Proceedings of the Edinburgh Mathematical Society (2016) **59**, 925–944
DOI:10.1017/S0013091515000395

SOLUTIONS FOR A KIRCHHOFF EQUATION WITH WEIGHT AND NONLINEARITY WITH SUBCRITICAL AND CRITICAL CAFFARELLI–KOHN–NIRENBERG GROWTH

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(Received 2 March 2014)

Abstract In this paper we study the multiplicity of non-trivial solutions to a class of nonlinear boundary-value problems of Kirchhoff type. We prove existence results when the problem has nonlinearities with subcritical and with critical Caffarelli–Kohn–Nirenberg exponent.

Keywords: variational methods; critical exponents; singular perturbations; Kirchhoff equation; nonlinear elliptic equations

2010 *Mathematics subject classification:* Primary 35A15; 35B33; 35B25; 35J60

1. Introduction

We are concerned with the existence of infinitely many solutions to the class of nonlinear boundary-value problems of Kirchhoff type

$$\left.\begin{aligned} L(u) = \lambda |x|^{-\delta} f(x,u) + |x|^{-\beta}|u|^{q-2}u \quad \text{in } \Omega,\\ u = 0 \quad \text{on } \partial\Omega, \end{aligned}\right\} \tag{1.1}$$

where

$$L(u) := -\left[M\left(\int_\Omega |x|^{-ap}|\nabla u|^p \,\mathrm{d}x\right)\right] \operatorname{div}(|x|^{-ap}|\nabla u|^{p-2}\nabla u)$$

and $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain with $N \geqslant 3$, $1 < p < N$, $a < (N-p)/p$, $p \leqslant q \leqslant p^*$, $\beta \leqslant (a+1)q + N(1-q/p)$, with $p^* = Np/(N-dp)$ the critical Caffarelli–Kohn–Nirenberg exponent, where $d = 1+a-b$ with $a \leqslant b < a+1$, and with $M\colon \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+$ a continuous function.

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Problem (1.1) with $a = b = \delta = \beta = 0$ and $p = 2$, that is,

$$\left.\begin{aligned} -M\biggl(\int_\Omega |\nabla u|^2\,\mathrm{d}x\biggr)\Delta u &= g(x,u) \quad \text{in } \Omega, \\ u &= 0 \qquad\quad \text{on } \partial\Omega, \end{aligned}\right\} \tag{1.2}$$

is called non-local because of the presence of the term $M(\int_\Omega |\nabla u|^2\,\mathrm{d}x)$, which implies that (1.2) is no longer a pointwise identity. This phenomenon causes some mathematical difficulties, which makes the study of such a class of problem particularly interesting. Besides this, this class of problem has a physical motivation. Indeed, the operator $M(\int_\Omega |\nabla u|^2\,\mathrm{d}x)\Delta u$ appears in the Kirchhoff equation, which arises in nonlinear vibrations, namely,

$$\left.\begin{aligned} u_{tt} - M\biggl(\int_\Omega |\nabla u|^2\,\mathrm{d}x\biggr)\Delta u &= g(x,u) \quad \text{in } \Omega\times(0,T), \\ u &= 0 \qquad\quad \text{on } \partial\Omega\times(0,T), \\ u(x,0) = u_0(x), \quad u_t(x,0) &= u_1(x) \quad\ \ \text{in } \Omega. \end{aligned}\right\} \tag{1.3}$$

Such a hyperbolic equation is a general version of the Kirchhoff equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \biggl(\frac{P_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 \mathrm{d}x\biggr)\frac{\partial^2 u}{\partial x^2} = 0, \tag{1.4}$$

presented by Kirchhoff in [**26**]. This equation extends the classical d'Alembert wave equation by considering the effects of the changes in the length of the strings during the vibrations. The parameters in (1.4) have the following meanings: L is the length of the string, h is the cross-sectional area, E is the Young's modulus of the material, ρ is the mass density and P_0 is the initial tension.

When an elastic string with fixed ends is subjected to transverse vibrations its length varies with time, which introduces changes in the tension in the string. This induced Kirchhoff to propose a nonlinear correction of the classical d'Alembert equation. Later on, Woinowsky-Krieger [**34**] (also Nash and Modeer [**31**]) incorporated this correction into the classical Euler–Bernoulli equation for a beam (plate) with hinged ends. See, for example, [**4**, **5**] and the references therein. The reader is referred to [**1**, **2**, **9**, **10**, **20**, **30**] and the references therein for more information on non-local problems.

To enunciate the main result we need to give some hypotheses on the functions M and f. The hypotheses on the continuous function $M\colon \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+$ are as follows.

There exists $m_0 > 0$ such that

(M_1) $M(t) \geqslant m_0$ for all $t \geqslant 0$,

(M_2) the function M is increasing.

The hypotheses on the Carathéodory function $f\colon \Omega\times\mathbb{R} \to \mathbb{R}$ are the following:

(f_1) $f(x,-t) = -f(x,t)$ $\forall (x,t) \in \Omega\times\mathbb{R}$;

(f_2) there exists $r \in [1, p^*)$ and positive constants C_1, C_2 with $C_1 < C_2$ such that

$$C_1|t|^{r-1} \leqslant f(x,t) \leqslant C_2|t|^{r-1} \quad \forall (x,t) \in \Omega\times(\mathbb{R}^+\cup\{0\});$$

https://doi.org/10.1017/S0013091515000395 Published online by Cambridge University Press

(f_3) the function f satisfies the well-known Ambrosetti–Rabinowitz superlinear condition, that is,

$$0 < \xi \int_0^t f(x,s)\,\mathrm{d}s \leqslant tf(x,t) \quad \forall (x,t) \in \Omega \times \mathbb{R}^+ \text{ for some } \xi \in (p, p^*).$$

Moreover, we suppose that $\delta \leqslant (a+1)r + N(1 - r/p)$.

The main results of this paper are listed below.

The first result gives us infinitely many solutions for problem (1.1) for the subcritical case.

Theorem 1.1. *Assume that (M_1), (f_1) and (f_2) hold and that $p \leqslant q < p^*$, $\beta < (a+1)q + N(1 - q/p)$ and $1 \leqslant r < p$. There then exists $\lambda_0 > 0$ such that problem (1.1) has infinitely many solutions for each $\lambda \in (0, \lambda_0)$.*

In the last two results, we obtain infinitely many solutions for problem (1.1) for the critical case.

Theorem 1.2. *Assume that (M_1), (M_2), (f_1) and (f_2) hold and that $q = p^*$, $\beta = bp^*$ and $1 \leqslant r < p$. There then exists $\lambda^* > 0$ such that problem (1.1) has infinitely many solutions for each $\lambda \in (0, \lambda^*)$.*

Theorem 1.3. *Assume that (M_1), (M_2), (f_1), (f_2) and (f_3) hold and that $q = p^*$, $\beta = bp^*$ and $p < r < p^*$. There then exists $\lambda^{**} > 0$ such that problem (1.1) has a non-trivial solution for each $\lambda \in (\lambda^{**}, +\infty)$.*

In recent years interest in the study of non-local problems of type (1.2) has grown exponentially. That was, probably, due to the difficulties existing in this class of problems that do not appear in the study of local problems, as well as due to their significance in applications. Without hope of being thorough, we mention some papers with multiplicity results and that are related to our main result. We will restrict our comments to the work that has emerged in the last four years.

Problem (1.2) was studied in [**20**]. The version with the p-Laplacian operator was studied in [**15**]. In both cases the authors showed a multiplicity result using genus theory. In [**18**, **23**–**25**, **29**, **32**] the authors showed a multiplicity result for problem (1.2) using the fountain theorem and the symmetric mountain pass theorem. The case with discontinuous nonlinearity was studied in [**16**], where Corrêa and Nascimento showed existence of two solutions via the mountain pass theorem and Ekeland's variational principle.

In [**13**] Chung and Quoc used the fountain theorem and showed a multiplicity result of solutions for a problem involving a non-local operator and nonlinearity of Caffarelli–Kohn–Nirenberg type and subcritical growth.

In this paper we study a different class of non-local operators to that considered in [**13**]. Moreover, our class of non-local operators includes, but is not restricted to, the type (1.4). Besides this, our results are true for the subcritical and critical case. For this, the arguments found in [**13**] could not be repeated and it was necessary to make a truncation on the function M in the critical case.

Our proofs of the main theorems are inspired by [**6**], where the authors showed a multiplicity result for a problem involving the p-Laplacian operator. But since we are working with a non-local operator, some refined estimates were necessary; for example, the proof of Lemma 4.1, which does not appear in [**6**].

Before concluding this introduction, it is very important to say that recently we found many papers in the literature in which the authors study the existence and multiplicity of solution to problems involving nonlinearities of Caffarelli–Kohn–Nirenberg type (see, for example, [**11**, **22**, **35**, **37**] and references therein).

This paper is organized as follows. In § 2 we provide some preliminary results on the Krasnoselskii genus and the variational framework. The Palais–Smale condition for the Euler–Lagrange functional associated with problem (1.1) and the proof of Theorem 1.1 are found in § 3. In § 4, by using the concentration–compactness principle, we prove the Palais–Smale condition, some auxiliary results and Theorems 1.2 and 1.3.

2. Preliminary results and variational framework

We start by considering some basic notions on the Krasnoselskii genus that we will use in the proofs of our main results.

Let E be a real Banach space. We denote by $\mathfrak{A}$ the class of all closed subsets $A \subset E\backslash\{0\}$ that are symmetric with respect to the origin, that is, $u \in A$ implies that $-u \in A$.

Definition 2.1. Let $A \in \mathfrak{A}$. The Krasnoselskii genus of A, $\gamma(A)$, is defined as being the least positive integer k such that there is an odd mapping $\phi \in C(A, \mathbb{R}^k)$ such that $\phi(x) \neq 0$ for all $x \in A$. If k does not exist, we set $\gamma(A) = +\infty$. Furthermore, by definition, $\gamma(\emptyset) = 0$.

In what follows we enunciate some results on the Krasnoselskii genus that may be found in [**3**, **12**, **17**, **27**].

Proposition 2.2. *Suppose that $E = \mathbb{R}^N$ and that $\partial\Omega$ is the boundary of an open symmetric and bounded subset $\Omega \subset \mathbb{R}^N$ with $0 \in \Omega$. Then $\gamma(\partial\Omega) = N$.*

Corollary 2.3. *If $\partial\Omega = S^{N-1}$ is the unit sphere in $\mathbb{R}^N$, then $\gamma(S^{N-1}) = N$.*

We now establish a result due to Clark [**14**].

Proposition 2.4. *Consider $\Phi \in C^1(X, \mathbb{R})$ a functional satisfying the Palais–Smale condition and suppose that*

(i) *Φ is bounded from below and even,*

(ii) *there is a compact set $K \in \mathfrak{A}$ such that $\gamma(K) = k$ and $\sup_{x\in K} \Phi(x) < \Phi(0)$.*

Then Φ possesses at least k distinct pairs of critical points and their corresponding critical values are less than $\Phi(0)$.

We point out that this result is a consequence of a basic multiplicity theorem involving an invariant functional under the action of a compact topological group.

Proposition 2.5. *If $K \in \mathfrak{A}$, $0 \notin K$ and $\gamma(K) \geqslant 2$, then K has infinitely many points.*

Now we will introduce the basic variational framework. Consider $\Omega \subset \mathbb{R}^N$ to be a bounded smooth domain with $0 \in \Omega$, $N \geqslant 3$, $1 < p < N$, $a < (N-p)/p$, $a \leqslant b < a+1$, and $p^* = Np/(N-dp)$, where $d = 1+a-b$. From [**8**, **36**] we have

$$\left(\int_\Omega |x|^{-\alpha}|u|^r \, \mathrm{d}x\right)^{p/r} \leqslant C\int_\Omega |x|^{-ap}|\nabla u|^p \, \mathrm{d}x \quad \forall u \in C_0^\infty(\Omega), \tag{2.1}$$

where $1 \leqslant r \leqslant Np/(N-p)$, $\alpha \leqslant (a+1)r + N(1-r/p)$, and $\mathcal{D}_a^{1,p}$ is the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\| = \left(\int_\Omega |x|^{-ap}|\nabla u|^p \, \mathrm{d}x\right)^{1/p},$$

that is, we have the continuous embedding of $\mathcal{D}_a^{1,p}$ in $L^r(\Omega, |x|^{-\alpha})$, where $L^r(\Omega, |x|^{-\alpha})$ is the weighted $L^r(\Omega)$ space with the norm

$$\|u\|_{r,\alpha} = \left(\int_\Omega |x|^{-\alpha}|u|^r \, \mathrm{d}x\right)^{1/r}.$$

Moreover, this embedding is compact if Ω is a bounded smooth domain, $1 \leqslant r < Np/(N-p)$, and $\alpha < (a+1)r + N(1-r/p)$. The best constant of the weighted Caffarelli–Kohn–Nirenberg type (see [**8**]) inequality will be denoted by $C_{a,p}^*$, which is characterized by

$$C_{a,p}^* = \inf_{u \in \mathcal{D}_a^{1,p}\setminus\{0\}} \frac{\int_\Omega |x|^{-ap}|\nabla u|^p \, \mathrm{d}x}{(\int_\Omega |x|^{-bp^*}|u|^{p^*} \, \mathrm{d}x)^{p/p^*}}.$$

We will look for solutions of problem (1.1) by finding critical points of the Euler–Lagrange functional $I\colon \mathcal{D}_a^{1,p} \to \mathbb{R}$ given by

$$I(u) = \frac{1}{p}\hat{M}(\|u\|^p) - \lambda\int_\Omega |x|^{-\delta}F(x,u)\, \mathrm{d}x - \frac{1}{q}\int_\Omega |x|^{-\beta}|u|^q \, \mathrm{d}x,$$

where $\hat{M}(t) := \int_0^t M(s)\, \mathrm{d}s$ and $F(x,t) = \int_0^t f(x,s)\, \mathrm{d}s$. Note that $I \in C^1$ and

$$\begin{aligned} I'(u)(\phi) = M(\|u\|^p)&\int_\Omega |x|^{-ap}|\nabla u|^{p-2}\nabla u \nabla\phi \, \mathrm{d}x \\ &- \lambda\int_\Omega |x|^{-\delta} f(x,u)\phi \, \mathrm{d}x - \int_\Omega |x|^{-\beta}|u|^{q-2}u\phi \, \mathrm{d}x \end{aligned}$$

for all $\phi \in \mathcal{D}_a^{1,p}$.

Theorems 1.1 and 1.2 will be proved by using Proposition 2.4. From (f_1) and (f_2) we have that I is even and $I(0) = 0$. However, we encounter the common difficulty of ensuring that I is bounded from below in $\mathcal{D}_a^{1,p}$, so we will use a modified functional to obtain the critical points of I. In the following we will construct the auxiliary functional.

We obtain, by (M_1), (f_1), (f_2) and the Caffarelli–Kohn–Nirenberg inequality (2.1), that

$$I(u) \geqslant \frac{m_0}{p}\|u\|^p - \lambda\tilde{C}_2\|u\|^r - \frac{1}{q}\tilde{C}\|u\|^q = g_\lambda(\|u\|^p),$$

where $g_\lambda\colon [0,+\infty) \to \mathbb{R}$ is given by

$$g_\lambda(t) = \frac{m_0}{p}t - \lambda\tilde{C}_2 t^{r/p} - \frac{1}{q}\tilde{C}t^{q/p}.$$

Note that $r < p \leqslant q \leqslant p^*$ implies that $\lim_{t\to+\infty} g_\lambda(t) = -\infty$. Hence, I is not bounded from below in $\mathcal{D}_a^{1,p}$ and we cannot apply Proposition 2.4. But there exists $\lambda_0 > 0$ such that for each $\lambda \in (0,\lambda_0)$ the function $g_\lambda(t)$ achieves a positive maximum, and there exist $t_1, t_2 \in (0,+\infty)$ with $t_1 < t_2$ and $g_\lambda(t_1) = g_\lambda(t_2) = 0$. Now, consider $\phi \in C_0^1([0,+\infty))$ with $0 \leqslant \phi \leqslant 1$, $\phi(t) = 1$ for all $t \in [0,t_1]$, $\phi(t) = 0$ for all $t \in [t_2,+\infty)$, and $\phi'(t) \leqslant 0$ for all $t \in [0,+\infty)$. Define the function $\bar{g}_\lambda\colon [0,+\infty) \to \mathbb{R}$ given by

$$\bar{g}_\lambda(t) = \frac{m_0}{p}t - \lambda\tilde{C}_2 t^{r/p} - \frac{\tilde{C}}{q}\phi(t)t^{q/p}.$$

Note that $\bar{g}_\lambda(0) = 0$, $\bar{g}_\lambda(t) \geqslant 0$ for all $t \geqslant t_1$, and

$$\lim_{t\to+\infty} \bar{g}_\lambda(t) = \lim_{t\to+\infty} \frac{m_0}{p}t - \lambda\tilde{C}_2 t^{r/p} = +\infty, \tag{2.2}$$

because $r/p < 1$ and $\phi(t) = 0$ for all $t \in [t_2,+\infty)$.

The auxiliary Euler–Lagrange functional that we will use is $J\colon \mathcal{D}_a^{1,p} \to \mathbb{R}$, given by

$$J(u) = \frac{1}{p}\hat{M}(\|u\|^p) - \lambda\int_\Omega |x|^{-\delta}F(x,u)\,\mathrm{d}x - \frac{\phi(\|u\|^p)}{q}\int_\Omega |x|^{-\beta}|u|^q\,\mathrm{d}x,$$

where $\hat{M}(t) := \int_0^t M(s)\,\mathrm{d}s$ and $\lambda \in (0,\lambda_0)$. Since $J(u) \geqslant \bar{g}_\lambda(\|u\|^p)$ and (2.2) hold, we obtain that J is coercive in $\mathcal{D}_a^{1,p}$, which implies that J is bounded from below in $\mathcal{D}_a^{1,p}$. Thus, the functional J is appropriate for proving Theorem 1.1.

The next lemma could be proved by using [**21**, Lemma 4.1]

Lemma 2.6 (S_+ condition). *Suppose that $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, that $0 \in \Omega$, $1 < p < N$, $-\infty < a < (N-p)/p$, $(u_n) \subset \mathcal{D}_a^{1,p}$ and $u \in \mathcal{D}_a^{1,p}$ are such that $u_n \rightharpoonup u$ as $n \to \infty$, and*

$$\limsup_{n\to\infty} \int_\Omega |x|^{-ap}|\nabla u_n|^{p-2}\nabla u_n \nabla(u_n - u)\,\mathrm{d}x \leqslant 0.$$

There then exists a subsequence strongly convergent in $\mathcal{D}_a^{1,p}$.

3. Subcritical case: Theorem 1.1

We will prove the next lemma, which states that a critical point of J with energy less than 0 is a critical point of I. We also note that the critical level from Proposition 2.4 is less than $I(0) = J(0) = 0$.

Lemma 3.1. *If u_0 is a critical point of J with $J(u_0) < 0$, then u_0 is a critical point of I.*

Proof. Let u_0 be a critical point of the functional J with $J(u_0) < 0$. Since J is continuous, there exists $R_0 > 0$ such that $J(u) < 0$ for all $u \in B(u_0, R_0) \subset \mathcal{D}_a^{1,p}$, which implies that $\|u\|^p < t_1$, because $J(u) \geqslant \bar{g}_\lambda(\|u\|^p)$ and $\bar{g}_\lambda(t) \geqslant 0$ if $t \geqslant t_1$. Therefore, for all $u \in B(u_0, R_0)$ we have $\phi(\|u\|^p) = 1$, and consequently $J(u) = I(u)$ for all $u \in B(u_0, R_0)$. In particular, it follows that u_0 is a critical point of I. □

Lemma 3.2. *Assume that (M_1), (f_1), (f_2) hold, and that $q < p^*$ and $\beta < (a+1)q + N(1 - q/p)$. Then J satisfies the Palais–Smale condition.*

Proof. Let $(u_n) \subset \mathcal{D}_a^{1,p}$ be a Palais–Smale sequence at level c, that is, $J(u_n) \to c$ and $J'(u_n) \to 0$ (in the dual of $\mathcal{D}_a^{1,p}$) as $n \to +\infty$. Since J is coercive, we have that $(u_n) \subset \mathcal{D}_a^{1,p}$ is bounded. Then, passing to a subsequence if necessary, we have $u \in \mathcal{D}_a^{1,p}$ such that

$$\left.\begin{aligned} u_n &\rightharpoonup u && \text{in } \mathcal{D}_a^{1,p}, \\ u_n &\to u && \text{in } L^s(\Omega, |x|^{-\sigma}), \\ u_n(x) &\to u(x) && \text{almost everywhere (a.e.) in } \Omega, \\ \|u_n\| &\to t_0 \geqslant 0 && \end{aligned}\right\} \tag{3.1}$$

as $n \to +\infty$, where $1 \leqslant s < p^*$ and $\sigma < (a+1)s + N(1 - s/p)$. Hence, $J'(u_n)(u_n - u) = o_n(1)$, that is,

$$\begin{aligned} M(\|u_n\|^p) \int_\Omega \frac{|\nabla u_n|^{p-2}\nabla u_n \nabla(u_n - u)}{|x|^{ap}} \,\mathrm{d}x - \lambda \int_\Omega \frac{f(x, u_n)(u_n - u)}{|x|^\delta} \,\mathrm{d}x \\ - \frac{p}{q}\phi'(\|u_n\|^p) \int_\Omega \frac{|u_n|^q}{|x|^\beta} \,\mathrm{d}x \int_\Omega \frac{|\nabla u_n|^{p-2}\nabla u_n \nabla(u_n - u)}{|x|^{ap}} \,\mathrm{d}x \\ - \phi(\|u_n\|^p) \int_\Omega \frac{|u_n|^{q-2}u_n(u_n - u)}{|x|^\beta} \,\mathrm{d}x = o_n(1). \end{aligned} \tag{3.2}$$

From Hölder's inequality, (3.1) and since ϕ is continuous, we obtain

$$\int_\Omega \frac{|u_n|^{q-2}u_n(u_n - u)}{|x|^\beta} \,\mathrm{d}x = o_n(1) \quad \text{and} \quad \phi(\|u_n\|^p) \int_\Omega \frac{|u_n|^{q-2}u_n(u_n - u)}{|x|^\beta} \,\mathrm{d}x = o_n(1).$$

By using (f_2), Hölder's inequality and (3.1), we obtain

$$\left| \int_\Omega |x|^{-\delta} f(x, u_n)(u_n - u) \,\mathrm{d}x \right| \leqslant C_2 \int_\Omega |x|^{-\delta} |u_n|^{r-1} |u_n - u| \,\mathrm{d}x = o_n(1).$$

Therefore, substituting into (3.2), we have

$$\left[M(\|u_n\|^p) - \frac{p}{q}\phi'(\|u_n\|^p) \int_\Omega \frac{|u_n|^q}{|x|^\delta} \,\mathrm{d}x \right] \int_\Omega \frac{|\nabla u_n|^{p-2}\nabla u_n \nabla(u_n - u)}{|x|^{ap}} \,\mathrm{d}x = o_n(1).$$

Since M and ϕ' are continuous functions, $M(t) \geqslant m_0$ and $\phi'(t) \leqslant 0$ for all $t \in [0, \infty)$, there exists $C > 0$ such that

$$m_0 \leqslant M(\|u_n\|^p) - \frac{p}{q}\phi'(\|u_n\|^p)\int_\Omega |x|^{-\beta}|u_n|^q \,\mathrm{d}x \leqslant C.$$

Thus,

$$\int_\Omega |x|^{-ap}|\nabla u_n|^{p-2}\nabla u_n \nabla(u_n - u)\,\mathrm{d}x = o_n(1).$$

Then, by using Lemma 2.6, the proof is finished. □

Proof of Theorem 1.1. We have that $\mathcal{D}_a^{1,p}$ is a reflexive and separable Banach space. Then, for any $k \in \mathbb{N}$, there is a k-dimensional linear subspace $\mathcal{X}_k$ of $\mathcal{D}_a^{1,p}$ with $\mathcal{X}_k \subset C_0^{+\infty}(\Omega)$. Therefore, all norms on $\mathcal{X}_k$ are equivalent. Hence, there exists a positive constant $C(k)$ that depends on k such that $rC(k)\|u\|^r \leqslant C_1\|u\|^r_{L^r(\Omega,|x|^{-\delta})}$ for all $u \in \mathcal{X}_k$. Hence, if $u \in \mathcal{X}_k$, we obtain by (f_2) that

$$\int_\Omega |x|^{-\delta}F(x,u)\,\mathrm{d}x \geqslant \frac{C_1}{r}\int_\Omega |x|^{-\delta}|u|^r \,\mathrm{d}x \geqslant C(k)\|u\|^r.$$

Thus, for all $u \in \mathcal{X}_k$ with $\|u\| \leqslant 1$, from continuity of the function M we conclude that there exists $C > 0$ such that

$$J(u) \leqslant C\|u\|^p - \lambda C(k)\|u\|^r.$$

Take $R = \min\{1, (\lambda C(k)/C)^{1/(p-r)}\}$ and consider $\mathcal{S} = \{u \in \mathcal{X}_k : \|u\| = s\}$ with $0 < s < R$. Since $1 \leqslant r < p$, for all $u \in \mathcal{S}$ we obtain

$$J(u) \leqslant s^r[Cs^{p-r} - \lambda C(k)] < 0 = J(0), \tag{3.3}$$

which implies that $\sup_{\mathcal{S}} J(u) < 0 = J(0)$.

Since $\mathcal{X}_k$ and $\mathbb{R}^k$ are isomorphic, and $\mathcal{S}$ and S^{k-1} are homeomorphic, we conclude that $\gamma(\mathcal{S}) = k$. Moreover, J is coercive, even and satisfies the Palais–Smale condition (see Lemma 3.2), so it follows from Proposition 2.4 that J has at least k pairs of different critical points. Since k is arbitrary, we obtain infinitely many critical points of J. Then, by using (3.3) and Lemma 3.1, we obtain infinitely many critical points of I. □

4. The critical case

In Theorems 1.2 and 1.3 we have $\beta = bp^*$, $q = p^*$, and by (M_2) that $M(t)$ is increasing. Since we are working with critical growth and a non-local operator without more information about the behaviour of function M at infinity, we need to make a truncation on the function M (see Lemmas 4.2 and 4.11). In the case in which $p < r < p^*$ the truncation is also necessary to prove Lemma 4.10.

Since $p < p^*$, we can obtain $\theta \in (p, p^*)$. From (M_2) there exists $t_0 > 0$ such that $m_0 \leqslant M(0) < M(t_0) < (\theta/p)m_0$ for the $1 \leqslant r < p$ case, and $m_0 \leqslant M(0) < M(t_0) < (\xi/p)m_0$

for the $p < r < p^*$ case, where ξ is given by (f_3). We set

$$M_0(t) := \begin{cases} M(t) & \text{if } 0 \leqslant t \leqslant t_0, \\ M(t_0) & \text{if } t \geqslant t_0. \end{cases}$$

From (M_2) we obtain

$$m_0 \leqslant M_0(t) \leqslant \frac{\theta}{p} m_0 \quad \forall t \geqslant 0 \tag{4.1}$$

and

$$m_0 \leqslant M_0(t) \leqslant \frac{\xi}{p} m_0 \quad \forall t \geqslant 0. \tag{4.2}$$

The proofs of the Theorems 1.2 and 1.3 are based on a careful study of solutions of the auxiliary problem

$$\left.\begin{aligned} L_0(u) = \lambda |x|^{-\delta} f(x,u) + |x|^{-\beta}|u|^{q-2}u \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{aligned}\right\} \tag{4.3}$$

where

$$L_0(u) :=: -\left[M_0\left(\int_\Omega |x|^{-ap}|\nabla u|^p \,\mathrm{d}x\right)\right] \operatorname{div}(|x|^{-ap}|\nabla u|^{p-2}\nabla u).$$

We will look for solutions of problem (4.3) by finding critical points of the Euler–Lagrange functional $I_\lambda \colon \mathcal{D}_a^{1,p} \to \mathbb{R}$ given by

$$I_\lambda(u) = \frac{1}{p}\hat{M}_0(\|u\|^p) - \lambda \int_\Omega |x|^{-\delta} F(x,u)\,\mathrm{d}x - \frac{1}{q}\int_\Omega |x|^{-\beta}|u|^q \,\mathrm{d}x,$$

where $\hat{M}_0(t) := \int_0^t M_0(s)\,\mathrm{d}s$. Note that I_λ is C^1 and for all $\phi \in \mathcal{D}_a^{1,p}$ we have

$$I'_\lambda(u)(\phi) = M_0(\|u\|^p)\int_\Omega \frac{|\nabla u|^{p-2}\nabla u \nabla\phi}{|x|^{ap}}\,\mathrm{d}x - \lambda\int_\Omega \frac{f(x,u)\phi}{|x|^\delta}\,\mathrm{d}x - \int_\Omega \frac{|u|^{q-2}u\phi}{|x|^\beta}\,\mathrm{d}x.$$

4.1. The $1 \leqslant r < p$ case

Lemma 4.1.

$$\lim_{\lambda\to 0^+} t_1(\lambda) = 0. \tag{4.4}$$

Proof. From $g_\lambda(t_1(\lambda)) = 0$ and $g'_\lambda(t_1(\lambda)) > 0$ we have

$$\frac{m_0}{p} = \lambda \tilde{C}_2(t_1(\lambda))^{(r-p)/p} + \frac{1}{p^*}\tilde{C}(t_1(\lambda))^{(p^*-p)/p} \tag{4.5}$$

and

$$m_0 > \lambda r \tilde{C}_2(t_1(\lambda))^{(r-p)/p} + \tilde{C}(t_1(\lambda))^{(p^*-p)/p} \tag{4.6}$$

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for all $\lambda \in (0, \lambda_0)$. From (4.5) and (4.6) we obtain

$$\lambda p\tilde{C}_2(t_1(\lambda))^{(r-p)/p} + \frac{p}{p^*}\tilde{C}(t_1(\lambda))^{(p^*-p)/p} > \lambda r\tilde{C}_2(t_1(\lambda))^{(r-p)/p} + \tilde{C}(t_1(\lambda))^{(p^*-p)/p},$$

which implies that

$$\lambda\tilde{C}_2(p-r)(t_1(\lambda))^{(r-p)/p} > \tilde{C}(1-\frac{p}{p^*})(t_1(\lambda))^{(p^*-p)/p}. \tag{4.7}$$

From (4.7) we conclude that

$$0 < t_1(\lambda) < \lambda^{p/(p^*-r)}\left[\frac{\tilde{C}_2(p-r)}{\tilde{C}(1-p/p^*)}\right]^{p/(p^*-r)}. \tag{4.8}$$

Since $p^* > r$, passing to the limit as $\lambda \to 0^+$ in (4.8), we conclude the proof. □

As in the subcritical case, we can construct an auxiliary functional $J_\lambda\colon \mathcal{D}_a^{1,p} \to \mathbb{R}$ given by

$$J_\lambda(u) = \frac{1}{p}\hat{M}_0(\|u\|^p) - \lambda\int_\Omega |x|^{-\delta}F(x,u)\,\mathrm{d}x - \frac{\phi(\|u\|^p)}{q}\int_\Omega |x|^{-bp^*}|u|^{p^*}\,\mathrm{d}x,$$

where $\hat{M}_0(t) := \int_0^t M_0(s)\,\mathrm{d}s$.

Lemma 4.2. *Let (u_n) be a bounded sequence in $\mathcal{D}_a^{1,p}$ such that*

$$I_\lambda(u_n) \to c \quad \text{and} \quad I'_\lambda(u_n) \to 0 \quad \text{in } (\mathcal{D}_a^{1,p})^{-1} \text{ as } n \to \infty.$$

Suppose that (M_1), (M_2), (f_1) and (f_2) hold, and that

$$c < \left(\frac{1}{\theta} - \frac{1}{p^*}\right)(m_0 C^*_{a,p})^{p^*/(p^*-p)} - \left[\frac{\lambda C C_2(1/r+1/\theta)}{1/\theta - 1/p^*}\right]^{p^*/(p^*-r)}\left[\left(\frac{r}{p^*}\right)^{r/(p^*-r)} - \left(\frac{r}{p^*}\right)^{p^*/(p^*-r)}\right],$$

where $C = (\int_\Omega (|x|^{-\delta+br})^{p^*/(p^*-r)}\,\mathrm{d}x)^{(p^*-r)/p^*}$. Then there exists a subsequence strongly convergent in $\mathcal{D}_a^{1,p}$.

Proof. Since (u_n) is bounded in $\mathcal{D}_a^{1,p}$, passing to a subsequence if necessary, we have

$$\begin{aligned} u_n &\rightharpoonup u && \text{in } \mathcal{D}_a^{1,p}, \\ u_n &\to u && \text{in } L^s(\Omega, |x|^{-\sigma}), \\ u_n(x) &\to u(x) && \text{a.e. in } \Omega, \\ \|u_n\| &\to t_0 \geqslant 0 \end{aligned}$$

as $n \to +\infty$, where $1 \leqslant s < p^*$ and $\sigma < (a+1)s + N(1-s/p)$. Moreover, using the concentration–compactness principle due to Lions (see [**28**, **36**]), we obtain an at most countable index set Λ, and sequences $(x_i) \subset \mathbb{R}^N$, $(\mu_i), (\nu_i) \subset (0, \infty)$ such that

$$|x|^{-ap}|\nabla u_n|^p \rightharpoonup |x|^{-ap}|\nabla u|^p + \mu \quad \text{and} \quad |x|^{-bp^*}|u_n|^{p^*} \rightharpoonup |x|^{-bp^*}|u|^{p^*} + \nu \tag{4.9}$$

as $n \to +\infty$ in the weak* sense of measures, where

$$\nu = \sum_{i\in\varLambda} \nu_i \delta_{x_i}, \quad \mu \geqslant \sum_{i\in\varLambda} \mu_i \delta_{x_i} \quad \text{and} \quad C^*_{a,p}\nu_i^{p/p^*} \leqslant \mu_i \tag{4.10}$$

for all $i \in \varLambda$, where δ_{x_i} is the Dirac mass at $x_i \in \varOmega$.

Now let $k \in \mathbb{N}$. Without loss of generality we can suppose that $B_2(0) \subset \varOmega$. Then for every $\varrho > 0$ we set $\psi_\varrho(x) := \psi((x - x_k)/\varrho)$, where $\psi \in C_0^\infty(\varOmega, [0,1])$ is such that $\psi \equiv 1$ on $B_1(0)$, $\psi \equiv 0$ on $\varOmega \setminus B_2(0)$, and $|\nabla\psi| \leqslant 1$. Observe that $(\psi_\varrho u_n)$ is bounded in $\mathcal{D}^{1,p}_a$. So we have $I'_\lambda(u_n)(\psi_\varrho u_n) \to 0$, that is,

$$\begin{aligned} M_0(\|u_n\|^p)&\int_\varOmega \frac{u_n|\nabla u_n|^{p-2}\nabla u_n \nabla\psi_\varrho}{|x|^{ap}}\,\mathrm{d}x + o_n(1) \\ &= -M_0(\|u_n\|^p)\int_\varOmega \frac{|\nabla u_n|^p\psi_\varrho}{|x|^{ap}}\,\mathrm{d}x + \lambda\int_\varOmega \frac{f(x,u_n)\psi_\varrho u_n}{|x|^\delta}\,\mathrm{d}x + \int_\varOmega \frac{\psi_\varrho |u_n|^{p^*}}{|x|^{bp^*}}\,\mathrm{d}x. \end{aligned}$$

It follows from (4.9) and (M_1) that

$$\begin{aligned} M_0(\|u_n\|^p)&\int_\varOmega \frac{u_n|\nabla u_n|^{p-2}\nabla u_n \nabla\psi_\varrho}{|x|^{ap}}\,\mathrm{d}x \\ &\leqslant -m_0\int_\varOmega \frac{|\nabla u|^p\psi_\varrho}{|x|^{ap}}\,\mathrm{d}x - m_0\int_\varOmega \psi_\varrho\,\mathrm{d}\mu + \lambda\int_\varOmega \frac{f(x,u_n)\psi_\varrho u_n}{|x|^\delta}\,\mathrm{d}x \\ &\quad + \int_\varOmega \frac{\psi_\varrho|u|^{p^*}}{|x|^{bp^*}}\,\mathrm{d}x + \int_\varOmega \psi_\varrho\,\mathrm{d}\nu + o_n(1). \end{aligned}$$

Since $u_n \to u$ in $L^r(\varOmega, |x|^{-\delta})$, by using (f_2) and the dominated convergence theorem we obtain

$$\lambda\int_\varOmega |x|^{-\delta} f(x,u_n)\psi_\varrho u_n\,\mathrm{d}x \to \lambda\int_\varOmega |x|^{-\delta} f(x,u)\psi_\varrho u\,\mathrm{d}x$$

as $n \to \infty$. Thus, we obtain

$$\begin{aligned} \limsup_{n\to\infty}&\left[M_0(\|u_n\|^p)\int_\varOmega \frac{u_n|\nabla u_n|^{p-2}\nabla u_n \nabla\psi_\varrho}{|x|^{ap}}\,\mathrm{d}x\right] \\ &\leqslant -m_0\int_\varOmega \frac{|\nabla u|^p\psi_\varrho}{|x|^{ap}}\,\mathrm{d}x - m_0\int_\varOmega \psi_\varrho\,\mathrm{d}\mu + \lambda\int_\varOmega \frac{f(x,u)\psi_\varrho u}{|x|^\delta}\,\mathrm{d}x \\ &\quad + \int_\varOmega \frac{\psi_\varrho|u|^{p^*}}{|x|^{bp^*}}\,\mathrm{d}x + \int_\varOmega \psi_\varrho\,\mathrm{d}\nu. \end{aligned}$$

From the dominated convergence theorem we obtain

$$\int_\varOmega \frac{|\nabla u|^p\psi_\varrho}{|x|^{ap}}\,\mathrm{d}x = o_\varrho(1), \quad \int_\varOmega \frac{f(x,u)\psi_\varrho u}{|x|^\delta}\,\mathrm{d}x = o_\varrho(1) \quad \text{and} \quad \int_\varOmega \frac{\psi_\varrho|u|^{p^*}}{|x|^{bp^*}}\,\mathrm{d}x = o_\varrho(1),$$

where $\lim_{\varrho\to0^+} o_\varrho(1) = 0$. So, we obtain

$$\begin{aligned} \lim_{\varrho\to0^+}&\left\{\limsup_{n\to\infty}\left[M_0(\|u_n\|^p)\int_\varOmega \frac{u_n|\nabla u_n|^{p-2}\nabla u_n \nabla\psi_\varrho}{|x|^{ap}}\,\mathrm{d}x\right]\right\} \\ &\leqslant \lim_{\varrho\to0^+}\left[\int_\varOmega \psi_\varrho\,\mathrm{d}\nu - m_0\int_\varOmega \psi_\varrho\,\mathrm{d}\mu\right]. \end{aligned} \tag{4.11}$$

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Now, we will show that

$$\lim_{\varrho\to 0^+}\left[\limsup_{n\to\infty} M_0(\|u_n\|^p)\int_\Omega |x|^{-ap}u_n|\nabla u_n|^{p-2}\nabla u_n\nabla\psi_\varrho\,\mathrm{d}x\right]=0. \tag{4.12}$$

Indeed, by Hölder's inequality,

$$\left|\int_\Omega \frac{u_n|\nabla u_n|^{p-2}\nabla u_n\nabla\psi_\varrho}{|x|^{ap}}\,\mathrm{d}x\right|\leqslant \|u_n\|^{p-1}\left(\int_\Omega \frac{|u_n\nabla\psi_\varrho|^p}{|x|^{ap}}\,\mathrm{d}x\right)^{1/p}.$$

Since u_n is bounded in $\mathcal{D}_a^{1,p}$, M_0 is continuous and $\operatorname{supp}(\psi_\varrho)\subset B(x_k;2\varrho)$, there exists $L>0$ such that

$$M_0(\|u_n\|^p)\int_\Omega \frac{u_n|\nabla u_n|^{p-2}\nabla u_n\nabla\psi_\varrho}{|x|^{ap}}\,\mathrm{d}x\leqslant L\left(\int_{B(x_k;2\varrho)}\frac{|u_n\nabla\psi_\varrho|^p}{|x|^{ap}}\,\mathrm{d}x\right)^{1/p}.$$

Using the dominated convergence theorem and Hölder's inequality we obtain

$$\begin{aligned}
&\limsup_{n\to\infty}\left[M_0(\|u_n\|^p)\int_\Omega \frac{u_n|\nabla u_n|^{p-2}\nabla u_n\nabla\psi_\varrho}{|x|^{ap}}\,\mathrm{d}x\right]\\
&\qquad\leqslant L\left(\int_{B(x_k;2\varrho)}\frac{|u|^p|\nabla\psi_\varrho|^p}{|x|^{ap}}\,\mathrm{d}x\right)^{1/p}\\
&\qquad\leqslant L\left(\int_{B(x_k;2\varrho)}|\nabla\psi_\varrho|^N\,\mathrm{d}x\right)^{1/N}\left(\int_{B(x_k;2\varrho)}\left(\frac{|u|^p}{|x|^{ap}}\right)^{N/(N-p)}\mathrm{d}x\right)^{(N-p)/Np}\\
&\qquad\leqslant L|B(x_k;2\varrho)|^{1/N}\left(\int_\Omega \chi_{B(x_k;2\varrho)}\left(\frac{|u|^p}{|x|^{ap}}\right)^{N/(N-p)}\mathrm{d}x\right)^{(N-p)/Np}.
\end{aligned}$$

Letting $\varrho\to 0^+$ in the above expression, we obtain (4.12). Thus, we conclude from (4.11) that

$$0\leqslant \lim_{\rho\to 0^+}\left[\int_\Omega \psi_\varrho\,\mathrm{d}\nu - m_0\int_\Omega \psi_\varrho\,\mathrm{d}\mu\right].$$

That is,

$$\begin{aligned}
0&\leqslant \lim_{\rho\to 0^+}\left[\int_{B(x_k;2\varrho)}\psi_\varrho\,\mathrm{d}\nu - m_0\int_{B(x_k;2\varrho)}\psi_\varrho\,\mathrm{d}\mu\right]\\
&=\nu(\{x_k\})-m_0\mu(\{x_k\})\\
&\leqslant \sum_{i\in\Lambda}\nu_i\delta_{x_i}(\{x_k\})-m_0\sum_{i\in\Lambda}\mu_i\delta_{x_i}(\{x_k\})\\
&=\nu_k-m_0\mu_k.
\end{aligned}$$

So, we have

$$m_0\mu_k\leqslant\nu_k.$$

It follows from (4.10) that

$$\nu_k\geqslant (m_0C^*_{a,p})^{p^*/(p^*-p)}\geqslant\left(\frac{1}{\theta}-\frac{1}{p^*}\right)(m_0C^*_{a,p})^{p^*/(p^*-p)}. \tag{4.13}$$

Now we shall prove that the above expression cannot occur, and therefore the set Λ is empty. Indeed, arguing by contradiction, let us suppose that (4.13) holds for some $k \in \Lambda$. Thus, since $m_0 \leqslant M_0(t) \leqslant (\theta/p)m_0$ for all $t \in \mathbb{R}$, and by using (f_1) and (f_2), we have

$$\begin{aligned}
c &= I_\lambda(u_n) - \frac{1}{\theta} I'_\lambda(u_n)(u_n) + o_n(1) \\
&\geqslant \left(\frac{m_0}{p} - \frac{\theta m_0}{\theta p}\right)\|u_n\|^p - \lambda \int_\Omega \frac{F(x,u_n) + (1/\theta)|x|^{-\delta} f(x,u_n)u_n}{|x|^\delta}\,\mathrm{d}x \\
&\quad + \left(\frac{1}{\theta} - \frac{1}{p^*}\right)\int_\Omega \frac{|u_n|^{p^*}}{|x|^{bp^*}}\,\mathrm{d}x + o_n(1) \\
&\geqslant -\lambda C_2\left(\frac{1}{r} + \frac{1}{\theta}\right)\int_\Omega \frac{|u_n|^r}{|x|^\delta}\,\mathrm{d}x + \left(\frac{1}{\theta} - \frac{1}{p^*}\right)\int_\Omega \frac{|u_n|^{p^*}\psi_\varrho}{|x|^{bp^*}}\,\mathrm{d}x + o_n(1).
\end{aligned}$$

Letting $n \to +\infty$ we obtain

$$\begin{aligned}
c &\geqslant -\lambda C_2\left(\frac{1}{r} + \frac{1}{\theta}\right)\int_\Omega \frac{|u|^r}{|x|^\delta}\,\mathrm{d}x + \left(\frac{1}{\theta} - \frac{1}{p^*}\right)\int_\Omega \frac{|u|^{p^*}\psi_\varrho}{|x|^{bp^*}}\,\mathrm{d}x + \left(\frac{1}{\theta} - \frac{1}{p^*}\right)\nu_k \\
&\geqslant -\lambda C_2\left(\frac{1}{r} + \frac{1}{\theta}\right)\int_\Omega \frac{|u|^r}{|x|^\delta}\,\mathrm{d}x + \left(\frac{1}{\theta} - \frac{1}{p^*}\right)\int_\Omega \frac{|u|^{p^*}\psi_\varrho}{|x|^{bp^*}}\,\mathrm{d}x \\
&\quad + \left(\frac{1}{\theta} - \frac{1}{p^*}\right)(m_0 C^*_{a,p})^{p^*/(p^*-p)}.
\end{aligned}$$

By Hölder's inequality,

$$\int_\Omega \frac{|u|^r}{|x|^\delta}\,\mathrm{d}x \leqslant C\left(\int_\Omega \frac{|u|^{p^*}}{|x|^{bp^*}}\,\mathrm{d}x\right)^{r/p^*},$$

where

$$C = \left(\int_\Omega (|x|^{-\delta+br})^{p^*/(p^*-r)}\,\mathrm{d}x\right)^{(p^*-r)/p^*} < \infty.$$

So, letting $\varrho \to +\infty$, we obtain

$$\begin{aligned}
c &\geqslant -\lambda C C_2\left(\frac{1}{r} + \frac{1}{\theta}\right)\left(\int_\Omega \frac{|u|^{p^*}}{|x|^{bp^*}}\,\mathrm{d}x\right)^{r/p^*} + \left(\frac{1}{\theta} - \frac{1}{p^*}\right)\int_\Omega \frac{|u|^{p^*}}{|x|^{bp^*}}\,\mathrm{d}x \\
&\quad + \left(\frac{1}{\theta} - \frac{1}{p^*}\right)(m_0 C^*_{a,p})^{p^*/(p^*-p)}.
\end{aligned}$$

Define $\varphi\colon \mathbb{R}^+ \to \mathbb{R}$ by

$$\varphi(t) = \left(\frac{1}{\theta} - \frac{1}{p^*}\right)t^{p^*} - \lambda C C_2\left(\frac{1}{r} + \frac{1}{\theta}\right)t^r.$$

This function attains its absolute minimum at the point

$$t_0 = \left(\frac{\lambda r C C_2(1/r + 1/\theta)}{p^*(1/\theta - 1/p^*)}\right)^{1/(p^*-r)} > 0.$$

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Thus, we conclude that

$$\begin{aligned} c &\geqslant \varphi(t_0) \\ &= \left(\frac{1}{\theta}-\frac{1}{p^*}\right)(m_0 C^*_{a,p})^{p^*/(p^*-p)} \\ &\quad - \left(\frac{1}{\theta}-\frac{1}{p^*}\right)\left[\frac{\lambda C C_2(1/r+1/\theta)}{(1/\theta-1/p^*)}\right]^{p^*/(p^*-r)}\left[\left(\frac{r}{p^*}\right)^{r/(p^*-r)}-\left(\frac{r}{p^*}\right)^{p^*/(p^*-r)}\right]. \end{aligned}$$

But this is a contradiction. Thus, Λ is empty and it follows that $u_n \to u$ in $L^{p^*}(\Omega, |x|^{-bp^*})$.

Now we will prove that $u_n \to u$ in $\mathcal{D}_a^{1,p}$.

Indeed, since $u_n \to u$ in $L^r(\Omega, |x|^{-\delta})$ and in $L^{p^*}(\Omega, |x|^{-bp^*})$, it follows from the dominated convergence theorem that

$$\lim_{n\to+\infty}\int_\Omega \frac{f(x,u_n)(u_n-u)}{|x|^\delta}\,\mathrm{d}x = \lim_{n\to+\infty}\int_\Omega \frac{|u_n|^{p^*-2}u_n(u_n-u)}{|x|^{bp^*}}\,\mathrm{d}x = 0.$$

Therefore, as (u_n) is bounded in $\mathcal{D}_a^{1,p}$, $I'_\lambda(u_n)(u_n-u) \to 0$ in $(\mathcal{D}_a^{1,p})^{-1}$, $\|u_n\| \to t_0$ as $n \to \infty$, and M is continuous and positive, we conclude that

$$\lim_{n\to\infty}\int_\Omega |x|^{-ap}|\nabla u_n|^{p-2}\nabla u_n \nabla(u_n-u)\,\mathrm{d}x = 0.$$

It follows from Lemma 2.6 that $u_n \to u$ in $\mathcal{D}_a^{1,p}$. □

Remark 4.3. Due to Lemma 4.1, we can consider $\lambda_0 > 0$ such that $t_1 = t_1(\lambda) \leqslant t_0$ for each $\lambda \in (0, \lambda_0)$. Also, we can obtain $\lambda^* \leqslant \lambda_0$ such that

$$\left(\frac{1}{\theta}-\frac{1}{p^*}\right)(m_0 C^*_{a,p})^{p^*/(p^*-p)} - \left[\frac{\lambda C C_2(1/r+1/\theta)}{1/\theta-1/p^*}\right]^{p^*/(p^*-r)}\left[\left(\frac{r}{p^*}\right)^{r/(p^*-r)}-\left(\frac{r}{p^*}\right)^{p^*/(p^*-r)}\right] > 0$$

for each $\lambda \in (0, \lambda^*)$.

Lemma 4.4. *If $J_\lambda(u) < 0$, then $\|u\|^p < t_1$ and $J_\lambda(v) = I_\lambda(v)$ for all v in a sufficiently small neighbourhood of u. Moreover, J_λ verifies a local Palais–Smale condition for $c < 0$ for all $\lambda \in (0, \lambda^*)$.*

Proof. Since $\bar{g}_\lambda(\|u\|^p) \leqslant J_\lambda(u) < 0$, arguing as in § 3 we conclude that $\|u\|^p < t_1$ and $J_\lambda(v) = I_\lambda(v)$ for all $v \in B(u; R_0)$. Moreover, if (u_n) is a sequence such that $J_\lambda(u_n) \to c < 0$ and $J'_\lambda(u_n) \to 0$ in $\mathcal{D}_a^{1,p}$, then for n sufficiently large, $I_\lambda(u_n) = J_\lambda(u_n) \to c < 0$ and $I'_\lambda(u_n) = J'_\lambda(u_n) \to 0$. Since J is coercive, we obtain that (u_n) is bounded in $\mathcal{D}_a^{1,p}$. It follows from Remark 4.3 that for $\lambda \in (0, \lambda^*)$,

$$c < 0 < \left(\frac{1}{\theta}-\frac{1}{p^*}\right)(m_0 C^*_{a,p})^{p^*/(p^*-p)} - \left[\frac{\lambda C C_2(1/r+1/\theta)}{1/\theta-1/p^*}\right]^{p^*/(p^*-r)}\left[\left(\frac{r}{p^*}\right)^{r/(p^*-r)}-\left(\frac{r}{p^*}\right)^{p^*/(p^*-r)}\right]$$

and from Lemma 4.2, up to a subsequence, (u_n) is strongly convergent in $\mathcal{D}_a^{1,p}$. □

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Now we will construct an appropriate minimax sequence of negative critical values for the functional J_λ.

Lemma 4.5. *Given $k \in \mathbb{N}$ there exists $\varepsilon = \varepsilon(k) > 0$ such that*

$$\gamma(J^{-\varepsilon}) \geqslant k,$$

where $J^{-\varepsilon} = \{u \in \mathcal{D}_a^{1,p} \colon J_\lambda(u) \leqslant -\varepsilon\}$.

Proof. Fix $k \in \mathbb{N}$ and let X_k be a k-dimensional subspace of $\mathcal{D}_a^{1,p}$. Thus, there exists $C(k) > 0$ such that $C(k)\|u\|^r \leqslant C_1\|u\|^r_{L^r(\Omega,|x|^{-\delta})}$ for all $u \in X_k$.

Considering $\bar{\rho} > 0$ such that $0 < \|u\| = \bar{\rho}$ and $0 < \|u\|^p < t_1$, we obtain that $J_\lambda(u) = I_\lambda(u)$. Arguing as in the proof of Theorem 1.1, we can take $R > 0$ such that

$$I_\lambda(u) < -\varepsilon$$

for all $u \in \mathcal{S} = \{u \in X_k \colon \|u\| = s\}$, with $0 < s < \min\{R, \bar{\rho}\}$. Hence, $\mathcal{S} \subset J^{-\varepsilon}$ and, since $J^{-\varepsilon}$ is symmetric and closed, from Corollary 2.3,

$$\gamma(J^{-\varepsilon}) \geqslant \gamma(\mathcal{S}) = k.$$

□

We define now, for each $k \in \mathbb{N}$, the sets

$$\varGamma_k = \{C \subset \mathcal{D}_a^{1,p}\backslash\{0\} \colon C \text{ is closed, } C = -C \text{ and } \gamma(C) \geqslant k\},$$
$$K_c = \{u \in \mathcal{D}_a^{1,p} \colon J'_\lambda(u) = 0 \text{ and } J_\lambda(u) = c\}$$

and the number

$$c_k = \inf_{C \in \varGamma_k} \sup_{u \in C} J_\lambda(u).$$

Lemma 4.6. *Given $k \in \mathbb{N}$, the number c_k is negative.*

Proof. From Lemma 4.5, for each $k \in \mathbb{N}$ there exists $\varepsilon > 0$ such that $\gamma(J^{-\varepsilon}) \geqslant k$. Moreover, $0 \notin J^{-\varepsilon}$ and $J^{-\varepsilon} \in \varGamma_k$. On the other hand,

$$\sup_{u \in J^{-\varepsilon}} J_\lambda(u) \leqslant -\varepsilon.$$

Hence,

$$-\infty < c_k = \inf_{C \in \varGamma_k} \sup_{u \in C} J_\lambda(u) \leqslant \sup_{u \in J^{-\varepsilon}} J_\lambda(u) \leqslant -\varepsilon < 0.$$

□

The next lemma allows us to prove the existence of critical points of J.

Lemma 4.7. *If $c = c_k = c_{k+1} = \cdots = c_{k+r}$ for some $r \in \mathbb{N}$, then*

$$\gamma(K_c) \geqslant r + 1$$

for all $\lambda \in (0, \lambda^*)$.

Proof. Let (u_n) be a sequence in K_c. Since Lemma 4.6 gives us $c = c_k = c_{k+1} = \cdots = c_{k+r} < 0$, from Lemma 4.4 we have that (u_n) is bounded and $J_\lambda(u_n) = I_\lambda(u_n)$ for all $n \in \mathbb{N}$. Thus, we can apply Lemma 4.2 and we obtain a subsequence strongly convergent in $\mathcal{D}_a^{1,p}$, that is, K_c is a compactness set. Moreover, $K_c = -K_c$. Suppose, by contradiction, that $\gamma(K_c) \leqslant r$, so there exists a closed and symmetric set U with $K_c \subset U$ such that $\gamma(U) = \gamma(K_c) \leqslant r$. Note that we can choose $U \subset J^0$ because $c < 0$. By the deformation lemma [**7**], we have an odd homeomorphism $\eta\colon \mathcal{D}_a^{1,p} \to \mathcal{D}_a^{1,p}$ such that $\eta(J^{c+\delta} - U) \subset J^{c-\delta}$ for some $\delta > 0$ with $0 < \delta < -c$. Thus, $J^{c+\delta} \subset J^0$ and by the definition of $c = c_{k+r}$ there exists $A \in \Gamma_{k+r}$ such that $\sup_{u\in A} J_\lambda(u) < c + \delta$, that is, $A \subset J^{c+\delta}$ and

$$\eta(A - U) \subset \eta(J^{c+\delta} - U) \subset J^{c-\delta}. \tag{4.14}$$

But $\gamma(\overline{A - U}) \geqslant \gamma(A) - \gamma(U) \geqslant k$ and $\gamma(\eta(\overline{A - U})) \geqslant \gamma(\overline{A - U}) \geqslant k$. Then $\eta(\overline{A - U}) \in \Gamma_k$ and this contradicts (4.14). Hence, the lemma is proved. □

Remark 4.8. If $-\infty < c_1 < c_2 < \cdots < c_k < \cdots < 0$, and since each c_k is a critical value of J_λ, then we obtain infinitely many critical points of J_λ, and hence problem (4.3) has infinitely many solutions.

On the other hand, if there are two constants satisfying $c_k = c_{k+r}$, then $c = c_k = c_{k+1} = \cdots = c_{k+r}$ and, from Lemma 4.7, there exists $\lambda^* > 0$ such that

$$\gamma(K_c) \geqslant r + 1 \geqslant 2.$$

From Proposition 2.5, K_c has infinitely many points, that is, problem (4.3) has infinitely many solutions.

Proof of Theorem 1.2. Let λ^* be as in Remark 4.3 and, for $0 < \lambda < \lambda^*$, let u_λ be the non-trivial solution of problem (4.3) found in Remark 4.8. Thus, $J_\lambda(u_\lambda) = I_\lambda(u_\lambda) < 0$. Hence, using Lemma 4.4 we have

$$\|u_\lambda\|^p \leqslant t_1 < t_0, \tag{4.15}$$

so we conclude that

$$M_0(\|u_\lambda\|^p) = M(\|u_\lambda\|^p)$$

and u_λ is a solution of problem (1.1). □

4.2. The $p < r < p^*$ case

In this section we adapt for our study the ideas in [**19**]. In what follows we prove that the functional I_λ has the mountain pass geometry.

Lemma 4.9. *Assume that conditions (M_1), (f_1) and (f_2) hold. There exist positive numbers ρ and α such that*

$$I_\lambda(u) \geqslant \alpha > 0 \quad \forall u \in \mathcal{D}_a^{1,p} \text{ with } \|u\| = \rho.$$

Proof. From (M_1), (f_1), (f_2) and the Caffarelli–Kohn–Nirenberg inequality, we obtain

$$I_\lambda(u) \geqslant \frac{m_0}{p}\|u\|^p - \lambda\tilde{C}_2\|u\|^r - \frac{1}{p^*}\tilde{C}\|u\|^{p^*}.$$

Since $p < r < p^*$, the result follows by choosing $\rho > 0$ small enough. □

Lemma 4.10. *For all $\lambda > 0$ there exists $e \in \mathcal{D}_a^{1,p}$ with $I_\lambda(e) < 0$ and $\|e\| > \rho$.*

Proof. Fix $v_0 \in \mathcal{D}_a^{1,p}\backslash\{0\}$ with $v_0 \geqslant 0$ in Ω and $\|v_0\| = 1$. Using (4.2) and (f_2) we obtain

$$I_\lambda(tv_0) \leqslant \frac{\xi}{p^2}m_0t^p\|v_0\|^p - \frac{\lambda C_1}{r}t^r\int_\Omega \frac{|v_0|^r}{|x|^\delta}\,\mathrm{d}x - \frac{t^{p^*}}{p^*}\int_\Omega \frac{|v_0|^{p^*}}{|x|^{bp^*}}\,\mathrm{d}x.$$

Since $p < r < p^*$, we have $\lim_{t\to+\infty} I_\lambda(tv_0) = -\infty$. Thus, for $\bar{t} > \rho$ large enough, $I_\lambda(\bar{t}v_0) < 0$. The result follows by considering $e = \bar{t}v_0$. □

Using a version of the mountain pass theorem without the Palais–Smale condition (see [**33**]), there exists a sequence $(u_n) \in \mathcal{D}_a^{1,p}$ satisfying

$$I_\lambda(u_n) \to c_\lambda \quad \text{and} \quad I'_\lambda(u_n) \to 0 \quad \text{in } (\mathcal{D}_a^{1,p})^{-1},$$

where

$$c_\lambda = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I_\lambda(\gamma(t))$$

and

$$\Gamma := \{\gamma \in C([0,1], \mathcal{D}_a^{1,p}) \colon \gamma(0) = 0,\ \gamma(1) = e\}.$$

Lemma 4.11. *If (M_1), (M_2) and (f_2) hold, then*

$$\lim_{\lambda\to+\infty} c_\lambda = 0.$$

Proof. Since the functional I_λ has the mountain pass geometry, it follows that there exists $t_\lambda > 0$ verifying $I_\lambda(t_\lambda v_0) = \max_{t\geqslant 0} I_\lambda(tv_0)$, where v_0 is given by Lemma 4.10. Hence, from (4.2) and (f_2) we obtain

$$0 = I'_\lambda(t_\lambda v_0)(t_\lambda v_0) \leqslant \frac{\xi}{p}m_0t_\lambda^p\|v_0\|^p - \lambda C_1t_\lambda^r\int_\Omega \frac{|v_0|^r}{|x|^\delta}\,\mathrm{d}x - t_\lambda^{p^*}\int_\Omega \frac{|v_0|^{p^*}}{|x|^{bp^*}}\,\mathrm{d}x,$$

that is,

$$\frac{\xi}{p}m_0t_\lambda^p \geqslant \lambda C_1t_\lambda^r\int_\Omega \frac{|v_0|^r}{|x|^\delta}\,\mathrm{d}x + t_\lambda^{p^*}\int_\Omega \frac{|v_0|^{p^*}}{|x|^{bp^*}}\,\mathrm{d}x \geqslant t_\lambda^{p^*}\int_\Omega \frac{|v_0|^{p^*}}{|x|^{bp^*}}\,\mathrm{d}x,$$

which implies that (t_λ) is bounded. Thus, there exists a sequence (λ_n) and $\beta_0 \geqslant 0$ such that $\lambda_n \to +\infty$ and $t_{\lambda_n} \to \beta_0$ as $n \to +\infty$. Consequently, there exists $D > 0$ such that

$$\frac{\xi}{p}m_0t_{\lambda_n}^p \leqslant D \quad \forall n \in \mathbb{N},$$

and so

$$\lambda_n C_1 t_{\lambda_n}^r \int_\Omega |x|^{-\delta}|v_0|^r \, dx + t_{\lambda_n}^{p^*} \int_\Omega |x|^{-bp^*}|v_0|^{p^*} \, dx \leqslant D \quad \forall n \in \mathbb{N}. \tag{4.16}$$

If $\beta_0 > 0$, we obtain

$$\lim_{n\to\infty} \left[\lambda_n C_1 t_{\lambda_n}^r \int_\Omega |x|^{-\delta}|v_0|^r \, dx + t_{\lambda_n}^{p^*} \int_\Omega |x|^{-bp^*}|v_0|^{p^*} \, dx\right] = +\infty,$$

which is a contradiction with (4.16). Thus, we conclude that $\beta_0 = 0$. Now, let us consider the path $\gamma_*(t) = te$ for $t \in [0, 1]$, which belongs to Γ, to obtain the estimate

$$0 < c_\lambda \leqslant \max_{t\in[0,1]} I_\lambda(\gamma_*(t)) = I_\lambda(t_\lambda v_0) \leqslant \frac{\xi}{p^2} m_0 t_\lambda^p.$$

In this way, observing that (c_λ) is a monotonous sequence, we conclude that

$$\lim_{\lambda\to+\infty} c_\lambda = 0.$$

□

Remark 4.12. Due to Lemma 4.11, there exists $\lambda_1 > 0$ such that

$$c_\lambda < \left(\frac{1}{p} m_0 - \frac{1}{\xi} M_0(t_0)\right) t_0$$

for all $\lambda > \lambda_1$.

Lemma 4.13. *Suppose that $\lambda > \lambda_1$ and that (M_1), (M_2), (f_2) and (f_3) hold. Let $(u_n) \in \mathcal{D}_a^{1,p}$ be a bounded sequence such that*

$$I_\lambda(u_n) \to c_\lambda \quad \text{and} \quad I'_\lambda(u_n) \to 0 \quad \text{in } (\mathcal{D}_a^{1,p})^{-1} \text{ as } n \to +\infty.$$

Then, for all $n \in \mathbb{N}$, we have

$$\|u_n\|^p \leqslant t_0.$$

Proof. Suppose by contradiction that for some $n \in \mathbb{N}$ we have $\|u_n\|^p > t_0$. Thus, for each $\lambda > \lambda_1$, from the definition of $M_0(t)$, (f_3) and (4.2), we have that

$$\begin{aligned} c_\lambda &= I_\lambda(u_n) - \frac{1}{\xi} I'_\lambda(u_n)(u_n) + o_n(1) \\ &\geqslant \frac{1}{p}\hat{M}_0(\|u_n\|^p) - \frac{1}{\xi} M_0(t_0)\|u_n\|^p + o_n(1) \\ &\geqslant \left(\frac{1}{p} m_0 - \frac{1}{\xi} M_0(t_0)\right)\|u_n\|^p + o_n(1). \end{aligned} \tag{4.17}$$

Since $m_0 < M(t_0) < (\xi/p)m_0$ we have $(1/p)m_0 - (1/\xi)M_0(t_0) > 0$. So we obtain

$$c_\lambda \geqslant \left(\frac{1}{p} m_0 - \frac{1}{\xi} M_0(t_0)\right) t_0 > 0.$$

But this contradicts Remark 4.12. Hence, we conclude that $\|u_n\|^p \leqslant t_0$. □

https://doi.org/10.1017/S0013091515000395 Published online by Cambridge University Press

Proof of Theorem 1.3. It follows from Lemma 4.11 that there exists $\lambda^{**} \geqslant \lambda_1 > 0$ such that

$$c_\lambda < \left(\frac{1}{\xi} - \frac{1}{p^*}\right)(m_0 C^*_{a,p})^{p^*/(p^*-p)} \tag{4.18}$$

for all $\lambda \geqslant \lambda^{**}$. Now, fix $\lambda \geqslant \lambda^{**}$. From Lemmas 4.9 and 4.10 there exists a bounded sequence $(u_n) \subset \mathcal{D}^{1,p}_a$ such that $I_\lambda(u_n) \to c_\lambda$ and $I'_\lambda(u_n) \to 0$ as $n \to \infty$.

Arguing as in Lemma 4.2, from (4.18) we conclude that, up to a subsequence, $u_n \to u_\lambda$ in $\mathcal{D}^{1,p}_a$. Thus, u_λ is a weak solution of problem (4.3). Moreover, by Lemma 4.13 we conclude that u_λ is a weak solution of problem (1.1). □

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