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COMPLEX OF RELATIVELY HYPERBOLIC GROUPS

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Abstract. In this paper, we prove a combination theorem for a complex of relatively hyperbolic groups. It is a generalization of Martin's (*Geom. Topology* **18** (2014), 31–102) work for combination of hyperbolic groups over a finite M_K-simplicial complex, where $k \leq 0$.

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1. Introduction. In [**5**], Dahmani showed that if G is the fundamental group of an acylindrical finite graph of relatively hyperbolic groups with edge groups fully quasi-convex in the respective vertex groups, then G is hyperbolic relative to the images of the maximal parabolic subgroups of vertex groups and their conjugates in G. By gluing the relative hyperbolic boundaries of each local groups, Dahmani constructed a compact metrizable space ∂G on which G has convergence action and the limit points are either conical or bounded parabolic. So, G is a relatively hyperbolic group due to Yaman [**26**]. Using these ideas, Martin [**7**] generalized this combination theorem for complex of hyperbolic groups. Let $G(\mathcal{Y})$ be a strictly developable non-positively curved simple complex of groups over a finite M_k simplicial complex with $k \leq 0$. Let G be the fundamental group of $G(\mathcal{Y})$ and X be a universal covering of $G(\mathcal{Y})$. Martin, in [**7**], proved that if X is hyperbolic, local groups are hyperbolic, local maps are quasi-convex embeddings and the action of G on X is acylindrical (i.e., there exists $K > 0$ such that any pair of points of diameter at least K in X has finite pointwise stabilizer, see Definition 3.4), then G is hyperbolic. In this paper, we prove a relative hyperbolic version of Martin's result.

THEOREM 1.1. *Let $\mathcal{G}(\mathcal{Y})$ be a strictly developable simple complex of finitely generated groups over a finite M_κ-simplicial complex Y with $k \leq 0$ and satisfying the following properties:*

- *For each vertex v of Y, the vertex group G_v is relatively hyperbolic to a maximal parabolic subgroup P_v.*
- *Local maps $\phi_{\sigma,\sigma'}$ are fully quasi-convex embeddings i.e. if $\sigma \subset \sigma'$ then $\phi_{\sigma,\sigma'}(G_{\sigma'})$ is fully quasi-convex in G_σ,*
- *The universal covering X of $\mathcal{G}(\mathcal{Y})$ is hyperbolic.*
- *The action of G, the fundamental group of $\mathcal{G}(\mathcal{Y})$, on X is acylindrical.*

Then G is hyperbolic relative to $\mathcal{P}$, where $\mathcal{P}$ is the collection of the images of P_v in G under the natural embedding $G_v \hookrightarrow G$ and there conjugates in G. Furthermore, local groups are fully quasi-convex in G.

The combination theorem of this sort for finite graph of hyperbolic groups was first given by Bestvina and Feighn [**1**]. Here, the edge groups embed quasi-isometrically into vertex groups and the graph of groups satisfies 'hallway flare condition'. This combination theorem was generalized by Mj and Reeves [**18**] for relatively hyperbolic case. Further, Mj and Sardar in [**20**] generalized these combination theorems for metric bundles with base space hyperbolic and fibers are uniformly hyperbolic metric spaces. Let S be a closed (not closed) orientable surface of negative Euler characteristic and $\phi : S \to S$ be a pseudo-Anosov homomorphism (fixing punctures and boundary pointwise, if non-empty). Let M_ϕ be the mapping torus, then it follows from combination theorem of Bestvina and Feighn in [**1**] (Mj and Reeves in [**18**]) that the fundamental group $\pi_1(M_\phi)$ is hyperbolic (relatively hyperbolic). However, in this case $\pi_1(S)$ being infinite index normal subgroup in $\pi_1(M)$ is not quasi-convex and $\pi_1(M_\phi)$ does not acts acylindrically on the Bass–Serre tree $\mathbb{R}$ (real numbers). Ilya Kapovich [**14**] proved that if finite graph of hyperbolic groups is acylindrically hyperbolic and satisfies quasi-isometrically embedded condition then fundamental group of graph of groups is hyperbolic and vertex groups are quasi-convex in the fundamental group. This was generalized to finite graph of relatively hyperbolic groups by Dahmani [**5**]. Martin in [**7**] generalized Ilya Kapovich's theorem for finite complex of hyperbolic groups.

An example to our interest can be constructed from Osin and Minasyan's work [**15**]. For instance, let M be a three-manifold and S be a punctured torus embedded in M. Suppose M splits over S with $M \setminus S$ having two components. Suppose G is fundamental group of M, A, B are fundamental groups of components and C is the fundamental group of punctured torus. Then, $G = A *_C B$. Let C_1 be the (cyclic) peripheral subgroup of C with respect to which C is hyperbolic relative to C_1. Now if A is hyperbolic relative to the subgroup C_1 and C is relatively quasi-convex in A. Then, G is acylindrically hyperbolic. Further, if B is hyperbolic relative to C_1 and C is fully quasi-convex in both A, B then G is hyperbolic relative to the collection of conjugates of C_1 in G.

We will adapt the strategies followed by Dahmani and Martin to prove the main theorem which is as follows:

- In our case local groups are relatively hyperbolic. In order to get a hyperbolic space on which the relatively hyperbolic group acts properly discontinuously, we will attach 'combinatorial horoballs' to each cosets of the peripheral subgroup. The resulting space is called Augmented space (See 2.8–2.10).
  We will construct a complex of spaces, EG (resp. boundary, ∂G) gluing the augmented spaces (see Definition 2.9) (resp. Bowditch boundaries) of the local groups similar to Martin's paper [**7**]. In Martin's paper local spaces are hyperbolic spaces on which local groups acts properly discontinuously and co-compactly. Here, we will take local spaces as augmented spaces and use the fact that relatively hyperbolic group acts on Bowditch boundary by convergence action. The topology defined on $EG \cup \partial G$ in [**7**] will work in our case and it will make $EG \cup \partial G$, a compact metrizable space.
- Next, we will prove that the action of G on ∂G is by convergence action. Since there are parabolic limit points in boundaries of local groups, we have to modify the proofs in [**7**] to work in our case.
- Last, we will show that all the limit points for this convergence action is either conical or bounded parabolic. Then by the Theorem 2.12 (due to Yaman, [**26**]), G will be relatively hyperbolic.

In Section 2, we will give several definitions of relatively hyperbolic groups due to Farb, Grooves and Manning and Bowditch. Convergence action, fully quasi-convex subgroups, convergence property and finite intersection properties are given in this section. Complex of groups is described in Section 3 and in the subsequent Section 4, the construction of boundary ∂G of fundamental group G of complex of groups $\mathcal{G}(\mathcal{Y})$ is provided. In Sections 5 and 6, we will prove Theorem 1.1.

2. Preliminaries on relative hyperbolicity.

2.1. Relative hyperbolicity. Relatively hyperbolic groups were first introduced by Gromov [**9**] to study hyperbolic manifolds with cusps. It was then studied by several people, we refer to the paper [**13**] by Hruska for several equivalent notions of relatively hyperbolic groups. For our purpose, we will require three equivalent definitions of relative hyperbolicity due to Farb [**6**], Bowditch [**3**] and Groves and Manning [**10**].

DEFINITION 2.1 (Hyperbolic Metric Space). Let $\delta \geq 0$. We say that a geodesic triangle Δ is δ-*slim* in a geodesic metric space if any side of the triangle Δ is contained in the δ- neighbourhood of the union of the other two sides. A geodesic metric space is said to be δ-*hyperbolic* if all the triangles are δ-slim. A geodesic metric space is said to be hyperbolic if it is δ-hyperbolic for some $\delta \geq 0$.

First, we give the definition of relative hyperbolicity due to Farb. Let G be a finitely generated group and H be a finitely generated subgroup of it. Also let Γ_G be the Cayley graph of G.

DEFINITION 2.2 (Coned-off Cayley Graph, [**6**]). The coned-off Cayley graph of G w.r.t. H, denoted by $\widehat{\Gamma_G}$, is obtained from Γ_G by adding an extra vertex $v(gH)$ for each left coset of H in G and an extra edge $e(gh)$ of length $1/2$ joining each $gh \in gH$ to $v(gH)$.

Given a path γ in Γ_G, the inclusion $\Gamma_G \to \widehat{\Gamma_G}$, gives a path $\tilde{\gamma}$ (after removing backtracks and loops of length 1) in $\widehat{\Gamma_G}$. If $\tilde{\gamma}$ goes through some $v(gH)$, then we say γ penetrates gH. We call γ to be a relative k-quasi geodesic if $\tilde{\gamma}$ is a k-quasi geodesic in $\widehat{\Gamma_G}$. Also, γ is said to be a path without backtracking if after going through a cone point $v(gH)$ it never return to gH.

DEFINITION 2.3 (Bounded Coset Penetration Property, [**6**]). (G, H) is said to have bounded coset penetration property if for each $k > 1$ there exists $c(k) > 0$ such that for any two relative k-quasi geodesics γ_1, γ_2 in Γ_G with $d_{\Gamma_G}(\gamma_1, \gamma_2) \leq 1$, the following holds,

(1) if γ_1 penetrates gH but γ_2 does not then γ_1 travels at most $c(k)$ distance in gH.

(2) if both γ_1, γ_2 penetrates gH then the entry points as well as the exit points of the paths are $c(k)$ close to each other in Γ_G.

DEFINITION 2.4 (Farb [**6**]). Let G be a finitely generated group and H be a finitely generated subgroup of it. G is said to strongly hyperbolic relative to H if $\widehat{\Gamma_G}$ is hyperbolic and (G, H) satisfy bounded coset penetration property.

The next definition by Bowditch gives a dynamical characterization of relative hyperbolicity, which we will essentially use to prove the main theorem. For that, we need the notion of convergence group.

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Definition 2.5 (Convergence Group). Let G acts on compact metrizable space M. The action is called convergence group action if for any sequence $\{g_n\}$ in G, there exists a subsequence $\{g_{\phi(n)}\}$ and $\xi^+, \xi^- \in M$ such that $g_{\phi(n)}(K)$ converges uniformly to ξ^+, for all compact sets $K \subset M \backslash \{\xi^-\}$.

Definition 2.6.

(1) (Bounded Parabolic Limit Points) An element $g \in G$ is called parabolic if it fixes exactly one point of M and the corresponding fixed point ξ(say) is said to be parabolic limit point. Furthermore, a parabolic limit point is said to be bounded parabolic if $Stab(\xi)$ acts properly discontinuously and co-compactly on $M \backslash \{\xi\}$.

(2) (Conical Limit Point) Let G has a convergence action on M. A point $\xi \in M$ is said to be conical limit point if there exists a sequence $\{g_n\}$ and $\xi^+ \neq \xi^- \in M$ such that $g_n\xi \to \xi^+$, $g_n\xi^{'} \to \xi^-$ for all $\xi^{'} \in M \backslash \{\xi\}$.

(3) (Geometrically Finite Action) Let G has a convergence action on a compact metrizable space M. The action is said to be geometrically finite if the limit points are either conical or bounded parabolic.

Next, we give the Bowditch's definition of Relative Hyperbolicity.

Definition 2.7 (Bowditch [3]). Let G be finitely generated group and $\mathcal{P}$ be a finite collection of finitely generated subgroups of it. G is said to hyperbolic relative to $\mathcal{P}$ if it acts properly discontinuously on a proper hyperbolic metric space $\widetilde{\Gamma}$ such that

- G acts on $\partial\widetilde{\Gamma}$ by convergence and geometrically finite action.
- the conjugates of the elements of $\mathcal{P}$ are precisely the maximal parabolic subgroups.

we call $\partial\widetilde{\Gamma}$ the Bowditch boundary of G.

Note that $\widehat{\Gamma_G}$ is locally infinite and the action of G on it, is not properly discontinuous unless H is finite. Groves and Manning have defined a proper metric space by gluing combinatorial horoballs along parabolic subgroups and their translates, similar to coned-off Cayley graph and it is called Augmented space. Also, G acts on its augmented space properly discontinuously by isometries. Let G be finitely generated group and $\mathcal{P}$ be a finite collection of subgroups of it. Let $\mathcal{S}$ be a finite generating set of G such that $\langle \mathcal{S} \cap P \rangle = P$ for all $P \in \mathcal{P}$ and Γ_G be the Cayley graph of G with respect to $\mathcal{S}$.

Definition 2.8 (Combinatorial Horoballs, [10]). Let C be a 1-complex with 0-skeleton C^0 and 1-skeleton C^1. We will construct a 1-complex $\mathcal{H}(C)$ following ways:

- 0-skeleton of $\mathcal{H}(C)$, $\mathcal{H}(C)^{(0)} := C^{(0)} \times (\{0, 1, 2, \ldots\})$,
- 1-skeleton of $\mathcal{H}(C)$, $\mathcal{H}(C)^{(1)} := \big\{[(v, 0), (w, 0)] : v, w \in C^{(0)}, [v, w] \in C^{(1)}\big\} \cup \big\{[(v, k), (w, k)] : v, w \in C^{(0)}, k > 0, d_C(v, w) \leq 2^k\big\} \cup \big\{[(v, k), (v, k+1)] : v \in C^{(0)}, k \geq 0\big\}$.

Definition 2.9 (Augmented Space, [13]). Let $G, \mathcal{P}, \mathcal{S}$ be as mentioned above. Also let $\mathcal{T}$ be the set of representative for distinct cosets of all $P \in \mathcal{P}$. The Cayley graph of P with respect to $P \cap \mathcal{S}$ embedded in Γ_G as a subcomplex. Let Γ_t, $t \in \mathcal{T}$, be the translates of these subcomplexes. We define

$$\Gamma_G^h := \Gamma_G \cup \big(\cup_{t\in\mathcal{T}} (\mathcal{H}(\Gamma_t))\big) / \simeq,$$

as augmented space, where $\mathcal{H}(\Gamma_t) \times \{0\}$'s are identified to subcomplexes Γ_t.

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DEFINITION 2.10 (Groves and Manning [**10**]). *G is said to hyperbolic relative to $\mathcal{P}$ if the augmented space Γ_G^h is hyperbolic for any appropriate choice of $\mathcal{S}$.*

REMARK 2.11. Due to equivalence of these definitions, we can take Γ^h as $\tilde{\Gamma}$ and $\partial\Gamma^h$ will be Bowditch boundary.

Next, we will state a theorem due to A. Yaman which is a generalization of Bowditch's result on characterization of hyperbolic groups [**2**].

THEOREM 2.12 (Yaman [**26**]). *Let G has a geometrically finite action on a perfect metrizable compact space M and $\mathcal{P}$ be the collection of maximal parabolic subgroups. Also let every parabolic subgroup be finitely generated and there are only finitely many orbits of bounded parabolic points. Then G is hyperbolic relative to $\mathcal{P}$ and M is equivariantly homeomorphic to its Bowditch boundary.*

We can omit the finiteness of the set of orbits of parabolic points by a theorem of Tukia ([**24**], Theorem 1B). As discussed in the introduction, we will use this characterization of relative hyperbolicity to prove the main theorem.

2.2. Fully quasi-convex subgroup. Fully quasi-convex subgroups of relatively hyperbolic group were introduced by Dahmani in [**5**]. It is a generalization of quasi-convex subgroups of hyperbolic group in the sense that it satisfies *limit set property, convergence property and finite intersection (finite height) property*, which is not in general true for quasi-convex subgroup of relatively hyperbolic group. The definition of fully quasi-convex subgroups, Remark 2.15 and Theorems 2.16 and 2.17 are taken from [**5**]. We refer to Section 1.2 of [**5**] for proofs.

DEFINITION 2.13 (Dahmani [**5**]). Let G be a relatively hyperbolic group with Bowditch boundary ∂G. A subgroup H of G is called quasi-convex if H has a geometrically finite action on ΛH. It is called fully quasi-convex if for any infinite sequence $\{g_n\}$, all comes from distinct cosets of H, $\bigcap_n(g_n\Lambda H)$ is empty.

REMARK 2.14. If H is fully quasi-convex, then gHg^{-1} is also fully quasi-convex, for all $g \in G$.

REMARK 2.15 ([**5**]). Let G be a relatively hyperbolic group. If H is fully quasi-convex in G, then each parabolic point for H in $\Lambda(H)$ is a parabolic point for G in ∂G and if P is the corresponding maximal parabolic subgroup in G then the corresponding maximal parabolic subgroup in H is precisely $P \cap H$.

The following two properties of fully quasi-convex subgroups are proved by Dahmani [**5**].

THEOREM 2.16 (Limit set property, [**5**]). *Let H_1 and H_2 are fully quasi-convex in G then $H_1 \bigcap H_2$ is fully quasi-convex. Moreover, $\Lambda(H_1 \bigcap H_2) = \Lambda H_1 \bigcap \Lambda H_2$.*

THEOREM 2.17 (Convergence property, [**5**]). *Let G be a relatively hyperbolic group and H be a fully quasi-convex subgroup in it. Let $\{g_n\}$ be a sequence of elements in G all comes from distinct cosets of H. Then there exists a subsequence $\{g_{\phi(n)}\}$ such that $g_{\phi(n)}\Lambda H$ uniformly converges to a point.*

LEMMA 2.18. *Let H be a finitely generated fully quasi-convex subgroup of finitely generated relatively hyperbolic group G. Then, Γ_H^h is quasi-convex in Γ_G^h.*

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Proof. Let H be generated by S and extend the generating set to generate G. Then the corresponding augmented spaces Γ_H^h will be a subgraph of Γ_G^h. Take two points x, y in Γ_H^h and join them by a geodesic c in Γ_G^h. Since H is relatively quasi-convex in G, by a theorem of Hruska [**13**], Γ_H is quasi-convex in $\widehat{\Gamma}_G$. Let $\widehat{c}$ be the image of c in $\widehat{\Gamma}_G$ after removing the backtracks and loops of length 1. Take the projection of $\widehat{c}$ onto Γ_H and call it $\tilde{c}$, hence $\tilde{c}$ is a quasi-geodesic in $\widehat{\Gamma}_G$. Note that image of $\widehat{c}$ and c are same outside horoballs. Also, Hausdorff distance between $\widehat{c}$ and $\tilde{c}$ is bounded outside horoballs and if they enter same horoballs then the distances between entry points, as well as exit points, are bounded in Γ_G and so in Γ_G^h. So if we can prove for any geodesic in Γ_G^h entirely lies in a horoball with starting and ending points close to H the geodesic is in bounded distance from Γ_H^h, then we are done. By Lemma 3.1 of Grooves and Manning [**10**] any geodesic in a horoball tracks a geodesic consists of two vertical segments and one horizontal segment (Hausdorff distance is at most 4). Now a geodesic consists of two vertical segments and one horizontal segment with starting and ending points close to H lie in a bounded neighbourhood of Γ_H^h. Hence, we are done. □

Next, we will prove that there are finitely many conjugates of a fully quasi-convex subgroup that have infinite total intersection.

PROPOSITION 2.19 (Finite intersection property). *Let G be a relatively hyperbolic group and H be a fully quasi-convex subgroup in it. Then there exists finitely many distinct left cosets $g_1H, g_2H \ldots g_mH$ in G for which $\bigcap_{k=1}^{m} g_kHg_k^{-1}$ is infinite.*

Proof. If possible, let there exists a infinite sequence $\{g_n\}$ all comes from distinct cosets of H such that $\bigcap_{n=1}^{\infty} g_nHg_n^{-1}$ is infinite, i.e., $\Lambda(\bigcap_{n=1}^{\infty} g_nHg_n^{-1})$ is non-empty. But $\Lambda(\bigcap_{n=1}^{\infty} g_nHg_n^{-1}) \subset \bigcap_{n=1}^{\infty} \Lambda(g_nHg_n^{-1})$ and the fact that $\Lambda(gHg^{-1}) = g\Lambda H$, we have $\bigcap_{n=1}^{\infty} g_n\Lambda H$ is non-empty that contradicts the second condition of fully quasi-convexity. □

3. Background on complex of groups.

Bass and Serre in [**21**] completely described the class of groups that act on trees without inversion. Such groups are fundamental group of graph of groups. Haefliger in [**12**] generalized this theory to the class of groups acting on simplicial complexes and it is called complex of groups. In this section, we will discuss the basics of complex of groups. For a detailed discussion on this topic, we refer to [**4**].

Let Y be a simplicial complex. We will denote the set of simplices and set of vertices of Y by $S(Y)$ and $V(Y)$, respectively. Let $\mathcal{Y}$ be the scwol (refer to [**4**]) corresponding to the first Barycentric subdivision of Y and its directed edge set is denoted by $\mathcal{E}^{\pm}(\mathcal{Y})$.

3.1. Complex of groups.

DEFINITION 3.1 (Complex of Groups, [**4**]). A simple complex of groups, $G(\mathcal{Y})$, over a simplicial complex Y consists of

(1) local groups G_σ for each $\sigma \in S(Y)$,

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(2) a monomorphism $\varphi_{\sigma,\sigma'} : G_{\sigma'} \to G_\sigma$ whenever $\sigma \subset \sigma'$,
(3) for $\sigma \subset \sigma' \subset \sigma''$, $\varphi_{\sigma,\sigma''} = \varphi_{\sigma,\sigma'} \circ \varphi_{\sigma',\sigma''}$.

Definition 3.2 (Fundamental Group of Complex of Groups, [**4**]). Let T be a maximal tree in 1-skeleton of $\mathcal{Y}$. The Fundamental Group of $G(\mathcal{Y})$ with respect to T, denoted by $\pi_1(G(\mathcal{Y}), T)$, is generated by $\bigsqcup_{\sigma \in S(Y)} G_\sigma \bigsqcup \mathcal{E}^{\pm}(\mathcal{Y})$ subject to

(1) relations of G_σ,
(2) $(a^+)^{-1} = a^-, (a^-)^{-1} = a^+$,
(3) $(ab)^+ = a^+b^+$
(4) $a^+ g a^- = \varphi_a(g)$,
(5) $a^+ = 1$ for all edge a of T.

In fact the above definition is independent of the choice of the maximal tree, and we will call it G in the subsequent sections. There is a canonical morphism, $\iota_T : G(\mathcal{Y}) \to G$ that takes $G_\sigma \to G$ and $a \mapsto a^+$. The natural homomorphisms $G_\sigma \to G$ is injective if and only if the complex of groups $G(\mathcal{Y})$ is developable. For definition of developability, see Definition 2.11 of [**4**].

Next, we will define a CW complex on which G will act naturally and the quotient space will be Y.

Definition 3.3 (Universal Covering, [**4**]). We define the universal covering of $G(\mathcal{Y})$ associated to ι_T as

$$X := \left(G \times \coprod_{\sigma \in S(Y)} \sigma \right) \Big/ \simeq,$$

where $(g, i_{\sigma,\sigma'}(x)) \simeq (g\iota_T([\sigma, \sigma'])^{-1}, x)$, $[\sigma, \sigma'] \in \mathcal{E}(\mathcal{Y})$, $i_{\sigma,\sigma'} : \sigma' \to \sigma$ is the embedding and $(gg', x) \simeq (g, x)$, $g' \in G_\sigma$, $g \in G$.

G acts naturally on X by left multiplication on the first factor.

Definition 3.4 (Acylindrical Action). Let $K > 0$. The action of G on a metric space (X, d) is said to be K-*acylindrical* if for any pair of points $x, y \in X$ with $d(x, y) \geq K$ the pointwise stabilizer of $\{x, y\}$ is finite. The action of G on X is said to be acylindrical if it is K-acylindrical for some $K > 0$.

3.2. Complex of spaces. Definition 3.5. A complex of spaces, $C(\mathcal{Y})$, over a simplicial complex Y consists of
(1) local spaces C_σ for each $\sigma \in S(Y)$,
(2) an embedding $\varphi_{\sigma,\sigma'} : C_{\sigma'} \to C_\sigma$ whenever $\sigma \subset \sigma'$,
(3) for $\sigma \subset \sigma' \subset \sigma''$, $\varphi_{\sigma,\sigma''} = \varphi_{\sigma,\sigma'} \circ \varphi_{\sigma',\sigma''}$.

Definition 3.6 (Realization of complex of spaces). Let $C(\mathcal{Y})$ be a complex of spaces over Y. We define the realization of $C(\mathcal{Y})$ to be the quotient space

$$|C(\mathcal{Y})| := \left(\coprod_{\sigma \in S(Y)} (\sigma \times C_\sigma) \right) \Big/ \simeq,$$

where $(i_{\sigma,\sigma'}(x), s) \simeq (x, \varphi_{\sigma,\sigma'}(s))$, $[\sigma, \sigma'] \in \mathcal{E}(\mathcal{Y})$.

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4. Construction of EG and ∂G. Let $G(\mathcal{Y})$ be a developable simple complex of group with fundamental group G as defined in 3.2. For each vertex v of Y, the vertex group G_v is relatively hyperbolic to the subgroup P_v. Local maps $\varphi_{\sigma,\sigma'}$ are fully quasi-convex embeddings i.e., if $\sigma \subset \sigma'$ then $\varphi_{\sigma,\sigma'}(G_{\sigma'})$ is fully quasi-convex in G_σ. Then by Remark 2.15, G_σ is relatively hyperbolic to the subgroup $P_v \cap G_\sigma$ for each $\sigma \in S(X)$. We call $P_v \cap G_\sigma$ as P_σ. By extending the generating set of $G_{\sigma'}$ to a generating set of G_σ, $\varphi_{\sigma,\sigma'} : G_{\sigma'} \to G_\sigma$ will induce a natural equivariant embeddings between the corresponding Cayley graphs and Augmented spaces. Also, $\varphi_{\sigma,\sigma'}$ naturally extends to the Bowditch boundaries of corresponding local groups.

Let X be the universal covering of $G(\mathcal{Y})$ associated to ι_T. Let Γ_σ be the Cayley graph of G_σ and Γ^h_σ be the augmented spaces on which G_σ acts properly discontinuously. Also, let ∂G_σ be the Bowditch Boundary of G_σ and $\overline{\Gamma^h_\sigma} = \Gamma^h_\sigma \cup \partial G_\sigma$.

Definition 4.1. We define a complex of spaces over X, EG (*resp*. EG^h) associated to $G(\mathcal{Y})$

$$EG := \left(G \times \coprod_{\sigma \in S(Y)} (\sigma \times \Gamma_\sigma) \right) \Big/ \simeq,$$

$$EG^h := \left(G \times \coprod_{\sigma \in S(Y)} (\sigma \times \Gamma^h_\sigma) \right) \Big/ \simeq,$$

where $(g, i_{\sigma,\sigma'}(x), s) \simeq (g\iota_T([\sigma, \sigma'])^{-1}, x, \varphi_{\sigma,\sigma'}(s))$, $[\sigma, \sigma'] \in \mathcal{E}(\mathcal{Y})$ and $(gg', x, s) \simeq (g, x, g's)$, $g' \in G_\sigma$, $g \in G$.

G has natural action on EG^h by left multiplication on the first factor. Also, there is an obvious projection map $p : EG^h \to X$, which injectively sends the first two factors and this map is G-equivariant.

Definition 4.2. We define the space

$$\partial_{stab} G := \left(G \times \coprod_{\sigma \in S(Y)} (\{\sigma\} \times \partial G_\sigma) \right) \Big/ \simeq,$$

where $(g, \{\sigma\}, s) \simeq (g\iota_T([\sigma, \sigma'])^{-1}, \{\sigma'\}, \varphi_{\sigma,\sigma'}(s))$, $[\sigma, \sigma'] \in \mathcal{E}(\mathcal{Y})$ and $(gg', \{\sigma\}, s) \simeq (g, \{\sigma\}, g's)$, $g' \in G_\sigma$, $g \in G$.

Now, we define the boundary of G as

$$\partial G := \partial_{stab} G \cup \partial X.$$

Also, we define $\overline{EG^h} := EG^h \cup \partial G$.

Here, we are taking the union of augmented spaces (respectively, boundaries) corresponding to vertex groups of X and gluing them along the augmented spaces (respectively, boundaries) of the local groups accordingly.

G also has natural action on ∂G and $\overline{EG^h}$ by left multiplication on the first factor. In the subsequent section, we will try to give a topology on $\overline{EG^h}$ such that $\overline{EG^h}$ and ∂G will be compact and action of G will be geometrically finite convergence action.

For simplicity of notation, we will denote G_σ as the stabilizer subgroup of the simplex σ in X (Note that $stab(\sigma)$ is actually conjugate of a local group of $G(\mathcal{Y})$). It is easy to see that the map $\partial G_\sigma \to \partial G$ is G_σ – equivariant for every simplex σ in X.

In the subsequent sections, we assume our complex of groups satisfies all the hypothesis of the main theorem. Then by 2.16, 2.17, and 2.19 $\mathcal{G}(\mathcal{Y})$ will satisfies *limit set property*, *convergence property,* and *finite intersection property*.

4.1. Domains and topology. This section is mostly taken from Sections 4 and 6 from Martin's paper [7]. Proofs of most of the propositions and theorems will work as it is by adapting to our setting.

DEFINITION 4.3. Let $\xi \in \partial_{stab} G$. We define *domain* of ξ, $D(\xi) := span\{\sigma \in S(X) : \xi \in \partial_{stab} G_\sigma\}$.

PROPOSITION 4.4 (Propositions 4.2, 4.4 of [7]).

(i) *For every vertex v, the quotient map $\partial G_v \to \partial G$ is injective,*

(ii) *For every $\xi \in \partial_{stab} G$, $D(\xi)$ is finite convex subcomplex of X uniformly bounded by the acylindricity constant.*

DEFINITION 4.5 (ξ-family, [7]). Let $\xi \in \partial_{stab} G$. A ξ-family is defined to be as a collection $\mathcal{U}$ of open sets U_v, where $v \in V(D(\xi))$ and U_v is a neighbourhood of representative of ξ in $\overline{\Gamma^h_v}$ such that for every two adjacent vertices v, v' we have

$$\varphi_{v,e}(\overline{\Gamma^h_e}) \cap U_v = \varphi_{v',e}(\overline{\Gamma^h_e}) \cap U_{v'},$$

where e is an edge between v and v'

Next, we give a topology on ∂G due to Martin [7].

Let us choose a basepoint $v_0 \in X$. For a given point $x \in X$(*resp.* $\eta \in \partial X$), we denote c_x(*resp.* c_η) to be the unique geodesic segment(respectively, geodesic ray) from v_0 to x(*resp.* η). We denote $D^\epsilon(\xi)$ to be the ϵ-neighbourhood of $D(\xi)$, where $\epsilon \in (0, 1)$.

A geodesic c is said to be *goes through* (reps. *enters*) $D^\epsilon(\xi)$ if $\exists\ t_0, t_1$ such that $c(t_0) \in D^\epsilon(\xi)$, $c(t_1) \in \overline{D^\epsilon(\xi)}$ and $\forall t > t_1$, $c(t) \notin D^\epsilon(\xi)$ (respectively, if $\exists t_0$ such that $c(t_0) \in D^\epsilon(\xi)$). If c_x or c_η goes through $D^\epsilon(\xi)$, the first simplex which is met by c_x or c_η after leaving $D^\epsilon(\xi)$ is said to be an *exit simplex* and is denoted by $\sigma_{\xi,\epsilon}(x)$. For $x \in D^\epsilon(\xi)$, we define $\sigma_{\xi,\epsilon}(x) := \sigma_x$

DEFINITION 4.6 (Martin [7]). Let $\xi \in \partial_{stab} G, \mathcal{U}$ a ξ-family and $\epsilon \in (0, 1)$. We define

(i) $Cone_{\mathcal{U},\epsilon}(\xi) := \{x \in \overline{X} \setminus D(\xi) : c_x$ goes through $D^\epsilon(\xi)$ and for all $v \in V(D(\xi) \cap \sigma_{\xi,\epsilon}(x))$, $\overline{\Gamma^h_{\sigma_{\xi,\epsilon}(x)}} \subset U_v$, in $\overline{\Gamma^h_v}\}$,

(ii) $\widetilde{Cone}_{\mathcal{U},\epsilon}(\xi) := \{x \in \overline{X} :\ c_x$ enters $D^\epsilon(\xi)$ and for all $v \in V(D(\xi) \cap \sigma_{\xi,\epsilon}(x))$, $\overline{\Gamma^h_{\sigma_{\xi,\epsilon}(x)}} \subset U_v$,in $\overline{\Gamma^h_v}\}$.

Martin, in [7], proved that the cones $Cone_{\mathcal{U},\epsilon}(\xi)$ and $\widetilde{Cone}_{\mathcal{U},\epsilon}(\xi)$ are open sets in $\overline{X}$.

Topology on $\overline{EG^h}$.

$\overline{EG^h}$ consists of three kind of elements $\tilde{x} \in EG^h$, $\eta \in \partial X$, and $\xi \in \partial_{\text{stab}} G$.

- For $\tilde{x} \in EG^h$: We define a basis of neighbourhood of $\tilde{x}$ in $\overline{EG^h}$ coming from the topology of EG^h as a CW complex and denote it by $\mathcal{O}_{\overline{EG^h}}(\tilde{x})$.

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- For $\eta \in \partial X$: Let $\mathcal{O}_{\overline{X}}(\eta)$ be the basis of neighbourhood of η in $\overline{X}$ and $U \in \mathcal{O}_{\overline{X}}(\eta)$. We define a neighbourhood of η in $\overline{EG^h}$.

$$V_U(\eta) = p^{-1}(U \cap X) \cup (U \cap \partial X) \cup \{\xi \in \partial_{\text{stab}} G | D(\xi) \subset U\}.$$

We define, $\mathcal{O}_{\overline{EG^h}}(\eta) := \{V_U(\eta) | U \in \mathcal{O}_{\overline{X}}(\eta)\}$, the basis of neighbourhood of η in $\overline{EG^h}$.
- For $\xi \in \partial_{\text{stab}} G$: Let $\mathcal{U}$ be ξ-family and $\epsilon \in (0, 1)$. We define four sets around ξ as follows:
$W_1 = \{\tilde{x} \in EG^h : p(\tilde{x}) = x \in D^\epsilon(\xi)$ and $\varphi_{v,\sigma_x}(\tilde{x}) \in U_v$ for all vertex $v \in D(\xi) \cap \sigma_x\}$,
$W_2 =$ the set of points in EG whose projection in X belongs to $Cone_{\mathcal{U},\epsilon}(\xi)$.
$W_3 := Cone_{\mathcal{U},\epsilon}(\xi) \cap \partial X$,
$W_4 := \{\xi' \in \partial_{\text{stab}} G : D(\xi') \backslash D(\xi) \subset \widetilde{Cone}_{\mathcal{U},\epsilon(\xi)}$ and $\xi' \in U_v$, for all vertex $v \in D(\xi) \cap D(\xi')\}$.

We define a neighbourhood around ξ as $W_{\mathcal{U},\epsilon}(\xi) := W_1 \cup W_2 \cup W_3 \cup W_4$. Let $\mathcal{O}_{\overline{EG^h}}(\xi) = \{W_{\mathcal{U},\epsilon}(\xi) : \mathcal{U}$ ξ-family and $\epsilon \in (0, 1)\}$. We give $\overline{EG^h}$ the topology generated by the sub-basis $\mathcal{O}_{\overline{EG^h}}(x)$, $x \in \overline{EG^h}$. In fact, Martin showed that $\mathcal{O}_{\overline{EG^h}}(x)$ is a basis for this topology. Martin, in [7], showed that the topology remains equivalent even if we change the base point. From Proposition 4.4 the map $\partial G_v \to \partial G$ is injective for all vertex v of X, moreover Martin proved that these maps are embedding (Proposition 6.19 [7]).

For hyperbolic case, that is, if we consider local groups to be hyperbolic and take Gromov boundary instead of Bowditch boundary then the Separability, Metrisability, and Compactness of $\overline{EG}$ are proved in [7]. The proof requires X to be $CAT(0)$, acylindrical action of G on X and convergence property of the local groups which are true in our case also, hence same proofs will work in proving the Separability, Metrisability, and Compactness of $\overline{EG^h}$. For instance, to prove sequentially compactness of $\overline{EG^h}$, we take a sequence $\{x_n\}$ of points in $\overline{EG^h}$. Now, due to Theorem 6.17 of [7], EG^h is dense in $\overline{EG^h}$. So, we can take the sequence $\{x_n\}$ in EG^h and let $a_n = p(x_n)$ be its image in X. For each n, let $\{\sigma_1^{(n)}, \sigma_2^{(n)} \dots, \sigma_{m(n)}^{(n)}\}$ be the path of simplices meet by the geodesics $[v_0, a_n]$(note that $\{\sigma_1^{(n)} = v_0\}$). Then, three cases can occur.
<u>Case 1</u> : $\{a_n\}$'s contained in finitely many simplices in X. Then, upto subsequences we can assume for all n, a_n's contained in the interior of a single simplex, σ (say). Hence, x_n will converges to some point of $\overline{\Gamma_\sigma^h} \hookrightarrow \overline{EG^h}$.
<u>Case 2</u> : Number of simplices in $\{\sigma_k^{(n)}\}_n$ is finite for all $k = 1, \dots, m(n)$. Then, upto subsequence $< a_n, a_{n'} >_{v_0} \to \infty$. Hence, $\{a_n\}$ converges to η, where $\eta \in \partial X$. From the definition of topology on ∂G, it can be proved that $\{x_n\}$ converges to η.
<u>Case 3</u> : Number of simplices in $\{\sigma_m^{(n)}\}$ is infinite for some m. Let m_0 be the first number such that the number of simplices in $\{\sigma_{m_0}^{(n)}\}$ is infinite. Now upto subsequence we can let $\sigma_1, \sigma_2, \dots, \sigma_{m_0-1}$ be the first $m_0 - 1$ number of simplices met by the geodesics $[v_0, a_n]$. Obviously, $\sigma_{m_0-1} \subset \sigma_{m_0}^{(n)}$ for all n. Then by convergence property $\partial G_{\sigma_{m_0}^{(n)}}$ converges to some point ξ in $\partial G_{\sigma_{m_0-1}}$. Then from the definition of topology on $\overline{EG^h}$, it can be shown that $\{x_n\}$ converges to ξ.

THEOREM 4.7 (Martin, Theorems 7.12 and 7.13 of [7]). *$\overline{EG^h}$ is separable, metrizable and is compact.*

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5. Convergence group action of G. In this section, we describe Martin's strategy (in [7]) to prove convergence action of G on ∂G. It is divided into following three propositions. As the proof of these propositions almost remains the same as given by Martin [7], we will not provide the full details but give the ideas and account for where it differs.

PROPOSITION 5.1 (by adapting Lemma 9.14 of [7]). *Let $\{g_n\}$ be an injective sequence in G and there exists v_0 and v_1 such that $g_n v_0 = v_1$ for infinitely many n. Then there exists $\xi^+, \xi^- \in \partial G$ and a subsequence $\{g_{n_r}\}$ of $\{g_n\}$ such that for any compact set K in $\partial G \backslash \{\xi^-\}$, $g_{n_r} K$ convergences to ξ^+ uniformly.*

We sketch the proof of Proposition 5.1. Without loss of generality, we can take $g_n v_0 = v_0$ for infinitely many n and v_0 as the base point of the topology on $\overline{EG^h}$. Then, g_n stabilizes the vertex space $\Gamma^h_{v_0}$. G_{v_0} has convergence action on $\overline{\Gamma^h_{v_0}}$. Thus, there exists a subsequence of $\{g_n\}$, and points $\xi_-, \xi_+ \in \partial G_{v_0}$ such that for every compact set K_{v_0} of $\overline{\Gamma^h_{v_0}} \setminus \{\xi_-\}$, the sequence of translates $g_n K_{v_0}$ converge uniformly to ξ_+. Let K be compact set in $\partial G \backslash \{\xi^-\}$ and $\overline{p}(K) = (\cup_{\xi \in K} D(\xi)) \cup (K \cap \partial X)$. We will be applying convergence criterion proved by Martin (Corollary 7.16 of [7]) in order to show that upto a subsequence, $g_n K$ converges uniformly to ξ_+. In order to do that let us first look into simplices in one simplicial neighbourhood of $D(\xi_+)$.

Let σ be a simplex in X such that $v_0 \in \sigma \cap D(\xi_+)$. This implies $\partial G_\sigma \subset \partial G_{v_0}$. If σ is not contained in $D(\xi_-)$, then $\xi_- \notin \partial G_\sigma$. Thus, up to a subsequence, convergence action of vertex group G_{v_0} implies that $g_n \partial G_\sigma$ converges uniformly to ξ_+ in G_{v_0}. Now, let σ be contained in the subcomplex $D(\xi_-)$ then $\xi_- \in \partial G_\sigma$.
Case I. Let ξ^- be parabolic in ∂G_{v_0}. Now suppose for some $v \in V(D(\xi^+) \cap D(\xi^-))$ fixed by all g_n, $g_n.\xi^- \to \xi^{'} (\neq \xi^+)$. Then, we got $\{g_n\}$ and $\xi^{'} (\neq \xi^+)$ such that $g_n.\xi^- \to \xi^{'}$ and $g_n.\tilde{\xi} \to \xi^+$ for all $\tilde{\xi} \neq \xi^-$, which contradicts the fact that ξ^- is parabolic. Hence,

1) For all $v \in V(D(\xi^+) \cap D(\xi^-))$ fixed by g_n's and for any compact set K_v in ∂G_v up to a subsequence, $g_n K_v$ converges to ξ^+ uniformly.

2) For any simplex σ not in $D(\xi^+) \cap D(\xi^-)$ but having a common vertex $v \in \sigma \cap (D(\xi^+) \cap D(\xi^-))$ fixed by all g_n's the following holds: if $\{g_n G_\sigma\}$ is an infinite collection of cosets then, up to a subsequence, $g_n \partial G_\sigma$ converges to ξ^+ uniformly by convergence property for fully quasi-convex subgroups.

Case II. ξ^- is not parabolic in ∂G_{v_0}. Recall, we have taken σ to be a simplex contained in $D(\xi^-)$ with the vertex $v_0 \in V(D(\xi^+) \cap D(\xi^-))$, then ∂G_σ contains at least two points, including ξ^-, otherwise G_σ would be a parabolic subgroup that implies ξ^- is parabolic in ∂G_{v_0}, a contradiction.

• Let the set of cosets $\{g_n G_\sigma : n \geq 1\}$ be infinite. For $x \in \partial G_\sigma$ other than ξ^-, up to a subsequence, $g_n x$ converges to ξ^+. This is due to convergence action of G_v. So, $g_n \partial G_\sigma$ converges uniformly to ξ_+.

• If the set of cosets $\{g_n G_\sigma : n \geq 1\}$ is finite then up to a subsequence of $\{g_n\}$, we can take $g_n \partial G_\sigma = g_N \partial G_\sigma$ and $g_n^{-1} g_N$ stabilizes σ. Replacing $g_n^{-1} g_N$ by g_n, we can assume g_n stabilizes each σ and hence $\xi_+ \in g_n \partial G_\sigma$.

Suppose τ is a simplex in $D(\xi_-) \cap D(\xi_+)$ fixed pointwise by each element of $\{g_n\}$. For each vertex $v \in \tau$, $\xi_-, \xi_+ \in \partial G_v$ and due to convergence property of G_v for any compact set C in $\partial G_v \setminus \{\xi_-\}$, up to a subsequence of $\{g_n\}$, $g_n C$ converges uniformly to ξ_+. Note that if ∂G_τ is a single point, then ξ_- is a parabolic point. In that case, $\xi_- = \xi_+$ and for any compact set C in ∂G_v, up to a subsequence of $\{g_n\}$, $g_n C$ converges uniformly to ξ_+. Now for any simplex σ with a vertex $v \in \sigma \cap \tau$ and $v \in D(\xi_+)$, we can continue the above process. Let A be a finite subcomplex in $D(\xi^+) \cap D(\xi^-)$ such that

1) A is fixed by g_n's pointwise.

2) For all simplex σ contained in the deleted simplicial neighbourhood of A, $g_n \partial G_\sigma$ converges to ξ^+ uniformly.

3) For all simplex σ in A and for all $v \in V(\sigma \cap A)$, $g_n K_\sigma$ convergences to ξ^+ uniformly, for any compact sets K_σ in $\partial G_\sigma \backslash \{\xi^-\}$.

Let K be a compact subset in $\partial G \backslash \{\xi^-\}$. Now if $K_v = K \cap \partial G_v$ is non-empty for some vertex v of A then as discussed above, up to a subsequence $g_n K_v$ converges to ξ^+ uniformly. And for any other point x of K, join v_0 to $g_n x$ by a geodesic $[v_0, g_n x]$. The exit simplex for the geodesic $[v_0, g_n x]$ from A will lie $N(A) \setminus A$, where $N(A)$ is one simplicial neighbourhood of A. Then by above reason, up to a subsequence, the sequence of translates of exit simplex by g_n's converges to ξ_+. By convergence criterion proved by Martin (Corollary 7.16 of [7]), it would imply that $g_n K$ convergences to ξ^+ uniformly.

Suppose Q is a relatively hyperbolic group then it acts on the augmented space Q^h properly discontinuously by isometries. The augmented space is proper and hyperbolic. Consider the Bowditch boundary ∂Q of Q. Let $\xi \in \partial Q$ and U be a neighbourhood of ξ in $\overline{Q^h}$. Let K be a compact set in Q^h. Consider a base point p in Q. The basis of neighbourhoods of ξ is given by the collection $V(\xi, r)$ of all $\alpha \in \overline{Q^h}$ such that if for some sequences $\{x_n\}, \{y_n\}$ with $\alpha = [(x_n)], \xi = [(y_n)]$ we have $\liminf_{i,j\to\infty}(x_i, y_j)_p \geq r$. There exists a sequence $\{r_n\}$ going to infinity such that $V(\xi, r_n) \subsetneq V(\xi, r_{n+1})$ for all n. For all large n, $V(\xi, r_n) \subset U$ and the distance between complement of U in Q and closure of $V(\xi, r_n)$ in Q goes to infinity as $n \to \infty$. Thus, there exists a natural number N such that if some translate of K intersects $V(\xi, r_N)$ then it must be contained in U. Thus, it amounts to say 'compact sets fade at infinity' in Q^h i.e., for any $\xi \in \partial Q$, for any neighbourhood U of ξ in $\overline{Q^h}$ and for any compact set K in Q^h, there exists a sub neighbourhood V of ξ such that if any Q translate of K intersects V then it must be contained in U.

Now for a complex of (relatively) hyperbolic groups, in each local groups compact set fade at infinity. Using this, Martin [7] proved that compact set in EG fade at infinity. (See Proposition 8.8 of [7]) The same thing hold in our case also where the same proof of Proposition 8.8 goes through.

Let $\{g_n\}$ be an injective sequence. Using compact set fade at infinity, we have for any compact set K in EG^h, up to a subsequence, $g_n K$ converges to ξ. This information is used by Martin [7] to prove the following two lemmas for complex of hyperbolic groups. The exact proof works in our case also.

PROPOSITION 5.2 (Lemma 9.15 of [7]). *Let $\{g_n\}$ be a injective sequence in G. Suppose $\{g_n v\}$ is bounded for some(hence any) vertex v and there do not exist v_0 and v_1 such that $g_n v_0 = v_1$ for infinitely many n. Then there exists $\xi^+, \xi^- \in \partial G$ such that for any compact set K in $\partial G \backslash \{\xi^-\}$, $g_n K$ converges to ξ^+ uniformly.*

PROPOSITION 5.3 (Lemma 9.16 of [7]). *Let $\{g_n\}$ be a injective sequence in G such that $d(g_n v_0, v_0) \to \infty$ for some(hence any) vertex v_0. Then there exists $\xi^+, \xi^- \in \partial G$ such that for any compact set K in $\partial G \backslash \{\xi^-\}$, $g_n K$ converges to ξ^+ uniformly.*

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Using above lemmas, we have the following theorem

THEOREM 5.4 (Corollary 9.17 of [**7**]). *G has convergence action on ∂G*

6. Main theorem. Let $\mathcal{G}(\mathcal{Y})$ be a strictly developable simple complex of groups over a finite M_κ-simplicial complex Y with $\kappa \leq 0$ and satisfying the hypothesis of Theorem 1.1. Let G be the fundamental group of $\mathcal{G}(\mathcal{Y})$.

Local groups G_v are relatively hyperbolic implies G_v has convergence action on the Bowditch boundary ∂G_v. Every point on ∂G_v is either conical limit point or bounded parabolic point for the action of G_v on ∂G_v.

LEMMA 6.1 (By adapting Lemma 9.18 of [**7**]). *The conical limit points of G are precisely the conical limit points of vertex stabilizers and boundary points of X*

Sketch of Proof: Consider a conical limit point α in ∂G_v for the action of G_v on ∂G_v. As the map $\partial G_v \to \partial G$ is embedding, the point α is conical limit point for the action of G_v on ∂G, As G has convergence action on ∂G, α is conical limit point for the action of G on ∂G.

Now let $\eta \in \partial X$. We need to find a sequence $\{g_n\}$ and $\xi^+ \neq \xi^- \in \partial G$ such that $g_n\eta \to \xi^+$, $g_n\xi^{'} \to \xi^-, \forall \xi^{'} \in \partial G \backslash \{\eta\}$. Since action of G on X is co-compact, we can choose a simplex σ and a sequence $\{g_n\}$ such that the sequence of simplices $\{g_n\sigma\}$ intersect with the geodesic $[v_0, \eta)$. Let v be a vertex of σ then $g_n\tilde{x}$ converges to η for all $\tilde{x} \in \partial G_v$. Choose v as the basepoint.

Consider the sequence $\{g_n^{-1}\}$ of group elements. Since $d(v, g_n^{-1}v) \to \infty$, let $g_n^{-1}v$ be converge to $\xi^-(\in \partial G)$. Also by Proposition 7.3 except for possibly one elements, g_n^{-1}-translates of boundary points will converges to ξ^-.

Suppose $\xi^- \in \partial X$. Note that $< g_n^{-1}v, g_n^{-1}g_m v >_v \; = d(g_n v, v) + d(g_m v, g_n v) - d(v, g_m v)$. Now taking projection of $g_m v$ and $g_n v$ onto geodesic $[v, \eta)$, we can check that $< g_n^{-1}v, g_n^{-1}g_m v >_v$ is uniformly bounded for all m and n. Hence, $g_n^{-1}\eta$ cannot converge to ξ^- and we are done.

Let $\xi^- \in \partial_{stab}G$ and $\tilde{x} \in \partial G_v$. Now translating the geodesic $[v, \eta)$ by isometry g_n^{-1}, we see that the vertex v lie uniformly closed to geodesic $[g_n^{-1}v, g_n^{-1}\eta)$. Hence, if $g_n^{-1}\eta$ converges to ξ^-, i.e., $g_n^{-1}\eta$ and $g_n^{-1}\tilde{x}$ converges to same point, then ξ^- must be belongs to ∂G_v. If we can show that there exists $\{h_n\}$ from G_v such that $(h_n g_n^{-1})v$ does not converge to a point of ∂G_v, then as $(g_n h_n^{-1})\tilde{x}$ still converges to η, replacing $\{g_n^{-1}\}$ with $\{h_n g_n^{-1}\}$, we are done.

Let for each n, $\sigma_1^{(n)}$ be the first simplex met by $[v, g_n^{-1}v]$ after leaving v. Then upto multiplying g_n^{-1} by an element from G_v on the left, we can let $[v, g_n^{-1}v]$ meet a single simplex, say, σ_1 after leaving v. Also, let τ_1 be the face of σ_1 which is met by $[v, g_n^{-1}v]$ after leaving σ_1. Similarly, let $\sigma_2^{(n)}$ be the first simplex met by $[v, g_n^{-1}v]$ after leaving τ_1. Since G_{σ_1} is fully quasi-convex in G_{τ_1}, by Lemma 2.18, $\Gamma^h_{\sigma_1}$ will be quasi-convex in $\Gamma^h_{\tau_1}$. Choose any $x_n \in \Gamma_{\sigma_2^{(n)}}$ and let y_n be its projection on $\Gamma^h_{\sigma_1}$, so y_n's will lie in Γ_{σ_1}. Then we can find $\{h_n\} \subset G_{\sigma_1} \subset G_v$ such that $h_n x_n$ project to 1(1 is the identity) for all n, since the action of G_{σ_1} on Γ_{σ_1} is transitive. Hence, $h_n\overline{\Gamma^h}_{\sigma_2^{(n)}}$ do not converge to a point of ∂G_{σ_1}. Also, since $\Gamma^h_{\sigma_1}$ is fixed by all h_n's $h_n\overline{\Gamma^h}_{\sigma_2^{(n)}}$ cannot converge to a point of ∂G_{τ_1}. Hence by convergence property upto subsequence, we can let $\sigma_2^{(n)}$ to be constant, say, σ_2. We replace $\{g_n^{-1}\}$ with $\{h_n g_n^{-1}\}$. Notice if $G_{\sigma_1} \cap G_{\sigma_2}$ is finite then the limit of $\{g_n^{-1}\tilde{x}\}$, i.e., ξ^- cannot be contained in ∂G_{σ_1} and so $\xi^- \notin \partial G_v$ because of the convexity of $D(\xi^-)$. If

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$G_{\sigma_1} \cap G_{\sigma_2}$ is infinite then we again follow the same process. Since action of G on X is acylindrical after finite number of steps intersection of stabilizers will be finite.

The central idea of the following lemma is due to Dahmani [**5**]:

LEMMA 6.2.

(i) *The image of a bounded parabolic point in vertex stabilizer's boundary is a bounded parabolic for G,*

(ii) *The corresponding maximal parabolic subgroup is the image of a maximal parabolic subgroup in the vertex stabilizer.*

Proof.

(i) Let $\tilde{\xi}$ be a bounded parabolic point of boundary of some vertex stabilizer and $\pi(\tilde{\xi}) = \xi$ be its image in ∂G. We will show ξ is bounded parabolic.

Let $P = stab(\xi)$ in G. Then P stabilizes $D(\xi)$, domain of ξ. Let $\xi_{v_i} \in \partial G_{v_i}$ be such that $\pi(\xi_{v_i}) = \xi$, where $\{v_1, \ldots, v_n\}$ is the set of vertices of $D(\xi)$. From construction of $\partial_{stab} G$, for each $i = 1, \ldots, n$, ξ_{v_i} is bounded parabolic point of G_{v_i} and let P_{v_i} be the maximal parabolic subgroup of G_{v_i} stabilizing ξ_{v_i}. From the construction of $D(\xi)$, $\xi_{v_1}, \ldots, \xi_{v_n}$ are the all which are identified to ξ. Thus, P_{v_i} also stabilizes $\{\xi_{v_1}, \ldots, \xi_{v_n}\}$ and hence it stabilizes $D(\xi)$. So, P_{v_i} is a subgroup of P. Let K_i be a compact fundamental domain in $\partial G_{v_i} \setminus \{\xi_{v_i}\}$ for co-compact action of P_{v_i} on $\partial G_{v_i} \setminus \{\xi_{v_i}\}$. Let $N(D(\xi))$ be one open simplicial neighbourhood of $D(\xi)$ in X and $S(N(D(\xi)) \setminus D(\xi))$ be collection of simplices in $N(D(\xi))$ of $D(\xi)$ that is not contained in $D(\xi)$. Let $S_i := \{\sigma \in S(N(D(\xi)) \setminus D(\xi)) : \partial G_\sigma \cap K_i \neq \emptyset\}$.

We claim that $\bigcup_{i=1}^{n} PS_i = S(N(D(\xi)) \setminus D(\xi))$. Let $\sigma \in S_i$ and $p \in P$. As P stabilizes $D(\xi)$, then $p\sigma \in S(N(D(\xi)) \setminus D(\xi))$. Conversely, let $\sigma \in (N(D(\xi)) \setminus D(\xi))$ and $v_i \in D(\xi) \cap \sigma$. Then $\partial G_\sigma \subset \partial G_{v_i} \setminus \{\xi_{v_i}\}$. But, K_i is a fundamental domain for P_{v_i}, hence there exists $p \in P_{v_i} \hookrightarrow P$ such that $p\partial G_\sigma = \partial G_{p\sigma}$ intersect with K_i. So, $p\sigma \in S_i$ and this proves our claim.

$D(\xi)$ is a finite closed convex subspace of the $CAT(0)$ space X and is stabilized by P. Hence, P has a fix point, say $\{x_0\}$, in $D(\xi)$. The topology on ∂G is independent of base point. Let us take x_0 to be the base point for the topology of ∂G. For $x \in \overline{X} \setminus D(\xi)$, there exists $0 < \epsilon_x < 1$ such that $x \in \overline{X} \setminus D_{\epsilon_x}(\xi)$. Let $\sigma_{x,\epsilon_x} \in S(N(D(\xi)) \setminus D(\xi))$ denote the exit simplex for the geodesic $[x_0, x]$.

- For each i, let $T_i := \{x \in \overline{X} \setminus D(\xi) : \sigma_{x,\epsilon_x} \in S_i\}$.
- Let $K_i' := \{\alpha \in \partial G : D(\alpha) \cap T_i \neq \emptyset\}$ and $\overline{K_i'}$ be its closure in ∂G.

For each i, $K_i \cup \overline{K_i'}$ being closed is compact in ∂G. We claim $\xi \notin (K_i \cup \overline{K_i'})$ for all i and $\bigcup_{i=1}^{n}(K_i \cup \overline{K_i'})$ is a compact fundamental domain for action of P on $\partial G \setminus \{\xi\}$.

Claim 1. $\xi \notin (K_i \cup \overline{K_i'})$.

For each i, $K_i \subset \partial G_{v_i} \setminus \{\xi_{v_i}\}$ implies $\xi \notin K_i$ and $D(\xi) \cap T_i = \phi$ implies $\xi \notin K_i'$. Now if possible let $\{\alpha_m\}$ be a sequence in K_i' for some i such that $\alpha_m \to \xi$. By the definition of the topology on ∂G, $D(\alpha_m) \setminus D(\xi) \subset \widetilde{Cone}_{\mathcal{U},\epsilon}(\xi)$ for any ξ-family $\mathcal{U}$ and $0 < \epsilon < 1$. Let $x_m \in D(\alpha_m) \cap T_i$ then by definition of K_i', $\partial G_{\sigma_{x_m,\epsilon_{x_m}}} \cap K_i \neq \phi$ for all m. Also, $\partial G_{\sigma_{x_m,\epsilon_{x_m}}} \subset \partial G_{v_i} \setminus \{\xi_{v_i}\}$, by convergence Property $\partial G_{\sigma_{x_m,\epsilon_{x_m}}} \to \xi_{v_i}$ uniformly. This implies $\xi_{v_i} \in K_i$, which is a contradiction.

Claim 2. $\bigcup_{i=1}^{n} P(K_i \cup \overline{K_i'}) = \partial G \setminus \{\xi\}$.

Let $\alpha(\neq \xi) \in \partial G$. If $\alpha \in \partial X$ then the claim is true since x_0 is fixed by P and $\bigcup_{i=1}^{n} PS_i = S(N(D(\xi))\backslash D(\xi))$. For $\alpha \in \partial_{stab}G$, we will divide the proof of the claim into two cases.

Case 1. $D(\alpha) \cap D(\xi) \neq \emptyset$. Then, $\alpha \in \partial G_{v_i}$ for some $v_i \in D(\alpha) \cap D(\xi)$. $\alpha \neq \xi_{v_i}$, now since K_i is a fundamental domain for the action of P_{v_i} in $\partial G_{v_i} \setminus \{\xi_{v_i}\}$, there exists $x \in K_i$ and $p \in P_{v_i} \hookrightarrow P$ such that $\alpha = px \in PK_i$

Case 2. $D(\alpha) \cap D(\xi) = \emptyset$. Let $x \in D(\alpha)$ and $\sigma_{x,\epsilon_x} \in S(N(D(\xi)) \setminus D(\xi))$ be the exit simplex for the geodesic $[x_0, x]$ in X. As $\bigcup_{i=1}^{n} PS_i = S(N(D(\xi)) \setminus D(\xi))$ and P fixes x_0 there exists $p \in P$ such that $\sigma_{px,\epsilon_x} = p\sigma_{x,\epsilon_x} \in S_i$ for some i. So, $px \in T_i$ and $px \in pD(\alpha) = D(p\alpha)$. So, $p\alpha \in K_i'$ and hence $\alpha \in PK_i'$.

(ii) Let $\tilde{\xi}$ be a bounded parabolic point of boundary of some vertex stabilizer and $\pi(\tilde{\xi}) = \xi$ be its image in ∂G, with $P = stab(\xi)$ in G. Then P stabilizes $D(\xi)$ and it fixes a point $x_0 \in D(\xi)$. Let σ be the simplex in $D(\xi)$ containing x_0 in the interior. From the definition of action of G on X, if some element of G fixes an interior point of a simplex then it fixes the whole simplex pointwise. So, P fixes σ pointwise. Without loss of generality, we can take x_0 to be a vertex v_i of σ. Thus, P fixes ξ_{v_i} and hence $P = P_{v_i}$. $\square$

Proof of Theorem 1.1: From Lemma 5.4, G has a convergence action on compact metrizable space ∂G. The limit points are either conical (by Lemma 6.1) or bounded parabolic (by Lemma 6.2). Hence, by Theorem 2.12 (due to Yaman, [**26**]), G is hyperbolic relative to $\mathcal{P}$, where $\mathcal{P}$ is the collection of the images of P_v in G under the natural embedding $G_v \hookrightarrow G$. This embedding extends to a G_v-equivariant embedding $\partial G_v \hookrightarrow \partial G$. Hence, the limit set for the action of G_v on ∂G is ∂G_v and this action is geometrically finite implies that G_v is quasi-convex in G. For $\xi \in \partial G_v \subset \partial G$, the domain $D(\xi)$ of ξ is finite implies that $\cap_{n\geq 1} g_n \partial G_v$ is empty for any sequence of infinite distinct left cosets $g_n G_v$ of G_v in G. Thus, G_v is fully quasi-convex in G. Local groups are fully quasi-convex in vertex groups implies that local groups are fully quasi-convex in G.

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