Glasgow Math. J. **63** (2021) 245–257. © Glasgow Mathematical Journal Trust 2020.
doi:10.1017/S0017089520000130.

FOUR-MANIFOLDS WITH POSITIVE CURVATURE

R. DIÓGENES

UNILAB, Instituto de Ciências Exatas e da Natureza, Campus dos Palmares, ROD. CE 060, KM 51, 62.785-000 Acarape, CE, Brazil
e-mail: rafaeldiogenes@unilab.edu.br

E. RIBEIRO

Departamento de Matemática, Universidade Federal do Ceará - UFC, CAMPUS do Pici, Av. Humberto Monte, Bloco 914, 60455-760 Fortaleza, CE, Brazil
e-mail: ernani@mat.ufc.br

and E. RUFINO

Departamento de Matemática, Universidade Federal de Roraima - UFRR, Campus Paricarana, Av. CAP. Ene Garcez, 2413, 69310-000 Boa Vista, RR, Brazil
e-mail: elzimar.rufino@ufrr.br

(Received 5 December 2018; revised 23 January 2020; accepted 13 March 2020;
first published online 6 April 2020)

Abstract. In this note, we prove that a four-dimensional compact oriented half-conformally flat Riemannian manifold M^4 is topologically $\mathbb{S}^4$ or $\mathbb{CP}^2$, provided that the sectional curvatures all lie in the interval $[\frac{3\sqrt{3}-5}{4}, 1]$. In addition, we use the notion of biorthogonal (sectional) curvature to obtain a pinching condition which guarantees that a four-dimensional compact manifold is homeomorphic to a connected sum of copies of the complex projective plane or the 4-sphere.

2010 *Mathematics Subject Classification*. Primary 53C25, 53C20, 53C21; Secondary 53C65

1. Introduction.

A classical topic in Riemannian geometry is to study manifolds with positive sectional curvature. The sectional curvature is the most natural generalization to higher dimensions of the Gaussian curvature of a surface, given that it controls the behavior of geodesics. However, very few topological obstructions to positive sectional curvature are known, and many conjectures about this subject remain open, as, for example, the Hopf conjecture on $\mathbb{S}^2 \times \mathbb{S}^2$, which is one of the oldest conjectures in global Riemannian geometry.

A compact (without boundary) Riemannian manifold (M^n, g) is said δ-pinched if the sectional curvature K satisfies

$$1 \geq K \geq \delta. \tag{1.1}$$

If the strict inequality holds, we say that M^n is strictly δ-pinched.

The notion of curvature pinching was introduced by Rauch [25] in 1951. In considering this notion, Rauch was able to show that a compact simply connected Riemannian

E. Ribeiro Jr. was partially supported by grants from CNPq/Brazil (Grant: 303091/2015-0), PRONEX-FUNCAP/CNPq/Brazil, and CAPES/Brazil – Finance Code 001.
E. Rufino was partially supported by CAPES/Brazil.

manifold which is strictly (3/4)-pinched is a topological sphere. This curvature pinching was improved to $\delta = 1/4$ in 1960 by Berger [1] and Klingenberg [19]. Such an improvement became known as the Topological Sphere Theorem. After almost 50 years, Brendle and Schoen [8] showed by an outstanding method that under the same curvature pinching of Berger and Klingenberg such a manifold must be diffeomorphic to the sphere, this result has been known as the Differentiable Sphere Theorem; see also [7]. Moreover, by combining the results of Berger [2] and Petersen and Tao [24], it is known that given a Riemannian manifold M^n, there is a real number ε (unknown) such that if M^n is $(\frac{1}{4} - \varepsilon)$-pinched, then M^n is either homeomorphic to $\mathbb{S}^n$ or diffeomorphic to a spherical space form of rank 1. In [18], Hulin used the classical Weitezenböck formula to show that if a four-dimensional connected manifold $(M^4,\ g)$ is $(\frac{1}{4} - \gamma)$-pinched, for $\gamma < 2,5 \,.\, 10^{-4}$, then the second Betti number of M^4 is less than or equal to 1. These results stimulated many interesting works. In the next subsection, we quickly review some related results in order to draw the state of the art and put our results in perspective.

1.1. Four-manifolds with positive sectional curvature. In the last decades, many mathematicians have been studied four-dimensional manifolds under suitable curvature pinching conditions. It is well known that four-manifolds display peculiar features. In large part, this is attributed to the fact that on a four-dimensional oriented compact Riemannian manifold (M^4, g), the bundle of 2-forms, denoted by $\Lambda^2 M$, can be invariantly decomposed as

$$\Lambda^2 M = \Lambda^+ M \oplus \Lambda^- M,$$

where $\Lambda^\pm M$ are the ± 1-eigenspaces of the Hodge star operator $*$. Hence, the space of harmonic 2-forms $H^2(M^4; \mathbb{R})$ can be split as $H^2(M^4; \mathbb{R}) = H^+(M^4; \mathbb{R}) \oplus H^-(M^4; \mathbb{R})$, where $H^\pm(M^4; \mathbb{R})$ stands for the space of positive and negative harmonic 2-forms, respectively. Furthermore, the second Betti number b_2 of M^4 can be written as $b_2 = b^+ + b^-$, where $b^\pm = \dim H^\pm(M^4; \mathbb{R})$.

For our purposes, it is important to recall that a four-dimensional Riemannian manifold M^4 is said to be *positive definite* if and only if $b^- = 0$. In particular, when the signature of M^4 is nonzero, we will always orient the manifold so as to make the signature positive.

It follows from Bourguignon [6] and Ville [31] that a $(\frac{4}{19} \approx 0.2105)$-pinched four-dimensional compact manifold is topologically the sphere $\mathbb{S}^4$ or the complex projective space $\mathbb{CP}^2$. This pinching constant was improved in 1991 by Seaman [27] to ≈ 0.1714. Recently, Diógenes and Ribeiro [13] were able to show that a four-dimensional compact oriented connected Riemannian manifold which is (≈ 0.16139)-pinched must be definite. In particular, they showed that a four-dimensional compact oriented Einstein manifold $\frac{1}{10}$-pinched is either topologically $\mathbb{S}^4$ or homothetically isometric to $\mathbb{CP}^2$. Besides, the main result in [26] implies that a four-dimensional compact Einstein manifold M^4 with normalized Ricci curvature $Ric = 1$ and sectional curvature $K \geq \frac{1}{12}$ must be isometric to either $\mathbb{S}^4$ or $\mathbb{CP}^2$; see also [9, 12, 33]. Indeed, it remains a challenging task to obtain new classification results under weaker curvature pinching conditions.

Before stating our first result, let us also recall that a metric on a four-dimensional manifold M^4 is *half-conformally flat* if it is sefdual or antiselfdual, namely, $W^- = 0$ or $W^+ = 0$, respectively, where W stands for the Weyl tensor. Typical examples of half-conformally flat manifolds include the standard sphere, the complex projective plane, or $K3$ surfaces with their Ricci-flat metrics. Other interesting examples were built by LeBrun [21]. For a nice overview on half-conformally flat manifolds, see, for instance, ([3], Chapter 13).

After these preliminary remarks, we may announce our first result as follows.

THEOREM 1. *Let (M^4, g) be a four-dimensional compact oriented connected half-conformally flat Riemannian manifold whose sectional curvatures all lie in the interval* $[\frac{3\sqrt{3}-5}{4}, 1]$. *Then, M^4 is topologically $\mathbb{S}^4$ or $\mathbb{CP}^2$.*

An important observation comes from the fact that $\mathbb{CP}^2\sharp\mathbb{CP}^2$ admits metrics with $K \geq 0$, and moreover, it is definite but has $b_2 > 1$. Indeed, it is very interesting to determine if $\mathbb{CP}^2\sharp\mathbb{CP}^2$ admits a metric with positive sectional curvature. It should also be emphasized that the lower bound for the sectional curvature considered in Theorem 1 (namely, ≈ 0.049) improves significantly the constant considered in Theorem 1 of [13] (namely, ≈ 0.16139). Besides, the conclusion of Theorem 1 is clearly an improvement.

As an attempt to better understand four-dimensional manifolds with positive sectional curvature, it is natural to investigate other curvature positivity conditions. In this perspective, we recall that, for each plane $P \subset T_pM$ at a point $p \in M^4$, *the biorthogonal (sectional) curvature* of P is defined by the following average of the sectional curvatures:

$$K^{\perp}(P) = \frac{K(P) + K(P^{\perp})}{2}, \tag{1.2}$$

where $P^{\perp}$ is the orthogonal plane to P. Indeed, the sum of two sectional curvatures on two orthogonal planes plays a very crucial role on four-dimensional manifolds. This notion appeared previously in works by Singer and Thorpe [30], Gray [16], Seaman [29], Noronha [23], Costa and Ribeiro Jr. [11], Bettiol [4], and many others. The positivity of the biorthogonal curvature is an intermediate condition between positive sectional curvature and positive scalar curvature s. Moreover, as it was observed by Singer and Thorpe [30], a four-dimensional Riemannian manifold (M^4, g) is Einstein if and only if $K^{\perp}(P) = K(P)$ for any plan $P \subset T_pM$ at any point $p \in M^4$. From Seaman [28] and Costa and Ribeiro [11], $\mathbb{S}^4$ and $\mathbb{CP}^2$ are the only compact simply connected four-dimensional manifolds with positive biorthogonal curvature that can have (weakly) $1/4$-pinched biorthogonal curvature, or nonnegative isotropic curvature, or satisfy $K^{\perp} \geq \frac{s}{24} > 0$. In addition, by using this approach, Costa and Ribeiro [11] showed that the Yau's Pinching Conjecture is true in dimension 4. In [5], Bettiol proved that the positivity of biorthogonal curvature is preserved under connected sums. In particular, he showed that $\mathbb{S}^4$, $\sharp^m\mathbb{CP}^2\sharp^n\overline{\mathbb{CP}}^2$, and $\sharp^n(\mathbb{S}^2 \times \mathbb{S}^2)$ admit metrics with positive biorthogonal curvature. For more details, see [4, 5, 11, 23] and [29].

Now we may state our next result.

THEOREM 2. *Let (M^4, g) be a four-dimensional compact oriented connected Riemannian manifold satisfying*

$$K^{\perp} \geq \frac{s^2}{24(3\lambda_1 + s)},$$

where λ_1 is the first eigenvalue of Laplacian operator and s stands for the scalar curvature of M^4. Then, M^4 must be definite.

Since $\mathbb{S}^2 \times \mathbb{S}^2$ is not definite, Theorem 2 implies, in particular, that $\mathbb{S}^2 \times \mathbb{S}^2$ does not admit a metric satisfying

$$K^{\perp} \geq \frac{s^2}{24(3\lambda_1 + s)}.$$

https://doi.org/10.1017/S0017089520000130 Published online by Cambridge University Press

We also point out that this result was also independently observed by Cao and Tran (see Remark 1.1 and Theorem 1.1 (3) in [10]). The methods designed for the proof of Theorem 2 were essentially inspired by [17] (see also [13]).

In the sequel, as an application of Theorem 2 combined with results by Freedman [15] and Donaldson [14], we get the following corollary.

COROLLARY 1. *Let (M^4, g) be a four-dimensional simply connected compact oriented Riemannian manifold satisfying*

$$K^{\perp} \geq \frac{s^2}{24(3\lambda_1 + s)}.$$

Then, M^4 is homeomorphic to a connected sum $\mathbb{CP}^2 \sharp \cdots \sharp \mathbb{CP}^2$ of b_2 copies of the complex projective plane (if $b_2 > 0$) or the 4-sphere (if $b_2 = 0$).

It may be interesting to compare Corollary 1 with Theorem 1.3 in [26]. In fact, our latter result requires the same pinching condition of Theorem 1.3 in [26]; however, it does not require conditions on the Weyl tensor and analyticity of the metric.

2. Background. Throughout this section, we review some information and present lemmas that will be useful in the proof of the main results. We start recalling that on a four-dimensional oriented Riemannian manifold M^4 the bundle of 2-forms can be invariantly decomposed as a direct sum

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-.$$

In particular, the Weyl curvature tensor W is an endomorphism of the bundle of 2-forms $\Lambda^2 M = \Lambda^+ M \oplus \Lambda^- M$ such that

$$W = W^+ \oplus W^-,$$

where $W^{\pm} : \Lambda^{\pm} M \longrightarrow \Lambda^{\pm} M$ are called of the *selfdual* and *antiselfdual* parts of W. Thus, we may fix a point $p \in M^4$ and diagonalize $W^{\pm}$ such that $w_i^{\pm}$, $1 \leq i \leq 3$, are their respective eigenvalues. In particular, they satisfy

$$w_1^{\pm} \leq w_2^{\pm} \leq w_3^{\pm} \quad \text{and} \quad w_1^{\pm} + w_2^{\pm} + w_3^{\pm} = 0. \tag{2.1}$$

Next, as it was pointed out in [11] as well as [26], the definition of biorthogonal curvature provides the following identities:

$$K_1^{\perp} = \frac{w_1^+ + w_1^-}{2} + \frac{s}{12} \tag{2.2}$$

and

$$K_3^{\perp} = \frac{w_3^+ + w_3^-}{2} + \frac{s}{12}, \tag{2.3}$$

https://doi.org/10.1017/S0017089520000130 Published online by Cambridge University Press

where $K_1^{\perp}(p) = \min\{K^{\perp}(P);\, P \subset T_pM\}$ and $K_3^{\perp}(p) = \max\{K^{\perp}(P);\, P \subset T_pM\}$. Furthermore, if $\mathcal{R}$ denotes the curvature of M^4, we get the following decomposition:

$$\mathcal{R} = \left(\begin{array}{c|c} W^+ + \frac{s}{12} Id & \mathring{R}ic \\ \hline \mathring{R}ic^{\star} & W^- + \frac{s}{12} Id \end{array}\right) = U + W^+ + W^- + Z, \tag{2.4}$$

where $U = \frac{s}{12} Id_{\Lambda^2}$, $Z = \begin{pmatrix} 0 & \mathring{R}ic \\ \mathring{R}ic^{\star} & 0 \end{pmatrix}$, and $\mathring{R}ic : \Lambda^- \to \Lambda^+$ stands for the traceless part of the Ricci curvature of M^4.

Proceeding, we also remember that if ω is a 2-form we have the following Weitzenböck formula:

$$\frac{1}{2}\Delta|\omega|^2 = \langle \Delta\omega, \omega\rangle + |\nabla\omega|^2 + \langle \mathcal{N}(\omega), \omega\rangle,$$

where $\mathcal{N}$ is the Weitzenböck operator given by

$$\begin{aligned} \langle \mathcal{N}(v_1 \wedge v_2), w_1 \wedge w_2\rangle = {} & Ric(v_1, w_1)\langle v_2, w_2\rangle + Ric(v_2, w_2)\langle v_1, w_1\rangle \\ & - Ric(v_1, w_2)\langle v_2, w_1\rangle - Ric(v_2, w_1)\langle v_1, w_2\rangle \\ & + 2\langle R(v_1, v_2)w_1, w_2\rangle. \end{aligned} \tag{2.5}$$

Here, v_i and w_i are tangent vectors; for more details, see [20] and [27]. We have adopted the opposite of the usual sign convention for the Laplacian, that is, $\Delta f = \mathrm{div}(\nabla f)$.

From now on, we assume that M^4 is a four-dimensional δ-pinched manifold, that is, the sectional curvature K of M^4 satisfies

$$1 \geq K \geq \delta. \tag{2.6}$$

With this condition, as a slight modification of the proof of an useful inequality by Berger [2] (see also [13], Lemma 1), we obtain the following lemma.

LEMMA 1. *Let (M^4, g) be a four-dimensional oriented Riemannian manifold. Then, we have*

$$\langle \mathcal{N}(\omega), \omega\rangle \geq 4K_1^{\perp}|\omega|^2 - \frac{1}{3}\left(s - 12K_1^{\perp}\right) \left||\omega_+|^2 - |\omega_-|^2\right|,$$

where $\omega = \omega_+ + \omega_-$ and $\omega_{\pm} \in \Lambda^{\pm}M$.

Proof. First of all, given a point $p \in M$, there exists an oriented orthonormal basis for T_pM $\{e_1, e_2, e_3, e_4\}$ satisfying $*(e_1 \wedge e_2) = e_3 \wedge e_4$ and such that

$$\omega = \frac{\sqrt{2}}{2}(|\omega_+| + |\omega_-|)e_1 \wedge e_2 + \frac{\sqrt{2}}{2}(|\omega_+| - |\omega_-|)e_3 \wedge e_4$$

https://doi.org/10.1017/S0017089520000130 Published online by Cambridge University Press

at point p. In view of the identity (2.5), we get

$$\begin{aligned}\langle \mathcal{N}(\omega), \omega\rangle &= |\omega|^2(K_{13}+K_{14}+K_{23}+K_{24}) - 2R_{1234}\left(|\omega_+|^2-|\omega_-|^2\right)\\ &= 2|\omega|^2(K^{\perp}_{13}+K^{\perp}_{14}) - 2R_{1234}\left(|\omega_+|^2-|\omega_-|^2\right)\\ &\geq 4K^{\perp}_1|\omega|^2 - 2R_{1234}\left(|\omega_+|^2-|\omega_-|^2\right), \end{aligned} \tag{2.7}$$

where $K^{\perp}_{ij}$ stands for the biorthogonal curvature of plane $e_i \wedge e_j$. Here, we used that $\langle \mathcal{N}(\omega_+), \omega_-\rangle = 0$. Next, we apply the Seaman's estimate

$$|R_{ijkl}| \leq \frac{2}{3}(K^{\perp}_3 - K^{\perp}_1)$$

in order to obtain

$$\langle \mathcal{N}(\omega), \omega\rangle \geq 4K^{\perp}_1|\omega|^2 - \frac{4}{3}(K^{\perp}_3 - K^{\perp}_1)\left(|\omega_+|^2-|\omega_-|^2\right). \tag{2.8}$$

On the other hand, it follows from (2.1) that

$$w^{\pm}_3 \leq -2w^{\pm}_1.$$

These data combined with (2.2) and (2.3) yield

$$\begin{aligned}K^{\perp}_3 &= \frac{w^+_3+w^-_3}{2} + \frac{s}{12}\\ &\leq -(w^+_1+w^-_1) + \frac{s}{12}\\ &= -2K^{\perp}_1 + \frac{s}{6} + \frac{s}{12},\end{aligned}$$

so that

$$K^{\perp}_3 \leq \frac{s}{4} - 2K^{\perp}_1. \tag{2.9}$$

Putting together (2.9) and (2.8), we infer

$$\langle \mathcal{N}(\omega), \omega\rangle \geq 4K^{\perp}_1|\omega|^2 - \frac{1}{3}(s-12K^{\perp}_1)||\omega_+|^2-|\omega_-|^2|.$$

This finishes the proof of the lemma. □

In order to introduce the next lemma, we need to fix notation. We consider the set given by

$$G = \{x\wedge y \in \Lambda^2;\quad x \text{ and } y \text{ are unitary and orthogonal}\}.$$

Besides, let $\mathcal{G} = G/\pm 1$ be the two-dimensional Grassmannian manifold of T_pM. From this, it follows that if $H \in \Lambda^+$ and $K \in \Lambda^-$, we have $\frac{H+K}{\sqrt{2}} \in \mathcal{G}$ if and only if $||H|| = ||K|| = 1$ (cf. Lemma 1 in [**32**]). Now, we may state a result obtained by Ville (cf. Lemma 2 in [**32**], see also [**31**]), which plays an important role in this paper.

LEMMA 2 ([**32**]). *Let M^4 be a four-dimensional oriented δ-pinched Riemannian manifold. Then,*

(1) for all $P \in \mathcal{G}$, we have $\delta \leq \big\langle \big(U+W\big)(P), P\big\rangle \leq 1$;
(2) for all $H \in \Lambda^+$, we have $\delta \leq u + \frac{1}{2}\left\langle W^+H, H\right\rangle \leq 1$, where $u = \frac{s}{12}$.

For what follows, since W^+ is a symmetric endomorphism of Λ^+, we may consider an orthonormal basis of Λ^+ given by $\{H_1, H_2, H_3\}$, such that

$$W^+H_i = w_i^+H_i, \quad \text{for } i = 1,\ 2 \text{ or } 3.$$

Moreover, we consider $K_i = \dfrac{\mathring{R}ic^\star(H_i)}{||\mathring{R}ic^\star(H_i)||} \in \Lambda^-$ and let $z_i = \langle \mathring{R}ic^\star(H_i), K_i\rangle$. Thus, we know that it holds

$$||\mathring{R}ic^\star(H_i)||^2 = \langle z_iK_i,\ z_iK_i\rangle = z_i^2.$$

Hereafter, we set $\lambda_i^- = \langle W^-K_i,\ K_i\rangle$ and $v_i = u + \frac{1}{2}w_i^+$. In particular, it follows from Lemma 2 that $\delta \le v_i \le 1$. With these settings, we have the following lemma due to Ville (cf. Lemma 3 in [**31**], see also [**32**]).

LEMMA 3 ([**31**]). *Let M^4 be a four-dimensional oriented δ-pinched Riemannian manifold. Then,*

$$||Z||^2 \le 2\sum_{i=1}^{3} A_i^2, \tag{2.10}$$

where $||Z||^2 = ||\mathring{R}ic^\star||^2 + ||\mathring{R}ic||^2$ *and* $A_i = \min\{(1 - v_i + \frac{1}{2}\lambda_i^-),\ (v_i + \frac{1}{2}\lambda_i^- - \delta)\}$.

Proof. Since the proof of this lemma is very short, we include it here for sake of completeness. First of all, we easily compute

$$\begin{aligned} ||Z||^2 &= ||\mathring{R}ic^\star||^2 + ||\mathring{R}ic||^2 = 2||\mathring{R}ic^\star||^2 \\ &= 2\sum_{i=1}^{3} ||\mathring{R}ic^\star(H_i)||^2 \\ &= 2\sum_{i=1}^{3} \langle \mathring{R}ic^\star(H_i),\ K_i\rangle^2. \end{aligned} \tag{2.11}$$

Now, we need to estimate the value of $\langle \mathring{R}ic^\star(H_i),\ K_i\rangle$. To do so, notice that by Lemma 2, we have

$$\delta \le \left\langle \mathcal{R}\Big(\frac{H_i \pm K_i}{\sqrt{2}}\Big),\ \frac{H_i \pm K_i}{\sqrt{2}}\right\rangle \le 1. \tag{2.12}$$

Moreover, one has

$$\begin{aligned} \left\langle \mathcal{R}\Big(\frac{H_i \pm K_i}{\sqrt{2}}\Big),\ \frac{H_i \pm K_i}{\sqrt{2}}\right\rangle &= u + \frac{1}{2}\langle W^+H_i,\ H_i\rangle + \frac{1}{2}\langle W^-K_i,\ K_i\rangle \\ &\quad + 2\left\langle \mathring{R}ic\Big(\frac{\pm K_i}{\sqrt{2}}\Big),\ \frac{H_i}{\sqrt{2}}\right\rangle \\ &= u + \frac{1}{2}w_i^+ + \frac{1}{2}\lambda_i^- \pm \langle \mathring{R}ic(K_i),\ H_i\rangle \\ &= v_i + \frac{1}{2}\lambda_i^- \pm \langle \mathring{R}ic^\star(H_i),\ K_i\rangle. \end{aligned} \tag{2.13}$$

This jointly with formula (2.12) gives

$$\delta - \nu_i - \frac{1}{2}\lambda_i^- \leq \pm\langle \mathring{Ric}^\star(H_i),\ K_i\rangle \leq 1 - \nu_i - \frac{1}{2}\lambda_i^-,$$

from which it follows that

$$|\langle \mathring{Ric}^\star(H_i),\ K_i\rangle| \leq A_i,$$

as wished. □

We are now in the position to present the proof of the main results.

3. Proof of the main results.

3.1. Proof of Theorem 1.

Proof. To begin with, since (M^4, g) is a compact half-conformally flat smooth manifold of positive scalar curvature, we can use Proposition 2.4 of [22] to infer that M^4 is definite. Moreover, it is known by Synge's theorem that M^4 is simply connected. Thus, from Donaldson [14] and Freedman [15], M^4 is homeomorphic to a connected sum $\mathbb{CP}^2\sharp\cdots\sharp\mathbb{CP}^2$ of b_2 copies of the complex projective plane (if $b_2 > 0$) or the 4-sphere (if $b_2 = 0$).

Now, it remains to prove that either $b_2 = 0$ or $b_2 = 1$. To this end, it suffices to prove that

$$|\tau(M)| < \frac{1}{2}\chi(M). \tag{3.1}$$

Indeed, without loss of generality, we may assume that $\tau(M) > 0$. Therefore, by using the classical Gauss–Bonnet–Chern formula

$$\chi(M) = \frac{1}{8\pi^2}\int_M \left(\frac{s^2}{24} + |W^+|^2 + |W^-|^2 - \frac{1}{2}|\mathring{Ric}|^2\right) dV_g,$$

jointly with the Hirzebrush's theorem

$$\tau(M) = \frac{1}{12\pi^2}\int_M \left(|W^+|^2 - |W^-|^2\right) dV_g,$$

we arrive at

$$\chi(M) - 2\tau(M) = \frac{1}{8\pi^2}\int_M \left(\frac{s^2}{24} - \frac{1}{3}|W^+|^2 + \frac{7}{3}|W^-|^2 - \frac{1}{2}|\mathring{Ric}|^2\right) dV_g.$$

Before proceeding, for simplicity, let us consider

$$\mathcal{F}(g) = \left(\frac{s^2}{24} - \frac{1}{3}|W^+|^2 + \frac{7}{3}|W^-|^2 - \frac{1}{2}|\mathring{Ric}|^2\right),$$

and hence, we easily see that

$$\chi(M) - 2\tau(M) = \frac{1}{8\pi^2}\int_M \mathcal{F}(g) dV_g. \tag{3.2}$$

So, we need to show the positivity of the right-hand side of (3.2). But, in view of the pinching condition $\delta \leq K \leq 1$, we may use equation (1) in [31] (see also Lemmas 1.3 and 1.4 in [31]) to obtain

$$\mathcal{F}(g) \geq \frac{10}{9}\left(\sum_{i=1}^{3} v_i\right)^2 - \frac{4}{3}\sum_{i=1}^{3} v_i^2 + \frac{7}{2}\alpha^2 - 2\sum_{i=1}^{3}\min\left\{\left(1 - v_i - \frac{\lambda_i^-}{2}\right)^2, \left(v_i + \frac{\lambda_i^-}{2} - \delta\right)^2\right\}, \tag{3.3}$$

where $\alpha = \max|\lambda_i^-|$. For more details, see [31].

From now on, since (M^4, g) is half-conformally flat, we assume that $W^- = 0$. Whence, it follows that $\lambda_i^- = 0$ for $i = 1,\ 2$ or 3 and consequently, $\alpha = 0$. Thus, by (3.3), one obtains

$$\frac{\mathcal{F}(g)}{2} \geq \frac{5}{9}\left(\sum_{i=1}^{3} v_i\right)^2 - \frac{2}{3}\sum_{i=1}^{3} v_i^2 - \sum_{i=1}^{3} m(v_i)^2, \tag{3.4}$$

where $m(x) = \min\{1 - x, x - \delta\}$.

In order to proceed, we introduce the function $f : \mathbb{R}^3 \to \mathbb{R}$ given by

$$f(x_1, x_2, x_3) = \frac{5}{9}\left(\sum_{i=1}^{3} x_i\right)^2 - \frac{2}{3}\sum_{i=1}^{3} x_i^2 - \sum_{i=1}^{3} m(x_i)^2,$$

and moreover, let us consider the set

$$E = \{(x_1, x_2, x_3) \in \mathbb{R}^3;\ \delta \leq x_1 \leq x_2 \leq x_3 \leq 1\}.$$

Then, easily one verifies that

$$Hess\, f = \frac{10}{9}\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix},$$

whose the eigenvalues are $\{-3, -3, 0\}$. Therefore, f is concave in E, and then, it suffices to show the positivity of f in the following points: (δ, δ, δ), $(\delta, \delta, 1)$, $(\delta, 1, 1)$, and $(1, 1, 1)$. In fact, it is not hard to check that

- $f(\delta, \delta, \delta) = 3\delta^2$;
- $f(\delta, \delta, 1) = \dfrac{8\delta^2 + 20\delta - 1}{9}$;
- $f(\delta, 1, 1) = \dfrac{-\delta^2 + 20\delta + 8}{9}$;
- $f(1, 1, 1) = 3$.

Whence, taking into account that $\delta \geq \frac{3\sqrt{3}-5}{4}$, we conclude that f is nonnegative, and we therefore have

$$\frac{\mathcal{F}(g)}{2} \geq f(v_1, v_2, v_3) \geq 0.$$

https://doi.org/10.1017/S0017089520000130 Published online by Cambridge University Press

Next, since there exist points where f is positive on E, we deduce

$$\int_M \mathcal{F}(g)dV_g > 0,$$

and hence, these data into (3.2) yield $|\tau(M)| < \frac{1}{2}\chi(M)$. Finally, it is straightforward to check that $b_2 = 0$ or $b_2 = 1$. Thereby, according to Freedman [15] (see also Donaldson [14]), M^4 is homeomorphic to the complex projective space $\mathbb{CP}^2$ or the 4-sphere $\mathbb{S}^4$.

So, the proof is completed. □

3.2. Proof of Theorem 2.

Proof. The first part of the proof follows the ideas outlined in [26], which was partially inspired by [17]. For the sake of completeness, we include here all details. Indeed, we argue by contradiction, assuming that M^4 is indefinite. In this case, there exist nonzero harmonics 2-forms ω_+ and ω_- such that

$$\int_M \left(\left(|\omega_+|^2 + \varepsilon\right)^{\frac{1}{4}} - t\left(|\omega_-|^2 + \varepsilon\right)^{\frac{1}{4}} \right) dV_g = 0,$$

for any $\varepsilon > 0$ and $t = t(\varepsilon)$. Next, following the same steps of the first part of the proof of Theorem 1 in [13] (see equation (3.6) in [13]), we achieve at

$$0 \geq \lambda_1 \int_M \left(|\omega_+|^{\frac{1}{2}} - t|\omega_-|^{\frac{1}{2}}\right)^2 dV_g \\ + \int_M \left(|\omega_+|^{-1}\langle \mathcal{N}(\omega_+), \omega_+\rangle + t^2|\omega_-|^{-1}\langle \mathcal{N}(\omega_-), \omega_-\rangle\right) dV_g. \tag{3.5}$$

Proceeding in analogy with [13], we choose $X_+ = |\omega_+|^{-\frac{1}{2}}\omega_+$ and $X_- = t|\omega_-|^{-\frac{1}{2}}\omega_-$. Hence, easily one verifies that $X_\pm \in \Lambda^\pm$, $|X_+| = |\omega_+|^{\frac{1}{2}}$ and $|X_-| = t|\omega_-|^{\frac{1}{2}}$. Therefore, by considering $X = X_+ + X_-$, it follows from (3.5) that

$$0 \geq \int_M \left\{\lambda_1 \left(|X_+| - |X_-|\right)^2 + \langle \mathcal{N}(X), X\rangle\right\} dV_g,$$

and then by Lemma 1, we deduce

$$0 \geq \int_M \left\{\lambda_1 \left(|X_+| - |X_-|\right)^2 + 4K_1^\perp|X|^2 - \frac{1}{3}(s - 12K_1^\perp)\left||X_+|^2 - |X_-|^2\right|\right\} dV_g.$$

Since $|X_+| = |\omega_+|^{\frac{1}{2}}$ and $|X_-| = t|\omega_-|^{\frac{1}{2}}$, the above expression can be written succinctly as

$$0 \geq \int_M \Big(\lambda_1 \left(|\omega_+| - 2t|\omega_+|^{\frac{1}{2}}|\omega_-|^{\frac{1}{2}} + t^2|\omega_-|\right) + 4K_1^\perp|\omega_+| \\ + 4K_1^\perp t^2|\omega_-| - \frac{1}{3}(s - 12K_1^\perp)\left||\omega_+| - t^2|\omega_-|\right|\Big) dV_g. \tag{3.6}$$

Notice that the integrand of (3.6) is a quadratic function of t. Thus, for simplicity, we may set

$$\mathcal{P}(t) = \lambda_1 \left(|\omega_+| - 2t|\omega_+|^{\frac{1}{2}}|\omega_-|^{\frac{1}{2}} + t^2|\omega_-|\right) + 4K_1^\perp|\omega_+| \\ + 4K_1^\perp t^2|\omega_-| - \frac{1}{3}\left(s - 12K_1^\perp\right)\left||\omega_+| - t^2|\omega_-|\right|.$$

Before proceeding, given a point $p \in M^4$, let us consider the sets

$$A = \left\{t;\quad |\omega_+| \geq t^2 |\omega_-| \text{ at } p\right\}$$

and

$$B = \left\{t;\quad |\omega_+| < t^2 |\omega_-| \text{ at } p\right\}.$$

Therefore, the definition of $\mathcal{P}(t)$ implies that

$$\begin{aligned}\mathcal{P}(t) = &\left[\lambda_1 + 4K_1^{\perp} - \frac{1}{3}(s - 12K_1^{\perp})\right] |\omega_+| - 2\lambda_1 |\omega_+|^{\frac{1}{2}} |\omega_-|^{\frac{1}{2}} t \\ &+ \left[\lambda_1 + 4K_1^{\perp} + \frac{1}{3}(s - 12K_1^{\perp})\right] |\omega_-| t^2, \end{aligned} \tag{3.7}$$

in A and

$$\begin{aligned}\mathcal{P}(t) = &\left[\lambda_1 + 4K_1^{\perp} + \frac{1}{3}(s - 12K_1^{\perp})\right] |\omega_+| - 2|\omega_+|^{\frac{1}{2}} |\omega_-|^{\frac{1}{2}} t \\ &+ \left[\lambda_1 + 4K_1^{\perp} - \frac{1}{3}(s - 12K_1^{\perp})\right] |\omega_-| t^2, \end{aligned} \tag{3.8}$$

in B. In both cases, the discriminant Δ of $\mathcal{P}(t)$ is given by

$$\begin{aligned}\Delta &= 4\lambda_1^2 |\omega_+||\omega_-| - 4\left[\lambda_1 + 4K_1^{\perp} - \frac{1}{3}\left(s - 12K_1^{\perp}\right)\right]\left[\lambda_1 + 4K_1^{\perp} + \frac{1}{3}(s - 12K_1^{\perp})\right] |\omega_+||\omega_-| \\ &= 4|\omega_+||\omega_-| \left\{\lambda_1^2 - \left[\left(\lambda_1 + 4K_1^{\perp}\right)^2 - \frac{1}{9}(s - 12K_1^{\perp})^2\right]\right\} \\ &= 4|\omega_+||\omega_-| \left[\lambda_1^2 - \left(\lambda_1^2 + 8\lambda_1 K_1^{\perp} + 16(K_1^{\perp})^2\right) + \frac{1}{9}\left(s^2 - 24K_1^{\perp} s + 144(K_1^{\perp})^2\right)\right] \\ &= \frac{4}{9}|\omega_+||\omega_-| \left(-72\lambda_1 K_1^{\perp} + s^2 - 24K_1^{\perp} s\right).\end{aligned}$$

Hence, the condition $K^{\perp} \geq \frac{s^2}{24(3\lambda_1 + s)}$ guarantees that $\Delta \leq 0$.

On the other hand, it follows from (3.7) that

$$\left[\lambda_1 + 4K_1^{\perp} + \frac{1}{3}(s - 12K_1^{\perp})\right] |\omega_-| \geq 0,$$

where we used that $s - 12K_1^{\perp} = -6(w_1^+ + w_1^-) \geq 0$. Otherwise, if $\mathcal{P}(t)$ is given by (3.8), it follows from our assumption that it holds

$$\begin{aligned}\left(\lambda_1 + 4K_1^{\perp} - \frac{1}{3}(s - 12K_1^{\perp})\right)|\omega_-| &\geq \left(\lambda_1 + \frac{s^2}{6(3\lambda_1 + s)} - \frac{1}{3}\left(s - \frac{s^2}{2(3\lambda_1 + s)}\right)\right) |\omega_-| \\ &= \frac{6\lambda_1(3\lambda_1 + s) + s^2 - 2(3\lambda_1 + s)s + s^2}{6(3\lambda_1 + s)}|\omega_-| \\ &= \frac{18\lambda_1^2 + 6\lambda_1 s + 2s^2 - 6\lambda_1 s - 2s^2}{6(3\lambda_1 + s)}|\omega_-| \\ &= \frac{18\lambda_1^2}{6(3\lambda_1 + s)}|\omega_-| \\ &\geq 0.\end{aligned}$$

https://doi.org/10.1017/S0017089520000130 Published online by Cambridge University Press

Thereby, in both cases, we have $\mathcal{P}(t) \geq 0$. Next, since p is an arbitrary point, we conclude from (3.6) that $\mathcal{P}(t) \equiv 0$. Now, it suffices to use (3.7) and (3.8) to deduce

$$\left[\lambda_1 + 4K_1^{\perp} \pm \frac{1}{3}(s - 12K_1^{\perp})\right] |\omega_-| = 0.$$

But, taking into account that $\lambda_1 + 4K_1^{\perp} \pm \frac{1}{3}(s - 12K_1^{\perp}) > 0$, we conclude $|\omega_-| = 0$. This yields the desired contradiction and therefore forces the intersection form of M^4 to be definite. To conclude, we invoke again Freedman [15] and Donaldson [14] to deduce that M^4 is homeomorphic to the complex projective space $\mathbb{CP}^2$ or the 4-sphere $\mathbb{S}^4$.

This finishes the proof of the theorem. □

ACKNOWLEDGEMENTS. The authors want to thank the referee for his careful reading, relevant remarks, and valuable suggestions. Moreover, the authors want to thank Xiaodong Cao, Hung Tran, Ezio Costa, and Qing Cui for their valuable comments and helpful conversations about this subject. In particular, we would like to thank Xiadong Cao and Qing Cui for pointing out the preprints [10] and [12].

REFERENCES

1. M. Berger, Les variétés Riemenniennes 1/4-pincées, *Ann. Scuola Norm. Sup. Pisa* **14** (1960), 161–170.

2. M. Berger, Sur quelques variétés Riemenniennes suffisament pincées, *Bull. Soc. Math. France* **88** (1960), 57–71.

3. A. Besse, *Einstein manifolds* (Springer-Verlag, New York, 2008).

4. R. Bettiol, Positive biorthogonal curvature on $\mathbb{S}^2 \times \mathbb{S}^2$, *Proc. Amer. Math. Soc.* **142** (2014), 4341–4353.

5. R. Bettiol, Four-dimensional manifolds with positive biorthogonal curvature, *Asian J. Math.* **21** (2017), 391–396.

6. J. Bourguignon, La conjecture de Hopf sur $\mathbb{S}^2 \times \mathbb{S}^2$ Geometrie Riemannienne en dimension 4, Seminaire Arthur Besse, CEDIC Paris, 1981, 347–355.

7. S. Brendle and R. Schoen, Curvature, sphere theorems, and the Ricci flow, *Bull. Amer. Math. Soc.* **48** (2010), 1–32.

8. S. Brendle and R. Schoen, Manifolds with 1/4-pinched curvature are space forms, *J. Amer. Math. Soc.* **22** (2009), 287–307.

9. X. Cao and H. Tran, Einstein four manifolds with pinched sectional curvature, *Adv. Math.* **335** (2018), 322–342.

10. X. Cao and H. Tran, Four-manifolds of pinched sectional curvature. Arxiv: 1809.05158v2 [math.DG] (2019).

11. E. Costa and E. Ribeiro, Four-dimensional compact manifolds with nonnegative biorthogonal curvature, *Michigan Math. J.* **63** (2014), 673–688.

12. Q. Cui and L. Sun, On the topology and rigidity of four-dimensional Einstein manifolds. Preprint (2018).

13. R. Diógenes and E. Ribeiro Jr., Four-dimensional manifolds with pinched positive sectional curvature, *Geom. Dedicata.* **200** (2019), 321–330.

14. K. Donaldson, An application of gauge theory to four dimensional topology, *J. Differential Geom.* **18** (1983) 279–315.

15. M. Freedman, The topology of four-dimensional manifolds, *J. Differential Geom.* **17** (1982), 327–454.

16. A. Gray, Invariants of curvature operators of four-dimensional Riemannian manifolds, in *Proceedings of 13th Biennial Seminar Canadian Mathematics Congress*, vol. 2 (1972), 42–65.

17. M. Gursky, Four-manifolds with $\delta W^+ = 0$ and Einstein constants of the sphere, *Math. Ann.* **318** (2000), 417–431.

18. D. Hulin, Le second nombre de Betti d'une variété riemannienne $(\frac{1}{4} - \varepsilon)$-pincée de dimension 4, *Annales de l'Institut Fourier.* **33**(2) (1983), 167–182.

19. W. Klingenberg, Über Riemannsche Mannigfaltigkeiten mit positiver Krümmung, *Comment. Math. Helv.* **35** (1961), 47–54.

20. K.-S. Ko, On 4-dimensional Einstein manifolds which are positively pinched, *Korean J. Math.* **3** (1995), 81–88.

21. C. LeBrun, Explicit self-fual metrics on $\mathbb{CP}^2 \sharp \cdots \sharp \mathbb{CP}^2$, *J. Diff. Geom.* **34** (1991), 223–253.

22. H. Noronha, Self-duality and 4-manifolds with nonnegative curvature on totally isotropic 2-planes, *Michigan Math. J.* **41** (1994), 3–12.

23. H. Noronha, Some results on nonnegatively curved four manifods, *Matemat. Contemp.* **9** (1995), 153–175.

24. P. Petersen and T. Tao, Classification of almost quarter-pinched manifolds, *Proc. Amer. Math. Soc.* **137** (2009), 2437–2440.

25. H. Rauch, A contribution to differential geometry in the large, *Ann. Math.* **54** (1951), 38–55.

26. E. Ribeiro Jr., Rigidity of four-dimensional compact manifolds with harmonic Weyl tensor, *Annali di Matematica Pura Appl.* **195** (2016), 2171–2181.

27. W. Seaman, Harmonic two-forms in four dimensions, *Proc. Amer. Math. Soc.* **112** (1991), 545–548.

28. W. Seaman, On manifolds with nonnegative curvature on totally isotropic 2-planes, *Trans. Amer. Math. Soc.* **338** (1993), 843–855.

29. W. Seaman, Orthogonally pinched curvature tensors and aplications, *Math. Scand.* **69** (1991), 5–14.

30. I. Singer and J. Thorpe, The curvature of 4-dimensional spaces, in *Global Analysis, Papers in Honor of K. Kodaira* (Princeton, 1969), 355–365.

31. M. Ville, Les variétés Riemannienes de dimension 4 $\frac{4}{19}$-pincées, *Ann. Inst. Fourier* **39** (1989), 149–154.

32. M. Ville, On 1/4-pinched 4-dimensional Riemannian manifolds of negative curvature, *Ann. Glob. Anal. Geom.* **3** (1985), 329–336.

33. P. Wu, A note on Einstein four-manifolds with positive curvature, *J. Geom. Phys.* **114** (2017) 19–22.