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Diffusive–dispersive travelling waves and kinetic relations V. Singular diffusion and nonlinear dispersion

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We consider scalar hyperbolic conservation laws with non-convex flux and vanishing, *nonlinear and possibly singular*, diffusion and dispersion terms. The diffusion has the form $R(u, u_x)_x$ and we cover, for instance, the singular diffusion $(|u_x|^p u_x)_x$, where $p \geqslant 0$ is arbitrary. We investigate the existence, uniqueness and various properties of *classical and non-classical travelling waves* and of the *kinetic function*. The latter serves to characterize non-classical shock waves, via an additional algebraic constrain called a kinetic relation. We discover that $p = \frac{1}{3}$ is a somewhat unexpected critical value. For $p \leqslant \frac{1}{3}$, we obtain properties that are qualitatively similar to those we established earlier for *regular and linear* diffusion. However, for $p > \frac{1}{3}$, the behaviour of the kinetic function is very different, as, for instance, non-classical shocks can have *arbitrary small* strength. The behaviour of the kinetic function near the origin is carefully investigated and depends on whether $p < \frac{1}{2}$, $p = \frac{1}{2}$ or $p > \frac{1}{2}$. In particular, in the special case of the cubic flux-function and for the regularization $(|u_x|^p u_x)_x$ with $p = 0$, $\frac{1}{2}$ or 1, the kinetic function can be computed *explicitly*. When $p = \frac{1}{2}$, the kinetic function is simply a *linear* function of its argument.

1. Introduction

This paper is part of a series by the authors [4–7] concerned with the effect of vanishing diffusion and dispersion terms on discontinuous solutions of hyperbolic (or hyperbolic–elliptic) systems of conservation laws. The existence of travelling waves and the properties of the associated shock waves were investigated for various models arising in fluid dynamics and solid mechanics. In the present paper, we investigate whether our earlier results [4] extend to rather general, strongly nonlinear and possibly *singular* diffusion terms. Our motivation is twofold. On the one

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hand, nonlinear, degenerate diffusion has been found useful in many applications of fluid and solid mechanics and has been studied mathematically (for instance, in [19, 21, 28]). On the other hand, the kinetic function generated by singular diffusion turns out to have unusual properties, which are of interest by themselves in connection with the general theory of non-classical shock waves recently developed in [18].

Consider the nonlinear hyperbolic conservation law

$$u_t + f(u)_x = 0, \quad u(x,t) \in \mathbb{R}, \quad x \in \mathbb{R}, \quad t > 0, \tag{1.1}$$

where the flux $f : \mathbb{R} \to \mathbb{R}$ is a given smooth function satisfying the (so-called) concave–convex property

$$uf''(u) > 0 \quad \text{for all } u \neq 0, \qquad \lim_{\pm\infty} f' = +\infty. \tag{1.2}$$

The case $f(u) = u^3$, for instance, is typical in applications. Solutions of (1.1) are discontinuous and fail to be uniquely determined from their initial data. It is customary to add a vanishing right-hand side to (1.1) based on second- (or higher-) order derivatives in order to regularize the solutions and, in the limit, select physically meaningful solutions. Here, we consider a regularization of the general form

$$u_t + f(u)_x = (R(u, \beta u_x))_x + \gamma(c_1(u)(c_2(u)u_x)_x)_x, \qquad u = u^{\beta,\gamma}(x,t), \tag{1.3}$$

where $\beta > 0$ and $\gamma \geqslant 0$ are some parameters tending to zero. The nonlinear dispersion coefficients $c_1, c_2 > 0$ are given, Lipschitz continuous functions. The diffusion function $R = R(u, v)$ is such that

(H1) $R(u, 0) = 0$;

(H2) $R(u, v)$ is Lipschitz continuous and monotone increasing in v, for every u;

(H3) there exists $p \geqslant 0$ such that, by setting $R(u, v) = b(u, v)|v|^p v$, the function b is continuous and, for all $\underline{u} < \bar{u}$, there exist constants $0 < \underline{b} < \bar{b}$ such that

$$\underline{b} \leqslant b(u, v) \leqslant \bar{b} \quad \text{for all } \underline{u} \leqslant u \leqslant \bar{u}, \ v \in \mathbb{R}.$$

Our objective is to generalize to the model (1.3) the results obtained earlier in [4] about the existence and properties of diffusive–dispersive travelling wave solutions. We will be able to cover general diffusions of the form described in (H1)–(H3) for arbitrary $p \geqslant 0$. Interestingly enough, when p is sufficiently large (specifically when $p > \frac{1}{3}$), the diffusion is 'so degenerate' that non-classical shocks of *arbitrary small amplitude* are found in the zero diffusion–dispersion limit. This is not so for smaller values $p \leqslant \frac{1}{3}$ (in particular, for the case $p = 0$ studied in [4]), in which all trajectories with *sufficiently small strength* are *always classical*.

For the terminology used in the present paper (non-classical shocks, undercompressive waves, kinetic relations, etc.), we refer the reader to the textbook [18]. It is well known that (second-order) diffusion terms have a smoothing effect on the solutions of (1.1), while (third-order) dispersion terms tend to generate high-frequency oscillations. The regime in which both effects are kept in balance is of particular interest and, from now on, we assume that

$$\text{the ratio } \gamma/\beta^2 \text{ is constant.}$$

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Consider the entropy–entropy flux pair (U, F) given by

$$U''(u) = \frac{c_2(u)}{c_1(u)}, \quad F(u) = \int_0^u U'(s) f'(s)\,\mathrm{d}s, \quad u \in \mathbb{R}.$$

In the entropy variable $\hat{u} = U'(u)$, we can rewrite equation (1.3) in the form

$$\partial_t u + \partial_x f(u) = (R(u, \beta u_x))_x + \gamma(c_1(u)(c_1(u)\hat{u}_x)_x)_x.$$

Clearly, any (smooth) solution of (1.3) satisfies the balance law:

$$\left.\begin{aligned} \partial_t U(u) + \partial_x F(u) = \Omega, \\ \Omega := (U'(u)R(u, \beta u_x))_x - U''(u)R(u, \beta u_x)u_x \\ + \gamma(\hat{u}c_1(u)(c_1(u)\hat{u}_x)_x - \tfrac{1}{2}c_1(u)^2|\hat{u}_x|^2)_x. \end{aligned}\right\} \tag{1.4}$$

Observe that, in the right-hand side of (1.4), the contribution from the diffusion decomposes into a conservative term and a non-positive one, while the contribution from the dispersion is entirely conservative. Henceforth, in the limit $\beta, \gamma \to 0$, the function $u = \lim u^{\beta,\gamma}$ is a solution of (1.1) satisfying the entropy inequality

$$\partial_t U(u) + \partial_x F(u) \leqslant 0. \tag{1.5}$$

This entropy inequality is going to play a central role in the present paper. It is now well recognized that the so-called *kinetic relation* must be prescribed to uniquely determine the propagation speed of non-classical shocks within general discontinuous solutions of (1.1). For further material on the entropy inequality in connection with classical and non-classical shock waves, we refer to [18] and the references therein.

Recall that a *travelling wave solution* of (1.3) is a smooth function of the form

$$u^{\beta,\gamma}(x, t) = u(y), \qquad y = x - \lambda t,$$

where the constant λ is the *wave speed*. It satisfies the ordinary differential equation

$$-\lambda u_y + f(u)_y = (R(u, \beta u_y))_y + \gamma(c_1(u)(c_2(u)u_y)_y)_y, \qquad u = u^{\beta,\gamma}, \tag{1.6}$$

together with the boundary conditions

$$\lim_{y\to\pm\infty} u(y) = u_\pm, \qquad \lim_{y\to\pm\infty} u_y(y) = \lim_{y\to\pm\infty} u_{yy}(y) = 0, \tag{1.7}$$

where $u_\pm$ are constant states. Letting $\beta, \gamma \to 0$, we see that the pointwise limit

$$u(x, t) = \lim_{\beta,\gamma\to 0} u^{\beta,\gamma}(x, t) = \begin{cases} u_-, & x < \lambda t, \\ u_+, & x > \lambda t, \end{cases} \tag{1.8}$$

is a weak solution of (1.1) satisfying the inequality (1.5). From (1.6) and (1.7), one deduces that u_-, u_+ and λ satisfy the *Rankine–Hugoniot relation*

$$-\lambda(u_+ - u_-) + f(u_+) - f(u_-) = 0. \tag{1.9}$$

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A travelling wave connecting two states u_- and u_+ and converging to a classical (respectively, non-classical) shock will be called a classical (respectively, non-classical) trajectory. The existence of travelling waves with *linear* diffusion and dispersion were studied by Bona and Schonbek [9], Jacobs *et al*. [16] and the authors in [4]. In particular, the paper [4] covers the class of diffusions

$$R(u, \beta u_x) = \beta b(u) u_x.$$

On the other hand, when $f(u) = u^3$ and with the *nonlinear* diffusion–dispersion term

$$R(u, \beta u_x) = \beta^2 |u_x| u_x,$$

Hayes and LeFloch [13] proved the existence of non-classical trajectories by exhibiting an *explicit* formula for the kinetic function (see § 3 below). Other works on the effect of dispersive terms on shock waves and the properties of kinetic functions *for systems* include Slemrod [24, 25], Truskinovsky [22, 26, 27], Abeyaratne and Knowles [1–3], Fan and Slemrod [12], Shearer *et al*. [8, 20, 23], LeFloch *et al*. [13–15, 17, 18], Colombo and Corli [10] and Corli and Fan [11]. See [18] for an overview of recent results.

Our objective in the present paper is to investigate to what extent the qualitative properties of Hayes–LeFloch's example are valid for more general concave–convex flux-functions f and for a large class of possibly singular regularizations. We will derive detailed information on the non-classical shocks in terms of the associated *kinetic function* $\varphi(u)$. In particular, we will prove the following results.

(1) If $0 \leqslant p \leqslant \frac{1}{3}$, then
  - (i) given any value of the ratio γ/β^2, all shocks with sufficiently small strength are always classical;
  - (ii) given any left-hand value u_-, the shocks are all classical if the ratio γ/β^2 is sufficiently small, i.e. if the diffusion is sufficiently large;
  - (iii) given any left-hand value u_-, there always exist non-classical trajectories leaving from u_- provided the ratio γ/β^2 is sufficiently large, i.e. provided the dispersion is sufficiently large.

(2) If $p > \frac{1}{3}$, then, for any given left-hand value u_-, there always exist non-classical trajectories leaving from u_-.

Furthermore, the arguments in [4] could also extend to show that, under some mild assumption on the flux f and for any given ratio γ/β^2, there always exist non-classical trajectories associated with shocks of sufficiently large strength.

We will also be able to describe the behaviour of the kinetic function near the origin. Assume that f satisfies $f'''(0) \neq 0$ and set $\alpha = \beta/\sqrt{\gamma}$. Then the kinetic function $\varphi(u) := \varphi^\flat_\alpha(u)$ has the following asymptotic behaviour in $u = 0$.

Case 1. If $0 \leqslant p < \frac{1}{2}$, then $\varphi^{\flat\prime}_{p\alpha}(0) = -\frac{1}{2}$.

Case 2. If $p = \frac{1}{2}$, then $\varphi^{\flat\prime}_{p\alpha}(0) \in (-1, -\frac{1}{2})$, with

$$\lim_{\alpha \to 0+} \varphi^{\flat\prime}_{p\alpha}(0) = -1 \quad \text{and} \quad \lim_{\alpha \to +\infty} \varphi^{\flat\prime}_{p\alpha}(0) = -\tfrac{1}{2}.$$

Case 3. If $\frac{1}{2} < p$, then $\varphi^{\flat\prime}_{p\alpha}(0) = -1$.

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In addition, we will describe three rather interesting, explicit examples of kinetic functions associated with the cubic flux $f(u) = u^3$. In fact, no explicit formula for the non-classical shocks is available, except in the cubic-flux case first treated in [16] for $p = 0$ and in [13] for $p = 1$ and, in the present paper, for $p = \frac{1}{2}$. In the latter case, we discover here that the kinetic function is *linear* in u.

An outline of this paper follows. The main results are stated in § 2. In § 3, we describe examples of kinetic functions. Sections 4 and 5 contain the proofs of the main results. The proofs consist of suitable extensions of the techniques developed earlier by Jacobs *et al*. [16] and by the authors [4]. There are several new difficulties. For instance, when $p > 1$, the trajectories may blow up even when u is bounded. On the other hand, the critical value $p = \frac{1}{3}$ is quite unexpected. (To have transition at $p = 1$ could have been expected.) In addition, the behaviour of the kinetic function at the origin is associated with yet another critical value, $p = \frac{1}{2}$.

2. Main results

Observe that, using (1.7), equation (1.6) can be integrated once:

$$\gamma c_1(u)(c_2(u)u_y)_y + R(u, \beta u_y) = -\lambda(u - u_-) + f(u) - f(u_-). \tag{2.1}$$

From (1.7) and (2.1), and by letting $y \to +\infty$, we obtain the Rankine–Hugoniot relation

$$-\lambda(u_+ - u_-) + f(u_+) - f(u_-) = 0. \tag{2.2}$$

Throughout this paper, we are interested in $\beta, \gamma > 0$, except in theorem 2.4, where we will take $\gamma = 0$. When $\gamma = 0$, equation (2.1) is an ordinary differential equation on the real line: all of the solutions are monotone and their behaviour is easily determined by straightforward monotonicity arguments (see theorem 2.4 below for a statement of the result in this case). The (more interesting) case $\gamma > 0$ requires a phase-plane analysis.

Since $c_2(u) > 0$, by the simple rescaling

$$\frac{\mathrm{d}}{\mathrm{d}y} \to \frac{1}{\sqrt{\gamma}c_2(u)}\frac{\mathrm{d}}{\mathrm{d}y},$$

we get the simpler equation

$$u_{yy} + \hat{R}(u, \alpha u_y) = c(u)(-\lambda(u - u_-) + f(u) - f(u_-)), \tag{2.3}$$

where

$$\alpha = \frac{\beta}{\sqrt{\gamma}}, \qquad c(u) = \frac{c_2(u)}{c_1(u)} = U''(u)$$

and

$$\hat{R}(u, v) = c(u)R(u, v/c_2(u)).$$

Clearly, the function $\hat{R}$ satisfies the same assumptions as R, which we restate here for convenience.

(H1) $\hat{R}(u, 0) = 0$.

(H2) $\hat{R}(u, v)$ is Lipschitz continuous and monotone increasing in v, for every u.

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(H3) $\hat{R}(u,v) = \hat{b}(u,v)|v|^p v$, where $\hat{b}$ is continuous in (u,v) and satisfies the following bounds: for all $\underline{u} < \bar{u}$, there exist constants $0 < \underline{b} < \bar{b}$ such that

$$\underline{b} \leqslant \hat{b}(u,v) \leqslant \bar{b} \quad \text{for all} \quad \underline{u} \leqslant u \leqslant \bar{u}, \quad v \in \mathbb{R}.$$

Here, the functions $\hat{b}$ and b are related by

$$\hat{b}(u,v) = \frac{1}{c_1(u)c_2(u)^p} b(u, v/c_2(u)).$$

In the rest of this paper, without loss of generality, we can therefore assume that

$$\gamma = 1, \qquad c_2(u) = 1,$$

and we set $c(u) = 1/c_1(u)$ and $\alpha = \beta$. We then rewrite (2.3) in the form

$$u_y = v, \tag{2.4 a}$$

$$v_y = -c(u)R(u, \alpha v) + c(u)(g(u,\lambda) - g(u_-,\lambda)) \tag{2.4 b}$$

and

$$\lim_{y\to\pm\infty} u(y) = u_\pm, \qquad \lim_{y\to\pm\infty} v(y) = 0, \tag{2.4 c}$$

where

$$g(u,\lambda) := f(u) - \lambda u. \tag{2.5}$$

The relation (2.2) then reads

$$g(u_-,\lambda) = g(u_+,\lambda).$$

We will need the following notation:

$$f'(\varphi^\natural(u)) = \frac{f(u) - f(\varphi^\natural(u))}{u - \varphi^\natural(u)} \quad \text{for all } u \neq 0. \tag{2.6}$$

Note that $u\varphi^\natural(u) < 0$ and, by continuity, $\varphi^\natural(0) = 0$. Thanks to (1.2), the map $\varphi^\natural : \mathbb{R} \to \mathbb{R}$ is strictly monotone decreasing and onto, and so is invertible. Its inverse function is denoted by $\varphi^{-\natural}$. Finally, for each u_-, we set

$$\lambda^\natural(u_-) = f'(\varphi^\natural(u_-)),$$

which is a *lower bound* for all shock speeds λ satisfying (2.2) for some u_+. We denote by $\lambda^{-\natural}$ the inverse function of $\lambda^\natural$.

Given u_-, let us define the *global entropy dissipation* function by

$$H(u_-,u_+) = \int_{u_-}^{u_+} c(u)(g(s,\lambda) - g(u_-,\lambda))\,\mathrm{d}s, \quad u_+ \in \mathbb{R}, \tag{2.7}$$

where

$$\lambda = \bar{a}(u_-,u_+) := \begin{cases} \dfrac{f(u_+) - f(u_-)}{u_+ - u_-}, & u_+ \neq u_-, \\ f'(u_-), & u_+ = u_-. \end{cases} \tag{2.8}$$

Based on (1.4), it can be checked (see [4]) that the following result holds.

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LEMMA 2.1. *If there exists a travelling wave of (2.4) connecting u_- to u_+, then*

$$H(u_-,u_+) \geqslant H(u_-,u_-) = 0,$$

where the inequality is strict if $u_- \neq u_+$ and $\beta > 0$. In addition, there exists a strictly monotone decreasing function $\varphi_0^\flat : \mathbb{R} \to \mathbb{R}$ such that, for all $u_- \neq 0$,

$$H(u_-,u_+) = 0 \quad \textit{and} \quad u_+ \neq u_- \quad \textit{if and only if } u_+ = \varphi_0^\flat(u_-).$$

For all u_-, u_+, we have

$$H(u_-,u_+) > 0 \quad \textit{if and only if } \operatorname{sgn}(u_-)\varphi_0^\flat(u_-) < \operatorname{sgn}(u_-)u_+ < \operatorname{sgn}(u_-)u_-$$

and

$$\operatorname{sgn}(u_-)\varphi^{-\natural}(u_-) \leqslant \operatorname{sgn}(u_-)\varphi_0^\flat(u_-) \leqslant \operatorname{sgn}(u_-)\varphi^\natural(u_-).$$

The function $\varphi_0^\flat$ corresponds to the *maximal negative* entropy dissipation. From lemma 2.1, we deduce that, if there exists a travelling wave of (2.4) connecting u_- to u_+, then

$$u_+ \text{ lies between } \varphi_0^\flat(u_-) \text{ and } u_-.$$

Define the function $\varphi_0^\sharp$ and λ_0 by the conditions $\lambda_0(0) = 0$ and, for $u_- \neq 0$,

$$\lambda_0(u_-) = \frac{f(u_-) - f(\varphi_0^\flat(u_-))}{u_- - \varphi_0^\flat(u_-)} = \frac{f(u_-) - f(\varphi_0^\sharp(u_-))}{u_- - \varphi_0^\sharp(u_-)} \tag{2.9}$$

and $\operatorname{sgn}(u_-)\varphi_0^\flat(u_-) \leqslant \operatorname{sgn}(u_-)\varphi_0^\sharp(u_-)$. The speed $\lambda_0(u_-)$ is the *maximal* admissible speed for the range of right-hand states u_+ included between $\varphi_0^\sharp(u_-)$ and $\varphi^\natural(u_-)$, at least. Recall that $\lambda^\natural(u_-)$ is a lower bound for the speeds.

To state the results, for each left-hand state u_-, we define the *shock set* generated by the equation (2.1) as

$$S_\alpha(u_-) := \{u_+ / \text{ there exists a travelling wave satisfying (1.6), (1.7)}\}.$$

THEOREM 2.2 (classical and non-classical shock waves). *Consider the travelling wave solutions of (1.6), (1.7) under the assumptions that the flux satisfies (1.2) and the diffusion–dispersion ratio $\alpha = \beta/\sqrt{\gamma}$ belongs to the interval $(0,\infty)$. Then there exists a function $\varphi_\alpha^\flat : \mathbb{R} \to \mathbb{R}$ satisfying*

$$\operatorname{sgn}(u)\varphi_0^\flat(u) < \operatorname{sgn}(u)\varphi_\alpha^\flat(u) \leqslant \operatorname{sgn}(u)\varphi^\natural(u) \quad \textit{for all } u \neq 0. \tag{2.10}$$

The shock set is given by

$$S_\alpha(u) = \begin{cases} \{\varphi_\alpha^\flat(u)\} \cup (\varphi_\alpha^\sharp(u), u], & u \geqslant 0, \\ [u, \varphi_\alpha^\sharp(u)) \cup \{\varphi_\alpha^\flat(u)\}, & u \leqslant 0. \end{cases} \tag{2.11}$$

Here, the function $\varphi_\alpha^\sharp$ is defined from $\varphi_\alpha^\flat$ by

$$\frac{f(u) - f(\varphi_\alpha^\sharp(u))}{u - \varphi_\alpha^\sharp(u)} = \frac{f(u) - f(\varphi_\alpha^\flat(u))}{u - \varphi_\alpha^\flat(u)} \quad \textit{for all } u \neq 0,$$

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with the constraint $\operatorname{sgn}(u)\varphi^\natural(u) \leqslant \operatorname{sgn}(u)\varphi^\sharp_\alpha(u)$. *Furthermore, the function* $\varphi^\flat_\alpha$ *is strictly monotone decreasing, and, in particular, the shock speed*

$$\lambda^\flat_\alpha(u) := \bar{a}(u, \varphi^\flat_\alpha(u))$$

is strictly monotone increasing for $u > 0$ *and strictly monotone decreasing for* $u < 0$.

Also, the kinetic function satisfies the following property. If $0 \leqslant p \leqslant \frac{1}{3}$, *then there exists a Lipschitz continuous function* $A^\natural_p : \mathbb{R} \to [0, \infty)$, $u_0 \mapsto A^\natural_p(u_0)$, *with* $A^\natural_p(0) = 0$, *called the threshold diffusion–dispersion function, such that*

$$\varphi^\flat_\alpha \equiv \varphi^\natural \quad \text{if and only if } \alpha \geqslant A^\natural_p(u_0). \tag{2.12}$$

If $p > \frac{1}{3}$, *then*

$$\varphi^\flat_\alpha(u) \neq \varphi^\natural(u) \quad \text{for all } u \neq 0. \tag{2.13}$$

The function $\varphi^\flat_\alpha : \mathbb{R} \to \mathbb{R}$ is called the *kinetic function* associated with the model (1.3). It completely characterizes the dynamics of the non-classical shock waves of the hyperbolic conservation law (1.1).

Theorem 2.3 (properties of the kinetic functions at the origin). *When the function* f *satisfies* $f'''(0) \neq 0$, *the kinetic functions has the following behaviour near the origin.*

(i) *For* $p = 0$, *we have*

$$\varphi^{\flat\prime}_{p\alpha}(0) = -\tfrac{1}{2}, \qquad A^\natural_p(0) = 0, \qquad |A^{\natural\prime}_p(0\pm)| > 0.$$

(ii) *For* $0 < p \leqslant \frac{1}{3}$, *we have*

$$\varphi^{\flat\prime}_{p\alpha}(0) = -\tfrac{1}{2}, \qquad A^\natural_p(0) = 0, \qquad A^{\natural\prime}_p(0\pm) = +\infty.$$

(iii) *For* $\frac{1}{3} < p < \frac{1}{2}$, *we have*

$$\varphi^{\flat\prime}_{p\alpha}(0) = -\tfrac{1}{2}.$$

(iv) *For* $p = \frac{1}{2}$, *we have*

$$\varphi^{\flat\prime}_{p\alpha}(0) \in (-1, -\tfrac{1}{2}),$$

$\varphi^{\flat\prime}_{p\alpha}(0)$ *depends on* α *and*

$$\lim_{\alpha \to 0+} \varphi^{\flat\prime}_{p\alpha}(0) = -1, \qquad \lim_{\alpha \to +\infty} \varphi^{\flat\prime}_{p\alpha}(0) = -\tfrac{1}{2}.$$

(v) *For* $\frac{1}{2} < p$, *we have*

$$\varphi^{\flat\prime}_{p\alpha}(0) = -1.$$

Theorems 2.2 and 2.3 are the main results of the present paper. The proofs are the subject of §§ 3–5 below. The dispersion-free case is much simpler and is dealt with in the next theorem.

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THEOREM 2.4 (diffusive shock waves). *For any $p \geqslant 0$, consider the travelling wave solutions of (1.6), (1.7) under the assumptions that the flux satisfies (1.2) and $\alpha := \beta/\sqrt{\gamma} = \infty$. Then the shock set is given by*

$$S_\infty(u) = \begin{cases} [\varphi^\natural(u), u], & u \geqslant 0, \\ [u, \varphi^\natural(u)], & u \leqslant 0. \end{cases} \tag{2.14}$$

Moreover, the travelling waves have 'compact support' if $p > 0$, in the sense that they are identically constant outside a bounded interval.

Proof. When $\gamma = 0$, equation (2.1) becomes

$$R(u, \beta u_y) = f(u) - f(u_0) - \lambda(u - u_0).$$

Assume that $u_0 < 0$. Then, thanks to (H2), the implicit function theorem shows that, for $u_0 < u < u_1$, we can write $u_y = K(u)$, where K is a smooth function. From this we deduce that

$$y(u) = \int_{\tilde{u}}^{u} \frac{\mathrm{d}u}{K(u)},$$

where it is convenient to choose $\tilde{u} := \frac{1}{2}(u_1 + u_0)$. Now, using (H3) and for some constants l_p, L_p, we obtain

$$\lim_{u \to u_0} y(u) = \begin{cases} l_p > -\infty, & p > 0, \\ -\infty, & p = 0, \end{cases}$$

and

$$\lim_{u \to u_1} y(u) = \begin{cases} L_p < +\infty, & p > 0, \\ +\infty, & p = 0. \end{cases}$$

This completes the proof of theorem 2.4. □

We complete this section with some further notation, which will be useful later on. For definiteness, we always suppose that the left-hand state is negative,

$$u_- < 0.$$

It will be convenient to set

$$u_0 = u_-.$$

Since the flux-function f is concave–convex, given any speed in the interval

$$\lambda \in (\lambda^\natural(u_-), f'(u_-)),$$

there exist exactly three distinct solutions u_0, u_1 and u_2 of (2.4) with

$$u_0 < u_1 < \varphi^\natural(u_0) < u_2. \tag{2.15}$$

Define the *local entropy dissipation* function by

$$G(u, u_0, \lambda) := \int_{u_0}^{u} (g(s, \lambda) - g(u_0, \lambda)) c(s) \, \mathrm{d}s.$$

Observe that $\partial_u G(u, u_0, \lambda) = 0$ if and only if u is an equilibrium point. The following properties of the function G were established by the authors in [4].

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Lemma 2.5. *Given* $u_0 < 0$ *and* λ *in the interval* $\lambda \in (\lambda^\natural(u_-), f'(u_-))$, *the function* $\tilde G(u) := G(u, u_0, \lambda)$ *satisfies*

$$\tilde G'(u) > 0 \quad \text{for all } u \in (u_0, u_1) \text{ or } u > u_2,$$
$$\tilde G'(u) < 0 \quad \text{for all } u < u_0 \text{ or } u \in (u_1, u_2).$$

Moreover, if $\lambda \in (\lambda^\natural(u_0), \lambda_0(u_0))$, *we have*

$$\tilde G(u_0) = 0 < \tilde G(u_2) < \tilde G(u_1).$$

If $\lambda = \lambda_0(u_0)$, *then*

$$\tilde G(u_0) = \tilde G(u_2) = 0 < \tilde G(u_1).$$

If $\lambda \in (\lambda_0(u_0), f'(u_0))$, *then*

$$\tilde G(u_2) < 0 = \tilde G(u_0) < \tilde G(u_1).$$

From lemmas 2.1 and 2.5, we conclude that

$$\text{if there exists a trajectory connecting } u_0 \text{ to } u_1, \text{ then } \lambda \in [\lambda^\natural(u_0), f'(u_0)] \tag{2.16}$$

and

$$\text{if there exists a trajectory connecting } u_0 \text{ to } u_2, \text{ then } \lambda \in [\lambda^\natural(u_0), \lambda_0(u_0)]. \tag{2.17}$$

3. Kinetic functions when $f(u) = u^3$ and $R(u, u_x) = |u_x|^p u_x$

When the flux is taken to be the cubic function $f(u) = u^3$ and when $c_1(u) = c_2(u) = 1$, and $R(u, v) = |v|^p v$, $p \geqslant 0$, then

$$\varphi^\natural(u) = -\tfrac{1}{2}u, \qquad \varphi_0^\flat(u) = -u. \tag{3.1}$$

In the case $p = 0$, Jacobs *et al*. [16] were able to derive an *explicit formula* for the kinetic function, precisely,

$$\varphi_\alpha^\flat(u) = \begin{cases} -u - \frac{1}{3}\sqrt{2}\alpha, & u \leqslant -\frac{2}{3}\sqrt{2}\alpha, \\ -\frac{1}{2}u, & |u| < \frac{2}{3}\sqrt{2}\alpha, \\ -u + \frac{1}{3}\sqrt{2}\alpha, & u \geqslant \frac{2}{3}\sqrt{2}\alpha. \end{cases} \tag{3.2}$$

They also proved that the threshold diffusion–dispersion ration is given by

$$A_0^\natural(u_0) = \frac{3}{2\sqrt{2}}|u_0|. \tag{3.3}$$

In the case $p = 1$, Hayes and LeFloch [13] discovered that the kinetic function is given by the (implicit) formula

$$\int_{u_0}^{\varphi_\alpha^\flat(u_0)} (f(u) - f(u_0) - \lambda(u - u_0)) \exp(2\alpha u)\, du = 0, \quad \text{where } \lambda = \frac{f(\varphi_\alpha^\flat(u_0)) - f(u_0)}{\varphi_\alpha^\flat(u_0) - u_0}. \tag{3.4}$$

In addition, when $p = 1$, the kinetic function satisfies $\varphi_\alpha^{\flat\prime}(0) = -1$.

Our result for general p is the following one.

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THEOREM 3.1. *When $f(u) = u^3$ and $R(u, \alpha u_x) = \alpha^{p+1}|u_x|^p u_x$, the critical diffusion satisfies*

$$A_p^\natural(u_0) = \begin{cases} a_p|u_0|^{(1-2p)/(p+1)}, & 0 < p \leqslant \frac{1}{3}, \\ +\infty, & p > \frac{1}{3}, \end{cases} \tag{3.5}$$

where a_p is a positive constant depending only on p.

The kinetic function satisfies (for all $\alpha > 0$)

$$\varphi_{p\alpha}^{\flat\prime}(0) = \begin{cases} -\frac{1}{2}, & 0 < p < \frac{1}{2}, \\ k_\alpha \in (-1, -\frac{1}{2}), & p = \frac{1}{2}, \\ -1, & p > \frac{1}{2}. \end{cases} \tag{3.6}$$

Moreover, when $p = \frac{1}{2}$, the kinetic function is linear,

$$\varphi_{p\alpha}^\flat(u) := k_\alpha u, \tag{3.7}$$

where the coefficient $k_\alpha < 0$ and $\alpha \mapsto k_\alpha$ is a strictly monotone increasing function of α, mapping $(0, +\infty)$ onto $(-1, -\frac{1}{2})$.

Proof. Equation (2.3) becomes

$$u_{yy} + \alpha^{p+1}|u_y|^p u_y = u^3 - u_0^3 - \lambda(u - u_0). \tag{3.8}$$

This equation is invariant by the transformation $u \to -u$. This allows us to restrict our attention to $u_0 < 0$, since the values $u_0 > 0$ are dealt with by a symmetry argument. Setting $z = u/u_0$, $z_0 = 1$, $z_2 = u_2/u_0$ and using the scaling $\xi = |u_0|y$, we obtain

$$z_{\xi\xi} + \kappa^{p+1}|z_\xi|^p z_\xi = z^3 - z_0^3 - \lambda^*(z - z_0), \tag{3.9}$$

where

$$\lambda^* = \frac{z_2^3 - z_0^3}{z_2 - z_0} = \frac{\lambda}{u_0^2} \tag{3.10}$$

and

$$\kappa = \alpha|u_0|^{(2p-1)/(p+1)}. \tag{3.11}$$

This means that for κ and α related by (3.11), we have

$$\frac{\varphi_\alpha^\flat(u_0)}{u_0} = \varphi_\kappa^\flat(1). \tag{3.12}$$

It can be checked, using theorem 5.1 with $u_0 = 1$, that, by setting $a_p = A_p^\natural(1)$ and considering (3.11), we can derive (3.5). On the other hand, using (3.11), (3.12) and the fact that

$$\lim_{\kappa\to 0} \varphi_\kappa^\flat(1) = \varphi_0^\flat(1) = -1, \qquad \lim_{\kappa\to+\infty} \varphi_\kappa^\flat(1) = \varphi^\natural(1) = -\tfrac{1}{2},$$

and that $\kappa = \alpha$ if $p = \frac{1}{2}$, we get (3.6) and (3.7). Finally, the monotonicity of $\alpha \mapsto k_\alpha$ is a direct consequence of the monotonicity of the mapping $\alpha \mapsto \varphi_\alpha^\flat(1)$, to be established later in § 4 (lemma 4.4). □

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4. Existence of non-classical trajectories

From now on, we suppose that $p > 0$, since the (regular) case $p = 0$ can be treated easily from the ideas in [4], where there is a minor difference that is that the diffusion coefficient b here depends not only on u, but also on v.

In this section, a propagation speed λ and a left-hand state $u_0 < 0$ are given, and we look for non-classical trajectories connecting u_0 to the associated equilibrium u_2 introduced in § 2. According to our earlier discussion (see the conclusion (2.17)), we necessarily have

$$u_2 \in [\varphi^{\natural}(u_0), \varphi_0^{\flat}(u_0)], \qquad \lambda \in [\lambda^{\natural}(u_0), \lambda_0(u_0)], \tag{4.1}$$

which will thus be assumed throughout the discussion. We want to prove that a trajectory connecting u_0 to u_2 exists if and only if the parameter $\alpha = \beta/\sqrt{\gamma}$ has a specific value, depending, of course, on u_0 and λ. For all $p > 0$, the eigenvalues associated with the system (2.4) linearized at the equilibrium points u_0, u_1 and u_2 satisfy

$$\mu^2 = c(u)(f'(u) - \lambda).$$

So we set

$$\underline{\mu}(u, \lambda) := -\sqrt{c(u)(f'(u) - \lambda)}, \qquad \bar{\mu}(u, \lambda) := \sqrt{c(u)(f'(u) - \lambda)}. \tag{4.2}$$

We have the following properties of equilibria u_0, u_1, and u_2, in the general case $\beta \geqslant 0$ and $\gamma \in \mathbb{R}\backslash\{0\}$. Note that (4.1) is not needed at this stage yet. If $f'(u) - \lambda > 0$, then $(u, 0)$ is a saddle point having, by definition, two real eigenvalues with opposite sign, $\underline{\mu} < 0 < \bar{\mu}$. If $f'(u) - \lambda < 0$, then $(u, 0)$ is a centre having two purely imaginary eigenvalues. We now state our main result in this section, relying now on the conditions (4.1).

Theorem 4.1 (non-classical trajectories). *Assume that $p > 0$. Given two states $u_0 < 0$ and $u_2 > 0$ corresponding to a propagation speed λ satisfying*

$$\lambda = \bar{a}(u_0, u_2) = \frac{f(u_2) - f(u_0)}{u_2 - u_0} \in (\lambda^{\natural}(u_0), \lambda_0(u_0)] \tag{4.3}$$

or, equivalently, $\lambda \in [\lambda_0(u_2), \lambda^{-\natural}(u_2))$, there is a unique value of the diffusion parameter $\alpha \geqslant 0$ such that u_0 can be connected to u_2 by a travelling wave solution of (2.4).

Since $\bar{\mu}(u_0) > 0$, by general properties of differential equations, there exist two trajectories leaving from u_0 at $y = -\infty$ and satisfying

$$\lim_{y \to -\infty} \frac{v(y)}{u(y) - u_0} = \bar{\mu}(u_0, \lambda) = \sqrt{c(u)(f'(u) - \lambda)}. \tag{4.4}$$

One trajectory approaches this point in the quadrant $Q_1 = \{u < u_0,\ v < 0\}$, while the other approaches it in the quadrant $Q_2 = \{u > u_0,\ v > 0\}$. On the other hand, there are two trajectories reaching u_2 at $y = +\infty$ and satisfying

$$\lim_{y \to +\infty} \frac{v(y)}{u(y) - u_2} = \underline{\mu}(u_2, \lambda) = -\sqrt{c(u)(f'(u) - \lambda)}. \tag{4.5}$$

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One trajectory approaches this point in the quadrant $Q_3 = \{u < u_2,\ v > 0\}$; the other approaches it in the quadrant $Q_4 = \{u > u_2,\ v < 0\}$. As in [4], it is easy to deduce that, in the phase plane, a travelling wave solution connecting u_0 to u_2 necessarily approaches the equilibrium $(u_0, 0)$ at $y = -\infty$ through the quadrant Q_2, and the equilibrium $(u_2, 0)$ at $y = +\infty$ through the quadrant Q_3. It is not difficult to check that any travelling wave is monotone in some limited range, at least of the variable u. More precisely, if $u = u(y)$ is a solution of (2.4) defined on an interval $(-\infty, \bar{y})$ and satisfies $\lim_{y\to-\infty} u(y) = u_0$ and $u_0 < u(y) < u_1$ for all $y < \bar{y}$, then we must have $u_y > 0$ on the interval $(-\infty, \bar{y})$. Similarly, if $u = u(y)$ is a solution of (2.4) defined on an interval $(\bar{y}, +\infty)$ and satisfying $\lim_{y\to+\infty} u(y) = u_2$ and $u_1 < u(y) < u_2$ for all $y > \bar{y}$, then we have $u_y > 0$ on the interval $(\bar{y}, +\infty)$.

The above property justifies and motivates the re-parametrization of the trajectories with the variable u instead of y for both of the semi-trajectories leaving from u_0 and from u_2, respectively.

LEMMA 4.2. *The trajectories leaving from u_0 in Q_2 cross the line $u = u_1$ for the 'first time' at some point $(u_1, v_1^-(\alpha))$. This part of the trajectory is the graph of a function*

$$[u_0, u_1] \ni u \mapsto v_-(u, \lambda, \alpha).$$

Moreover, for all fixed $u \in (u_0, u_1)$, the function $[0, +\infty) \ni \alpha \mapsto v_-(u, \lambda, \alpha)$ is strictly monotone decreasing.

Proof. Multiplying (2.4 b) by u_y, using (H1) and (H2), and finally integrating over $(-\infty, y]$, we get

$$0 \leqslant v \leqslant \sqrt{2G(u, u_0, \lambda)}.$$

Thus the trajectory defining in the neighbourhood of $(u_0, 0)$ remains bounded in Q_2 for all $u \in [u_0, u_1]$, and this concludes the first statement of lemma 4.2. Now, combining (2.4 a) and (2.4 b), we see that the function $u \mapsto v_-(u, \lambda, \alpha)$ satisfies the following key equation in the phase plane:

$$v\frac{\mathrm{d}v}{\mathrm{d}u} + c(u)R(u, \alpha v) = G_u(u, u_0, \lambda). \tag{4.6}$$

Take two diffusion values $0 \leqslant \underline{\alpha} < \bar{\alpha}$ and consider the corresponding trajectories issuing from u_0, say, $\underline{v}(u) = v_-(u, \lambda, \underline{\alpha})$ and $\bar{v}(u) = v_-(u, \lambda, \bar{\alpha})$, respectively. Then, using (4.4), we get that, for all $u \in (u_0, u_0 + \epsilon]$ with $\epsilon \ll 1$,

$$\begin{aligned} &R(u, \underline{\alpha}\underline{v}) - R(u, \bar{\alpha}\bar{v}) \\ &\qquad \sim b(u, 0)(\underline{\alpha}^{p+1} - \bar{\alpha}^{p+1})(u - u_0)^{p+1} c(u)^{(p+1)/2} (f'(u_0) - \lambda)^{(p+1)/2} \\ &\qquad < 0. \end{aligned}$$

Thus we obtain from (4.6) that

$$\frac{\mathrm{d}}{\mathrm{d}u}(\underline{v}^2 - \bar{v}^2) > 0,$$

and then

$$\underline{v}(u) > \bar{v}(u) \quad \text{for } u \in (u_0, u_0 + \epsilon].$$

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Now assume that there exists a first point $u^* \in [u_0 + \epsilon, u_1)$ such that $\underline{v}(u^*) = \bar{v}(u^*) = v^*$. Then, on the one hand,

$$\frac{\mathrm{d}\underline{v}}{\mathrm{d}u}(u^*) \leqslant \frac{\mathrm{d}\bar{v}}{\mathrm{d}u}(u^*),$$

and, on the other hand, from (H2), we get

$$R(u^*, \underline{\alpha} v^*) < R(u^*, \bar{\alpha} v^*).$$

But this contradicts the identity obtained from (4.6),

$$v^* \left(\frac{\mathrm{d}\underline{v}}{\mathrm{d}u}(u^*) - \frac{\mathrm{d}\bar{v}}{\mathrm{d}u}(u^*) \right) + c(u^*)(R(u^*, \underline{\alpha} v^*) - R(u^*, \bar{\alpha} v^*)) = 0.$$

This completes the proof of lemma 4.2. □

LEMMA 4.3. *For each $p > 0$, there exists a constant $\alpha_p \in (0, +\infty]$ satisfying*

$$\alpha_p = \begin{cases} +\infty, & 0 < p \leqslant 1, \\ < +\infty, & p > 1, \end{cases}$$

such that, for all $\alpha < \alpha_p$, the trajectory converging to u_2 in Q_3 crosses the line $u = u_1$ for the 'last' time at some point of the form $(u_1, v_1^+(\alpha))$. This part of the trajectory is the graph of a function

$$[u_1, u_2] \ni u \mapsto v_+(u, \lambda, \alpha).$$

Moreover, for each $u \in [u_1, u_2)$, the function $[0, \alpha_p) \ni \alpha \mapsto v_+(u, \lambda, \alpha)$ is strictly monotone increasing and satisfies

$$\lim_{\alpha \to \alpha_p} v_1^+(\alpha) = +\infty.$$

Proof. First, the monotonicity of $\alpha \mapsto v_+(u, \lambda, \alpha)$ is obtained in the same manner as the function $\alpha \mapsto v_-(u, \lambda, \alpha)$. In comparison with our earlier results [4], we note that, when $p \geqslant 1$, the existence of $v_1^+(\alpha)$ is not guaranteed for all values of $\alpha \geqslant 0$.

Thanks to (H3), and to the continuity of the function $u \mapsto c(u)$, for some positive constants $\underline{d}$ and $\bar{d}$, we have

$$\underline{d} \leqslant c(u) b(u, v) \leqslant \bar{d} \quad \text{for all } (u, v) \in [u_1, u_2] \times \mathbb{R}.$$

We distinguish between three cases.

CASE 1 ($0 < p < 1$). Integrating equation (4.6) over $[u, u_2]$ with $u_1 \leqslant u \leqslant u_2$, we get

$$\tfrac{1}{2} v^2(u) = G(u, u_2, \lambda) + \int_u^{u_2} c(u) R(u, \alpha v) \, \mathrm{d}u.$$

But, since $G_u \leqslant 0$ on $[u_1, u_2]$, the function $u \mapsto v(u)$ is necessarily monotone decreasing on the same interval. Then

$$\tfrac{1}{2} v^2(u) \leqslant \alpha^{p+1} \bar{d} (u_2 - u) v^{p+1}(u) + G(u, u_2, \lambda),$$

which clearly implies that v remains bounded and $v_1^+(\alpha)$ exists for all $\alpha \geqslant 0$.

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Consider now equation (4.6) divided by v^{p+1}, that is,

$$\frac{1}{v^p}\frac{\mathrm{d}v}{\mathrm{d}u} + \alpha^{p+1}c(u)b(u,\alpha v) = \frac{G_u(u,u_0,\lambda)}{v^{p+1}}. \tag{4.7}$$

Since $G_u \leqslant 0$ on $[u_1,u_2]$, integrating (4.7) over the latter interval yields

$$\frac{1}{1-p}(0 - v_1^+(\alpha)^{1-p}) \leqslant -\underline{d}\alpha^{p+1}(u_2 - u_1).$$

Thus we obtain that

$$v_1^+(\alpha)^{1-p} \geqslant (1-p)\underline{d}\alpha^{p+1}(u_2-u_1)$$

and $\lim_{\alpha\to+\infty} v_1^+(\alpha) = +\infty$.

Case 2 ($p = 1$). Let us define the two functions

$$z(u) := v^2(u)\exp(2\underline{d}\alpha^2 u), \qquad w(u) := v^2(u)\exp(2\bar{d}\alpha^2 u).$$

Then, on the one hand,

$$\frac{\mathrm{d}w}{\mathrm{d}u}(u) = 2\exp(2\bar{d}\alpha^2 u)\left(v\frac{\mathrm{d}v}{\mathrm{d}u}(u) + \bar{d}\alpha^2 v^2\right) \geqslant 2\exp(2\bar{d}\alpha^2 u)G_u(u,u_2,\lambda),$$

and a simple integration over $[u,u_2]$ gives that w is bounded and thus v is also bounded. The existence of $v_1^+(\alpha)$ is ensured for all α. On the other hand, the function z satisfies

$$\frac{\mathrm{d}z}{\mathrm{d}u}(u) = 2\exp(2\underline{d}\alpha^2 u)\left(v\frac{\mathrm{d}v}{\mathrm{d}u}(u) + \underline{d}\alpha^2 v^2\right) \leqslant 2\exp(2\underline{d}\alpha^2 u)G_u(u,u_2,\lambda).$$

Then we get

$$\begin{aligned} z(u_1) &\geqslant 2\int_{u_1}^{u_2} \exp(2\underline{d}\alpha^2 s)|G_u(s,u_2,\lambda)|\,\mathrm{d}s &(4.8)\\ &\geqslant 2\int_0^{u_2-u_1} \exp(2\underline{d}\alpha^2(s+u_1))|G_u(s+u_1,u_2,\lambda)|\,\mathrm{d}s &(4.9)\\ &\geqslant 2\exp(2\underline{d}\alpha^2 u_1)\int_{(u_2-u_1)/2}^{u_2-u_1} \exp(2\underline{d}\alpha^2 s)|G_u(s+u_1,u_2,\lambda)|\,\mathrm{d}s. &(4.10)\end{aligned}$$

Finally,

$$\begin{aligned} v_1^+(\alpha)^2 &\geqslant 2\int_{(u_2-u_1)/2}^{u_2-u_1} \exp(2\underline{d}\alpha^2 s)|G_u(s+u_1,u_2,\lambda)|\,\mathrm{d}s &(4.11)\\ &\geqslant 2\exp(\underline{d}\alpha^2(u_2-u_1))\int_{(u_2-u_1)/2}^{u_2-u_1} |G_u(s+u_1,u_2,\lambda)|\,\mathrm{d}s &(4.12)\\ &\geqslant C\exp(\underline{d}\alpha^2(u_2-u_1)), &(4.13)\end{aligned}$$

where C is some positive constant. Then $\lim_{\alpha\to+\infty} v_1^+(\alpha) = +\infty$.

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CASE 3 ($p > 1$). First, note that, since the function $u \mapsto v_+(u, \lambda, 0)$ is bounded in $[u_1, u_2]$, by continuity, this property remains valid for the function $u \mapsto v_+(u, \lambda, \alpha)$, for small values of α. Taking $\tilde{u} = \frac{1}{2}(u_1 + u_2)$ and $\tilde{v}_\alpha = v_+(\tilde{u}, \lambda, \alpha)$, integration of (4.7) over the interval $[u_1, \tilde{u}]$ gives

$$\frac{1}{p-1}(\tilde{v}_\alpha^{1-p} - v_1^+(\alpha)^{1-p}) \geqslant \alpha^{p+1}\underline{d}(\tilde{u} - u_1).$$

Then

$$v_1^+(\alpha)^{1-p} \leqslant \tilde{v}_\alpha^{1-p} - \tfrac{1}{2}(p-1)\underline{d}\alpha^{p+1}(u_2 - u_1).$$

Thanks to the monotonicity of $\alpha \to \tilde{v}_\alpha$, we have

$$v_1^+(\alpha)^{1-p} \leqslant \tilde{v}_0^{1-p} - \tfrac{1}{2}(p-1)\underline{d}\alpha^{p+1}(u_2 - u_1).$$

Since the right-hand side of the last inequality is negative for large values of α, we obtain the existence of a finite α_p for which $v_1^+(\alpha)$ exists for all $\alpha \in [0, \alpha_p)$ and, at the same time, we obtain that $\lim_{\alpha \to \alpha_p} v_1^+(\alpha) = +\infty$. This completes the proof of lemma 4.3. □

Proof of theorem 4.1. The continuous function

$$[0, +\infty) \ni \alpha \mapsto v_\pm(\alpha) := v_+(u_1, \lambda, \alpha) - v_-(u_1, \lambda, \alpha) = v_1^+(\alpha) - v_1^-(\alpha)$$

measures the distance (in the phase plane) between the two trajectories when they reach the value $u = u_1$. Therefore, the condition $v_\pm(\alpha) = 0$ characterizes the travelling wave solution of interest connecting u_0 to u_2.

Suppose first that $\alpha = 0$. Integrating (4.6), on the one hand, with $v = v_-$ and over the interval $[u_0, u_1]$, and on the other hand with $v = v_+$ and over the interval $[u_1, u_2]$, we get

$$\tfrac{1}{2}(v_1^-(\alpha))^2 = G(u_1, u_0, \lambda)$$

and

$$\tfrac{1}{2}(v_1^+(\alpha))^2 = G(u_1, u_0, \lambda) - G(u_2, u_0, \lambda) = G(u_1, u_2, \lambda),$$

respectively. Since $G(u_2, u_0, \lambda) > 0$ by lemma 2.5, we conclude that $v_\pm(0) < 0$. But, from lemmas 4.2 and 4.3, we have that $\alpha \mapsto v_\pm(\alpha)$ is strictly monotone increasing and

$$\lim_{\alpha \to \alpha_p} v_\pm(\alpha) = +\infty.$$

Together with $v_\pm(0) < 0$, this completes the proof of theorem 4.1. □

The existence of the non-classical travelling waves is thus established. We can also prove the following result.

LEMMA 4.4 (properties of the critical diffusion–dispersion ratio). *Define*

$$\begin{aligned}\varDelta &= \{(u_0, u_2) \in \mathbb{R}_- \times \mathbb{R}_+ / u_2 \in (\varphi^\natural(u_0), \varphi_0^\flat(u_0)]\} \\ &= \{(u_0, u_2) \in \mathbb{R}_- \times \mathbb{R}_+ / u_0 \in (\varphi^{-\natural}(u_2), \varphi_0^\flat(u_2)]\}\end{aligned}$$

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and consider the function

$$\Delta \ni (u_0, u_2) \mapsto A(u_0, u_2),$$

which associates the (unique) value α such that there is a (non-classical) travelling wave connecting u_0 to u_2 (theorem 4.1). Then we have the following.

(1) *$A(u_0, u_2)$ is a strictly monotone decreasing function of u_2, mapping the interval $(\varphi^\natural(u_0), \varphi_0^\flat(u_0)]$ onto some interval of the form $[0, A_p^\natural(u_0))$, where $A_p^\natural(u_0) \in (0, \infty]$.*

(2) *$A(u_0, u_2)$ is a strictly monotone decreasing function of u_0, mapping the interval $(\varphi^{-\natural}(u_2), \varphi_0^\flat(u_2)]$ onto the interval $[0, A_p^\natural(\varphi^{-\natural}(u_2)))$.*

Following [18, ch. III], we refer to $A(u_0, u_2)$ as the *critical diffusion–dispersion ratio* at (u_0, u_2), where the value $A_p^\natural(u_0)$ is called the *threshold diffusion–dispersion ratio* at u_0: non-classical trajectories leaving from u_0 exists only when $\alpha < A_p^\natural(u_0)$.

Proof. We will prove the first statement; the proof of the second one being similar. We fix $u_0 < 0$ and $u_0 < u_2 < u_2^*$ so that

$$\lambda^\natural(u_0) < \lambda = \bar{a}(u_0, u_2) < \lambda^* = \bar{a}(u_0, u_2^*) \leqslant \lambda_0(u_0),$$

and, in particular,

$$\bar{\mu}(u_0, \lambda) > \bar{\mu}(u_0, \lambda^*).$$

Proceeding by contradiction, we assume that

$$\alpha^* := A(u_0, u_2^*) \geqslant \alpha := A(u_0, u_2).$$

Let $v = v(u)$ and $v^* = v^*(u)$ be the solutions of (4.6) associated with α and α^*, respectively, and connecting u_0 to u_2 and u_0 to u_2^*, respectively. By continuity, there would exist some $u_3 \in (u_0, u_2)$ such that

$$v(u_3) = v^*(u_3), \qquad \frac{\mathrm{d}v^*}{\mathrm{d}u}(u_3) \geqslant \frac{\mathrm{d}v}{\mathrm{d}u}(u_3).$$

Combining (4.6) for v and v^*, we get

$$v(u_3)\biggl(\frac{\mathrm{d}v^*}{\mathrm{d}u}(u_3) - \frac{\mathrm{d}v}{\mathrm{d}u}(u_3)\biggr) + c(u_3)(R(u_3, \alpha^* v(u_3)) - R(u_3, \alpha v(u_3))) \\ = c(u_3)(\lambda^* - \lambda)(u_0 - u_3),$$

which leads to a contradiction, since the left-hand side is positive and the right-hand side is negative. Namely, we have, on the one hand,

$$v(u_3) > 0 \quad \text{and} \quad \frac{\mathrm{d}v^*}{\mathrm{d}u}(u_3) - \frac{\mathrm{d}v}{\mathrm{d}u}(u_3) \geqslant 0,$$

and on the other hand, thanks to (H3),

$$R(u_3, \alpha^* v(u_3)) - R(u_3, \alpha v(u_3)) \geqslant 0.$$

Furthermore, $\lambda^* - \lambda > 0$ and $u_0 - u_3 < 0$. We conclude that $\alpha^* < \alpha$. □

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5. Properties of the kinetic function and classical trajectories

In this section, we derive some important asymptotic properties of the non-classical trajectories. First we check that the threshold diffusion–dispersion ratio does not remain finite for all values of p.

THEOREM 5.1. *With the notation of lemma 4.4, we have the following.*

CASE 1. *If* $0 < p \leqslant \frac{1}{3}$, *then*

$$A_p^\natural(u_0) < \infty \quad \textit{for all } u_0.$$

Moreover, there exists a travelling wave connecting u_0 *to* $u_2 = \varphi^\natural(u_0)$ *for the critical diffusion* $\alpha = A_p^\natural(u_0)$.

CASE 2. *If* $p > \frac{1}{3}$, *then*

$$A_p^\natural(u_0) = +\infty \quad \textit{for all } u_0.$$

Proof. Consider some value $u_0 < 0$ together with the corresponding values $\varphi^\natural(u_0)$ and

$$\lambda^\natural(u_0) = \frac{f(\varphi^\natural(u_0)) - f(u_0)}{\varphi^\natural(u_0) - u_0}.$$

By theorem 4.1, for each given $\lambda \in (\lambda^\natural(u_0), \lambda_0(u_0)]$, there exists a unique non-classical trajectory, denoted by $u \mapsto v(u)$ and connecting u_0 to some u_2 satisfying

$$v\frac{\mathrm{d}v}{\mathrm{d}u} + c(u)R(u, \alpha v) = G_u(u, u_0, \lambda), \tag{5.1}$$

with

$$\lambda = \bar{a}(u_0, u_2) = \frac{f(u_2) - f(u_0)}{u_2 - u_0}, \qquad u_2 > \varphi^\natural(u_0), \qquad \alpha = A(u_0, u_2).$$

Then it is easy to see that, for all $u \geqslant u_0$,

$$G_u(u, u_0, \lambda) \leqslant G_u(u, u_0, \lambda^\natural(u_0)). \tag{5.2}$$

Now, setting

$$C = \tfrac{1}{2}\sup\{c(u)\}\sup\{f''(u)^2\}, \quad u \in [u_0, \varphi^\natural(u_0)],$$

and using (5.1) and (5.2), we obtain

$$v\frac{\mathrm{d}v}{\mathrm{d}u} + R(u, \alpha v) \leqslant Ch(u)^2 \quad \text{for all } u \in [u_0, \varphi^\natural(u_0)], \tag{5.3}$$

where

$$h(u) = \varphi^\natural(u_0) - u. \tag{5.4}$$

Consider the case $0 < p \leqslant \frac{1}{3}$. Then there exists δ such that

$$\frac{1}{1-p} \leqslant \delta \leqslant \frac{2}{p+1}. \tag{5.5}$$

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Consider the function

$$u \to v^*(u) = (\varphi^\natural(u_0) - u)^\delta = h(u)^\delta.$$

For a given α^*, we can write

$$v^* \frac{\mathrm{d}v^*}{\mathrm{d}u} + c(u)R(u, \alpha^* v^*) = -\delta h(u)^{2\delta-1} + \alpha^{*p+1} c(u) b(u, \alpha^* v^*) h(u)^{(p+1)\delta}.$$

Let us now introduce the positive constants $\underline{d}$ and $\bar{d}$ such that

$$\underline{d} \leqslant c(u)b(u,v) \leqslant \bar{d} \quad \text{for all } (u,v) \in [u_0, \varphi^\natural(u_0)] \times \mathbb{R}.$$

Then, for all $u \in [u_0, \varphi^\natural(u_0)]$,

$$v^* \frac{\mathrm{d}v^*}{\mathrm{d}u} + c(u)R(u, \alpha^* v^*) \geqslant h(u)^{(p+1)\delta}(-\delta h(u)^{(1-p)\delta-1} + \underline{d}\alpha^{*p+1}). \tag{5.6}$$

It is clear that the curve $u \to v(u)$ crosses the curve $u \to v^*(u)$ at some point u_3 such that

$$u_0 < u_3 < \varphi^\natural(u_0), \qquad v(u_3) = v^*(u_3), \qquad \frac{\mathrm{d}v}{\mathrm{d}u}(u_3) \geqslant \frac{\mathrm{d}v^*}{\mathrm{d}u}(u_3).$$

Combining inequalities (5.3) and (5.6), we have

$$\begin{aligned} v(u_3)\biggl(\frac{\mathrm{d}v^*}{\mathrm{d}u}(u_3) - \frac{\mathrm{d}v}{\mathrm{d}u}(u_3)\biggr) + c(u_3)(R(u_3, \alpha^* v^*(u_3)) - R(u_3, \alpha v(u_3))) \\ \geqslant h(u_3)^{(p+1)\delta}(-\delta h(u_3)^{(1-p)\delta-1} - Ch(u_3)^{2-(p+1)\delta} + \underline{d}\alpha^{*p+1}). \end{aligned}$$

Now, using the monotonicity of h, since $(1-p)\delta - 1 \geqslant 0$ and $2 - (p+1)\delta \geqslant 0$ by (5.5) and by taking α^* sufficiently large such that

$$\underline{d}\alpha^{*p+1} \geqslant \delta h(u_0)^{(1-p)\delta-1} + Ch(u_0)^{2-(p+1)\delta},$$

we get

$$v(u_3)\biggl(\frac{\mathrm{d}v^*}{\mathrm{d}u}(u_3) - \frac{\mathrm{d}v}{\mathrm{d}u}(u_3)\biggr) + c(u_3)(R(u_3, \alpha^* v^*(u_3)) - R(u_3, \alpha v(u_3))) \geqslant 0,$$

and, by assumption (H2), we obtain that, necessarily,

$$\alpha \leqslant \alpha^*.$$

Finally, since $A(u_0, u_2)$ remains bounded as u_2 tends to $\varphi^\natural(u_0)$ and thanks to the continuity of the travelling wave $u \to v(u)$ with respect to the parameters λ and α, we can define the travelling wave connecting u_0 to $\varphi^\natural(u_0)$ by

$$v(\cdot, u_0, A_p^\natural(u_0)) = \lim_{u_2 \to \varphi^\natural(u_0)} v(\cdot, u_0, A(u_0, u_2)).$$

Consider now the case $p > \frac{1}{3}$. Assume, by contradiction, that $A_p^\natural(u_0) < \infty$. Then, by continuity, there exists a travelling wave $u \to v(u)$ connecting u_0 to $\varphi^\natural(u_0)$ for $\alpha = A_p^\natural(u_0)$. Clearly, since

$$\underline{\mu}(u_2, \lambda) = -\sqrt{c(u_2)(f'(u_2) - \lambda)},$$

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passing to the limit when $\lambda \to \lambda^\natural(u_0)$, such a curve satisfies

$$\frac{\mathrm{d}v}{\mathrm{d}u}(\varphi^\natural(u_0)) = \lim_{\lambda \to \lambda^\natural(u_0)} \underline{\mu}(u_2, \lambda) = 0.$$

Furthermore, we have that $v(u) > 0$ for $u \in (u_0, \varphi^\natural(u_0))$. The last property allows us to construct a strictly monotone increasing sequence (u_n) in the interval $(\frac{1}{2}\varphi^\natural(u_0), \varphi^\natural(u_0)) \in (u_0, \varphi^\natural(u_0))$ such that

$$\frac{\mathrm{d}v}{\mathrm{d}u}(u_n) \leqslant 0 \quad \text{and} \quad \lim_{n\to\infty} u_n = \varphi^\natural(u_0). \tag{5.7}$$

On the other hand, the constant

$$K = \tfrac{1}{2} \inf\{c(u)\} \inf\{f''(u)^2\}, \quad u \in [\tfrac{1}{2}\varphi^\natural(u_0), \varphi^\natural(u_0)] \tag{5.8}$$

is positive and we have

$$G_u(u, u_0, \lambda^\natural(u_0)) \geqslant K h(u)^2 \quad \text{for all } \in [\tfrac{1}{2}\varphi^\natural(u_0), \varphi^\natural(u_0)], \tag{5.9}$$

where h is defined by (5.4). Combining (5.7) and (5.9) and setting $v_n = v(u_n)$, we get

$$\bar{d}\alpha^{p+1} v_n^{p+1} \geqslant c(u_n) R(u_n, \alpha v_n) \geqslant K h(u_n)^2,$$

which gives

$$v_n \geqslant C h(u_n)^{2/(p+1)}, \tag{5.10}$$

where C is some positive constant.

On the other hand, using (4.7) and (H3), and since $G'_u(u, u_0, \lambda^\natural(u_0)) \geqslant 0$ for $u \geqslant u_0$, we get

$$\frac{1}{v(u)^p} \frac{\mathrm{d}v}{\mathrm{d}u} + \underline{d}\alpha^{p+1} \geqslant 0 \quad \text{for all } u \geqslant u_0. \tag{5.11}$$

Note that it is easy to check that if $p \geqslant 1$, any solution of (5.11) cannot satisfy $v(\varphi^\natural(u_0)) = 0$. Now, when $\frac{1}{3} \leqslant p < 1$, an integration of (5.11) over $[u, \varphi^\natural(u_0)]$, for $u \in [u_0, \varphi^\natural(u_0)]$, gives

$$\frac{1}{1-p}(0 - v(u)^{1-p}) + \underline{d}\alpha^{p+1} h(u) \geqslant 0 \quad \text{for } u \in [u_0, \varphi^\natural(u_0)].$$

Then

$$v_n \leqslant C' h(u_n)^{1/(1-p)}, \tag{5.12}$$

where C' is some positive constant. Finally, combining (5.10) and (5.12), we obtain that, for some positive constant C'',

$$h(u_n)^{(2/(p+1) - 1/(1-p))} \leqslant C''.$$

But this is not possible, since

$$\frac{2}{p+1} - \frac{1}{1-p} = \frac{1-3p}{1-p^2} < 0$$

and $h(u_n) \to h(\varphi^\natural(u_0)) = 0$ when $n \to \infty$. This completes the proof of theorem 5.1. □

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We now study classical trajectories. Given some $u_0 < 0$ and $\alpha > 0$, we study the existence of the (classical) travelling waves connecting $u_- = u_0$ to $u_+ = u_1$ (see the notation (2.15)). The shock speed here lies in the interval $\lambda \in (\lambda^\natural(u_0), f'(u_0))$. According to the results in § 4, for $u_0 < 0$ fixed, we obtain a critical diffusion value $A(\lambda, u_0)$ for each speed (in some range), for which a non-classical trajectory exists from u_0 to u_2. So we can consider the mapping

$$\lambda \mapsto A(\lambda, u_0),$$

which is defined and strictly decreasing from the interval $(\lambda^\natural(u_0), \lambda_0(u_0)]$ onto $[0, A_p^\natural(u_0))$. Hence this mapping admits an inverse function

$$\alpha \mapsto \Lambda(u_0, \alpha)$$

defined from the interval $[0, A_p^\natural(u_0))$ onto $(\lambda^\natural(u_0), \lambda_0(u_0)]$. By construction, given any $\alpha \in [0, A_p^\natural(u_0))$, there exists a non-classical trajectory (associated with the shock speed $\Lambda(\alpha, u_0)$) leaving from u_0 and solving the equation with the prescribed diffusion α.

Theorem 5.2 (classical trajectories). *Fix some $u_0 < 0$ and $\alpha > 0$. For every speed satisfying $\Lambda(u_0, \alpha) < \lambda \leqslant f'(u_0)$, there exists a unique travelling wave connecting $u_- = u_0$ to $u_+ = u_1$. Moreover, in the case that $\alpha \geqslant A_p^\natural(u_0)$ (when $p \leqslant \frac{1}{3}$), there also exists a travelling wave connecting $u_- = u_0$ to $u_+ = u_1$ for all $\lambda \in [\lambda^\natural(u_0), f'(u_0)]$.*

If $\lambda^\natural(u_0) < \lambda < \Lambda(u_0, \alpha)$, then there is no travelling wave connecting $u_- = u_0$ to $u_+ = u_1$.

The proof of theorem 5.2 is similar to the case $p = 0$, given in [4, theorems 5.1 and 5.2].

We now return to the non-classical travelling waves, regarding now α as a fixed parameter. We define the *kinetic function* for non-classical shocks,

$$(u_0, \alpha) \mapsto \varphi_\alpha^\flat(u_0) = u_2,$$

where u_2 denotes the right-hand state of the non-classical trajectory. So we have

$$\frac{f(u_0) - f(u_2)}{u_0 - u_2} = \Lambda(u_0, \alpha) = \lambda_\alpha^\flat(u_0).$$

Note that, when $p \leqslant \frac{1}{3}$, $\varphi_\alpha^\flat(u_0)$ makes sense for all $u_0 \in \mathbb{R}$ but $\alpha < A_p^\natural(u_0)$. According to theorem 5.2, this function can be extended to all values of α by setting

$$\varphi_\alpha^\flat(u_0) = \varphi^\natural(u_0) \quad \text{for all } \alpha \geqslant A_p^\natural(u_0).$$

For the same reason, function $\Lambda(\alpha, u_0)$ may be extended to arbitrary values α by setting

$$\Lambda(\alpha, u_0) = \lambda^\natural(u_0) \quad \text{for all } \alpha \geqslant A_p^\natural(u_0).$$

The following is an important property of the kinetic function.

Lemma 5.3. *For each $\alpha > 0$, the mapping $u_0 \mapsto \varphi_\alpha^\flat(u_0)$ is strictly monotone decreasing.*

The proof of lemma 5.3 is similar to the case $p = 0$, given in [4, theorem 5.3].

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Proof of theorem 2.2. The result in theorem 4.1 provides us with the existence and uniqueness of the non-classical shocks, while theorem 5.2 is concerned with the classical trajectories. These results prove that the shock set (see § 2) is given by (2.11). On the other hand, the monotonicity properties of the kinetic function is provided by lemma 5.3, from which we can also immediately deduce the monotonicity of the shock speed as a function of u_0. Equations (2.12) and (2.13) are a consequence of theorem 5.1. □

We now give the proof of theorem 2.3. Assume that f satisfies $f'''(0) > 0$ and set $k := \frac{1}{6} f'''(0)$. Given $\alpha \geqslant 0$, there exists $u_2 = \varphi_\alpha^\flat(u_0)$ such that there exists a travelling wave solution of (2.4) connecting u_0 to u_2.

Since the cubic function satisfies theorem 3.1, the proof of theorem 2.3 is the consequence of the following two lemmas.

Lemma 5.4. *Suppose that $0 < p \leqslant \frac{1}{3}$. Consider the flux function $f^*(u) = ku^3$ and the constant coefficients $c_1^* = c_1(0)$, $c_2^* = c_2(0)$ and $b^* = b(0,0)$. Define*

$$A_p^\natural(u_0) = A_p^\natural(u_0, f, b, c) \quad \textit{and} \quad A_p^{\natural *}(u_0) = A_p^\natural(u_0, f^*, b^*, c^*)$$

as the threshold diffusions at u_0 for equation (2.4), with corresponding flux and coefficients. Then we have

$$A_p^\natural(u_0) \sim A_p^{\natural *}(u_0) \quad \textit{when } u_0 \to 0.$$

The case $p = 0$ was already treated in [4, theorem 4.2]. It is clear that the proof therein, modulo minor changes, immediately extends for all $0 < p \leqslant \frac{1}{3}$. In particular, one relies here on the fact that the threshold diffusion is finite.

Lemma 5.5. *With the notation of lemma 5.4, consider the corresponding kinetic functions,*

$$\varphi_\alpha^\flat(u_0) = \varphi_\alpha^\flat(u_0, f, b, c) \quad \textit{and} \quad \varphi_\alpha^{\flat *}(u_0) = \varphi_\alpha^\flat(u_0, f^*, b^*, c^*).$$

Then we have

$$\varphi_\alpha^{\flat\prime}(0) = \begin{cases} -\frac{1}{2} & \textit{if } 0 < p < \frac{1}{2}, \\ \varphi_\alpha^{\flat *\prime}(0) & \textit{if } p = \frac{1}{2}, \\ -1 & \textit{if } p > \frac{1}{2}. \end{cases} \tag{5.13}$$

Remark 5.6. Note that (5.13) is equivalent to saying that $\varphi_\alpha^{\flat\prime}(0) = \varphi_\alpha^{\flat *\prime}(0)$ for all $p \geqslant 0$.

Proof. First, when $0 \leqslant p \leqslant \frac{1}{3}$, we clearly have that $\varphi_\alpha^{\flat\prime}(0) = -\frac{1}{2}$. Indeed, in this case, thanks to lemma 5.4, we have $A_p^\natural(0) = 0$. Then, by continuity, $\alpha > A_p^\natural(u_0)$ for small u_0 and $\varphi_\alpha^\flat(u_0) = \varphi^\natural(u_0)$, which imply that

$$\varphi_\alpha^{\flat\prime}(0) = {\varphi^\natural}'(u_0) = -\tfrac{1}{2}.$$

For $p > \frac{1}{3}$, $A_p^\natural(u_0) = +\infty$ and the previous properties are not valid. We give a rigorous proof, which is also valid for all $p > 0$. Without loss of generality, we can assume that $f(0) = f'(0) = 0$. Given $\epsilon > 0$, there exists $\eta > 0$ such that, for $|u| \leqslant \eta$, we have

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(i) $|\varphi^\natural(u) + \frac{1}{2}u| \leqslant \frac{1}{2}\epsilon^3|u|$;

(ii) $|f''(u) - 6ku| \leqslant 6\epsilon^3 k|u|$;

(iii) $|c(u) - c(0)| \leqslant \epsilon^4 c(0)$.

Also, using the continuity of b on $(u, v) = (0, 0)$ and since the velocity satisfies

$$0 \leqslant v(u) \leqslant \sqrt{2G(u, u_0, \lambda)} \leqslant \sqrt{2G(\varphi^\natural(u_0), u_0, \lambda^\natural(u_0))}, \quad u \in [u_0, u_2],$$

and

$$\lim_{u_0 \to 0} G(\varphi^\natural(u_0), u_0, \lambda^\natural(u_0)) = 0,$$

we can choose η sufficiently small such that, for all $|u| \leqslant |u_0| \leqslant \eta$, and

$$|v| \leqslant \sqrt{2G(\varphi^\natural(u_0), u_0, \lambda^\natural(u_0))},$$

then

(iv) $|b(u, v) - b(0, 0)| \leqslant \epsilon b(0, 0)$.

Fix $-\eta \leqslant u_0 < 0$ (the proof being similar for $0 < u_0 \leqslant \eta$). Given $\alpha \geqslant 0$, there exists $u_2 = \varphi^\flat_\alpha(u_0)$ such that there exists a travelling wave solution of (2.4) connecting u_0 to u_2. Consider the functions

$$f_+(u) = k(1 + 2\epsilon^3)u^3 = k^+ u^3 \quad \text{and} \quad f_-(u) = k(1 - 2\epsilon^3)u^3 = k^- u^3$$

and the constant functions c_1^+, c_2^+, b^+ and c_1^-, c_2^-, b^- defined by

$$\begin{aligned} c^+ &= c(0)(1 + \epsilon^4), & b^+ &= b(0, 0)(1 + \epsilon), \\ c^- &= c(0)(1 - \epsilon^4), & b^- &= b(0, 0)(1 - \epsilon). \end{aligned}$$

Thus we may define

$$u_2^+ = \varphi_\alpha^{\flat +}(u_0) = \varphi^\flat_\alpha(u_0, f_+, b^+, c^+), \qquad u_2^- = \varphi_\alpha^{\flat -}(u_0) = \varphi^\flat_\alpha(u_0, f_-, b^-, c^-).$$

We also define the speeds

$$\lambda = \frac{f(u_2) - f(u_0)}{u_2 - u_0}, \quad \lambda^+ = \frac{f_+(u_2^+) - f_+(u_0)}{u_2^+ - u_0} \quad \text{and} \quad \lambda^- = \frac{f_-(u_2^-) - f_-(u_0)}{u_2^- - u_0}.$$

Step 1. We begin by proving the inequality

$$\lambda^+ - \lambda \leqslant \max(f'_+(u_0) - f'(u_0), f'_+(u_2) - f'(u_2)) + k\epsilon^3 u_0^2. \tag{5.14}$$

Thanks to (ii), and since $u_0 < 0$ and $u_2 > 0$, we obtain that $f(u_2) \geqslant k(1 - \epsilon^3)u_2^3$ and $f(u_0) \leqslant k(1 - \epsilon^3)u_0^3$. Thus

$$\lambda \geqslant k(1 - \epsilon^3)\frac{u_2^3 - u_0^3}{u_2 - u_0}$$

and we can write

$$\lambda^+ - \lambda \leqslant k(1 + 2\epsilon^3)\frac{{u_2^+}^3 - u_0^3}{u_2^+ - u_0} - k(1 - \epsilon^3)\frac{{u_2}^3 - u_0^3}{u_2 - u_0}$$

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and

$$\leqslant k(1-\epsilon^3)\bigg(\frac{u_2^{+3}-u_0^3}{u_2^+-u_0}-\frac{u_2^3-u_0^3}{u_2-u_0}\bigg)+3k\epsilon^3\frac{u_2^{+3}-u_0^3}{u_2^+-u_0}.$$

But, since $u_2^+ \leqslant -u_0$, we obtain

$$\lambda^+ - \lambda \leqslant k(1-\epsilon^3)\bigg(\frac{u_2^{+3}-u_0^3}{u_2^+-u_0}-\frac{{u_2}^3-u_0^3}{u_2-u_0}\bigg)+3k\epsilon^3u_0^2 \tag{5.15}$$

On the other hand, thanks to (ii), we have

$$f'_+(u_0)-f'(u_0) \geqslant 3k(1+2\epsilon^3)u_0^2 - 3k(1+\epsilon^3)u_0^2 \geqslant 3k\epsilon^3u_0^2. \tag{5.16}$$

Assume, by contradiction, that (5.14) does not hold. Then

$$\lambda^+ - \lambda > f'_+(u_0)-f'(u_0)+k\epsilon^3u_0^2 \quad \text{and} \quad \lambda^+ - \lambda > f'_+(u_2)-f'(u_2)+k\epsilon^3u_0^2. \tag{5.17}$$

Combining (5.15)–(5.17), we get

$$\frac{u_2^{+3}-u_0^3}{u_2^+-u_0}-\frac{u_2^3-u_0^3}{u_2-u_0} > 0.$$

We can distinguish between two situations: if $u_2 < -\frac{1}{2}u_0$, then, since $u_2^+ \geqslant -\frac{1}{2}u_0$, we have $u_2^+ > u_2$.

If $u_2 \geqslant -\frac{1}{2}u_0$, using the monotonicity of the speed for $u \geqslant -\frac{1}{2}u_0$, we also get that $u_2^+ > u_2$. Consider now the two travelling waves solutions of (2.4) connecting u_0 to u_2 and u_0 to u_2^+, respectively. The corresponding curves in the phase plan $u \mapsto v(u)$ and $u \mapsto v^+(u)$ satisfy

$$v\frac{\mathrm{d}v}{\mathrm{d}u} + c(u)R(u,\alpha v) = c(u)(f(u)-f(u_0)-\lambda(u-u_0)) = c(u)l(u) \tag{5.18}$$

and

$$v^+\frac{\mathrm{d}v^+}{\mathrm{d}u} + c^+R^+(u,\alpha v^+) = c^+(f_+(u)-f_+(u_0)-\lambda_+(u-u_0)) = c^+l^+(u), \tag{5.19}$$

respectively. Here, R^+ is clearly defined by

$$R^+(u,\alpha v^+) = b^+\alpha^{p+1}|v|^p v.$$

We have

$$\frac{\mathrm{d}v}{\mathrm{d}u}(u_0) = \sqrt{c(u_0)(f'(u_0)-\lambda)} > \sqrt{c^+(f'_+(u_0)-\lambda_+)} = \frac{\mathrm{d}v^+}{\mathrm{d}u}(u_0).$$

Indeed, using (5.17), we have

$$\begin{aligned} c^+(f'_+(u_0)-\lambda_+) - c(u_0)(f'(u_0)-\lambda) \\ &= c^+((f'_+(u_0)-\lambda_+)-(f'(u_0)-\lambda)) + (c^+-c(u_0))(f'(u_0)-\lambda) \\ &< -c^+k\epsilon^3u_0^2 + (c^+-c(u_0))(f'(u_0)-\lambda). \end{aligned}$$

With our assumptions, $(c^+-c(u_0))(f'(u_0)-\lambda) \leqslant C\epsilon^4u_0^2$. Then, for small $\epsilon > 0$, the right-hand side is negative.

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Now, since

$$u_2^+ > u_2 \quad \text{and} \quad \frac{\mathrm{d}v}{\mathrm{d}u}(u_0) > \frac{\mathrm{d}v^+}{\mathrm{d}u}(u_0),$$

the two curves meet at least at some value u_3, $u_0 < u_3 < u_2$, such that

$$\frac{\mathrm{d}v^+}{\mathrm{d}u}(u_3) \geqslant \frac{\mathrm{d}v}{\mathrm{d}u}(u_3).$$

Combining (5.18) and (5.19), we get

$$\begin{aligned} v(u_3)\bigg(\frac{\mathrm{d}v^+}{\mathrm{d}u}(u_3) - \frac{\mathrm{d}v}{\mathrm{d}u}(u_3)\bigg) &+ (c^+b^+ - c(u_3)b(u_3, v(u_3)))v(u_3)^{p+1} \\ &= c^+l^+(u_3) - c(u_3)l(u_3) \\ &= (c^+ - c(u_3))l^+(u_3) + c(u_3)(l^+(u_3) - l(u_3)). \quad (5.20) \end{aligned}$$

We can write

$$l^+(u_3) - l(u_3) = \int_{u_0}^{u_3} (f'_+(u) - f'(u) - \lambda_+ + \lambda)\,\mathrm{d}u = \int_{u_0}^{u_3} g(u)\,\mathrm{d}u.$$

Since $g(u_0) < -k\epsilon^3 u_0^2$ and $g(u_2) < -k\epsilon^3 u_0^2$ by (5.17), and

$$ug'(u) = u(f''_+(u) - f''(u)) > 0,$$

we deduce that $g(u) < -k\epsilon^3 u_0^2$ for all $u \in [u_0, u_2]$, and particularly for $u \in [u_0, u_3]$. We obtain that

$$l^+(u_3) - l(u_3) < -k\epsilon^3 u_0^2(u_3 - u_0). \quad (5.21)$$

On the other hand,

$$(c^+ - c(u_3))l^+(u_3) \leqslant 2\epsilon^4 c(u_0)|l^+(u_3)| \leqslant 2\epsilon^4 c(u_0)(u_3 - u_0)|f'_+(u_4) - \lambda_+|$$

for some $u_4 \in (u_0, u_3)$. Then

$$(c^+ - c(u_3))l^+(u_3) \leqslant 2\epsilon^4 c(u_0)(u_3 - u_0)|f'_+(u_0) - \lambda^\natural(u_0)| \leqslant C(u_3 - u_0)\epsilon^4 u_0^2, \quad (5.22)$$

where C is a positive constant. Using (5.21) and (5.22), we obtain that, for small $\epsilon > 0$, the right-hand side in (5.20) is negative, while, by assumption, the left-hand side is non-negative. This gives us a contradiction and we conclude that (5.14) holds.

STEP 2. We now prove that inequality (5.14) implies that

$$u_2^+ \leqslant u_2(1 + \epsilon).$$

Indeed, as in step 1, using (ii) and since $u_0 < 0$ and $u_2 > 0$, we obtain

$$f(u_2) \leqslant k(1 + \epsilon^3)u_2^3 \quad \text{and} \quad f(u_0) \geqslant k(1 + \epsilon^3)u_0^3.$$

Thus we have

$$\begin{aligned} \lambda^+ - \lambda &\geqslant k(1 + 2\epsilon^3)\frac{u_2^{+3} - u_0^3}{u_2^+ - u_0} - k(1 + \epsilon^3)\frac{u_2^3 - u_0^3}{u_2 - u_0} \\ &\geqslant k(1 + \epsilon^3)\bigg(\frac{u_2^{+3} - u_0^3}{u_2^+ - u_0} - \frac{u_2^3 - u_0^3}{u_2 - u_0}\bigg) + k\epsilon^3\frac{u_2^{+3} - u_0^3}{u_2^+ - u_0} \end{aligned}$$

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and, since $u_2^+ \geqslant -\frac{1}{2}u_0$,

$$\lambda^+ - \lambda \geqslant k(1+\epsilon^3)\left(\frac{u_2^{+3} - u_0^3}{u_2^+ - u_0} - \frac{u_2^3 - u_0^3}{u_2 - u_0}\right) + \tfrac{3}{4}k\epsilon^3 u_0^2. \tag{5.23}$$

On the other hand, using (ii),

$$f'_+(u_0) - f'(u_0) \leqslant 3k(1+2\epsilon^3)u_0^2 - 3k(1-\epsilon^3)u_0^2 \leqslant 9k\epsilon^3 u_0^2. \tag{5.24}$$

Now, by (5.14), assume that

$$\lambda_+ - \lambda \leqslant f'_+(u_0) - f'(u_0) + k\epsilon^3 u_0^2, \tag{5.25}$$

the case $\lambda_+ - \lambda \leqslant f'_+(u_2) - f'(u_2) + k\epsilon^3 u_0^2$ being similar.

Combining (5.23), (5.24) and (5.25), we obtain

$$(1+\epsilon^3)\left(\frac{u_2^{+3} - u_0^3}{u_2^+ - u_0} - \frac{u_2^3 - u_0^3}{u_2 - u_0}\right) \leqslant C\epsilon^3 u_0^2,$$

which we can rewrite in the form

$$(u_2^+ - u_2)(u_2^+ + u_2 + u_0) \leqslant C\epsilon^3 u_0^2$$

for small $\epsilon > 0$. Now assume, by contradiction, that $u_2^+ > u_2(1+\epsilon)$. Then, on the one hand, using (i), we get

$$u_2^+ + u_2 + u_0 \geqslant (2+\epsilon)u_2 + u_0 \geqslant (2+\epsilon)(-\tfrac{1}{2}u_0)(1-\epsilon^3) + u_0 \geqslant C|u_0|\epsilon$$

and, on the other hand,

$$u_2^+ - u_2 > \epsilon u_2 \geqslant C|u_0|\epsilon.$$

Combining the last three inequalities, we obtain a contradiction of the form $\epsilon^2 \leqslant C\epsilon^3$. We conclude that

$$u_2^+ \leqslant u_2(1+\epsilon).$$

Finally, it is clear that, via some symmetry considerations, we also obtain that $u_2^- \geqslant u_2(1-\epsilon)$ for small $\epsilon > 0$. Finally, we have

$$\frac{u_2^+}{1+\epsilon} \leqslant u_2 \leqslant \frac{u_2^-}{1-\epsilon}. \tag{5.26}$$

Step 3. Using (5.26), we now prove (5.13).

As we have seen in the proof of theorem 3.1, this results from the property of scaling of the cubic function. Indeed, the travelling waves connecting u_0 to $u_2^* = \varphi_\alpha^{\flat *}(u_0)$ and u_0 to $u_2^+ = \varphi_\alpha^{\flat +}(u_0)$, with the corresponding coefficients, result in, respectively,

$$u_{yy} + \alpha^{p+1}c^* b^* |u_y|^p u_y = c^* k(u^3 - u_0^3 - \lambda(u - u_0))$$

and

$$u_{yy} + \alpha^{p+1}c^+ b^+ |u_y|^p u_y = c^+ k^+(u^3 - u_0^3 - \lambda(u - u_0)).$$

Then, with the notation given earlier, a simple rescaling gives

$$u_2^+ = \varphi_\alpha^{\flat +}(u_0) = \varphi_{\alpha^+}^{\flat *}(u_0), \tag{5.27 a}$$

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where

$$\alpha^+ = \alpha(1+\epsilon)^{1/(p+1)}(1+2\epsilon^3)^{(p-1)/(2(1+p))}\sqrt{1+\epsilon^4}. \tag{5.27 b}$$

In addition, the transformation (3.11), (3.12) remains valid for f^* and, using (5.27), we get

$$\frac{u_2^+}{u_0} = \varphi^{\flat *}_{\kappa^+}(1), \tag{5.28 a}$$

where

$$\begin{aligned}\kappa^+ &= \alpha^+|u_0|^{(2p-1)/(p+1)}\\ &= \alpha(1+\epsilon)^{1/(p+1)}(1+2\epsilon^3)^{(p-1)/(2(1+p))}\sqrt{1+\epsilon^4}|u_0|^{(2p-1)/(p+1)}.\end{aligned} \tag{5.28 b}$$

Similarly, we have

$$\frac{u_2^-}{u_0} = \varphi^{\flat *}_{\kappa^-}(1), \tag{5.29 a}$$

where

$$\kappa^- = \alpha(1-\epsilon)^{1/(p+1)}(1-2\epsilon^3)^{(p-1)/(2(1+p))}\sqrt{1-\epsilon^4}|u_0|^{(2p-1)/(p+1)}. \tag{5.29 b}$$

Finally, since $u_0 < 0$, we can rewrite (5.26) in the form

$$\frac{1}{1-\epsilon}\varphi^{\flat *}_{\kappa^-}(1) \leqslant \frac{u_2}{u_0} \leqslant \frac{1}{1+\epsilon}\varphi^{\flat *}_{\kappa^+}(1). \tag{5.30}$$

To conclude, we distinguish between three cases.

Case 1 ($0 \leqslant p < \frac{1}{2}$). Here,

$$\lim_{u_0\to 0}\kappa^+ = \lim_{u_0\to 0}\kappa^- = +\infty.$$

Thus, using (5.28) and (5.29),

$$\lim_{u_0\to 0}\varphi^{\flat *}_{\kappa^+}(1) = \lim_{u_0\to 0}\varphi^{\flat *}_{\kappa^-}(1) = \varphi^{\flat *}_{\infty}(1) = \varphi^{\natural *}(1) = -\tfrac{1}{2}.$$

Choosing $\delta \leqslant \eta$, we obtain that, for all $|u_0| \leqslant \delta$,

$$-3\epsilon \leqslant \frac{u_2}{u_0} + \tfrac{1}{2} \leqslant 3\epsilon,$$

which implies that $\varphi^{\flat\prime}_{\alpha}(0) = -\frac{1}{2}$.

Case 2 ($p > \frac{1}{2}$). Here,

$$\lim_{u_0\to 0}\kappa^+ = \lim_{u_0\to 0}\kappa^- = 0.$$

Thus, similarly, using (5.28) and (5.29), we get $\varphi^{\flat\prime}_{\alpha}(0) = -1$.

Case 3 ($p = \frac{1}{2}$). This case is a little bit more difficult, since κ^+ and κ^- do not depend on u_0. We proceed as follows. Given $\epsilon' > 0$, there exists $\delta > 0$ such that, if $|\gamma - \alpha| \leqslant \delta$, then $|\varphi^{\flat *}_{\gamma}(1) - \varphi^{\flat *}_{\alpha}(1)| \leqslant \frac{1}{3}\epsilon'$. On the other hand, using (5.30), we get

$$\begin{aligned}\frac{\epsilon}{1-\epsilon}\varphi^{\flat *}_{\alpha}(1) - \frac{1}{1-\epsilon}|\varphi^{\flat *}_{\kappa_-}(1) - \varphi^{\flat *}_{\alpha}(1)| &\leqslant \frac{u_2}{u_0} - \varphi^{\flat *}_{\alpha}(1)\\ &\leqslant \frac{1}{1+\epsilon}|\varphi^{\flat *}_{\kappa_+}(1) - \varphi^{\flat *}_{\alpha}(1)| - \frac{\epsilon}{1+\epsilon}\varphi^{\flat *}_{\alpha}(1).\end{aligned}$$

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Choosing $0 < \epsilon < \frac{1}{3}$ such that

$$|\kappa_+ - \alpha| \leqslant \delta, \quad |\kappa_- - \alpha| \leqslant \delta \quad \text{and} \quad \frac{\epsilon}{1-\epsilon}|\varphi_\alpha^{\flat *}(1)| \leqslant \tfrac{1}{2}\epsilon',$$

we obtain that

$$\left|\frac{u_2}{u_0} - \varphi_\alpha^{\flat *}(1)\right| \leqslant \epsilon',$$

and the proof of lemma 5.5 is completed. □

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