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Infinitely many non-radial solutions for the polyharmonic Hénon equation with a critical exponent

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We study the following polyharmonic Hénon equation:

$$(-\Delta)^m u = K(|y|)u^{(m)^*-1}, \quad u > 0 \quad \text{in } \boldsymbol{B}_1(0), \quad u \in \mathcal{D}_0^{m,2}(\boldsymbol{B}_1(0)),$$

where $(m)^* = 2N/(N-2m)$ is the critical exponent, $\boldsymbol{B}_1(0)$ is the unit ball in $\mathbb{R}^N$, $N \geqslant 2m+2$ and $K(|y|)$ is a bounded function. We prove the existence of infinitely many non-radial positive solutions, whose energy can be made arbitrarily large.

Keywords: polyharmonic Hénon equation; critical exponent; infinitely many non-radial solutions

2010 *Mathematics subject classification:* Primary 35J65; 35B05; 35B45

1. Introduction

We consider the following polyharmonic equation with critical exponent $(m)^* = 2N/(N-2m)$:

$$(-\Delta)^m u = K(|y|)u^{(m)^*-1}, \quad u > 0 \quad \text{in } \boldsymbol{B}_1(0), \quad u \in \mathcal{D}_0^{m,2}(\boldsymbol{B}_1(0)), \tag{1.1}$$

where $\boldsymbol{B}_1(0)$ is the unit ball in $\mathbb{R}^N$, $N \geqslant 2m+2$ and $K\colon [0,1] \to \mathbb{R}$ is a bounded function. $\mathcal{D}_0^{m,2}(\boldsymbol{B}_1(0))$ denotes the closure of $C_0^\infty(\boldsymbol{B}_1(0))$ with respect to the norm

$$\|u\| = \begin{cases} |\Delta^{m/2}u|_2 & \text{if } m \text{ is even}, \\ |\nabla\Delta^{(m-1)/2}u|_2 & \text{if } m \text{ is odd}, \end{cases} \tag{1.2}$$

where $|\cdot|_p$ denotes the L^p norm on $\boldsymbol{B}_1(0)$.

When $K(|y|) = |y|^\alpha$ and $m = 1$, (1.1) is reduced to the classical Hénon equation,

$$\left.\begin{aligned} -\Delta u &= |x|^\alpha u^{p-1} && \text{in } \boldsymbol{B}_1(0), \\ u &> 0 && \text{in } \boldsymbol{B}_1(0), \\ u &= 0 && \text{on } \partial\boldsymbol{B}_1(0), \end{aligned}\right\} \tag{1.3}$$

with $p = 2N/(N-2)$, which was first introduced by Hénon [15] in the study of astrophysics. Mathematically, due to the lack of compactness, solving problem (1.3) for $L^{2N/(N-2)}(\boldsymbol{B}_1(0))$ is more difficult than solving it for $H_0^1(\boldsymbol{B}_1(0))$. Ni [16] observed that the non-autonomous term $|y|^\alpha$ changes the global homogeneity of the equation and also shifts the original critical exponent $p = 2N/(N-2)$ up to a new exponent $p^\alpha = 2(N+\alpha)/(N-2)$. Ni proved that for any $\alpha > 0$ problem (1.3) admits a radial solution.

It is natural to ask whether (1.3) has a non-radial solution. Smets *et al*. [21] studied problem (1.3) when $N = 2$, with exponent very near to critical, i.e. $-\Delta u = |y|^\alpha u^{(N+2)/(N-2)-\varepsilon}$ with $\varepsilon > 0$ small. They proved that there exists a constant $\alpha^* > 0$ such that problem (1.3) admits at least one non-radial solution for any $\alpha > \alpha^*$. Cao and Peng [6] proved that when $N \geqslant 3$ the mountain-pass solution for (1.3) is non-radial and blows up as $\varepsilon \to 0$. For the purely critical case $p = 2^*$, Serra [20] proved that (1.3) has at least one non-radial solution provided that $N \geqslant 4$ and $\alpha > 0$ is sufficiently large. Recently, Wei and Yan [24] considered the multiplicity for problem (1.3); they proved that there exist infinitely many non-radial solutions for $N \geqslant 4$ and any $\alpha > 0$.

On the other hand, when $m = 1$ and K is defined in the entire space $\mathbb{R}^N$, problem (1.1) turns out to be the limit case

$$-\Delta u = K(y)u^{(N+2)/(N-2)} u > 0 \quad \text{in } \mathbb{R}^N, \quad u \in \mathcal{D}^{1,2}(\mathbb{R}^N), \tag{1.4}$$

where $\mathcal{D}^{1,2}(\mathbb{R}^N)$ is the completion of $C_0^\infty(\mathbb{R}^N)$ under the norm

$$\|u\| := \left(\int_{\mathbb{R}^N} |\nabla u|^2\right)^{1/2}.$$

In this case, it is known (see [4]) that any solution of (1.4) is radially symmetric if there is an $r_0 > 0$ such that $K(r)$ is non-increasing in $(0, r_0]$ and non-decreasing in $[r_0, +\infty)$.

It is natural to ask whether or not there exist non-radial solutions to (1.4) under some other assumptions on the function $K(y)$. This question was first raised by Bianchi himself [4]. Wei and Yan [23] obtained infinitely many non-radial solutions in the elliptic case by constructing a large number of bubbles. This result was later extended to the polyharmonic case by Guo and Li [13]. For the non-radial solutions in [13, 23], the alternative assumption on the function $K(y)$ satisfies the following condition:

$$K(r) = K(1) - K_0|r-1|^t + o(|r-1|^{t+\theta}) \text{ as } r \to 1, \quad \text{where } t \in [2, N-2m),\ \theta > 0. \tag{1.5}$$

As far as we know, there are few results for the polyharmonic Hénon equation on the unit ball $\boldsymbol{B}_1(0)$. The aim of this paper is to prove the existence of infinitely many non-radial solutions for the polyharmonic Hénon equation on the unit ball $\boldsymbol{B}_1(0)$. The polyharmonic operators have long been of interest due to their application in conformal geometry and elastic mechanics. For example, the conformal covariant operator P_4 ($m = 2$) was first introduced by Paneitz [17] in 1983 when studying smooth 4-manifolds, and the application of P_4 was generalized to any N-manifold by Branson [5] in 1993. Problems relating to polyharmonic operators to the elliptic operator (when $m = 1$) present new challenges. For more interesting results related to polyharmonic operators, we refer the reader to [1–3,7,11,12,18] and the references therein.

Before stating the main results, we recall (see [22]) that the family of functions

$$\left\{U_{x,\Lambda}(y) = P_{m,N}^{(N-2m)/4m}\left(\frac{\Lambda}{1+\Lambda^2|y-x|^2}\right)^{(N-2m)/2} \;\middle|\; x \in \mathbb{R}^N,\ \Lambda > 0\right\}$$

are the only radial solutions (usually called *bubbles*) of the following problem:

$$(-\Delta)^m u = u^{(N+2m)/(N-2m)}, \quad u > 0 \quad \text{in } \mathbb{R}^N, \tag{1.6}$$

where

$$P_{m,N} = \prod_{h=-m}^{m-1}(N+2h)$$

is a constant, $\Lambda > 0$ is the scaling parameter and $x \in \mathbb{R}^N$.

For any fixed positive integer $k \geqslant k_0$ with k_0 large enough, we define the scaling parameter $\mu_k := k^{(N-2m+l)/(N-2m)}$, $N \geqslant 2m+2$, $l \in (0,2]$. Using the transformation

$$u(y) \mapsto \mu_k^{-(N-2m)/2} u\left(\frac{y}{\mu_k}\right),$$

problem (1.1) becomes

$$(-\Delta)^m u = K\left(\frac{|y|}{\mu_k}\right)u^{(N+2m)/(N-2m)}, \quad u > 0 \quad \text{in } \boldsymbol{B}_{\mu_k}(0),\ u \in \mathcal{D}_0^{m,2}(\boldsymbol{B}_{\mu_k}(0)). \tag{1.7}$$

We define

$$H_{s,\mu_k} := \{u \in D_0^{m,2}(\boldsymbol{B}_{\mu_k}(0)) \mid u(\bar{y}, y'') = u(\mathrm{e}^{2\pi\mathrm{i}/k}\bar{y}, y''),\ \bar{y} \in \mathbb{R}^2,\ y'' \in \mathbb{R}^{N-2}\}.$$

Choose $\{x_j\}_{j=1}^k$ as the k vertices of the regular k-polygon inside $\boldsymbol{B}_{\mu_k}(0)$, where

$$x_j = \left(r_k \cos\left(\frac{2(j-1)\pi}{k}\right), r_k \sin\left(\frac{2(j-1)\pi}{k}\right), \mathbf{0}\right),$$
$$\mathbf{0} \in \mathbb{R}^{N-2},\ r_k \in \left(\mu_k\left(1-\frac{r_0}{k}\right),\ \mu_k\left(1-\frac{r_1}{k}\right)\right),\ r_0 > r_1$$

are positive constants. Let $P_k U_{x_j,\Lambda_k}$ denote the solution of the following Dirichlet problem (1.8) on $\boldsymbol{B}_{\mu_k}(0)$:

$$\left.\begin{aligned} (-\Delta)^m (P_k U_{x_j,\Lambda_k}) &= U_{x_j,\Lambda_k}^{(2m)^*-1} \quad \text{in } \boldsymbol{B}_{\mu_k}(0), \\ (P_k U_{x_j,\Lambda_k}) &\in D_0^{m,2}(\boldsymbol{B}_{\mu_k}(0)). \end{aligned}\right\} \tag{1.8}$$

Let $W_{r_k,\Lambda_k}(y) := \sum_{j=1}^k P_k U_{x_j,\Lambda_k}(y)$ be the approximate solution.

Our main result is as follows.

Theorem 1.1. *Suppose $N \geqslant 2m+2$. If $K(1) > 0$ and $K'(1) > 0$, then there exists an integer $k_0 > 0$ such that for any integer $k \geqslant k_0$ the boundary-value problem (1.7) has a solution $u_k = W_{r_k,\Lambda_k} + \phi_k$, where $\phi_k \in H_{s,\mu_k}$, $\|\phi_k\|_{L^\infty(B_{\mu_k}(0))} \to 0$ as $k \to \infty$ and $L_0 \leqslant \Lambda_k \leqslant L_1$ for some large constants $L_0, L_1 > 0$.*

As a consequence, we obtain the following.

THEOREM 1.2. *Suppose $N \geqslant 2m+2$. If $K(1) > 0$ and $K'(1) > 0$, then there exist infinitely many non-radial solutions for the polyharmonic problem (1.1).*

REMARK 1.3. The solutions of theorem 1.2 are constructed with bubbles near the boundary of the unit ball $\boldsymbol{B}_1(0)$, and the bubbles are all constrained in $\boldsymbol{B}_1(0)$ instead of diverging to infinity in $\mathbb{R}^N$ (see [13]).

Since there is no small parameter in (1.1), in order to prove the main theorem we follow the idea in [24]: we use the scaling parameter Λ_k as the blow-up parameter. More precisely, we place a large number of bubbles inside a k-polygon in the domain $\boldsymbol{B}_1(0)$ but near the boundary $\partial\boldsymbol{B}_1(0)$. Then the scaling parameter will be determined by the number of bubbles. The proof of the theorem consists of linearizing the equation around an approximation solution (the sum of the k bubbles) and studying the linearized problem. This is done in § 2. Section 3 is devoted to the energy expansion. Then the solution of the problem is reduced to finding the critical points of a perturbed energy functional with parameters μ_k and Λ_k. The proof of the theorem is completed in § 4.

2. Finite-dimensional reduction

In this section, we study the linearized problem by using the Lyapunov–Schmidt reduction method.

Let

$$Z_{i,1} = \frac{\partial P_k U_{x_i,\Lambda_k}(y)}{\partial \gamma_k}, \quad Z_{i,2} = \frac{\partial P_k U_{x_i,\Lambda_k}(y)}{\partial \Lambda_k}, \quad \gamma_k = |x_i|, \quad i = 1, 2, \dots, k.$$

We introduce the Banach space

$$X := \{u \in H_{s,\mu_k} \mid \langle U_{x_i,\Lambda_k}^{(m)^*-2} Z_{i,l}, u\rangle = 0,\ i = 1, 2, \dots, k,\ l = 1, 2;\ \|u\|_* < +\infty\},$$

with the norm

$$\|u\|_* := \sup_{y \in B_{\mu_k}(0)} \left[\sum_{j=1}^{k} \frac{1}{(1+|y-x_j|)^{(N-2m)/2+\tau}}\right]^{-1} |u(y)|,$$

and the Banach space

$$Y = \{h \in H_{s,\mu_k} \mid \langle h, Z_{i,l}\rangle = 0,\ i = 1, 2, \dots, k,\ l = 1, 2;\ \|h\|_{**} < \infty\}$$

with the norm

$$\|h\|_{**} := \sup_{y \in B_{\mu_k}(0)} \left[\sum_{j=1}^{k} \frac{1}{(1+|y-x_j|)^{(N+2m)/2+\tau}}\right]^{-1} |h(y)|,$$

where

$$\langle u, v\rangle = \int_{B_{\mu_k}(0)} uv, \qquad \tau = \frac{N-2m}{N-2m+l}.$$

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We consider the following linearized problem:

$$\left.\begin{aligned} L_k(\phi_k) = h_k + \sum_{j=1}^{2} c_j \sum_{i=1}^{k} U_{x_i,\Lambda_k}^{(m)^*-2} Z_{i,j} \quad \text{in } \boldsymbol{B}_{\mu_k}(0), \\ \phi_k \in X, \qquad h_k \in Y, \end{aligned}\right\} \tag{2.1}$$

where

$$L_k := (-\Delta)^m - ((m)^* - 1)K\left(\frac{|y|}{\mu_k}\right)W_{r_k,\Lambda_k}^{m^*-2}(y).$$

Then it is known that (see [2, theorem 2.1])

$$\mathrm{span}\{Z_{i,1} \mid i = 1, 2, \ldots, k,\ l = 1, 2\}$$

is the kernel space of the linear operator L_k.

LEMMA 2.1. *Assume $N \geqslant 2m + 2$. Then, for any constant $\sigma \in (0, N - 2m)$, there is a constant $C > 0$ such that*

$$\int_{\mathbb{R}^N} \frac{\mathrm{d}z}{|y - z|^{N-2m}(1 + |z|)^{2m+\sigma}} \leqslant \frac{C}{(1 + |y|)^{\sigma}}.$$

LEMMA 2.2. *Assume $N \geqslant 2m + 2$, $\tau \in (0, 2)$. Then there exists a small $\theta > 0$ such that*

$$\int_{\mathbb{R}^N} \sum_{j=1}^{k} \frac{W_{r_k,\Lambda_k}^{4m/(N-2m)}(z)}{|y - z|^{N-2m}(1 + |z - x_j|)^{(N-2m)/2+\tau}}\,\mathrm{d}z \leqslant C \sum_{j=1}^{k} \frac{1}{(1 + |z - x_j|)^{(N-2m)/2+\tau+\theta}}.$$

The proofs of lemmas 2.1 and 2.2 can be found in [13].

PROPOSITION 2.3. *Assume that ϕ_k solves (2.1) for given values of h_k and that $\|h_k\|_{**} \to 0$ as $k \to \infty$. Then $\|\phi_k\|_* \to 0$ as $k \to \infty$.*

Proof. We argue by contradiction. Without loss of generality, we may assume that $\|\phi_k\|_* \equiv 1$ and $\|h_k\|_{**} \to 0$ as $k \to \infty$. By the potential theory, we have

$$\begin{aligned} \phi_k(y) &= ((m)^* - 1) \int_{\boldsymbol{B}_{\mu_k}(0)} \frac{K(|z|/\mu_k)}{|y - z|^{N-2m}} W_{r_k,\Lambda_k}^{(m)^*-2}(z)\phi_k(z)\,\mathrm{d}z \\ &\quad + \int_{\boldsymbol{B}_{\mu_k}(0)} \frac{1}{|y - z|^{N-2m}}\left[h_k(z) + \sum_{j=1}^{2} c_j \sum_{i=1}^{k} U_{x_i,\Lambda_k}^{(m)^*-2}(z)Z_{i,j}(z)\right]\mathrm{d}z. \end{aligned} \tag{2.2}$$

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For the first term on the right-hand side of (2.2), we make use of lemma 2.2 such that

$$
\begin{aligned}
&|((m)^*-1)\int_{\boldsymbol{B}_{\mu_k}(0)}\frac{K(|z|/\mu_k)}{|y-z|^{N-2m}}W_{r_k,\Lambda_k}^{(m)^*-2}(z)\phi_k(z)\,\mathrm{d}z| \\
&\qquad\leqslant C\|\phi_k\|_*\int_{\boldsymbol{B}_{\mu_k}(0)}\sum_{j=1}^{k}\frac{W_{r_k,\Lambda_k}^{(m)^*-2}(z)}{|y-z|^{N-2m}(1+|z-x_j|)^{(N-2m)/2+\tau}}\,\mathrm{d}z \\
&\qquad\leqslant C\|\phi_k\|_*\int_{\mathbb{R}^N}\left(\sum_{j=1}^{k}\frac{W_{r_k,\Lambda_k}^{(m)^*-2}(z)}{|y-z|^{N-2m}(1+|z-x_j|)^{(N-2m)/2+\tau}}\right)\mathrm{d}z \\
&\qquad\leqslant C\|\phi_k\|_*\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{(N-2m)/2+\tau+\theta}}.
\end{aligned}\tag{2.3}
$$

For the second term on the right-hand side of (2.2), we use lemma 2.1 and obtain that

$$
\begin{aligned}
&\left|\int_{\boldsymbol{B}_{\mu_k}(0)}\frac{1}{|y-z|^{N-2m}}\left[h_k(z)+\sum_{j=1}^{2}c_j\sum_{i=1}^{k}U_{x_k,\Lambda_k}^{(m)^*-2}(z)Z_{i,j}(z)\right]\mathrm{d}z\right| \\
&\quad\leqslant\int_{\boldsymbol{B}_{\mu_k}(0)}\frac{|h_k(z)|}{|y-z|^{N-2m}}\,\mathrm{d}z \\
&\qquad+\sum_{j=1}^{2}|c_j|\sum_{i=1}^{k}\int_{\boldsymbol{B}_{\mu_k}(0)}\frac{1}{|y-z|^{N-2m}}U_{x_i,\Lambda_k}^{(m)^*-2}(z)|Z_{i,j}(z)|\,\mathrm{d}z \\
&\quad\leqslant\|h_k\|_{**}\int_{\boldsymbol{B}_{\mu_k}(0)}\frac{1}{|y-z|^{N-2m}}\left(\sum_{j=1}^{k}\frac{1}{(1+|z-x_j|)^{(N+2m)/2+\tau}}\right)\mathrm{d}z \\
&\qquad+\left(\sum_{j=1}^{2}|c_j|\right)\sum_{i=1}^{k}\frac{1}{(1+|y-x_k|)^{(N-2m)/2+\tau+\theta}}.
\end{aligned}\tag{2.4}
$$

Now we estimate c_j, $j=1,2$, as follows. Multiply both sides of the linearized equation (2.1) by the function $Z_{i,l}$, $l=1,2$, and integrate both sides on the ball $\boldsymbol{B}_{\mu_k}(0)$, such that

$$
\begin{aligned}
&\int_{\boldsymbol{B}_{\mu_k}(0)}\left[(-\Delta)^m\phi_k-((m)^*-1)K\left(\frac{|z|}{\mu_k}\right)W_{r_k,\Lambda_k}^{(m)^*-2}(z)\phi_k(z)\right]Z_{1,l}(z)\,\mathrm{d}z \\
&\qquad\qquad\qquad\qquad-\int_{\boldsymbol{B}_{\mu_k}(0)}h_k(z)Z_{1,l}(z)\,\mathrm{d}z \\
&\qquad=\sum_{j=1}^{2}c_j\sum_{i=1}^{k}\int_{\boldsymbol{B}_{\mu_k}(0)}U_{x_i,\Lambda_k}^{(m)^*-2}(z)Z_{i,j}(z)Z_{1,l}(z)\,\mathrm{d}z \\
&\qquad=[\bar{c}+o_k(1)]c_l.
\end{aligned}\tag{2.5}
$$

The left-hand side of (2.5) can be decomposed into three terms, i.e.

$$\int_{\boldsymbol{B}_{\mu_k}(0)}\left[(-\Delta)^m\phi_k(z)-((m)^*-1)K\left(\frac{|z|}{\mu_k}\right)W_{r_k,\Lambda_k}^{(m)^*-2}(z)\phi_k(z)\right]Z_{i,l}(z)\,\mathrm{d}z \\ -\int_{\boldsymbol{B}_{\mu_k}(0)}h_k(z)Z_{i,l}(z)\,\mathrm{d}z=\mathrm{I}+\mathrm{II}+\mathrm{III}, \quad (2.6)$$

where

$$\mathrm{I}=((m)^*-1)\int_{\boldsymbol{B}_{\mu_k}(0)}[U_{x_i,\Lambda_k}^{(m)^*-2}(z)-W_{r_k,\Lambda_k}^{(m)^*-2}(z)]Z_{i,l}(z)\phi_k(z)\,\mathrm{d}z,$$

$$\mathrm{II}=((m)^*-1)\int_{\boldsymbol{B}_{\mu_k}(0)}\left[1-K\left(\frac{|z|}{\mu_k}\right)\right]W_{r_k,\Lambda_k}^{(m)^*-2}(z)\phi_k(z)Z_{i,l}(z)\,\mathrm{d}z,$$

$$\mathrm{III}=-\int_{\boldsymbol{B}_{\mu_k}(0)}h_k(z)Z_{i,l}(z)\,\mathrm{d}z.$$

For the first term on the right-hand side of (2.6), we estimate that

$$\begin{aligned}|\mathrm{I}|&=\left|((m)^*-1)\int_{\boldsymbol{B}_{\mu_k}(0)}[U_{x_i,\Lambda_k}^{(m)^*-2}(z)-W_{r_k,\Lambda_k}^{(m)^*-2}(z)]Z_{i,l}(z)\phi_k(z)\,\mathrm{d}z\right|\\ &\leqslant C\sum_{j\neq i}\int_{\boldsymbol{B}_{\mu_k}(0)}\frac{|x_j-x_i||\phi_k(z)|\,\mathrm{d}z}{(1+|z-x_j|)^{2m}(1+|z-x_i|)^N}\\ &\leqslant C\|\phi_k\|_*\sum_{j\neq i}\int_{\mathbb{R}^N}\left[\sum_{l=1}^k(|x_j-x_i|)((1+|z-x_j|)^{2m}(1+|z-x_i|)^N\right.\\ &\qquad\qquad\left.\times(1+|z-x_l|)^{(N-2m)/2+\tau})^{-1}\right]\mathrm{d}z\\ &\leqslant o(\|\phi_k\|_*). \end{aligned} \quad (2.7)$$

For the second term on the right-hand side of (2.6), we choose an annular region

$$A_k=\{z\in\boldsymbol{B}_{\mu_k}(0)\mid |z|-\mu_k r_0|\leqslant\mu_k^{1/2}\}$$

such that

$$\begin{aligned}|\mathrm{II}|&=\left|((m)^*-1)\int_{\boldsymbol{B}_{\mu_k}(0)}\left[1-K\left(\frac{|z|}{\mu_k}\right)\right]W_{r_k,\Lambda_k}^{(m)^*-2}(z)\phi_k(z)Z_{i,l}(z)\,\mathrm{d}z\right|\\ &\leqslant C\|\phi_k\|_*\left(\int_{\boldsymbol{B}_{\mu_k}(0)\cap A_k}\sum_{j=1}^k\frac{|1-K(|z|/\mu_k)|W_{r_k,\Lambda_k}^{(m)^*-2}(z)}{(1+|z-x_i|)^{N-2m}(1+|z-x_j|)^{(N-2m)/2+\tau}}\,\mathrm{d}z\right.\\ &\qquad\left.+\int_{\boldsymbol{B}_{\mu_k}(0)\backslash A_k}\sum_{j=1}^k\frac{|1-K(|z|/\mu_k)|W_{r_k,\Lambda_k}^{(m)^*-2}(z)}{(1+|z-x_i|)^{N-2m}(1+|z-x_j|)^{(N-2m)/2+\tau}}\,\mathrm{d}z\right). \end{aligned} \quad (2.8)$$

Recall that $K(1)=1$, $K'(1)>0$ in lemma 2.2 and the right-hand side of (2.8) are all bounded, with

$$\begin{aligned}
&\|\phi_k\|_*\bigg(\int_{\boldsymbol{B}_{\mu_k}(0)\cap A_k}\sum_{j=1}^{k}\frac{|1-K(|z|/\mu_k)|W_{r_k,\Lambda_k}^{(m)^*-2}(z)}{(1+|z-x_i|)^{N-2m}(1+|z-x_j|)^{(N-2m)/2+\tau}}\,\mathrm{d}z\\
&\qquad+\int_{\boldsymbol{B}_{\mu_k}(0)\setminus A_k}\sum_{j=1}^{k}\frac{|1-K(|z|/\mu_k)|W_{r_k,\Lambda_k}^{(m)^*-2}(z)}{(1+|z-x_i|)^{N-2m}(1+|z-x_j|)^{(N-2m)/2+\tau}}\,\mathrm{d}z\bigg)\\
&\leqslant C\|\phi_k\|_*\int_{\mathbb{R}^N}\frac{W_{r_k,\Lambda_k}^{(m)^*-2}(z)}{(1+|z-x_i|)^{N-2m}}\\
&\qquad\qquad\times\sum_{j=1}^{k}\bigg[\frac{\mu_k^{-l}}{(1+|z-x_j|)^{(N-2m)/2+\tau}}+\frac{\mu_k^{-\sigma}}{(1+|z-x_j|)^{(N-2m)/2+\tau-2\sigma}}\bigg]\,\mathrm{d}z\\
&\leqslant C\|\phi_k\|_*\sum_{j=1}^{k}\bigg[\frac{\mu_k^{-l/2}}{(1+|x_i-x_j|)^{(N-2m)/2+\tau+\theta}}+\frac{\mu_k^{-\sigma}}{(1+|x_i-x_j|)^{(N-2m)/2+\tau-2\sigma+\theta}}\bigg]\\
&\leqslant C\|\phi_k\|_*[\mu_k^{-l/2}+\mu_k^{-\sigma}]. \qquad (2.9)
\end{aligned}$$

For the third term on right-hand side of (2.6), we estimate that

$$\begin{aligned}
|\mathrm{III}|&=\bigg|\int_{\boldsymbol{B}_{\mu_k}(0)}h_k(z)Z_{i,l}(z)\,\mathrm{d}z\bigg|\\
&\leqslant C\|h_k\|_{**}\int_{\boldsymbol{B}_{\mu_k}(0)}\frac{1}{(1+|z-x_j|)^{N-2m}}\sum_{j=1}^{k}\frac{1}{(1+|z-x_j|)^{(N+2m)/2+\tau}}\,\mathrm{d}z\\
&\leqslant C\|h_k\|_{**}\sum_{j=1}^{k}\frac{1}{(1+|x_i-x_j|)^{(N-2m)/2+\tau}}\\
&\leqslant C\|h_k\|_{**}. \qquad (2.10)
\end{aligned}$$

Combining (2.7)–(2.10), and considering the assumption $\|\phi_k\|_*\equiv 1$, we get the weighted estimate of ϕ_k such that

$$\begin{aligned}
1&=\sup_{y\in\boldsymbol{B}_{\mu_k}(0)}\bigg\{|\phi_k(y)|\bigg[\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{(N-2m)/2+\tau}}\bigg]^{-1}\bigg\}\\
&\leqslant\sup_{y\in\boldsymbol{B}_{\mu_k}(0)}\bigg\{((m)^*-1)\int_{\boldsymbol{B}_{\mu_k}(0)}\frac{K(|z|/\mu_k)W_{r_k,\Lambda_k}^{(m)^*-2}(z)|\phi_k(z)|}{|y-z|^{N-2m}}\,\mathrm{d}z\\
&\qquad\qquad\times\bigg[\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{(N-2m)/2+\tau}}\bigg]^{-1}\\
&\qquad\qquad+\int_{\boldsymbol{B}_{\mu_k}(0)}\frac{h_k(z)}{|y-z|^{N-2m}}\,\mathrm{d}z\bigg[\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{(N-2m)/2+\tau}}\bigg]^{-1}
\end{aligned}$$

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$$+\sum_{l=1}^{2}|c_l|\int_{\boldsymbol{B}_{\mu_k}(0)}\frac{\sum_{i=1}^{k}U_{x_i,\Lambda_k}^{(m)^*-2}(z)Z_{i,l}(z)}{|y-z|^{N-2m}}\,\mathrm{d}z \times\left[\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{(N-2m)/2+\tau}}\right]^{-1}\Bigg\}$$

$$\leqslant C\left[o_k(1)+\|h_k\|_{**}+\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{(N-2m)/2+\tau+\theta}} \times\left(\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{(N-2m)/2+\tau}}\right)^{-1}\right]. \qquad (2.11)$$

We claim that there exist some $i_0\in\{1,2,\dots,k\}$, $a>0$ and $R>0$ large enough such that $\|\phi_k\|_{L^\infty(\boldsymbol{B}_R(x_{i_0})\cap\boldsymbol{B}_{\mu_k}(0))}\geqslant a>0$. Otherwise, there exist some k large enough, $a\in(0,\frac{1}{2})$, $R>0$ such that $C[o_k(1)+\|h_k\|_{**}]<\frac{1}{2}$ and

$$\|\phi_k\|_{L^\infty(\boldsymbol{B}_R(x_i)\cap\boldsymbol{B}_{\mu_k}(0))}<a<1 \quad \text{for all } i=1,2,\dots,k.$$

Therefore, at some point $y\in\boldsymbol{B}_{\mu_k}(0)\setminus\bigcup_{i=1}^{k}\boldsymbol{B}_R(x_i)$, we get

$$C\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{(N-2m)/2+\tau+\theta}}\left(\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{(N-2m)/2+\tau}}\right)^{-1} \leqslant C\max_{1\leqslant j\leqslant k}\frac{1}{(1+|y-x_j|)^{\theta}}\leqslant\frac{C}{(1+R)^{\theta}}<\frac{1}{2}. \qquad (2.12)$$

Thus, $\|\phi_k\|_*$ can be bounded by a number strictly less than 1, such that

$$1=\|\phi_k\|_* \leqslant C\sup_{y\in\boldsymbol{B}_{\mu_k}(0)\setminus\bigcup_{i=1}^{k}\boldsymbol{B}_R(x_i)}\left[o_k(1)+\|h_k\|_{**} +\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{(N-2m)/2+\tau+\theta}} \times\left(\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{(N-2m)/2+\tau}}\right)^{-1}\right]<\frac{1}{2}, \qquad (2.13)$$

which is a contradiction. Hence, we have proved that $\|\phi_k\|_{L^\infty(\boldsymbol{B}_R(x_{i_0})\cap\boldsymbol{B}_{\mu_k}(0))}\geqslant a>0$ for some positive a and R. Therefore, the translated form $\bar{\phi}_k(y)=:\phi_k(y-x_{i_0})$ converges to a non-trivial $\bar{\phi}$ with $\|\bar{\phi}\|_{L^\infty(\mathbb{R}^N)}\geqslant a>0$, and $\bar{\phi}$ solves the eigenvalue problem

$$(-\Delta)^m\bar{\phi}-((m)^*-1)U_{0,\Lambda_0}^{(m)^*-2}\bar{\phi}=0 \text{ in } \mathbb{R}^N \quad \text{for some } \Lambda_0\in[L_0,L_1], \qquad (2.14)$$

which means $\bar{\phi}\in\operatorname{Ker}(L)$, where $L:=[(-\Delta)^m-((m)^*-1)U_{0,\Lambda_0}^{(m)^*-2}I]$.

In addition, by passing to the limit $k\to\infty$ in

$$\int_{\boldsymbol{B}_{\mu_k}(0)}U_{x_i,\Lambda_k}^{(m)^*-2}(y)Z_{i,l}(y)\phi_k(y)\,\mathrm{d}y=0,$$

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we have

$$\int_{\mathbb{R}^N} U_{0,\Lambda_0}^{(2m)^*-2}(y) Z_p(y)\bar{\phi}(y)\,\mathrm{d}y = 0, \quad p = 1, 2, \ldots, N+1,$$

where

$$\mathrm{Ker}(L) = \mathrm{span}\left\{ Z_i(y) := \frac{\partial U_{0,\Lambda_0}}{\partial y_i},\ i = 1, 2, \ldots, N; \right.$$
$$\left. Z_{N+1}(y) := y \cdot \nabla U_{0,\Lambda_0}(y) + \frac{N-2m}{2} U_{0,\Lambda_0}(y) \right\}.$$

Hence, $\bar{\phi} \in \mathrm{Ker}(L) \cap \mathrm{Ker}(L)^{\perp} = \{0\}$, which contradicts the condition $\|\bar{\phi}\|_{L^\infty(\mathbb{R}^N)} \geqslant a > 0$. □

Similarly to the proofs of [8, proposition 4.1] and [14, proposition 3.1], we can find the unique solution of (2.1) by the Fredholm alternative and Riesz representation arguments. The existence result can be stated as follows.

Proposition 2.4. *There exist some $k_0 > 0$ and a constant $C > 0$, both independent of k, such that, for all $k \geqslant k_0$ and all $h \in L^\infty(\mathbb{R}^N)$, the linearized problem (2.1) has a unique solution $\phi_k = L_k^{-1}(h)$ with $\|L_k^{-1}(h)\|_* \leqslant C\|h\|_{**}$.*

Next we consider the following problem in terms of μ_k and Λ_k:

$$(-\Delta)^m (W_{r_k,\Lambda_k} + \phi_k) = K\left(\frac{|y|}{\mu_k}\right)(W_{r_k,\Lambda_k} + \phi_k)^{(m)^*-1} + \sum_{j=1}^{2} c_j \sum_{i=1}^{k} U_{x_i,\Lambda_k}^{(m)^*-2} Z_{i,j} \quad \text{in } \boldsymbol{B}_{\mu_k}(0). \tag{2.15}$$

Proposition 2.5. *There exists an integer $k_0 > 0$ such that, for each $k \geqslant k_0$, $\Lambda_k \in [L_0, L_1]$ and*

$$r_k \in \left[\mu_k\left(1 - \frac{r_0}{\mu_k}\right), \mu_k\left(1 - \frac{r_1}{k}\right)\right],$$

the perturbation problem (2.15) has a unique solution ϕ_k satisfying

$$\|\phi_k\|_* \leqslant C\left(\frac{1}{\mu_k}\right)^{l/2+\sigma},$$

where $\sigma > 0$ is a small constant.

In order to prove proposition 2.5, we rewrite the perturbation problem (2.15) as the following linearized problem:

$$L_k(\phi_k) = N_k(\phi_k) + l_k + \sum_{j=1}^{2} c_j \sum_{i=1}^{k} U_{x_i,\Lambda_k}^{(m)^*-2} Z_{i,j} \quad \text{in } \boldsymbol{B}_{\mu_k}(0), \tag{2.16}$$

where

$$N_k(\phi_k) = K\left(\frac{|y|}{\mu_k}\right)[(W_{r_k,\Lambda_k} + \phi_k)^{(m)^*-1} - W_{r_k,\Lambda_k}^{(m)^*-1} - ((m)^* - 1)W_{r_k,\Lambda_k}^{(m)^*-2}\phi_k]$$

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is the nonlinear term dependent on ϕ_k, while

$$l_k = K\left(\frac{|y|}{\mu_k}\right) W_{r_k,\Lambda_k}^{(m)^*-1}(y) - \sum_{j=1}^{k} U_{x_j,\Lambda_k}^{(m)^*-1}(y)$$

is the other nonlinear term, which is independent of ϕ_k.

LEMMA 2.6. *If $N \geqslant 2m+2$, then $\|N_k(\phi_k)\|_{**} \leqslant C\|\phi_k\|_*^{\min\{(m)^*-1,2\}}$.*

Proof. Observe that the number $(m)^* - 1 = (N+2m)/(N-2m)$ is less than or equal to 2 if $N \geqslant 6m$ and remains greater than 2 if $2m+2 < N \leqslant 6m-1$. By applying the mean-value theorem twice, there exists some $s \in (0,1)$ such that

$$\begin{aligned} |N_k(\phi_k)| &= \tfrac{1}{2}((m)^*-1)((m)^*-2)|W_{r_k,\Lambda_k} + s\phi_k|^{(m)^*-3}|\phi_k|^2 \\ &\leqslant \begin{cases} C|\phi_k|^{(m)^*-1}, & N \geqslant 6m, \\ CW_{r_k,\Lambda_k}^{(m)^*-3}|\phi_k|^2, & N < 6m. \end{cases} \end{aligned} \tag{2.17}$$

In the following, we make use of the discrete version of the Hölder inequality,

$$\sum_{j=1}^{k} a_j b_j \leqslant \left(\sum_{j=1}^{k} a_j^p\right)^{1/p} \left(\sum_{j=1} b_j^q\right)^{1/q}, \quad a_j, b_j \geqslant 0, \quad \frac{1}{p} + \frac{1}{q} = 1, \tag{2.18}$$

and discuss the estimates for $N_k(\phi_k)$ for two cases.

CASE 1 ($N \geqslant 6m$). Observe that

$$\tau\frac{N-2m}{N-2m+l} \leqslant C \quad \text{and} \quad \sum_{j=1}^{k} \frac{1}{(1+|y-x_j|)^{\tau}} \leqslant C.$$

It then follows from (2.17) and (2.18) that

$$\begin{aligned} |N_k(\phi_k)(y)| &\leqslant C\|\phi_k\|_*^{(m)^*-1}\Bigg[\left(\sum_{j=1}^{k} \frac{1}{(1+|y-x_j|)^{(N+2m)/2+\tau}}\right)^{(N-2m)/(N+2m)} \\ &\qquad \times \left(\sum_{j=1}^{k} \frac{1}{(1+|y-x_j|)^{\tau}}\right)^{4m/(N+2m)}\Bigg]^{(N+2m)/(N-2m)} \\ &\leqslant C\|\phi_k\|_*^{(m)^*-1} \sum_{j=1}^{k} \frac{1}{(1+|y-x_j|)^{(N+2m)/2+\tau}}. \end{aligned} \tag{2.19}$$

Hence, $\|N_k(\phi_k)\|_{**} \leqslant C\|\phi_k\|_*^{(m)^*-1}$.

CASE 2 ($2m+2 \leqslant N \leqslant 6m-1$). By the same reasoning as case 1, we have

$$\begin{aligned} |N_k(\phi_k)| &\leqslant C\|\phi_k\|_*^2 \left(\sum_{j=1}^{k} \frac{1}{(1+|y-x_j|)^{(N-2m)/2+\tau}}\right)^2 \\ &\qquad \times \left[\sum_{j=1}^{k} \frac{1}{(1+|y-x_j|)^{N-6m}}\right]^{(6m-N)/(N-2m)} \end{aligned}$$

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$$\leqslant C\|\phi_k\|_*^2\bigg(\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{(N-2m)/2+\tau}}\bigg)^{(m)^*-1}$$

$$\leqslant C\|\phi_k\|_*^2\bigg(\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{(N+2m)/2+\tau}}\bigg). \tag{2.20}$$

Thus, we get $\|N_k(\phi_k)\|_{**} \leqslant C\|\phi_k\|_*^2$, as desired. □

LEMMA 2.7. *Assume* $N \geqslant 2m+2$, $r_k \in [\mu_k(1-r_0/\mu_k), \mu_k(1-r_1/k)]$. *Then*

$$\|l_k\|_{**} \leqslant C\bigg(\frac{1}{\mu_k}\bigg)^{l/2+\sigma}.$$

Proof. We divide the ball region $\boldsymbol{B}_{\mu_k}(0)$ into k slices:

$$\Omega_j := \bigg\{y \in \boldsymbol{B}_{\mu_k}(0) \;\bigg|\; y=(y',y'') \in \mathbb{R}^2 \times \mathbb{R}^{N-2},\ \bigg\langle \frac{y'}{|y'|}, \frac{x_j}{|x_j|}\bigg\rangle \geqslant \cos\bigg(\frac{\pi}{k}\bigg)\bigg\}, \quad j=1,2,\dots,k.$$

Then we set $l_k = J_0 + J_1 + J_2$, where

$$\left.\begin{aligned} J_0(y) &:= K\bigg(\frac{|y|}{\mu_k}\bigg)(W_{r_k,\Lambda_k}^{(m)^*-1} - \sum_{j=1}^{k}(P_{\mu_k}U_{x_j,\Lambda_k})^{(m)^*-1}),\\ J_1(y) &:= K\bigg(\frac{|y|}{\mu_k}\bigg)\bigg(\sum_{j=1}^{k}(P_{\mu_k}U_{x_j,\Lambda_k})^{(m)^*-1} - \sum_{j=1}^{k}U_{x_j,\Lambda_k}^{(m)^*-1}\bigg),\\ J_2(y) &:= \sum_{j=1}^{k}U_{x_j,\Lambda_k}^{(m)^*-1}\bigg(K\bigg(\frac{|y|}{\mu_k}\bigg)-1\bigg). \end{aligned}\right\} \tag{2.21}$$

Note that the ball region $\boldsymbol{B}_{\mu_k}(0)$ is evenly divided by k slices $\Omega_1, \Omega_2, \dots, \Omega_k$. It is sufficient to consider Ω_1; the discussion on the other regions $\Omega_2, \dots, \Omega_k$ is similar.

Let $y \in \Omega_1$. Then

$$|y-x_j| \geqslant |y-x_1| \quad \text{and} \quad \frac{1}{1+|y-x_j|} \leqslant \frac{C}{|x_j-x_1|} \quad \text{for all } j \neq 1.$$

Choosing some $\alpha \in (\max\{\frac{1}{2}(N-2m),1\}, \min\{4m, N-2m\})$, we have

$$\begin{aligned} |J_0(y)| &\leqslant C\sum_{j=2}^{k}\frac{1}{(1+|y-x_1|)^{4m}(1+|y-x_j|)^{N-2m}}\\ &\quad + C\bigg(\sum_{j=2}^{k}\frac{1}{(1+|y-x_j|)^{N-2m}}\bigg)^{(m)^*-1}\\ &\leqslant C\frac{1}{(1+|y-x_1|)^{N+2m-\alpha}}\frac{1}{|x_j-x_i|^{\alpha}}\\ &\quad + C\sum_{j=2}^{k}\frac{1}{(1+|y-x_j|)^{(N+2m)/2+\tau}} \end{aligned}$$

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$$\times \left[\sum_{j=2}^{k} \frac{1}{(1+|y-x_j|)^{((N+2m)/4)((N-2m)/2-\tau(N-2m)/(N+2m))}} \right]^{4/(N-2m)}$$
$$\leqslant C \sum_{j=1}^{k} \frac{1}{(1+|y-x_j|)^{(N+2m)/2+\tau}} \left(\frac{1}{\mu_k}\right)^{l/2+\sigma}. \tag{2.22}$$

Thus, $\|J_0\|_{**} \leqslant C(1/\mu_k)^{l/2+\sigma}$.

On the other hand, it is known that there exists a Green function $G_m\colon (\boldsymbol{B}_1(0) \times \boldsymbol{B}_1(0)) \to \mathbb{R}$ (see [9, ch. 4] for Boggio's formula) such that

$$\left.\begin{aligned} (-\Delta_x)^m G_m(x,y) &= \delta(x), \\ G_m(x,y) &= G_m(y,x) \quad \forall x,y \in \boldsymbol{B}_1(0), \\ D^\gamma G_m(x,y)|_{\partial B_1(0)} &= 0, \quad |\gamma| \leqslant m-1, \end{aligned}\right\} \tag{2.23}$$

where δ represents the Dirac function on the unit ball $\boldsymbol{B}_1(0)$.

We define the regular part $H_m\colon (\boldsymbol{B}_1(0) \times \boldsymbol{B}_1(0)) \to \mathbb{R}$ such that

$$H_m(x,y) = \frac{P_{m,N}}{|x-y|^{N-2m}} - G_m(x,y).$$

Let $\bar{x}_j = x_j/\mu_k, \bar{y} = y/\mu_k \in \boldsymbol{B}_1(0)$. Then it holds (see [10, proposition 3.1] and [3, 19]) that

$$\frac{H_m(\bar{y},\bar{x}_j)}{\mu_k^{N-2m}} = \frac{C}{\mu_k^{N-2m}|\bar{y}-\bar{x}_j|^{N-2m}} \leqslant \frac{C}{(1+|y-x_j|)^{N-2m}}, \tag{2.24}$$

$$U_{x_j,\Lambda_k}(y) - P_k U_{x_j,\Lambda_k}(y) = \frac{H_m(\bar{y},\bar{x}_j)}{\mu_k^{N-2m}} + O\left(\frac{1}{d_k^N \mu_k^N}\right), \quad d_k = \left(1 - \frac{r_k}{\mu_k}\right)^{1/l}. \tag{2.25}$$

Using (2.24) and (2.25), and letting $t = 1/2 + \sigma/l$, we have

$$\begin{aligned} |J_1(y)| &\leqslant C \sum_{j=1}^{k} \frac{|P_k U_{x_j,\Lambda_k}(y) - U_{x_j,\Lambda_k}(y)|}{(1+|y-x_j|)^{4m}} \\ &\leqslant \sum_{j=1}^{k} \frac{C}{(1+|y-x_j|)^{4m}} \left[\frac{H_m(\bar{y},\bar{x}_j)}{\mu_k^{N-2m}} + O\left(\frac{1}{d_k^N \mu_k^N}\right) \right] \\ &\leqslant \sum_{j=1}^{k} \frac{C}{(1+|y-x_j|)^{4m+(N-2m)t}} \left(\frac{H_m(\bar{y},\bar{x}_j)}{\mu_k^{N-2m}} \right)^t + \sum_{j=1}^{k} \frac{C}{d_k^N \mu_k^N (1+|y-x_j|)^{4m}} \\ &\leqslant \frac{C}{\mu_k^{(N-2m)t}} \sum_{j=1}^{k} \frac{1}{(1+|y-x_j|)^{4m+(N-2m)t}} + \frac{C}{d_k^N \mu_k^N} \sum_{j=1}^{k} \frac{1}{(1+|y-x_j|)^{4m}} \\ &\leqslant C\left(\frac{1}{\mu_k}\right)^{lt} \sum_{j=1}^{k} \frac{1}{(1+|y-x_j|)^{4m+(N-2m)t}} + C\left(\frac{1}{\mu_k}\right)^N \sum_{j=1}^{k} \frac{1}{(1+|y-x_j|)^{4m}} \\ &\leqslant C\left(\frac{1}{\mu_k}\right)^{l/2+\sigma} \sum_{j=1}^{k} \frac{1}{(1+|y-x_j|)^{(N+2m)/2+\tau}}. \end{aligned} \tag{2.26}$$

Hence, $\|J_1\|_{**} \leqslant C/\mu_k^{l/2+\sigma}$.

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Now we define

$$J_2 := \sum_{j=2}^{k}\left(K\left(\frac{|y|}{\mu_k}\right)-1\right)U_{x_j,\Lambda_k}^{(m)^*-1}(y)+K\left(\frac{|y|}{\mu_k}\right)U_{x_1,\Lambda_k}^{(m)^*}(y)$$

for the following two cases.

Case 1 $((K(|y|/\mu_k)-1)U_{x_j,\Lambda_k}^{(m)^*}(y),\ j=2,3,\dots,k)$. Using that $|y-x_j|\geqslant|y-x_1|$ and $1/(1+|y-x_j|)\leqslant C/|x_j-x_1|$ for $j\neq 1$, we have

$$\left|\left(K\left(\frac{|y|}{\mu_k}\right)-1\right)U_{x_j,\Lambda_k}^{(m)^*}(y)\right| \leqslant \frac{C}{(1+|y-x_j|)^{(N+2m)/2+\tau}(1+|y-x_j|)^{(N+2m)/2-\tau}} \leqslant \frac{C}{(1+|y-x_1|)^{(N+2m)/2+\tau}|x_1-x_j|^{(N+2m)/2-\tau}}. \tag{2.27}$$

Case 2 $((K(|y|/\mu_k)-1)U_{x_j,\Lambda_k}^{(m)^*-1}(y),\ j=1)$. In this case, we divide Ω_1 by I and II, where $\mathrm{I} := \{y\in\Omega_1 \mid ||y|-\mu_k|\geqslant\sigma\mu_k\}$ and $\mathrm{II} := \{y\in\Omega_1 \mid ||y|-\mu_k|<\delta\mu_k\}$. For $y\in\mathrm{I}\subset\Omega_1$, we observe that $||y|-\mu_k|\geqslant\sigma\mu_k$, where $\sigma>0$ is a fixed constant. Then $||y|-|x_1||\geqslant||y|-\mu_k|-||x_1|-\mu_k|\geqslant\frac{1}{2}\sigma\mu_k$, and hence

$$\left|\left(K\left(\frac{|y|}{\mu_k}\right)-1\right)U_{x_1,\Lambda_k}^{(m)^*-1}(y)\right| \leqslant \frac{C}{(1+|y-x_1|)^{N+2m}} \leqslant \frac{C}{(1+|y-x_1|)^{(N+2m)/2+\tau}}\left(\frac{1}{\mu_k}\right)^{(N+2m)/2-\tau} \leqslant \frac{C}{(1+|y-x_1|)^{(N+2m)/2+\tau}}\left(\frac{1}{\mu_k}\right)^{l/2+\sigma}. \tag{2.28}$$

For $y\in\mathrm{II}\subset\Omega_1$, we notice that $||y|-|x_1||\leqslant||y|-\mu_k|+|\mu_k-|x_1||\leqslant 2\sigma\mu_k$, and

$$\frac{||y|-|x_1||^l}{\mu_k^l(1+|y-x_1|)^{N+2m}} \leqslant \frac{C|y-x_1|^{l/2+\sigma}}{\mu_k^{l/2+\sigma}(1+|y-x_1|)^{N+2m}} \leqslant \frac{C}{\mu_k^{l/2+\sigma}(1+|y-x_1|)^{N+2m-l/2-\sigma}} \leqslant \frac{C}{(1+|y-x_1|)^{(N+2m)/2+\tau}}\left(\frac{1}{\mu_k}\right)^{l/2+\sigma}. \tag{2.29}$$

Therefore, the estimate on subregion II is as follows:

$$\left|\left(K\left(\frac{|y|}{\mu_k}\right)-1\right)U_{x_1,\Lambda_k}^{(m)^*-1}(y)\right| \leqslant \frac{C||y|-|x_1||^l}{\mu_k^l(1+|y-x_1|)^{N+2m}}+\frac{C}{\mu_k^{l/2+\sigma}(1+|y-x_1|)^{(N+2m)/2+\tau}}$$

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$$\leqslant \frac{C}{\mu_k^{l/2+\sigma}(1+|y-x_1|)^{(N+2m)/2+\tau}} + \frac{C}{\mu_k^{l/2+\sigma}(1+|y-x_1|)^{(N+2m)/2+\tau}}$$
$$\leqslant \frac{C}{(1+|y-x_1|)^{(N+2m)/2+\tau}}\left(\frac{1}{\mu_k}\right)^{l/2+\sigma}. \tag{2.30}$$

Combining (2.28) and (2.30), the pointwise estimate of J_2 in the slice Ω_1 can be written as

$$|J_2(y)| \leqslant \sum_{j=2}^{k}\left|\left(K\left(\frac{|y|}{\mu_k}\right)-1\right)U_{x_j,\Lambda_k}^{m^*-1}(y)\right| + \left|K\left(\frac{|y|}{\mu_k}-1\right)U_{x_1,\Lambda_k}^{m^*-1}(y)\right|$$
$$\leqslant C\left(\sum_{j=2}^{k}\frac{1}{|x_1-x_j|^{(N+2m)/2-\tau}} + \frac{1}{\mu_k^{l/2+\sigma}}\right)\frac{1}{(1+|y-x_1|)^{(N+2m)/2+\tau}}$$
$$\leqslant \frac{C}{\mu_k^{l/2+\sigma}(1+|y-x_1|)^{(N+2m)/2+\tau}}. \tag{2.31}$$

By symmetry, we have that the pointwise estimate (2.31) holds for all $y \in \boldsymbol{B}_{\mu_k}(0)$. Thus,

$$\|l_k\|_{**} \leqslant \|J_0\|_{**} + \|J_1\|_{**} + \|J_2\|_{**} \leqslant C\left(\frac{1}{\mu_k}\right)^{l/2+\sigma},$$

as desired. □

Now we are ready to prove proposition 2.5.

Proof of proposition 2.5. Let $E_k = \{\xi_k \in X \mid \|\xi_k\|_* \leqslant C(1/\mu_k)^{l/2+\sigma}\}$. Then the linearized problem (2.16) is equivalent to the fixed-point problem $\phi_k = A_k(\phi_k) := L_k^{-1}(N(\phi_k)) + L_k^{-1}(l_k)$. By propositions 2.3 and 2.4, we see that L_k^{-1} is a bounded operator from $(Y, \|\cdot\|_{**})$ to $(X, \|\cdot\|_*)$. Considering lemmas 2.6 and 2.7, for any $\phi_k \in E_k$ we have

$$\|A_k(\phi_k)\|_* \leqslant C[\|N_k(\phi_k)\|_{**} + \|l_k\|_{**}]$$
$$\leqslant C(\|\phi_k\|_*^{\min\{m^*-1,2\}} + \mu_k^{-(l/2+\sigma)})$$
$$\leqslant C\left(\frac{1}{\mu_k}\right)^{l/2+\sigma}, \tag{2.32}$$

which implies that A_k maps E_k to E_k itself. By the fixed-point theory, it is sufficient to prove that A_k is a contraction map. Choose any two different elements ψ_1, ψ_2 in E_k. Then, by applying the mean-value theorem twice, there exist some $s, t \in (0,1)$ such that

$$\|A_k(\psi_1) - A_k(\psi_2)\|_* = \|L_k^{-1}(N_k(\psi_1) - N_k(\psi_2))\|_*$$
$$\leqslant C\|N_k(\psi_1) - N_k(\psi_2)\|_{**}$$
$$\leqslant C\|N_k'(s\psi_1 + (1-s)\psi_2)(\psi_1 - \psi_2)\|_{**}, \tag{2.33}$$

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where

$$|N_k'(s\psi_1+(1-s)\psi_2)| = |(W_{r_k,\Lambda_k}+s\psi_1+(1-s)\psi_2)^{(m)^*-2} - W_{r_k,\Lambda_k}^{(m)^*-2}|$$
$$\leqslant C|W_{r_k,\Lambda_k}+ts\psi_1+t(1-s)\psi_2|^{(m)^*-3}|s\psi_1+(1-s)\psi_2|$$
$$\leqslant \begin{cases} C(|\psi_1|+|\psi_2|)^{(m)^*-2} & \text{if } N \geqslant 6m, \\ CW_{r_k,\Lambda_k}^{(m)^*-3}(|\psi_1|+|\psi_2|) & \text{if } N < 6m. \end{cases} \tag{2.34}$$

Similarly to the arguments in lemma 2.6, we estimate the difference $A_k(\psi_1) - A_k(\psi_2)$ by

$$\|A_k(\psi_1)-A_k(\psi_2)\|_* \leqslant \begin{cases} C(\|\psi_1\|_*^{(m)^*-2}+\|\psi_2\|_*^{(m)^*-2})\|\psi_1-\psi_2\|_* & \text{if } N \geqslant 6m, \\ C(\|\psi_1\|_*+\|\psi_2\|_*)\|\psi_1-\psi_2\|_* & \text{if } N < 6m. \end{cases} \tag{2.35}$$

Choose k sufficiently large that, for all $\psi_1, \psi_2 \in E_k$,

$$C(\|\psi_1\|_*^{(m)^*-2}+\|\psi_2\|_*^{(m)^*-2}) < \tfrac{1}{2}.$$

Then A_k is a contraction map from E_k to itself. By the Banach fixed-point theorem, there is a unique fixed point $\phi_k \in E_k$ such that $\phi_k = A_k(\phi_k)$ and

$$\|\phi_k\|_* = \|A_k(\phi_k)\|_* \leqslant C\left(\|\phi_k\|_*^{\min\{(m)^*-1,2\}} + \left(\frac{1}{\mu_k}\right)^{l/2+\sigma}\right)$$
$$\leqslant C\left(\frac{1}{\mu_k}\right)^{l/2+\sigma}. \tag{2.36}$$

□

3. Energy expansion

The idea of the energy expansion comes from the observation that the nonlinear energy can be approximated by a linear combination of simple terms with the parameters μ_k and Λ_k for k large enough. We define a perturbed energy functional $F_k\colon \mathbb{R}^2 \to \mathbb{R}$ by

$$F_k(d_k,\Lambda_k) := I_k(W_{r_k,\Lambda_k}+\phi_k),$$

where $d_k = (1-r_k/\mu_k)^{1/l}$, ϕ_k is the perturbed solution obtained in the linearized problem (2.1) and $I_k\colon \mathcal{D}^{m,2}(\boldsymbol{B}_{\mu_k}(0)) \to \mathbb{R}$ is defined by

$$I_k(u_k) = \begin{cases} \dfrac{1}{2}\displaystyle\int_{\boldsymbol{B}_{\mu_k}(0)} |\Delta^{m/2}u_k|^2 - \dfrac{1}{(m)^*}\displaystyle\int_{\boldsymbol{B}_{\mu_k}(0)} K\left(\dfrac{|y|}{\mu_k}\right)|u_k|^{(m)^*}, & m \text{ even}, \\ \dfrac{1}{2}\displaystyle\int_{\boldsymbol{B}_{\mu_k}(0)} |\nabla\Delta^{(m-1)/2}u_k|^2 - \dfrac{1}{(m)^*}\displaystyle\int_{\boldsymbol{B}_{\mu_k}(0)} K\left(\dfrac{|y|}{\mu_k}\right)|u_k|^{(m)^*}, & m \text{ odd}. \end{cases} \tag{3.1}$$

PROPOSITION 3.1. *If $N \geqslant 2m+2$, we have*

$$I_k(W_{r_k,\Lambda_k}) = k\biggl[A + \frac{B_1 H(\bar{x}_1,\bar{x}_1)}{\Lambda_k^{N-2m}\mu_k^{N-2m}} + B_2 K'(1) d_k^l - \sum_{i=1}^{k} \frac{B_3 G_m(\bar{x}_i,\bar{x}_1)}{\Lambda_k^{N-2m}\mu_k^{N-2m}} + O\biggl(\biggl(\frac{1}{\mu_k}\biggr)^{l/2+\sigma}\biggr)\biggr], \tag{3.2}$$

where A, B_1, B_2 and B_3 are positive constants.

Proof. Since $P_k U_{x_i,\Lambda_k}$ satisfies the projected problem (1.8), we can write

$$\sum_{j=1}^{k}\sum_{i=1}^{k}\int_{\boldsymbol{B}_{\mu_k}(0)} U_{x_j,\Lambda_k}^{(m)^*-1} P_k U_{x_i,\Lambda_k} = \begin{cases} \displaystyle\int_{\boldsymbol{B}_{\mu_k}(0)} |\Delta^{m/2} W_{r_k,\Lambda_k}|^2, & m \text{ even}, \\ \displaystyle\int_{\boldsymbol{B}_{\mu_k}(0)} |\nabla\Delta^{(m-1)/2} W_{r_k,\Lambda_k}|^2, & m \text{ odd}. \end{cases} \tag{3.3}$$

Then the energy $I_k(W_{r_k,\Lambda_k})$ can be split into two parts: the positive kinetic energy

$$K_k(W_{r_k,\Lambda_k}) := \frac{1}{2}\sum_{j=1}^{k}\sum_{i=1}^{k}\int_{\boldsymbol{B}_{\mu_k}(0)} U_{x_j,\mu_k}^{(m)^*-1} P_k U_{x_i,\Lambda_k}, \tag{3.4}$$

and the potential energy

$$P_k(W_{r_k,\Lambda_k}) := -\frac{1}{(m)^*}\int_{\boldsymbol{B}_{\mu_k}(0)} K\biggl(\frac{|y|}{\mu_k}\biggr)|W_{r_k,\Lambda_k}|^{(m)^*}. \tag{3.5}$$

By changing variables, we have the following expansion of the kinetic term:

$$\begin{aligned} K_k(W_{r_k,\Lambda_k}) &= \tfrac{1}{2}k\biggl[\int_{\boldsymbol{B}_{\mu_k}(x_1)} U_{0,1}^{(m)^*} - \int_{\boldsymbol{B}_{\mu_k}(0)} U_{x_1,\Lambda_k}^{(m)^*-1}(U_{x_1,\Lambda_k} - P_k U_{x_1,\Lambda_k}) \\ &\qquad + \sum_{i=2}^{k}\int_{\boldsymbol{B}_{\mu_k}(0)} U_{x_1,\Lambda_k}^{(m)^*-1} P_k U_{x_i,\Lambda_k}\biggr] \\ &= \tfrac{1}{2}k\biggl[\int_{\mathbb{R}^N} U_{0,1}^{(m)^*} + O\biggl(\int_{\mathbb{R}^N\setminus\boldsymbol{B}_{\mu_k/k}(0)} U_{0,1}^{(m)^*}\biggr) \\ &\qquad - \int_{\boldsymbol{B}_{\mu_k}(0)} U_{x_1,\Lambda_k}^{(m)^*-1}(y)\frac{H_m(\bar{y},\bar{x}_1)}{\Lambda_k^{N-2m}\mu_k^{N-2m}}\,\mathrm{d}y \\ &\qquad + O\biggl(\frac{1}{\mu_k^N}\biggr) + \frac{1}{\Lambda_k^{N-2m}\mu_k^{N-2m}} \\ &\qquad \times\biggl(\sum_{i=2}^{N}\int_{\boldsymbol{B}_{\mu_k}(0)} U_{x_i,\Lambda_k}(y) G_m(\bar{x}_i,\bar{y}_i)\,\mathrm{d}y\biggr)\biggr] \end{aligned}$$

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$$= \tfrac{1}{2}k\biggl[\bar{A} + O\biggl(\frac{1}{\mu_k^{l(m)^*/2}}\biggr) - \frac{\bar{B}_1 H_m(\bar{x}_1,\bar{x}_1)}{\Lambda_k^{N-2m}\mu_k^{N-2m}} + O\biggl(\frac{1}{\mu_k^N}\biggr) + \frac{\bar{B}_2\sum_{i=2}^k G_m(\bar{x}_i,\bar{x}_i)}{\Lambda_k^{N-2m}\mu_k^{N-2m}}\biggr]. \tag{3.6}$$

Next, we discuss the potential part, $P_k(W_{r_k,\Lambda_k})$. By the symmetry of the integrations on the slices Ω_j, $j = 1, 2, \dots, k$, we have

$$\begin{aligned} P_k(W_{r_k,\Lambda_k}) &= -\frac{k}{(m)^*}\int_{\Omega_1} K\biggl(\frac{|y|}{\mu_k}\biggr)|W_{r_k,\Lambda_k}|^{(m)^*} \\ &= -\frac{k}{(m)^*}\biggl\{\int_{\Omega_1} K\biggl(\frac{|y|}{\mu_k}\biggr)(P_k U_{x_1,\Lambda_k})^{(m)^*} \\ &\quad + (m)^*\int_{\Omega_1}(P_k U_{x_1,\Lambda_k})^{(m)^*-1}\biggl(\sum_{j=2}^k P_k U_{x_j,\Lambda_k}\biggr) \\ &\quad + O\biggl(\int_{\Omega_1}\biggl|K\biggl(\frac{|y|}{\mu_k}\biggr) - 1\biggr|\sum_{j=2}^k U_{x_1,\Lambda_k}^{(m)^*-1}U_{x_j,\Lambda_k} \\ &\quad + \int_{\Omega_1} U_{x_1,\Lambda_k}^{N/(N-2m)}\biggl(\sum_{j=2}^k U_{x_j,\Lambda_k}^{N/(N-2m)}\biggr)\biggr)\biggr\}. \end{aligned} \tag{3.7}$$

The first term on the right-hand side of (3.7) can be calculated as

$$\begin{aligned} &\int_{\Omega_1} K\biggl(\frac{|y|}{\mu_k}\biggr)(P_k U_{x_1,\Lambda_k})^{(m)^*} \\ &\quad = \int_{\Omega_1} U_{x_1,\Lambda_k}^{(m)^*} + \int_{\Omega_1} K\biggl(\frac{|y|}{\mu_k}\biggr)[(P_k U_{x_1,\Lambda_k})^{(m)^*} - U_{x_1,\Lambda_k}^{(m)^*}] \\ &\qquad + \int_{\Omega_1}\biggl(K\biggl(\frac{|y|}{\mu_k}\biggr) - 1\biggr)U_{x_1,\Lambda_k}^{(m)^*} \\ &\quad = \bar{A} - (m)^*\frac{\bar{B}_1 H_m(\bar{x}_1,\bar{x}_1)}{\Lambda_k^{N-2m}\mu_k^{N-2m}} + (K(|x_1|) - 1)\bar{B}_3 + O\biggl(\biggl(\frac{1}{\mu_k}\biggr)^{l/2+\sigma}\biggr) \\ &\quad = \bar{A} - (m)^*\frac{\bar{B}_1 H_m(\bar{x}_1,\bar{x}_1)}{\Lambda_k^{N-2m}\mu_k^{N-2m}} - K'(1)d^l\bar{B}_3 + O\biggl(\biggl(\frac{1}{\mu_k}\biggr)^{l/2+\sigma}\biggr). \end{aligned} \tag{3.8}$$

By the definition of the Green function G_m, the second term can be estimated as

$$\begin{aligned} \int_{\Omega_1}(P_k U_{x_1,\Lambda_k})^{(m)^*-1}\biggl(\sum_{j=2}^k P_k U_{x_j,\Lambda_k}\biggr) &= \frac{\bar{B}_2\sum_{j=2}^k G_m(\bar{x}_j,\bar{x}_1)}{\Lambda_k^{N-2m}\mu_k^{N-2m}} + O\biggl(\frac{k^N}{\mu_k^N}\biggr) \\ &= \frac{\bar{B}_2\sum_{j=2}^k G_m(\bar{x}_j,\bar{x}_1)}{\Lambda_k^{N-2m}\mu_k^{N-2m}} + O\biggl(\biggl(\frac{1}{\mu_k}\biggr)^{l/2+\sigma}\biggr). \end{aligned} \tag{3.9}$$

Since $|y - x_j| \geqslant |y - x_1|$ for all $y \in \Omega_1$, we can find some s close to $N - 2m$ such that the last term can be estimated as follows:

$$\begin{aligned}
&\int_{\Omega_1} \left(K\left(\frac{|y|}{\mu_k}\right) - 1\right) \sum_{j=2}^{k} U_{x_1,\Lambda_k}^{(m)^*-1} U_{x_j,\Lambda_k} + U_{x_1,\Lambda_k}^{(m)^*/2} \left(\sum_{j=2}^{k} U_{x_j,\Lambda_k}\right)^{(m)^*/2} \\
&\quad \leqslant C \int_{\Omega_1} \frac{U_{x_1,\Lambda_k}^{(m)^*-1}}{(1 + |y - x_1|)^{N-2m-s}} \,\mathrm{d}y \left(\sum_{j=2}^{k} \frac{1}{|x_j - x_1|^s}\right) \\
&\qquad + C \int_{\Omega_1} \frac{U_{x_1,\Lambda_k}^{N/(N-2m)}}{(1 + |y - x_1|)^{(N-2m-s)N/(N-2m)}} \,\mathrm{d}y \left(\sum_{j=1}^{k} \frac{1}{|x_j - x_1|^s}\right)^{N/(N-2m)} \\
&\quad \leqslant C \left(\frac{1}{\mu_k}\right)^{l/2+\sigma}.
\end{aligned} \tag{3.10}$$

We now combine the estimates (3.8)–(3.10) for the three terms on the right-hand side of (3.7), such that the potential $P_k(W_{r_k,\Lambda_k})$ can be expanded in the following form:

$$\begin{aligned}
P_k(W_{r_k,\Lambda_k}) = -\frac{k}{(m)^*} \bigg[\bar{A} - (m)^* \frac{\bar{B}_1 H_m(\bar{x}_1, \bar{x}_1)}{\Lambda_k^{N-2m} \mu_k^{N-2m}} + (m)^* \sum_{j=2}^{N} \frac{\bar{B}_2 G_m(\bar{x}_j, \bar{x}_1)}{\Lambda_k^{N-2m} \mu_k^{N-2m}} \\
- K'(1) \,\mathrm{d}^l \bar{B}_3 + O\left(\left(\frac{1}{\mu_k}\right)^{l/2+\sigma}\right)\bigg].
\end{aligned} \tag{3.11}$$

Considering the kinetic expansion (3.6) and potential expansion (3.11), we get

$$\begin{aligned}
I_k(W_{r_k,\Lambda_k}) &= K_k(W_{r_k,\Lambda_k}) + P_k(W_{r_k,\Lambda_k}) \\
&= \tfrac{1}{2} k \bigg[\bar{A} - \frac{\bar{B}_1 H_m(\bar{x}_1, \bar{x}_1)}{\Lambda_k^{N-2m} \mu_k^{N-2m}} + \frac{\bar{B}_2 \sum_{i=2}^{k} G_m(x_1, \bar{x}_i)}{\Lambda_k^{N-2m} \mu_k^{N-2m}} + O(\mu_k^{-l(m)^*/2}) \bigg] \\
&\quad - \frac{k}{(m)^*} \bigg[\bar{A} - (m)^* \frac{\bar{B}_1 H_m(\bar{x}_1, \bar{x}_1)}{\Lambda_k^{N-2m} \mu_k^{N-2m}} \\
&\qquad + (m)^* \sum_{j=2}^{k} \frac{\bar{B}_2 G_m(\bar{x}_j, \bar{x}_1)}{\Lambda_k^{N-2m} \mu_k^{N-2m}} + O\left(\left(\frac{1}{\mu_k}\right)^{l/2+\sigma}\right) \bigg] \\
&= k \bigg[A + \frac{B_1 H_m(\bar{x}_1, \bar{x}_1)}{\Lambda_k^{N-2m} \mu_k^{N-2m}} \\
&\qquad - \frac{B_2 \sum_{j=2}^{k} G_m(\bar{x}_j, \bar{x}_1)}{\Lambda_k^{N-2m} \mu_k^{N-2m}} + B_3 K'(1) \,\mathrm{d}^l + O\left(\left(\frac{1}{\mu_k}\right)^{l/2+\sigma}\right) \bigg],
\end{aligned} \tag{3.12}$$

where

$$A = \left(\frac{1}{2} - \frac{1}{(m)^*}\right) \bar{A}, \qquad B_1 = \frac{\bar{B}_1}{2}, \qquad B_2 = \frac{\bar{B}_2}{2}, \qquad B_3 = \frac{\bar{B}_3}{(m)^*}.$$

□

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In the spirit of the expansion in proposition 3.1, we derive similar expansions for

$$\frac{\partial I_k(W_{r_k,\Lambda_k})}{\partial \Lambda_k} \quad \text{and} \quad \frac{\partial I_k(W_{r_k,\Lambda_k})}{\partial \gamma_k}.$$

PROPOSITION 3.2. *If $N \geqslant 2m+2$, then, for k large enough, we have*

$$\frac{\partial I_k(W_{r_k,\Lambda_k})}{\partial \Lambda_k} = k(N-2m)\biggl[-\frac{B_1 H_m(\bar{x}_1,\bar{x}_1)}{\Lambda_k^{N+1-2m}\mu_k^{N-2m}} + \frac{B_3\sum_{j=2}^k G_m(\bar{x}_j,\bar{x}_1)}{\Lambda_k^{N+1-2m}\mu_k^{N-2m}} + O\biggl(\biggl(\frac{1}{\mu_k}\biggr)^{l/2+\sigma}\biggr)\biggr], \quad (3.13)$$

$$\frac{I_k(W_{r_k,\Lambda_k})}{\partial \gamma_k} = k\biggl[\frac{B_1 \partial H_m(\bar{x}_1,\bar{x}_1)/\partial\gamma_k}{\Lambda_k^{N-2m}\mu_k^{N-2m}} - \frac{B_2 K_k'(1) l d^{l-1}}{\mu_k} - \sum_{j=2}^{k}\frac{B_3 \partial G_m(\bar{x}_j,\bar{x}_1)/\partial\gamma_k}{\Lambda_k^{N+1-2m}\mu_k^{N-2m}} + O\biggl(\biggl(\frac{1}{\mu_k}\biggr)^{l/2+\sigma}\biggr)\biggr]. \quad (3.14)$$

Now we study the respective expansions for the perturbed energy $F_k(d_k,\Lambda_k)$.

PROPOSITION 3.3. *If $N \geqslant 2m+2$, then*

$$F_k(d_k,\Lambda_k) = k\biggl[A + \frac{B_1 H_m(\bar{x}_1,\bar{x}_1)}{\Lambda_k^{N-2m}\mu_k^{N-2m}} + B_2 K_k'(1)\,\mathrm{d}^l - \sum_{j=2}^{k}\frac{B_3 G_m(\bar{x}_1,\bar{x}_j)}{\Lambda_k^{N-2m}\mu_k^{N-2m}} + O\biggl(\biggl(\frac{1}{\mu_k}\biggr)^{l/2+\sigma}\biggr)\biggr]. \quad (3.15)$$

Proof. Observe that ϕ_k is a solution of the linearized problem (2.1). Therefore, $u_k = W_{r_k,\Lambda_k} + \phi_k$ satisfies the equation

$$\langle I_k'(W_{r_k,\Lambda_k}+\phi_k),\phi_k\rangle = 0.$$

Applying the mean-value theorem to $F(d_k,\Lambda_k)$ twice, there exist some $t\in(0,1)$, $s\in(0,1)$ such that

$$\begin{aligned}
&F_k(d_k,\Lambda_k)\\
&= I_k(W_{r_k,\Lambda_k}) - \tfrac{1}{2}\langle D^2 I_k(W_{r_k,\Lambda_k}+t\phi_k)(\phi_k),\phi_k\rangle\\
&= I_k(W_{r_k,\Lambda_k}) + \frac{(m)^*-1}{2}\int_{\boldsymbol{B}_{\mu_k}(0)} K\biggl(\frac{|y|}{\mu_k}\biggr)[(W_{r_k,\Lambda_k}+t\phi_k)^{(m)^*-2} - W_{r_k,\Lambda_k}^{(m)^*-2}]\phi_k^2\\
&\quad - \frac{1}{2}\int_{\boldsymbol{B}_{\mu_k}(0)} (N_k(\phi_k)+l_k)\phi_k\\
&= I_k(W_{r_k,\Lambda_k})\\
&\quad + \frac{((m)^*-1)((m)^*-2)}{2}\int_{\boldsymbol{B}_{\mu_k}(0)} K\biggl(\frac{|y|}{\mu_k}\biggr) t(W_{r_k,\Lambda_k}+ts\phi_k)^{(m)^*-3}\phi_k^3\\
&\quad - \frac{1}{2}\int_{\boldsymbol{B}_{\mu_k}(0)} (N_k(\phi_k)+l_k)\phi_k
\end{aligned}$$

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$$
= \begin{cases} I_k(W_{r_k,\Lambda_k}) \\ \quad + O\biggl(\displaystyle\int_{\boldsymbol{B}_{\mu_k}(0)} (|\phi_k|^{(m)^*} + |N_k(\phi_k)||\phi_k| + |l_k||\phi_k|)\biggr) & \text{if } N \geqslant 6m, \\ I_k(W_{r_k,\Lambda_k}) \\ \quad + O\biggl(\displaystyle\int_{\boldsymbol{B}_{\mu_k}(0)} W_{r_k,\Lambda_k}^{(m)^*-3}|\phi_k|^3 + |N_k(\phi_k)||\phi_k| + |l_k||\phi_k|\biggr) & \text{if } N < 6m. \end{cases} \tag{3.16}
$$

Using lemma 2.6, we find the remainder term of $F(d_k, \Lambda_k)$ for two cases.

CASE 1 ($N \geqslant 6m$). The main part of the remainder term,

$$
\int_{\boldsymbol{B}_{\mu_k}(0)} |\phi_k|^{(m)^*},
$$

is given by

$$
\begin{aligned}
&\int_{\boldsymbol{B}_{\mu_k}(0)} |\phi_k|^{(m)^*} \\
&\quad \leqslant C\|\phi_k\|_*^{(m)^*} \int_{\boldsymbol{B}_{\mu_k}(0)} \biggl(\sum_{j=1}^{k} \frac{1}{(1+|y-x_j|)^{(N-2m)/2+\tau}}\biggr)^{(m)^*} \\
&\quad \leqslant Ck\|\phi_k\|_*^{(m)^*} \biggl[\int_{\Omega_1} \biggl(\sum_{j=1}^{k} \frac{1}{(1+|y-x_j|)^{(N-2m)/2+\tau}}\biggr)^{(m)^*}\biggr] \\
&\quad \leqslant Ck\|\phi_k\|_*^{(m)^*} \biggl\{\int_{\Omega_1} \frac{1}{(1+|y-x_1|)^{N+\tau}} \\
&\qquad\qquad + \int_{\Omega_1} \biggl[\sum_{j=2}^{k} \frac{1}{(1+|y-x_j|)^{(N-2m)/2+\tau}}\biggr]^{(m)^*}\biggr\} \\
&\quad \leqslant Ck\|\phi_k\mu_k^{-\tau}\|_*^{(m)^*} + Ck\|\phi_k\|_*^{(m)^*} \biggl(\int_{\Omega_1} \frac{1}{(1+|y-x_1|)^{N}}\biggr)\biggl(\sum_{j=2}^{k} \frac{1}{|x_j-x_1|^{\tau}}\biggr)^{(m)^*} \\
&\quad \leqslant Ck\ln k\biggl(\frac{1}{\mu_k}\biggr)^{(l/2+\sigma)(m)^*} \\
&\quad \leqslant Ck\ln k\biggl(\frac{1}{\mu_k}\biggr)^{l/2+\sigma}.
\end{aligned} \tag{3.17}
$$

Meanwhile, the remaining two nonlinear terms related to $N_k(\phi_k)$ and l_k are estimated as follows:

$$
\begin{aligned}
&|N_k(\phi_k)||\phi_k| + |l_k||\phi_k| \\
&\qquad \leqslant Ck\|\phi_k\|_*(\|N_k(\phi_k)\|_{**} + \|l_k\|_{**}) \\
&\qquad\quad \times \biggl[\int_{\boldsymbol{B}_{\mu_k}(0)} \frac{\mathrm{d}y}{(1+|y-x_1|)^{N+2\tau}} + \int_{\boldsymbol{B}_{\mu_k}(0)} \frac{\mathrm{d}y}{(1+|y-x_1|)^{N+\tau}}\biggr]
\end{aligned}
$$

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$$
\begin{aligned}
&\leqslant Ck\|\phi_k\|_*(\|N_k(\phi_k)\|_{**}+\|l_k\|_{**})\\
&\quad\times\biggl[\mu_k^{-2\tau}+\biggl(\sum_{j=2}^{k}\frac{1}{|x_j-x_1|^{(N-2m)/2+\tau}}\biggr)\int_{\boldsymbol{B}_{\mu_k}(0)}\frac{\mathrm{d}y}{(1+|y-x_1|)^{N+\tau}}\biggr]\\
&\leqslant Ck\|\phi_k\|_*(\|N_k(\phi_k)\|_{**}+\|l_k\|_{**})\\
&\leqslant Ck\biggl(\frac{1}{\mu_k}\biggr)^{l/2+\sigma}. \qquad (3.18)
\end{aligned}
$$

Case 2 ($2m+2\leqslant N\leqslant 6m-1$). It is sufficient to estimate the integral

$$
\int_{\boldsymbol{B}_{\mu_k}(0)} W_{r_k,\Lambda_k}^{(m)^*-3}|\phi_k|^3.
$$

Observing that $(m)^*-3>0$ for $N\leqslant 6m-1$, we have

$$
\begin{aligned}
&\int_{\boldsymbol{B}_{\mu_k}(0)} W_{r_k,\Lambda_k}^{(m)^*-3}|\phi_k|^3\\
&\quad\leqslant C\|\phi_k\|_*^3\int_{\boldsymbol{B}_{\mu_k}(0)} W_{r_k,\Lambda_k}^{(m)^*-3}\biggl(\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{(N-2m)/2+\tau}}\biggr)^3\\
&\quad\leqslant C\|\phi_k\|_*^3\int_{\boldsymbol{B}_{\mu_k}(0)}\biggl(\sum_{i=1}^{k}\frac{1}{(1+|y-x_i|)^{6m-N}}\biggr)\\
&\qquad\qquad\times\biggl(\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{(3N-6m)/2+3\tau}}\biggr)\\
&\quad\leqslant Ck\|\phi_k\|_*^3\int_{\boldsymbol{B}_{\mu_k}(0)}\frac{1}{(1+|y-x_1|)^{6m-N}}\\
&\qquad\qquad\times\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{(3N-6m)/2+3\tau}}\,\mathrm{d}y\\
&\quad\leqslant Ck\|\phi_k\|_*^3\biggl[\int_{\boldsymbol{B}_{\mu_k}(0)}\frac{\mathrm{d}y}{(1+|y-x_1|)^{(N+6m)/2+3\tau}}\\
&\qquad\qquad+\sum_{j=2}^{k}\int_{\boldsymbol{B}_{\mu_k}(0)}\frac{\mathrm{d}y}{(1+|y-x_1|)^{6m-N}(1+|y-x_j|)^{(3N-6m)/2+3\tau}}\biggr]\\
&\quad\leqslant Ck\|\phi_k\|_*^3\biggl[\mu_k^{-((6m-N)/2+3\tau)}\\
&\qquad\qquad+\sum_{j=2}^{k}\frac{1}{|x_1-x_j|^{\tau}}\int_{\boldsymbol{B}_{\mu_k}(0)}\frac{\mathrm{d}y}{(1+|y-x_1|)^{(N+6m)/2+2\tau}}\biggr]\\
&\quad\leqslant Ck\|\phi_k\|_*^3[\mu_k^{-((6m-N)/2+3\tau)}+\mu_k^{-((6m-N)/2+2\tau)}]\\
&\quad\leqslant C\biggl(\frac{1}{\mu_k}\biggr)^{l/2+\sigma}. \qquad (3.19)
\end{aligned}
$$

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Thus, the desired result is obtained by combining (3.17)–(3.19) with the expansion (3.2) for $I_k(W_{r_k,\Lambda_k})$. □

4. Proof of theorem 1.1

Observe that

$$\frac{\partial}{\partial d_k} = -l\mu_k d_k^{l-1}\frac{\partial}{\partial r_k} \sim -l\mu_k\frac{\partial}{\partial r_k}.$$

Recall the energy expansion (3.15) for the perturbed energy functional $F_k(d_k,\Lambda_k)$. Then we have

$$\begin{aligned}
\frac{\partial F_k(d_k,\Lambda_k)}{\partial d_k} &= -l\mu_k d_k^{l-1}\frac{\partial F_k(d_k,\Lambda_k)}{\partial r_k}\\
&= -\mu_k l\Big[\frac{\partial I_k(W_{r_k,\Lambda_k})}{\partial r_k} + O\Big(\Big(\frac{1}{\mu_k}\Big)^{l/2+\sigma}\Big)\Big]\\
&= -\mu_k lk\Big[\frac{B_1\partial H_m(\bar{x}_1,\bar{x}_1)/\partial r_k}{\Lambda_k^{N-2m}\mu_k^{N-2m}} - \frac{B_2K'(1)\,\mathrm{d}_k^{l-1}l}{\mu_k}\\
&\qquad - \sum_{j=2}^{k}\frac{B_3\partial G_m(\bar{x}_j,\bar{x}_1)/\partial r_k}{\Lambda_k^{N-2m}\mu_k^{N-2m}} + O\Big(\Big(\frac{1}{\mu_k}\Big)^{l/2+\sigma}\Big)\Big]\\
&= k\Big[\frac{lB_1\partial H_m(\bar{x}_1,\bar{x}_1)/\partial d_k}{\Lambda_k^{N-2m}\mu_k^{N-2m}} + B_2K'(1)l^2d_k^{l-1}\\
&\qquad - \sum_{j=2}^{k}\frac{B_3l\partial G_m(\bar{x}_j,\bar{x}_1)/\partial d_k}{\Lambda_k^{N-2m}\mu_k^{N-2m}} + O\Big(\Big(\frac{1}{\mu_k}\Big)^{l/2+\sigma}\Big)\Big]\\
&= k\Big[\frac{B_1l\partial H_m(\bar{x}_1,\bar{x}_1)/\partial d_k}{\Lambda_k^{N-2m}\mu_k^{N-2m}} + B_2K'(1)\,\mathrm{d}_k^{l-1}l^2\\
&\qquad - \frac{B_3l\sum_{j=2}^{k}\partial G_m(\bar{x}_j,\bar{x}_1)/\partial d_k}{\Lambda_k^{N-2m}\mu_k^{N-2m}} + O\Big(\Big(\frac{1}{\mu_k}\Big)^{l/2+\sigma}\Big)\Big].
\end{aligned}\tag{4.1}$$

Similarly, we obtain

$$\begin{aligned}
&\frac{\partial F_k(d_k,\Lambda_k)}{\partial \Lambda_k}\\
&\quad= \frac{I_k(W_{r_k,\Lambda_k})}{\partial\Lambda_k} + O\Big(\Big(\frac{1}{\mu_k}\Big)^{l/2+\sigma}\Big)\\
&\quad= k(N-2m)\Big[-\frac{B_1H_m(\bar{x}_1,\bar{x}_1)}{\Lambda_k^{N+1-2m}\mu_k^{N-2m}} + \frac{B_2\sum_{i=2}^{k}G_m(\bar{x}_i,\bar{x}_1)}{\Lambda_k^{N+1-2m}\mu_k^{N-2m}} + O\Big(\Big(\frac{1}{\mu_k}\Big)^{l/2+\sigma}\Big)\Big].
\end{aligned}\tag{4.2}$$

Since $\bar{x}_j = x_j/\mu_k \in B_1(0)$, $j=1,2,\ldots,k$, letting $\bar{x}_1^* = (1/(1-d_k),\mathbf{0})$ be the reflection of $\bar{x}_1$ with respect to the unit sphere, we have the following asymptotic

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estimates for H_m and G_m:

$$H_m(y,\bar{x}_1)=\frac{1}{|y-\bar{x}_1^*|^{N-2m}}(1+O(d_k)), \tag{4.3}$$

$$H_m(\bar{x}_1,\bar{x}_1)=\frac{1+O(d_k)}{2^{N-2m}\,\mathrm{d}_k^{N-2m}}, \tag{4.4}$$

$$G_m(y,\bar{x}_1)=\frac{1}{|y-\bar{x}_1|^{N-2m}}-\frac{(1+O(d_k))}{|y-\bar{x}_1^*|}. \tag{4.5}$$

For $j=2,3,\dots,k$, there exists some positive constant $B_4>0$ such that

$$\begin{aligned}\sum_{j=2}^{k}&G_m(\bar{x}_j,\bar{x}_1)\\ &=\sum_{j=2}^{k}\left[\frac{1}{|\bar{x}_j-\bar{x}_1|^{N-2m}}-\frac{1+O(d_k)}{|\bar{x}_j-\bar{x}^*|^{N-2m}}\right]\\ &=\sum_{j=2}^{k}\frac{k^{N-2m}}{|j-1|^{N-2m}|\bar{x}_1|^{N-2m}}\\ &\qquad\times\left(1-(1+O(d_k))\left(1+\frac{4d_k^2+4d_k|\bar{x}_j-\bar{x}_1|\sin((j-1)\pi/k)}{|\bar{x}_i-\bar{x}_1|^2}\right)^{2/(N-2m)}\right)\\ &=B_4k^{N-2m}+O(k^{N-2m}\,\mathrm{d}_k).\end{aligned} \tag{4.6}$$

Let

$$A_1=\frac{B_1}{2^{N-2m}},\quad A_2=B_2K'(1)\quad\text{and}\quad A_3=B_3B_4.$$

Utilizing (4.4) and (4.6), we can obtain a more precise expansion for $F(d_k,\Lambda_k)$ and $\partial F(d_k,\Lambda_k)/\partial\Lambda_k$:

$$\begin{aligned}F_k(d_k,\Lambda_k)=k\biggl[A&+\frac{A_1}{d_k^{N-2m}\Lambda_k^{N-2m}\mu_k^{N-2m}}+A_2d_k^l\\ &-\frac{A_3k^{N-2m}}{\Lambda_k^{N-2m}\mu_k^{N-2m}}+O\biggl(\biggl(\frac{1}{\mu_k}\biggr)^{l/2+\sigma}\biggr)\biggr],\end{aligned} \tag{4.7}$$

$$\begin{aligned}\frac{\partial F_k(d_k,\Lambda_k)}{\partial\Lambda_k}=k\biggl[&-\frac{A_1(N-2m)}{d_k^{N-2m}\Lambda_k^{N+1-2m}\mu_k^{N-2m}}\\ &+\frac{A_3k^{N-2m}(N-2m)}{\Lambda_k^{N+1-2m}\mu_k^{N-2m}}+O\biggl(\biggl(\frac{1}{\mu_k}\biggr)^{l/2+\sigma}\biggr)\biggr],\end{aligned} \tag{4.8}$$

$$\begin{aligned}\frac{\partial F_k(d_k,\Lambda_k)}{\partial d_k}=k\biggl[&-\frac{A_1(N-2m)}{d_k^{N-2m+1}\Lambda_k^{N-2m}\mu_k^{N-2m}}\\ &+lA_2d_k^{l-1}+O\biggl(\biggl(\frac{1}{\mu_k}\biggr)^{l/2+\sigma}\biggr)\biggr].\end{aligned} \tag{4.9}$$

Thus, the pair (d_k, Λ_k) is the critical point of the perturbed functional F if and only if (d_k, Λ_k) satisfies the following system:

$$\left.\begin{aligned} -\frac{A_1(N-2m)}{d_k^{N-2m}\Lambda_k^{N+1-2m}\mu_k^{N-2m}} + \frac{A_3 k^{N-2m}(N-2m)}{\Lambda_k^{N+1-2m}\mu_k^{N-2m}} + O\biggl(\biggl(\frac{1}{\mu_k}\biggr)^{l/2+\sigma}\biggr) = 0, \\ -\frac{A_1(N-2m)}{d_k^{N-2m+1}\Lambda_k^{N-2m}\mu_k^{N-2m}} + lA_2 d_k^{l-1} + O\biggl(\biggl(\frac{1}{\mu_k}\biggr)^{l/2+\sigma}\biggr) = 0. \end{aligned}\right\} \tag{4.10}$$

Letting $D_k := kd_k$, we define a vector functional $F = (f_1, f_2)$ as the principal part of system (4.10):

$$\left.\begin{aligned} f_1(D_k, \Lambda_k) &= -\frac{A_1(N-2m)}{\Lambda_k^{N-2m+1}D_k^{N-2m}} + \frac{A_3(N-2m)}{\Lambda_k^{N-2m+1}}, \\ f_2(D_k, \Lambda_k) &= -\frac{A_1(N-2m)}{\Lambda_k^{N-2m}D_k^{N-2m+l}} + lA_2. \end{aligned}\right\} \tag{4.11}$$

Then $F = 0$ has a solution

$$(D_k^0, \Lambda_k^0) = \biggl(\biggl(\frac{A_1}{A_3}\biggr)^{1/(N-2m)}, \biggl(\frac{(N-2m)A_3^{(N-2m+l)/(N-2m)}}{lA_1^{l/(N-2m)}A_2}\biggr)^{1/(N-2m)}\biggr),$$

with

$$\frac{\partial f_1}{\partial D_k}\bigg|_{(D_k^0,\Lambda_k^0)} > 0, \quad \frac{\partial f_2}{\partial \Lambda_k}\bigg|_{(D_k^0,\Lambda_k^0)} > 0, \quad \frac{\partial f_1}{\partial \Lambda_k}\bigg|_{(D_k^0,\Lambda_k^0)} = 0, \quad \frac{\partial f_2}{\partial D_k}\bigg|_{(D_k^0,\Lambda_k^0)} > 0.$$

Hence, by (4.9) and (4.8), the Jacobian of the perturbed function F_k at (D_k^0, Λ_k^0) is strictly positive. The implicit function theorem implies that there exists some (D_k, Λ_k) near (D_k^0, Λ_k^0) that solves (4.10) for any $k \geqslant k_0$, where k_0 is sufficiently large. Therefore, we obtain infinitely many solutions $\{u_k = W_{r_k,\Lambda_k} + \phi_k\}_{k \geqslant k_0}$ in accordance with the infinite series $\{(D_k, \Lambda_k)\}_{k \geqslant k_0}$. □

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