

$$\frac{\mathcal{SM}_C(E) + \mathcal{SM}_C(\mathbb{R}^n \setminus E)}{2}.$$

Therefore, we expect that

$$\mathcal{M}_C(S) = \frac{1}{2}\int_S (h_C(\nu_S) + h_C(-\nu_S))\,\mathrm{d}\mathcal{H}^{n-1},$$

where ν_S is a unit normal to S. We are able to prove such a formula under suitable conditions on S similar to those one must assume for the isotropic case (see theorems 3.4 and 3.7). Actually, we need to show the anisotropic Minkowski content, as is this essential, following our approach, in order to prove a general formula for the anisotropic outer Minkowski content $\mathcal{SM}_C(E)$ that also takes into account points with density 0, as done in the isotropic case: such a general formula is stated in theorem 4.4, coincides with (1.3) when $C = B$ and holds for a suitable class of sets that is stable under finite unions.

2. Notation and preliminaries

2.1. Notation

Let $n \geqslant 1$ be integer. Given a measurable set $A \subset \mathbb{R}^n$, we shall denote by $|A|$ its Lebesgue measure. If $k \in \{0,\dots,n\}$, the k-dimensional Hausdorff measure of $S \subset \mathbb{R}^n$ will be denoted by $\mathcal{H}^k(S)$. We shall use the notation $x\cdot y$ for the standard scalar product between x and y in $\mathbb{R}^n$, and $B_r(x)$ for the closed ball of radius r centred in x. For each $k \in \mathbb{N}$ with $k \leqslant n$ we denote by $\boldsymbol{G}_k$ the set of unoriented k-planes on $\mathbb{R}^n$; for any $\pi \in \boldsymbol{G}_k$ we denote by $\pi^\perp \in \boldsymbol{G}_{n-k}$ the $(n-k)$-plane orthogonal to π. Finally, if μ is a positive, real or vector measure on some space X and $f\colon X \to Y$ is measurable, we define the measure $f_\sharp\mu$ on Y as $f_\sharp\mu(F) := \mu(f^{-1}(F))$ for any F measurable in Y; a positive and real measure μ on X is said to be a probability measure if $\mu(X) = 1$.

2.2. Geometric measure theory

In this subsection we recall some basic notions of geometric measure theory that we shall need; for details we refer the reader to [1, 10, 15]. Let $n \geqslant 1$ be integer and let $k \in \mathbb{N}$ with $k \leqslant n$. The following general property of Radon measures holds; here ω_k is the volume of the k-dimensional unit ball.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^n$ be an open set and let μ be a positive Radon measure on Ω. Then, for any $t > 0$ and for any B Borel set in Ω, the following implications* *hold:*

$$\limsup_{\rho\to 0}\frac{\mu(B_\rho(x))}{\omega_k\rho^k}\geqslant t\quad \forall x\in B \implies \mu\geqslant t\mathcal{H}^k\llcorner B,$$

$$\limsup_{\rho\to 0}\frac{\mu(B_\rho(x))}{\omega_k\rho^k}\leqslant t\quad \forall x\in B \implies \mu\leqslant 2^k t\mathcal{H}^k\llcorner B.$$

A very useful consequence of theorem 2.1 turns out to be the following:

$$B \text{ Borel in } \Omega \quad \text{with } \mu(B)=0$$
$$\implies \lim_{\rho\to 0}\frac{\mu(B_\rho(x))}{\omega_k\rho^k}=0 \quad \text{for } \mathcal{H}^k\text{-almost every (a.e.) } x\in B. \tag{2.1}$$

Now let $S\subset\mathbb{R}^n$. We say that S is k-rectifiable if there exist a bounded set $B\subset\mathbb{R}^k$ and a Lipschitz function $f\colon B\to\mathbb{R}^n$ such that $S=f(B)$. We say that $S\subset\mathbb{R}^n$ is countably $\mathcal{H}^k$-rectifiable if there exist countably many Lipschitz functions $f_h\colon\mathbb{R}^k\to\mathbb{R}^n$ such that

$$\mathcal{H}^k\biggl(S\setminus\bigcup_{h=0}^{+\infty}f_h(\mathbb{R}^k)\biggr)=0.$$

If, in addition, $\mathcal{H}^k(S)<+\infty$, then S is said to be $\mathcal{H}^k$-rectifiable. A classical rectifiability criterium says that a Borel set $S\subset\mathbb{R}^n$ with $\mathcal{H}^k(S)<+\infty$ is $\mathcal{H}^k$-rectifiable if and only if

$$\lim_{\rho\to 0}\frac{\mathcal{H}^k(S\cap B_\rho(x))}{\omega_k\rho^k}=1\quad\text{for } \mathcal{H}^k\text{-a.e. } x\in S. \tag{2.2}$$

It turns out that if S is countably $\mathcal{H}^k$-rectifiable, then for $\mathcal{H}^k$-almost any point $x_0\in S$, the approximate tangent space $\operatorname{Tan}^k(S,x_0)\in\boldsymbol{G}_k$ is well defined, i.e.

$$\lim_{\rho\to 0}\frac{1}{\rho^k}\int_S\phi\biggl(\frac{x-x_0}{\rho}\biggr)\,\mathrm{d}\mathcal{H}^k(x)=\int_{\operatorname{Tan}^k(S,x_0)}\phi(y)\,\mathrm{d}\mathcal{H}^k(y)\quad\forall\phi\in C_c^\infty(\mathbb{R}^n).$$

In particular, if $k=n-1$, then $\operatorname{Tan}^{n-1}(S,x_0)^\perp$ is generated by some unit vector denoted by ν_S.

We recall that by a Lipschitz k-graph we mean the graph of a Lipschitz function $\phi\colon\pi\to\pi^\perp$, where $\pi\in\boldsymbol{G}_k$. Given a countably $\mathcal{H}^k$-rectifiable set S, it is well known that S can be covered, up to a $\mathcal{H}^k$-negligible set, by a countable family of pairwise disjoint compact subsets of S that are contained in some Lipschitz k-graph and with finite k-dimensional Hausdorff measure. We now recall the notion of a k-dimensional Jacobian and the area formula. Let $L\colon\mathbb{R}^k\to\mathbb{R}^n$ be a linear map. The k-dimensional Jacobian of L is defined by $\boldsymbol{J}_kL:=\sqrt{\det(L^*\circ L)}$, where $L^*\colon\mathbb{R}^n\to\mathbb{R}^k$ denotes the transpose of L. The Jacobian is related, as is well known in the smooth case, to the change-of-variable formula for multiple integrals: more precisely, if $f\colon\mathbb{R}^k\to\mathbb{R}^n$ is Lipschitz, then for any measurable set $E\subset\mathbb{R}^k$ the multiplicity function $y\mapsto\mathcal{H}^0(E\cap f^{-1}(\{y\}))$ is measurable and the area formula

$$\int_{\mathbb{R}^n}\sum_{x\in E\cap f^{-1}(\{y\})}g(x)\,\mathrm{d}\mathcal{H}^k(y)=\int_E g(x)\boldsymbol{J}_k\,\mathrm{d}f_x\,\mathrm{d}x \tag{2.3}$$

holds for each $g\colon E \to \mathbb{R}$ Borel, where $\mathrm{d}f_x\colon \mathbb{R}^k \to \mathbb{R}^n$ is the differential of f at x, which exists for a.e. $x \in \mathbb{R}^k$ by Rademacher's theorem.

There is another useful formula, known as the coarea formula: if Ω is open in $\mathbb{R}^n$, $f\colon \Omega \to \mathbb{R}$ is Lipschitz continuous and $g\colon \Omega \to [0, +\infty]$ is Borel, then

$$\int_\Omega g(x)|\nabla f(x)|\,\mathrm{d}x = \int_{-\infty}^{+\infty}\int_{\Omega\cap\{f=t\}} g(y)\,\mathrm{d}\mathcal{H}^{n-1}(y)\,\mathrm{d}t. \tag{2.4}$$

Now, let $E \subset \mathbb{R}^n$ be a measurable set and let $\Omega \subset \mathbb{R}^n$ be an open domain; we denote by χ_E the characteristic function of E. We say that E has finite perimeter in Ω if the distributional derivative of χ_E, denoted by $D\chi_E$, is an $\mathbb{R}^n$-valued Radon measure on Ω with finite total variation; the perimeter of E in Ω is defined by $\mathcal{P}(E;\Omega) := |\mathrm{D}\chi_E|(\Omega)$, where $|\mathrm{D}\chi_E|$ denotes the total variation of $\mathrm{D}\chi_E$. We also let $\mathcal{P}(E) := \mathcal{P}(E;\mathbb{R}^n)$. For sufficiently smooth boundaries, the perimeter coincides with the $(n-1)$-dimensional Hausdorff measure of the topological boundary. An interesting situation is the following: given a Lipschitz map $f\colon A \to \mathbb{R}$, with A open and bounded in $\mathbb{R}^{n-1}$, the subgraph of f turns out to be a set with finite perimeter in $A\times\mathbb{R}$, and its perimeter coincides with the $(n-1)$-dimensional Hausdorff measure of the graph of f. The upper and lower n-dimensional densities of E at x are respectively defined by

$$\Theta^*_n(E,x) := \limsup_{\rho\to 0}\frac{|E\cap B_\rho(x)|}{\omega_n\rho^n}, \qquad \Theta_{*n}(E,x) := \liminf_{\rho\to 0}\frac{|E\cap B_\rho(x)|}{\omega_n\rho^n}.$$

If $\Theta^*_n(E,x) = \Theta_{*n}(E,x)$, their common value is denoted by $\Theta_n(E,x)$. For every $t \in [0,1]$ we define $E^t := \{x \in \mathbb{R}^n\colon \Theta_n(E,x) = t\}$. The essential boundary of E is defined as $\partial^*E := \mathbb{R}^n \setminus (E^0 \cup E^1)$. It turns out that if E has a finite perimeter in Ω, then $\mathcal{H}^{n-1}(\partial^*E \setminus E^{1/2}) = 0$, and $\mathcal{P}(E;\Omega) = \mathcal{H}^{n-1}(\partial^*E\cap\Omega)$. Moreover, one can define a subset of $E^{1/2}$ as the set of points x where there exists a unit vector $\nu_E(x)$ such that

$$\frac{E-x}{\rho} \to \{y\in\mathbb{R}^n\colon y\cdot\nu_E(x)\leqslant 0\} \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}^n) \text{ as } \rho\to 0,$$

which is referred to as the outer normal to E at x. The set where $\nu_E(x)$ exists is called the reduced boundary and is denoted by $\mathcal{F}E$: roughly speaking, if we zoom in around a point on the reduced boundary (more properly, *blow up*), we see something flat and precisely a half-space orthogonal to what we call normal at that point. Note that the standard definition of reduced boundary, which can be found in [1], is different from that presented here, and in [1, definition 3.54] the reduced boundary is defined precisely in terms of the distributional derivative of χ_E and its behaviour with respect to the total variation of such a derivative; we admit that the distributional definition is more intuitive, but the definition given here, based on a result obtained by De Giorgi in 1954, turns out to be more practical. One can show that $\mathcal{H}^{n-1}(\partial^*E\setminus\mathcal{F}E) = 0$. Moreover, one has the decomposition $D\chi_E = (-\nu_E)\mathcal{H}^{n-1}\llcorner\mathcal{F}E$. We also introduce the set $\partial^2E := \{x\in\partial E\cap E^0\colon \exists\mathrm{Tan}^{n-1}(\partial E,x)\}$. Let us collect some elementary properties of sets with a countably $\mathcal{H}^{n-1}$-rectifiable boundary and with finite perimeter in Ω; for any $E\subseteq\mathbb{R}^n$ we let $E^{\mathrm{c}} := \mathbb{R}^n\setminus E$.

Lemma 2.2. *Assume that E has a finite perimeter in Ω and that ∂E is countably $\mathcal{H}^{n-1}$-rectifiable. Then the following hold:*

$$\mathcal{H}^{n-1}(\mathcal{F}E) = \mathcal{H}^{n-1}(\mathcal{F}E^{c}); \tag{2.5}$$

$$\mathcal{H}^{n-1}(\partial E \setminus \mathcal{F}E) = \mathcal{H}^{n-1}(\partial^2 E) + \mathcal{H}^{n-1}(\partial^2 E^{c}); \tag{2.6}$$

$$\nu_{E^{c}}(x) = -\nu_E(x) \quad \textit{for any } x \in \mathcal{F}E; \tag{2.7}$$

$$\mathcal{H}^{n-1}(\partial E) = \mathcal{H}^{n-1}(\partial^2(\partial E)). \tag{2.8}$$

Proof. Properties (2.5), (2.7) and (2.8) are trivial. In order to prove (2.6) we note that

$$\begin{aligned}\mathcal{H}^{n-1}(\partial E) &= \mathcal{H}^{n-1}(\partial E^{c}) \\ &= \mathcal{H}^{n-1}(\mathcal{F}E^{c}) + \mathcal{H}^{n-1}(\partial E^{c} \cap (E^{c})^1) + \mathcal{H}^{n-1}(\partial E^{c} \cap (E^{c})^0) \\ &= \mathcal{H}^{n-1}(\mathcal{F}E) + \mathcal{H}^{n-1}(\partial E \cap E^0) + \mathcal{H}^{n-1}(\partial^2 E^{c}) \\ &= \mathcal{H}^{n-1}(\mathcal{F}E) + \mathcal{H}^{n-1}(\partial^2 E) + \mathcal{H}^{n-1}(\partial^2 E^{c}).\end{aligned}$$

Therefore, $\mathcal{H}^{n-1}(\partial E \setminus \mathcal{F}E) = \mathcal{H}^{n-1}(\partial^2 E) + \mathcal{H}^{n-1}(\partial^2 E^{c})$, which is (2.6). □

Remark 2.3. Using lemma 2.2, we may observe that if E is such that its topological boundary ∂E is a set countably $\mathcal{H}^{n-1}$-rectifiable and bounded, then one of the following holds for $\mathcal{H}^{n-1}$-a.e. $x \in \partial E$:

(1) $x \in \mathcal{F}E$, and the outer normal $\nu_E(x)$ to E at x exists;

(2) $x \in E^1 \cap \partial E$ (in such a case no outer normal exists);

(3) $x \in \partial^2 E$, and two outer normals to E at x exist, say $\nu_E(x)$ and $-\nu_E(x)$.

This is in accordance with known results in the literature for sets with positive reach (see, for example, [9]); namely, it can be shown that the topological boundary of a compact subset E of $\mathbb{R}^n$ with positive reach is $(n-1)$-rectifiable, and that, for $\mathcal{H}^{n-1}$-a.e. $x \in \partial E$, either $x \in \mathcal{F}E$ or $x \in \partial^2 E$ (see also [2]).

We now recall the well-known Besicovitch theorem.

Theorem 2.4. *Let $A \subset \mathbb{R}^n$ be a bounded set and let $\rho\colon A \to (0, +\infty)$ be a function. There exists a set $S \subset A$ at most countable such that*

$$A \subset \bigcup_{x \in S} B_{\rho(x)}(x).$$

Moreover, every point of $\mathbb{R}^n$ belongs to at most ξ balls $B_{\rho(x)}(x)$ centred at a point S, where ξ is a constant depending only on n.

We shall also need the following variant of the Vitali covering theorem concerning the Lebesgue measure: if $A \subset \mathbb{R}^n$, we say that $\mathcal{F}$ is a fine cover of A if for each $x \in A$ there exist balls in $\mathcal{F}$ centred at x and with arbitrarily small radii.

Theorem 2.5. *Let $A \subset \mathbb{R}^n$ be a bounded and Borel set and let $\mathcal{F}$ be a fine cover of A. Then, for any positive Radon measure μ in $\mathbb{R}^n$, there exists a disjoint family $\mathcal{F}' \subset \mathcal{F}$ such that*

$$\mu\biggl(A \setminus \bigcup_{F \in \mathcal{F}'} F\biggr) = 0.$$

3. The anisotropic Minkowski content

Let $C \subset \mathbb{R}^n$ be a convex body, i.e. a compact and convex subset of $\mathbb{R}^n$ with 0 in its interior; here and in what follows $a, b \in \mathbb{R}$ are such that $0 < a < b$ and $B_a(0) \subset C \subset B_b(0)$. We denote by h_C its support function, i.e.

$$h_C(v) := \sup_{x \in C} x \cdot v, \quad v \in \mathbb{R}^n.$$

Define, for any $S \subset \mathbb{R}^n$ closed,

$$\mathcal{M}^*_C(S) := \limsup_{\varepsilon \to 0} \frac{|S + \varepsilon C|}{2\varepsilon}, \qquad \mathcal{M}_{*C}(S) := \liminf_{\varepsilon \to 0} \frac{|S + \varepsilon C|}{2\varepsilon}.$$

If $\mathcal{M}^*_C(S) = \mathcal{M}_{*C}(S)$, their common value is denoted by $\mathcal{M}_C(S)$. As we saw in § 1, there exists a relation between $\mathcal{M}_{B_1(0)}(\partial E)$ and $\mathcal{SM}(E)$. To be more precise, let

$$\mathcal{SM}(E; \Omega) := \lim_{\varepsilon \to 0} \frac{|\{x \in \Omega\colon \operatorname{dist}(x, E) \leqslant \varepsilon\} \setminus E|}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{|(E + \varepsilon B_1(0)) \setminus E|}{\varepsilon}$$

whenever such a limit exists; of course we have $\mathcal{SM}(E) = \mathcal{SM}(E; \mathbb{R}^n)$. Hence, the following theorem holds (see [2]).

Theorem 3.1. *If E has finite perimeter in Ω and $\mathcal{M}_{B_1(0)}(\partial E) = \mathcal{P}(E; \Omega)$, then $\mathcal{SM}(E; \Omega) = \mathcal{P}(E; \Omega)$.*

Moreover, we need to give the precise statement of the main result in [7]; let

$$\mathcal{SM}_C(E; \Omega) := \lim_{\varepsilon \to 0} \frac{|(E + \varepsilon C) \cap \Omega| - |E|}{\varepsilon}.$$

Note that in this case we also have $\mathcal{SM}_C(E) = \mathcal{SM}_C(E; \mathbb{R}^n)$.

Theorem 3.2 (Chambolle *et al*. [7, theorem 3.4]). *If E has finite perimeter in Ω and $\mathcal{SM}(E; \Omega) = \mathcal{P}(E; \Omega)$, then*

$$\mathcal{SM}_C(E; \Omega) = \int_{\mathcal{FE}} h_C(\nu_E) \, \mathrm{d}\mathcal{H}^{n-1}.$$

Lemma 3.3. *Let $A \subset \mathbb{R}^{n-1}$ be open and bounded and let $f\colon A \to \mathbb{R}$ be Lipschitz continuous. Let G be the graph of f, i.e. $G := \{(x, y) \in A \times \mathbb{R} : y = f(x)\}$. Then $\mathcal{M}_C(G)$ exists and*

$$\mathcal{M}_C(G) = \int_G \phi_C(\nu_G) \, \mathrm{d}\mathcal{H}^{n-1},$$

where

$$\phi_C(v) := \frac{h_C(v) + h_C(-v)}{2} \quad \forall v \in \mathbb{R}^n.$$

Proof. Let $E := \{(x,y) \in A \times \mathbb{R} \colon y < f(x)\}$ be the subgraph of f. Since f is Lipschitz, E has finite perimeter in $A \times \mathbb{R}$ and $\mathcal{M}^{n-1}(\partial E) = \mathcal{M}_{B_1(0)}(\partial E) = \mathcal{P}(E; A \times \mathbb{R})$. Applying theorem 3.1, we deduce that also $\mathcal{SM}(E;\Omega) = \mathcal{P}(E;\Omega)$. Thus, using (2.7) and theorem 3.2 we get

$$\begin{aligned}\mathcal{M}_C(G) &= \frac{\mathcal{SM}_C(E;\Omega) + \mathcal{SM}_C(\Omega \setminus E;\Omega)}{2}\\ &= \int_{\mathcal{FE}} \frac{h_C(\nu_E) + h_C(-\nu_E)}{2}\,\mathrm{d}\mathcal{H}^{n-1}\\ &= \int_G \phi_C(\nu_G)\,\mathrm{d}\mathcal{H}^{n-1},\end{aligned}$$

which yields the conclusion. □

We are ready to prove the first main theorem of this section, which may be seen as the generalization to the anisotropic case of theorem 2.104 in [1]. Of course, the next proofs are similar to those for the classical Minkowski content in [1], which are anyway not elementary, but for the convenience of the reader we shall give details.

THEOREM 3.4. *Let $S \subset \mathbb{R}^n$ be a compact and countably $\mathcal{H}^{n-1}$-rectifiable set such that*

$$\eta(B_r(x)) \geqslant \gamma r^{n-1}$$

holds for all $x \in S$ and for all $r \in (0,1)$ for some $\gamma > 0$ and some Radon measure η on $\mathbb{R}^n$ that is absolutely continuous with respect to $\mathcal{H}^{n-1}$. Then $\mathcal{M}_C(S)$ exists and

$$\mathcal{M}_C(S) = \int_S \phi_C(\nu_S)\,\mathrm{d}\mathcal{H}^{n-1}. \tag{3.1}$$

Proof. Let $\{S_h\}_{h\in\mathbb{N}}$ be a countable family of pairwise disjoint compact subsets of S that covers S, up to a $\mathcal{H}^{n-1}$-negligible set and that is contained in some Lipschitz $(n-1)$-graph and with finite $\mathcal{H}^{n-1}$ measure. Applying lemma 3.3 and using the subadditivity of the lim inf operator, we get, for any $N \in \mathbb{N}$,

$$\begin{aligned}\mathcal{M}_{*C}(S) &\geqslant \liminf_{\varepsilon\to 0} \frac{|\bigcup_{h=1}^N (S_h + \varepsilon C)|}{2\varepsilon} \geqslant \sum_{h=0}^N \liminf_{\varepsilon\to 0} \frac{|S_h + \varepsilon C|}{2\varepsilon}\\ &= \sum_{h=0}^N \mathcal{M}_{*C}(S_h) = \sum_{h=0}^N \int_{S_h} \phi_C(\nu_S)\,\mathrm{d}\mathcal{H}^{n-1}.\end{aligned}$$

Passing to the limit as $N \to +\infty$, we find the estimate from below:

$$\mathcal{M}_{*C}(S) \geqslant \int_S \phi_C(\nu_S)\,\mathrm{d}\mathcal{H}^{n-1}. \tag{3.2}$$

The main point of the proof concerns the proof of the estimate from above, i.e.

$$\mathcal{M}^*_C(S) \leqslant \int_S \phi_C(\nu_S)\,\mathrm{d}\mathcal{H}^{n-1}. \tag{3.3}$$

The idea is to use a suitable covering argument to control what remains outside a region covered by a finite number of subsets of S, where we can say that $\mathcal{M}_C$ exists by lemma 3.3. Fix $\sigma \in (0,1)$. We can find a finite number N of pairwise disjoint compact subsets S_h of S that are contained in some Lipschitz $(n-1)$-graph, with finite $\mathcal{H}^{n-1}$-measure and such that

$$\eta(S) \leqslant \sigma + \sum_{h=1}^{N} \eta(S_h).$$

Consider the set

$$E := S \setminus \bigcup_{h=1}^{N} S_h$$

and, for any $\varepsilon \in (0,1)$,

$$S_{\sigma,\varepsilon} := \left\{ x \in S \colon \operatorname{dist}\left(x, \bigcup_{h=1}^{N} S_h \right) \geqslant \sigma^{1/n}\varepsilon \right\}.$$

Using Besicovitch's theorem (theorem 2.4) we are able to cover $S_{\sigma,\varepsilon}$ by many balls $\{B_{a\sigma^{1/n}\varepsilon}(x_j)\}_{j\in J}$ (recall that a has been chosen such that $B_a(0) \subset C$) with $x_j \in S_{\sigma,\varepsilon}$ for each $j \in J$, and such that, for ε small enough, using the assumption on η, we have

$$\sum_{j\in J} \gamma(a\sigma^{1/n}\varepsilon)^{n-1} \leqslant \sum_{j\in J} \eta(B_{a\sigma^{1/n}\varepsilon}(x_j)) \leqslant \xi\eta\left((S + \sigma^{1/n}\varepsilon C) \setminus \bigcup_{h=1}^{N} S_h \right) \leqslant \xi\sigma,$$

where ξ is as in theorem 2.4. As a consequence we get the estimate

$$\mathcal{H}^0(J) \leqslant \frac{\xi\sigma^{1/n}}{\gamma a^{n-1}\varepsilon^{n-1}}.$$

Therefore, recalling that $C \subset B_b(0)$, we obtain

$$\begin{aligned} |S_{\sigma,\varepsilon} + (1+\sigma^{1/n})\varepsilon C| \leqslant \sum_{j\in J} |B_{b(1+2\sigma^{1/n})\varepsilon}(x_j)| &\leqslant \frac{\omega_n b^n (1+2\sigma^{1/n})^n \sigma^{1/n}\xi\varepsilon}{\gamma a^{n-1}} \\ &\leqslant \frac{\omega_n b^n 3^n \sigma^{1/n}\xi\varepsilon}{\gamma a^{n-1}}. \end{aligned}$$

Now, since it holds that

$$S + \varepsilon C \subset (E + \varepsilon C) \cup \bigcup_{h=1}^{N} (S_h + \varepsilon C) \subset (S_{\sigma,\varepsilon} + (1+\sigma^{1/n})\varepsilon C) \cup \bigcup_{h=1}^{N} (S_h + \varepsilon C),$$

we deduce that, by lemma 3.3,

$$\begin{aligned} \mathcal{M}^*_C(S) &= \limsup_{\varepsilon\to 0} \frac{|S+\varepsilon C|}{2\varepsilon} \\ &\leqslant \limsup_{\varepsilon\to 0} \frac{|S_{\sigma,\varepsilon} + (1+\sigma^{1/n})\varepsilon C|}{2\varepsilon} + \sum_{h=1}^{N} \limsup_{\varepsilon\to 0} \frac{|S_h + \varepsilon C|}{2\varepsilon} \\ &\leqslant \frac{\omega_n b^n 3^n \sigma^{1/n}\xi}{2\gamma a^{n-1}} + \int_S \phi_C(\nu_S)\, \mathrm{d}\mathcal{H}^{n-1}, \end{aligned}$$

and (3.3) follows by sending $\sigma \to 0$. □

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We now move towards the case of $(n-1)$-rectifiability, as the first existence result for the Minkowski content, which can be found in [10, p. 275].

THEOREM 3.5. *If S is compact and $(n-1)$-rectifiable, then $\mathcal{M}_{B_1(0)}(S) = \mathcal{H}^{n-1}(S)$.*

The next lemma can be found in [1, lemma 2.105].

LEMMA 3.6. *Let $K \subset \mathbb{R}^{n-1}$ be a compact set and let $f\colon K \to \mathbb{R}^n$ be Lipschitz. Assume that $\boldsymbol{J}_{n-1}\,\mathrm{d}f_x = 0$ a.e. $x \in K$. Then $\mathcal{M}_{B_1(0)}(f(K)) = 0$.*

THEOREM 3.7. *Let $S \subset \mathbb{R}^n$ be compact and $(n-1)$-rectifiable. Then $\mathcal{M}_C(S)$ exists and*

$$\mathcal{M}_C(S) = \int_S \phi_C(\nu_S)\,\mathrm{d}\mathcal{H}^{n-1}.$$

Proof. The estimate from below,

$$\mathcal{M}_{*C}(S) \geqslant \int_S \phi_C(\nu_S)\,\mathrm{d}\mathcal{H}^{n-1},$$

can be proved as in theorem 3.4. Let $f\colon K \to \mathbb{R}^n$ be Lipschitz with $K \subset \mathbb{R}^{n-1}$ compact such that $S = f(K)$ and fix $\sigma \in (0,1)$. Consider the subset of K given by

$$F := \{x \in K\colon \exists \mathrm{d}f_x \text{ and } \boldsymbol{J}_{n-1}\,\mathrm{d}f_x > 0\}.$$

Let $K' \subset K \setminus F$ be compact and such that $\mathcal{L}^{n-1}(K \setminus (F \cup K')) < \sigma$, where $\mathcal{L}^{n-1}$ is the Lebesgue measure of dimension $n-1$. Moreover, let $S_0 := f(K')$. Combining theorem 3.5 with lemma 3.6, we get $\mathcal{H}^{n-1}(S_0) = 0$; thus, we obtain

$$\mathcal{M}_C(S_0) \leqslant \limsup_{\varepsilon \to 0} \frac{|S_0 + \varepsilon B_b(0)|}{2\varepsilon} = b\mathcal{M}^{n-1}(S_0) = b\mathcal{H}^{n-1}(S_0) = 0,$$

which means that $\mathcal{M}_C(S_0) = 0$. Now, consider the measure $\eta := f_\sharp(\mathcal{L}^{n-1} \llcorner F)$. By definition, η is concentrated on $f(K) = S$; moreover, if $S' \subset S$ is $\mathcal{H}^{n-1}$-negligible, then by the area formula we deduce that

$$\int_{F \cap f^{-1}(S')} \boldsymbol{J}_{n-1}\,\mathrm{d}f_x\,\mathrm{d}x = \int_{S'} \mathcal{H}^0(F \cap f^{-1}(\{y\}))\,\mathrm{d}\mathcal{H}^{n-1}(y) = 0.$$

Since, by the very definition of F, it holds that $\boldsymbol{J}_{n-1}\,\mathrm{d}f_x > 0$ on F, we get $\mathcal{L}^{n-1}(F \cap f^{-1}(S')) = 0$, which proves that η is absolutely continuous with respect to $\mathcal{H}^{n-1}$. Now we are ready to use the same covering argument as in the proof of theorem 3.4 in order to control the 'bad' part of S using the properties of the measure η. More precisely, we can find a finite number N of pairwise disjoint compact subsets S_h of S that are contained in some Lipschitz $(n-1)$-graph, with finite $\mathcal{H}^{n-1}$-measure and such that

$$\eta(S) \leqslant \sigma + \sum_{h=1}^{N} \eta(S_h).$$

Consider the set

$$E := S \setminus \bigcup_{h=0}^{N} S_h.$$

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Now, note that $\eta(E) < \sigma$ and $f^{-1}(E) \setminus F \subset K \setminus (F \cup K')$ since $E \cap S_0 = \emptyset$; we deduce that $\mathcal{L}^{n-1}(f^{-1}(E)) < 2\sigma$. If L now denotes the Lipschitz constant of f and we choose $\bar{\varepsilon} > 0$ such that $\mathcal{L}^{n-1}((K + \bar{\varepsilon}L^{-1}C) \setminus K) < \sigma$, we can consider, for any $\varepsilon \in (0, \bar{\varepsilon})$, the set

$$S_{\sigma,\varepsilon} := \left\{x \in S\colon \operatorname{dist}\left(x, \bigcup_{h=0}^{N} S_h\right) \geqslant \sigma^{1/n}\varepsilon\right\}.$$

Applying Besicovitch's theorem, we are able to cover $S_{\sigma,\varepsilon}$ by many balls

$$\{B_{a\sigma^{1/n}\varepsilon}(x_j)\}_{j\in J}$$

centred at points of $S_{\sigma,\varepsilon}$ and such that, for ε small enough,

$$\begin{aligned}\sum_{j\in J}\omega_{n-1}(aL^{-1}\sigma^{1/n}\varepsilon)^{n-1} &\leqslant \sum_{j\in J}\mathcal{L}^{n-1}((K+\varepsilon L^{-1}C)\cap f^{-1}(B_{a\sigma^{1/n}\varepsilon}(x_j)))\\ &\leqslant \xi\mathcal{L}^{n-1}\left((K+\varepsilon L^{-1}C)\cap f^{-1}\left(\bigcup_{j\in J}B_{a\sigma^{1/n}\varepsilon}(x_j)\right)\right)\\ &\leqslant \xi(\mathcal{L}^{n-1}(f^{-1}(E)) + \mathcal{L}^{n-1}((K+\bar{\varepsilon}L^{-1}C)\setminus K))\\ &\leqslant 3\xi\sigma,\end{aligned}$$

where ξ is as in Besicovitch's theorem. Therefore,

$$\mathcal{H}^0(J) \leqslant \frac{3\xi\sigma^{1/n}}{\omega_{n-1}a^{n-1}L^{1-n}\varepsilon^{n-1}};$$

hence,

$$\begin{aligned}|S_{\sigma,\varepsilon} + (1+\sigma^{1/n})\varepsilon C| &\leqslant \sum_{j\in J}|B_{b(1+2\sigma^{1/n})\varepsilon}(x_j)|\\ &\leqslant \frac{\omega_n b^n(1+2\sigma^{1/n})^n\sigma^{1/n}3\xi\varepsilon}{\omega_{n-1}a^{n-1}L^{1-n}}\\ &\leqslant \frac{\omega_n b^n 3^{n+1}\sigma^{1/n}\xi\varepsilon}{\omega_{n-1}a^{n-1}L^{1-n}}.\end{aligned}$$

Using

$$S+\varepsilon C \subset (E+\varepsilon C)\cup\bigcup_{h=0}^{N}(S_h+\varepsilon C) \subset (S_{\sigma,\varepsilon}+(1+\sigma^{1/n})\varepsilon C)\cup\bigcup_{h=0}^{N}(S_h+\varepsilon C)$$

again, we deduce that, by lemma 3.3,

$$\begin{aligned}\mathcal{M}^*_C(S) &= \limsup_{\varepsilon\to 0}\frac{|S+\varepsilon C|}{2\varepsilon}\\ &\leqslant \limsup_{\varepsilon\to 0}\frac{|S_{\sigma,\varepsilon}+(1+\sigma^{1/n})\varepsilon C|}{2\varepsilon} + \sum_{h=0}^{N}\limsup_{\varepsilon\to 0}\frac{|S_h+\varepsilon C|}{2\varepsilon}\\ &\leqslant \frac{\omega_n b^n 3^{n+1}\sigma^{1/n}\xi}{2\omega_{n-1}a^{n-1}L^{1-n}} + \int_S \phi_C(\nu_S)\,\mathrm{d}\mathcal{H}^{n-1}\end{aligned}$$

and the conclusion follows. □

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4. A more general formula for $\mathcal{SM}_C$

In this section we prove the generalization of (4.1). First, let us introduce the classes $\mathcal{O}$ and $\mathcal{O}'_C$.

DEFINITION 4.1. Let $\mathcal{O}$ be the class of Borel sets E of $\mathbb{R}^n$ such that

(i) ∂E is a countably $\mathcal{H}^{n-1}$-rectifiable bounded set,

(ii) there exist $\gamma > 0$ and a probability measure η in $\mathbb{R}^n$ absolutely continuous with respect to $\mathcal{H}^{n-1}$ such that $\eta(B_r(x)) \geqslant \gamma r^{n-1}$ for all $x \in \partial E$ and for all $r \in (0, 1)$.

Moreover, let $\mathcal{O}'_C$ be the class of Borel sets E of $\mathbb{R}^n$ such that

(i′) ∂E is a countably $\mathcal{H}^{n-1}$-rectifiable bounded set and

$$\mathcal{M}_C(\partial E) = \int_{\partial E} \phi_C(\nu_{\partial E})\,\mathrm{d}\mathcal{H}^{n-1},$$

(ii′) there exist $\gamma > 0$ and a probability measure η in $\mathbb{R}^n$ such that $\eta(B_r(x)) \geqslant \gamma r^{n-1}$ for all $x \in \partial E$ and for all $r \in (0, 1)$.

REMARK 4.2. Condition (ii′), and therefore also condition (ii), implies, by theorem 2.1, that $\mathcal{H}^{n-1}(\partial E)$ is finite; in particular, any set in $\mathcal{O}$ or $\mathcal{O}'_C$ has a finite perimeter.

We now recall the main result of [16].

THEOREM 4.3 (Villa [16, theorem 3.1]). *The classes $\mathcal{O}$ and $\mathcal{O}'_{B_1(0)}$ are stable under finite unions, and, for any $E \in \mathcal{O}$ (or $\mathcal{O}'_{B_1(0)}$), it holds that*

$$\mathcal{SM}(E) = \mathcal{P}(E) + 2\mathcal{H}^{n-1}(\partial E \cap E^0). \tag{4.1}$$

Now we are ready to state the main result of this section.

THEOREM 4.4. *The classes $\mathcal{O}$ and $\mathcal{O}'_C$ are stable under finite unions, and, for any $E \in \mathcal{O}$ (or $\mathcal{O}'_C$), it holds that*

$$\mathcal{SM}_C(E) = \int_{\mathcal{F}E} h_C(\nu_E)\,\mathrm{d}\mathcal{H}^{n-1} + 2\int_{\partial^2 E} \phi_C(\nu_E)\,\mathrm{d}\mathcal{H}^{n-1}. \tag{4.2}$$

REMARK 4.5. If $E \in \mathcal{O}$ (or $E \in \mathcal{O}'_C$) is such that $\mathcal{SM}(E) = \mathcal{P}(E)$, then by (4.1) it follows that $\mathcal{H}^{n-1}(E^0 \cap \partial E) = 0$, and so $\mathcal{H}^{n-1}(\partial^2 E) = 0$. As a consequence, from (4.2) we get

$$\mathcal{SM}_C(E) = \int_{\mathcal{F}E} h_C(\nu_E)\,\mathrm{d}\mathcal{H}^{n-1},$$

in accordance with theorem 3.2. Moreover, if

$$\frac{\mathcal{SM}(E) + \mathcal{SM}(E^{\mathrm{c}})}{2} = \mathcal{P}(E),$$

then by (4.1) it follows that $\mathcal{H}^{n-1}(E^0 \cap \partial E) + \mathcal{H}^{n-1}(E^1 \cap \partial E) = 0$, and so

$$\mathcal{H}^{n-1}(\partial^2 E) = \mathcal{H}^{n-1}(\partial^2 E^{\mathrm{c}}) = 0.$$

In this case, again as a consequence of (4.2), we get

$$\begin{aligned}\frac{\mathcal{SM}_C(E) + \mathcal{SM}_C(E^{\mathrm{c}})}{2} &= \frac{1}{2}\biggl(\int_{\mathcal{F}E} h_C(\nu_E)\,\mathrm{d}\mathcal{H}^{n-1} + \int_{\mathcal{F}E^{\mathrm{c}}} h_C(\nu_{E^{\mathrm{c}}})\,\mathrm{d}\mathcal{H}^{n-1}\biggr) \\ &= \int_{\mathcal{F}E} \frac{h_C(\nu_E) + h_C(-\nu_E)}{2}\,\mathrm{d}\mathcal{H}^{n-1},\end{aligned}$$

in accordance with [7, theorem 3.7].

We shall now prove theorem 4.4 by using the arguments in the proof of [16, theorem 3.1]; for the reader's convenience we shall also give complete proofs for the series of auxiliary lemmas that we need. First, it is easy to observe that $E \cap W \in \mathcal{O}$ for any $E \in \mathcal{O}$ and any closed $W \subset \mathbb{R}^n$; the following lemma implies that the same also holds for the class $\mathcal{O}'_C$.

Lemma 4.6. *If $S \subset \mathbb{R}^n$ is a countably $\mathcal{H}^{n-1}$-rectifiable compact set such that*

$$\mathcal{M}_C(S) = \int_S \phi_C(\nu_S)\,\mathrm{d}\mathcal{H}^{n-1},$$

then

$$\mathcal{M}_C(S \cap W) = \int_{S\cap W} \phi_C(\nu_S)\,\mathrm{d}\mathcal{H}^{n-1}$$

for all $W \subset \mathbb{R}^n$ closed.

Proof. Since $S \cap W$ is countably $\mathcal{H}^{n-1}$-rectifiable and compact, by (3.2) we know that

$$\mathcal{M}_{*C}(S \cap W) \geqslant \int_{S\cap W} \phi_C(\nu_S)\,\mathrm{d}\mathcal{H}^{n-1}.$$

Let us show that the opposite inequality holds for $\mathcal{M}^*_C(S \cap W)$. Consider the sequence $\{W_h\}_{h\in\mathbb{N}}$ of closed sets $W_h := \{x \in W^{\mathrm{c}} \colon \operatorname{dist}(x, W) \geqslant b/h\}$, which implies that

$$\left(x + \frac{1}{h}C\right) \cap W = \emptyset$$

for any $x \in W^{\mathrm{c}}$, since

$$x + \frac{1}{h}C \subseteq B_{b/h}(x).$$

Note that $W_h \nearrow W^{\mathrm{c}}$ as h goes to infinity. Let us observe that

$$S + \varepsilon C \supseteq ((S \cap W) + \varepsilon C) \cup ((S \cap W_h) + \varepsilon C)$$

and

$$((S \cap W) + \varepsilon C) \cap ((S \cap W_h) + \varepsilon C) = \emptyset$$

for all ε sufficiently small. Hence, for all $h \in \mathbb{N}$,

$$\mathcal{M}^*_C(S\cap W) \leqslant \mathcal{M}^*_C(S) - \mathcal{M}_{*C}(S\cap W_h) \leqslant \int_S \phi_C(\nu_S)\,\mathrm{d}\mathcal{H}^{n-1} - \int_{S\cap W_h} \phi_C(\nu_S)\,\mathrm{d}\mathcal{H}^{n-1}.$$

Now, by taking the limit for h, which goes to infinity, we get

$$\mathcal{M}^*_C(S\cap W) \leqslant \int_{S\cap W} \phi_C(\nu_S)\,\mathrm{d}\mathcal{H}^{n-1}$$

and so prove the assertion. □

Observing that for each $G \subset \mathbb{R}^n$ Borel set and for each $r, \rho > 0$ it holds that

$$|(\partial G + rC) \cap G \cap B_\rho(x)| \subset |(\partial G + rbB_1(0)) \cap G \cap B_\rho(x)|,$$

the following assertion is a direct application of [2, lemma 2].

LEMMA 4.7. *Let $G \subset \mathbb{R}^n$ be a Borel set and assume that there exist $\gamma > 0$ and a probability measure η on $\mathbb{R}^n$ such that $\eta(B_r(x)) \geqslant \gamma r^{n-1}$ for all $x \in \partial G$ and for all $r \in (0,1)$. Then*

$$\limsup_{\varepsilon\to 0} \frac{|(\partial G + \varepsilon C) \cap G \cap B_\rho(x)|}{\varepsilon} = o(\rho^{n-1})$$

for $\mathcal{H}^{n-1}$-a.e. $x \in G^0 \cap \partial G$.

For each Borel subset A of $\mathbb{R}^n$ let $\mathcal{SM}_{*C}(E;A)$ and $\mathcal{SM}^*_C(E;A)$ be given by

$$\mathcal{SM}_{*C}(E;A) := \liminf_{\varepsilon\to 0} \frac{|((E+\varepsilon C)\setminus E)\cap A|}{\varepsilon},$$
$$\mathcal{SM}^*_C(E;A) := \limsup_{\varepsilon\to 0} \frac{|((E+\varepsilon C)\setminus E)\cap A|}{\varepsilon}.$$

We also let $\mathcal{SM}_{*C}(E) := \mathcal{SM}_{*C}(E;\mathbb{R}^n)$ and $\mathcal{SM}^*_C(E) := \mathcal{SM}^*_C(E;\mathbb{R}^n)$.

LEMMA 4.8. *For any $E \in \mathcal{O}$ (or $\mathcal{O}'_C$), the following hold:*

$$\mathcal{SM}_{*C}(E;B_\rho(x)) = o(\rho^{n-1}) \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in E^1 \cap \partial E, \tag{4.3}$$

$$\mathcal{SM}_{*C}(E;B_\rho(x)) \geqslant \int_{\mathcal{F}E\cap \operatorname{int} B_\rho(x)} h_C(\nu_E)\,\mathrm{d}\mathcal{H}^{n-1} \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in E^{1/2}, \tag{4.4}$$

$$\mathcal{SM}_{*C}(E;B_\rho(x)) \geqslant 2\int_{\partial^2 E\cap \operatorname{int} B_\rho(x)} \phi_C(\nu_E)\,\mathrm{d}\mathcal{H}^{n-1} + o(\rho^{n-1}) \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in \partial^2 E. \tag{4.5}$$

Proof. Equality (4.3) follows directly from lemma 4.7 with the choice $G := E^{\mathrm{c}}$, and by taking into account that

$$|\partial E + \varepsilon C| = |(E+\varepsilon C)\setminus E| + |(E^{\mathrm{c}}+\varepsilon C)\setminus E^{\mathrm{c}}|. \tag{4.6}$$

Equality (4.4) can be found in [7]. It remains to prove (4.5). Since $0 \in \operatorname{int} C$, for any closed set $W \subset\subset B_\rho(x)$ there exists $\tilde\varepsilon > 0$ such that $W + \varepsilon C \subset B_\rho(x)$ for all $\varepsilon < \tilde\varepsilon$. So, noting that $\partial E \cap W$ satisfies the assumption of theorem 3.4 if $E \in \mathcal{O}$ (or

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(i′) in the definition of the class $\mathcal{O}'_C$ if $E \in \mathcal{O}'_C$), we get that

$$2\int_{\partial E\cap W} \phi_C(\nu_E)\,\mathrm{d}\mathcal{H}^{n-1} = \liminf_{\varepsilon\to 0} \frac{|(\partial E\cap W)+\varepsilon C|}{\varepsilon} \leqslant \liminf_{\varepsilon\to 0} \frac{|(\partial E+\varepsilon C)\cap(W+\varepsilon C)|}{\varepsilon} \leqslant \liminf_{\varepsilon\to 0} \frac{|(\partial E+\varepsilon C)\cap B_\rho(x)|}{\varepsilon}.$$

Let $\{W_k\}_{k\in\mathbb{N}}$ be an increasing sequence of closed sets with $W_k \subset\subset B_\rho(x)$ and such that $W_k \nearrow \operatorname{int} B_\rho(x)$. By taking the limit as k tends to ∞, we obtain that

$$\liminf_{\varepsilon\to 0} \frac{|(\partial E+\varepsilon C)\cap B_\rho(x)|}{\varepsilon} \geqslant 2\lim_{k\to\infty}\int_{\partial E\cap W_k} \phi_C(\nu_E)\,\mathrm{d}\mathcal{H}^{n-1} = 2\int_{\partial E\cap \operatorname{int} B_\rho(x)} \phi_C(\nu_E)\,\mathrm{d}\mathcal{H}^{n-1}. \tag{4.7}$$

Finally, we have that

$$\mathcal{SM}_{*C}(E;B_\rho(x)) = \liminf_{\varepsilon\to 0} \frac{|(\partial E+\varepsilon C)\cap B_\rho(x)| - |(\partial E+\varepsilon C)\cap E\cap B_\rho(x)|}{\varepsilon} \geqslant \liminf_{\varepsilon\to 0} \frac{|(\partial E+\varepsilon C)\cap B_\rho(x)|}{\varepsilon} - \limsup_{\varepsilon\to 0} \frac{|(\partial E+\varepsilon C)\cap E\cap B_\rho(x)|}{\varepsilon}.$$

Thus (4.5) follows from (4.7), lemma 4.7 and by taking into account that

$$\mathcal{H}^{n-1}(\partial E\cap E^0) = \mathcal{H}^{n-1}(\partial^2 E).$$

□

Now we are ready to prove the main result of this section.

Proof of theorem 4.4. Let $E \in \mathcal{O}$ (or $E \in \mathcal{O}'_C$). Let us show that the following lower bound for $\mathcal{SM}_{*C}(E)$ holds:

$$\mathcal{SM}_{*C}(E) \geqslant \int_{\mathcal{F}E} h_C(\nu_E)\,\mathrm{d}\mathcal{H}^{n-1} + 2\int_{\partial^2 E} \phi_C(\nu_E)\,\mathrm{d}\mathcal{H}^{n-1}. \tag{4.8}$$

Let μ be the measure in $\mathbb{R}^n$ defined by

$$\mu(A) := \int_{\mathcal{F}E\cap A} h_C(\nu_E)\,\mathrm{d}\mathcal{H}^{n-1} + 2\int_{\partial^2 E\cap A} \phi_C(\nu_E)\,\mathrm{d}\mathcal{H}^{n-1} \quad A\subset\mathbb{R}^n \text{ Borel}.$$

By rectifiability we can say that

$$\lim_{\rho\to 0} \frac{\mathcal{H}^{n-1}(\partial E\cap E^0\cap B_\rho(x))}{\rho^{n-1}} = \begin{cases} \omega_{n-1} & \text{for } \mathcal{H}^{n-1}\text{-a.e. } x\in\partial E\cap E^0, \\ 0 & \text{for } \mathcal{H}^{n-1}\text{-a.e. } x\in(\partial E\cap E^0)^{\mathrm{c}}; \end{cases}$$

the same conclusions hold for the quantities

$$\frac{\mathcal{H}^{n-1}(\partial E\cap E^1\cap B_\rho(x))}{\rho^{n-1}} \quad\text{and}\quad \frac{\mathcal{H}^{n-1}(\mathcal{F}E\cap B_\rho(x))}{\rho^{n-1}}.$$

Taking into account all this information and using lemma 4.8 we get that, for any $\varepsilon > 0$ and for $\mathcal{H}^{n-1}$-a.e. $x \in \partial E$,

$$\liminf_{\rho\to 0} \frac{\mathcal{SM}_{*C}(E; B_\rho(x)) + \varepsilon\mathcal{H}^{n-1}(E^1 \cap \partial E \cap B_\rho(x))}{\mu(\operatorname{int} B_\rho(x))} \geqslant 1. \tag{4.9}$$

(Note that the term $\varepsilon\mathcal{H}^{n-1}(E^1 \cap \partial E \cap B_\rho(x))$ in the above fraction is to avoid an indetermination of type $0/0$ at points $x \in E^1 \cap \partial E$.) Since the family of closed balls $B_\rho(x)$ with $\mu(\partial B_\rho(x)) = 0$ is a fine cover of ∂E, by the Vitali–Besicovitch covering theorem (theorem 2.5) for any $\delta > 0$ there exist finitely many disjoint closed balls $W_1, \dots, W_N$ with $\mu(\partial W_i) = 0$ such that

$$\mu\biggl(\partial E \setminus \bigcup_{i=1}^{N} W_i\biggr) < \delta.$$

The balls W_i can be chosen with centres in ∂E and such that

$$\mathcal{SM}_{*C}(E; W_i) + \varepsilon\mathcal{H}^{n-1}(E^1 \cap \partial E \cap W_i) \overset{(4.9)}{\geqslant} (1-\delta)\mu(W_i), \quad i = 1, \dots, N.$$

Then, the following chain of inequalities holds:

$$\begin{aligned}
&\mathcal{SM}_{*C}(E) + \varepsilon\mathcal{H}^{n-1}(E^1 \cap \partial E) \\
&\qquad \geqslant \mathcal{SM}_{*C}\biggl(E; \bigcup_{i=1}^{N} W_i\biggr) + \varepsilon\mathcal{H}^{n-1}\biggl(E^1 \cap \partial E \cap \bigcup_{i=1}^{N} W_i\biggr) \\
&\qquad \geqslant \sum_{i=1}^{N} (\mathcal{SM}_{*C}(E; W_i) + \varepsilon\mathcal{H}^{n-1}(E^1 \cap \partial E \cap W_i)) \\
&\qquad \geqslant (1-\delta) \sum_{i=1}^{N} \mu(W_i) = (1-\delta)\biggl(\mu(\mathbb{R}^n) - \mu\biggl(\mathbb{R}^n \setminus \bigcup_{i=1}^{N} W_i\biggr)\biggr) \\
&\qquad \geqslant (1-\delta)\biggl(\int_{\mathcal{F}E} h_C(\nu_E)\,\mathrm{d}\mathcal{H}^{n-1} + 2\int_{\partial^2 E} \phi_C(\nu_E)\,\mathrm{d}\mathcal{H}^{n-1} - \delta\biggr).
\end{aligned}$$

By taking the limit first as $\delta \to 0$ and then as $\varepsilon \to 0$, we obtain the inequality (4.8).

Observing now that E^{c} also belongs to $\mathcal{O}$ (respectively, $\mathcal{O}'_C$), we can also claim that

$$\mathcal{SM}_{*C}(E^{\mathrm{c}}) \geqslant \int_{\mathcal{F}\mathcal{E}^{\mathrm{c}}} h_C(\nu_{E^{\mathrm{c}}})\,\mathrm{d}\mathcal{H}^{n-1} + \int_{\partial^2 E^{\mathrm{c}}} \phi_C(\nu_{E^{\mathrm{c}}})\,\mathrm{d}\mathcal{H}^{n-1}. \tag{4.10}$$

Let us now define, for any $\varepsilon > 0$,

$$a_\varepsilon := \frac{|(E + \varepsilon C) \setminus E|}{\varepsilon}, \qquad b_\varepsilon := \frac{|(E^{\mathrm{c}} + \varepsilon C) \setminus E^{\mathrm{c}}|}{\varepsilon}.$$

Observe that, by taking into account (2.6) for $\mathcal{H}^{n-1}$-a.e. $x \in \partial E \setminus \mathcal{F}E$,

$$\int_{\partial E \setminus \mathcal{F}E} \phi_C(\nu_E)\,\mathrm{d}\mathcal{H}^{n-1} = \int_{\partial^2 E} \phi_C(\nu_E)\,\mathrm{d}\mathcal{H}^{n-1} + \int_{\partial^2 E^{\mathrm{c}}} \phi_C(\nu_{E^{\mathrm{c}}})\,\mathrm{d}\mathcal{H}^{n-1}. \tag{4.11}$$

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Moreover, it holds that

$$\liminf_{\varepsilon\to 0} a_\varepsilon = \mathcal{SM}_{*C}(E) \overset{(4.8)}{\geqslant} \int_{\mathcal{F}E} h_C(\nu_E)\,\mathrm{d}\mathcal{H}^{n-1} + 2\int_{\partial^2 E}\phi_C(\nu_E)\,\mathrm{d}\mathcal{H}^{n-1} =: a,$$

and, using also (2.5) and (2.7),

$$\begin{aligned}\liminf_{\varepsilon\to 0} b_\varepsilon &= \mathcal{SM}_{*C}(E^{\mathrm{c}})\\ &\overset{(4.10)}{\geqslant} \int_{\mathcal{F}E} h_C(-\nu_E)\,\mathrm{d}\mathcal{H}^{n-1} + 2\int_{\partial^2 E^{\mathrm{c}}}\phi_C(\nu_{E^{\mathrm{c}}})\,\mathrm{d}\mathcal{H}^{n-1} =: b.\end{aligned}$$

By (4.6) and by (3.1) if $E \in \mathcal{O}$ (respectively, by (i′) in the definition of the class $\mathcal{O}'_C$ if $E \in \mathcal{O}'_C$) it follows that

$$\begin{aligned}\limsup_{\varepsilon\to 0}(a_\varepsilon + b_\varepsilon) &= \limsup_{\varepsilon\to 0}\frac{|\partial E + \varepsilon C|}{\varepsilon}\\ &= 2\mathcal{M}_C(\partial E) = 2\int_{\partial E}\phi_C(\nu_{\partial E})\,\mathrm{d}\mathcal{H}^{n-1}\\ &= 2\int_{\partial E\cap\mathcal{F}E}\phi_C(\nu_E)\,\mathrm{d}\mathcal{H}^{n-1} + 2\int_{\partial E\setminus\mathcal{F}E}\phi_C(\nu_E)\,\mathrm{d}\mathcal{H}^{n-1}\\ &\overset{(4.11)}{=} \int_{\mathcal{F}E}(h_C(\nu_E) + h_C(-\nu_E))\,\mathrm{d}\mathcal{H}^{n-1}\\ &\qquad + 2\int_{\partial^2 E}\phi_C(\nu_E)\,\mathrm{d}\mathcal{H}^{n-1} + 2\int_{\partial^2 E^{\mathrm{c}}}\phi_C(\nu_{E^{\mathrm{c}}})\,\mathrm{d}\mathcal{H}^{n-1}\\ &= a + b.\end{aligned}$$

Since

$$\limsup_{\varepsilon\to 0}(a_\varepsilon + b_\varepsilon) \leqslant a + b, \quad \liminf_{\varepsilon\to 0} a_\varepsilon \geqslant a \in \mathbb{R} \quad\text{and}\quad \liminf_{\varepsilon\to 0} b_\varepsilon \geqslant b \in \mathbb{R}$$

imply $a_\varepsilon \to a$ and $b_\varepsilon \to b$, (4.2) follows.

To conclude the proof it remains only to show that the class $\mathcal{O}'_C$ is stable under finite unions, since the stability of the class $\mathcal{O}$ under finite unions has already been proved in [16]. Let $E_1, E_2 \in \mathcal{O}'_C$ and let $E := E_1 \cup E_2$. As $\partial E \subseteq \partial E_1 \cup \partial E_2$, it is clear that ∂E is a countably $\mathcal{H}^{n-1}$-rectifiable bounded set, and that (ii′) in the definition of the class $\mathcal{O}'_C$ is fulfilled. We know that, from (3.2),

$$\mathcal{M}_{*C}(\partial E) \geqslant \int_{\partial E}\phi_C(\nu_E)\,\mathrm{d}\mathcal{H}^{n-1}. \tag{4.12}$$

Next, we have to prove that

$$\mathcal{M}^*_C(\partial E) \leqslant \int_{\partial E}\phi_C(\nu_E)\,\mathrm{d}\mathcal{H}^{n-1}. \tag{4.13}$$

We first localize the lower and upper anisotropic Minkowski content: if S is compact and $\mathcal{H}^{n-1}$-rectifiable, $A \subset \mathbb{R}^n$ is closed and $B \subset \mathbb{R}^n$ is open, let

$$\mathcal{M}^*_C(S; A) := \limsup_{\varepsilon\to 0}\frac{|(S + \varepsilon C)\cap A|}{2\varepsilon}, \qquad \mathcal{M}_{*C}(S; B) := \liminf_{\varepsilon\to 0}\frac{|(S + \varepsilon C)\cap B|}{2\varepsilon}.$$

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Of course, we obtain $\mathcal{M}^*_C(S;\mathbb{R}^n) = \mathcal{M}^*_C(S)$ and $\mathcal{M}_{*C}(S;\mathbb{R}^n) = \mathcal{M}_{*C}(S)$; furthermore, following the proof of (3.2), we can see that, for any $B \subset \mathbb{R}^n$ open, the following holds:

$$\mathcal{M}_{*C}(S;B) \geqslant \int_{S\cap B} \phi_C(\nu_S)\,\mathrm{d}\mathcal{H}^{n-1}. \tag{4.14}$$

By observing that $\chi_{\partial E+\varepsilon C} + \chi_{(\partial E_1\cap\partial E_2)+\varepsilon C} \leqslant \chi_{\partial E_1+\varepsilon C} + \chi_{\partial E_2+\varepsilon C}$ and using (4.14), we get, for any $x \in \mathbb{R}^n$ and for $\mathcal{H}^1$-a.e. $\rho > 0$,

$$\begin{aligned}\mathcal{M}^*_C(\partial E;B_\rho(x)) &\leqslant \int_{\partial E_1\cap B_\rho(x)} \phi_C(\nu_{E_1})\,\mathrm{d}\mathcal{H}^{n-1} + \int_{\partial E_2\cap B_\rho(x)} \phi_C(\nu_{E_2})\,\mathrm{d}\mathcal{H}^{n-1}\\ &\qquad - \int_{\partial E_1\cap\partial E_2\cap \operatorname{int} B_\rho(x)} \phi_C(\nu_{E_1\cap E_2})\,\mathrm{d}\mathcal{H}^{n-1}\\ &= \int_{(\partial E_1\cup\partial E_2)\cap B_\rho(x)} \phi_C(\nu_{E_1\cup E_2})\,\mathrm{d}\mathcal{H}^{n-1}\\ &\qquad + \int_{\partial E_1\cap\partial E_2\cap B_\rho(x)} \phi_C(\nu_{E_1\cap E_2})\,\mathrm{d}\mathcal{H}^{n-1}\end{aligned} \tag{4.15}$$

and thus

$$\begin{aligned}\mathcal{M}^*_C(\partial E;B_\rho(x)) &\leqslant \int_{\partial E\cap B_\rho(x)} \phi_C(\nu_E)\,\mathrm{d}\mathcal{H}^{n-1}\\ &\qquad + \int_{(\partial E_1\cup\partial E_2)\cap(\partial E)^c\cap B_\rho(x)} \phi_C(\nu_E)\,\mathrm{d}\mathcal{H}^{n-1}\\ &\qquad + \int_{\partial E_1\cap\partial E_2\cap\partial B_\rho(x)} \phi_C(\nu_{E_1\cap E_2})\,\mathrm{d}\mathcal{H}^{n-1}.\end{aligned} \tag{4.16}$$

We note that now, for $\mathcal{H}^{n-1}$-a.e. $x \in \partial E$,

$$\int_{(\partial E_1\cup\partial E_2)\cap(\partial E)^c\cap B_\rho(x)} \phi_C(\nu_{E_1\cup E_2})\,\mathrm{d}\mathcal{H}^{n-1} = o(\rho^{n-1}),$$

since we can apply (2.1) to the Radon measure η given by

$$\eta(D) := \int_{(\partial E_1\cup\partial E_2)\cap(\partial E)^c\cap D} \phi_C(\nu_{E_1\cup E_2})\,\mathrm{d}\mathcal{H}^{n-1}, \quad D \text{ Borel in } \mathbb{R}^n.$$

Moreover, observe that $\mathcal{H}^{n-1}(\partial E_1\cap\partial E_2\cap\partial B_\rho(x)) = 0$ for $\mathcal{H}^1$-a.e. $\rho > 0$. Indeed, if for any $\rho \in A$ with $\mathcal{L}^1(A) > 0$ we had $\mathcal{H}^{n-1}(\partial E_1\cap\partial E_2\cap\partial B_\rho(x)) > 0$, then, by the coarea formula,

$$\begin{aligned}\mathcal{L}^n(\partial E_1\cap\partial E_2) &= \int_0^{+\infty} \mathcal{H}^{n-1}(\partial E_1\cap\partial E_2\cap\partial B_\rho(x))\,\mathrm{d}\rho\\ &\geqslant \int_A \mathcal{H}^{n-1}(\partial E_1\cap\partial E_2\cap\partial B_\rho(x))\,\mathrm{d}\rho\end{aligned}$$

and therefore $\mathcal{L}^n(\partial E_1 \cap \partial E_2) > 0$, which implies that $\mathcal{H}^{n-1}(\partial E_1 \cap \partial E_2) = +\infty$, which is a contradiction (see remark 4.2). Thus,

$$\int_{\partial E_1 \cap \partial E_2 \cap \partial B_\rho(x)} \phi_C(\nu_{E_1 \cap E_2}) \, \mathrm{d}\mathcal{H}^{n-1} = 0, \quad \mathcal{H}^1\text{-a.e. } \rho > 0.$$

From this we obtain, by (4.16), the key estimate

$$\mathcal{M}_C^*(\partial E; B_\rho(x)) \leqslant \int_{\partial E \cap B_\rho(x)} \phi_C(\nu_E) \, \mathrm{d}\mathcal{H}^{n-1} + o(\rho^{n-1}) \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in \partial E. \tag{4.17}$$

Now the assertion follows on applying theorem 2.5. For any $D \subset \mathbb{R}^n$ Borel, let

$$\sigma(D) := \int_{\partial E \cap D} \phi_C(\nu_E) \, \mathrm{d}\mathcal{H}^{n-1}.$$

As a consequence of (4.17) and (2.2) we may claim that

$$\limsup_{\rho \to 0} \frac{\mathcal{M}_C^*(\partial E; B_\rho(x))}{\sigma(B_\rho(x))} \leqslant 1 \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in \partial E.$$

Since ∂E is bounded, for any $\delta > 0$ there exists a finite covering $B_1, \dots, B_N$ of ∂E, where B_i are disjoint closed balls in $\mathbb{R}^n$ with

$$\sigma\left(\partial E \setminus \bigcup_{i=1}^N B_i\right) < \delta;$$

note that the balls B_i can be assumed to have centres in ∂E, such that $\sigma(\partial B_i) = 0$ and

$$\frac{\mathcal{M}_C^*(\partial E; B_i)}{\sigma(B_i)} \leqslant 1 + \delta.$$

Finally, let $B := \mathbb{R}^n \setminus \bigcup_{i=1}^N \operatorname{int} B_i$. Note that $\mathcal{M}_C^*(\partial E; B) = 0$, and thus

$$\begin{aligned} \mathcal{M}_C^*(\partial E) &\leqslant \mathcal{M}_C^*(\partial E, B) + \mathcal{M}_C^*\left(\partial E, \bigcup_{i=1}^N B_i\right) \\ &\leqslant \sum_{i=1}^N \mathcal{M}_C^*(\partial E, B_i) \leqslant (1+\delta) \sum_{i=1}^N \sigma(B_i) \\ &= (1+\delta)\sigma\left(\bigcup_{i=1}^N B_i\right) \leqslant (1+\delta) \int_{\partial E} \phi_C(\nu_E) \, \mathrm{d}\mathcal{H}^{n-1}. \end{aligned}$$

Inequality (4.13) follows by sending $\delta \to 0$, and this completes the proof. □

Acknowledgements

The authors were supported by Project GNAMPA 2014 'Perimetri anisotropi e ottimizzazione di forma' of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni of the Istituto Nazionale di Alta Matematica (INdAM, Rome). They thank the anonymous referee for valuable comments and suggestions to improve the readability of the paper, and Luigi Ambrosio (Scuola Normale Superiore, Pisa) for useful discussions and suggestions.

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