Combinatorics, Probability and Computing (2013) 22, 346–350. © Cambridge University Press 2013
doi:10.1017/S0963548313000059

Towards a Weighted Version of the Hajnal–Szemerédi Theorem

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Received 21 August 2011; revised 25 June 2012; first published online 28 February 2013

For a positive integer $r \geqslant 2$, a K_r-factor of a graph is a collection vertex-disjoint copies of K_r which covers all the vertices of the given graph. The celebrated theorem of Hajnal and Szemerédi asserts that every graph on n vertices with minimum degree at least $(1 - \frac{1}{r})n$ contains a K_r-factor. In this note, we propose investigating the relation between minimum degree and existence of perfect K_r-packing for edge-weighted graphs. The main question we study is the following. Suppose that a positive integer $r \geqslant 2$ and a real $t \in [0, 1]$ is given. What is the minimum weighted degree of K_n that guarantees the existence of a K_r-factor such that every factor has total edge weight at least $t\binom{r}{2}$? We provide some lower and upper bounds and make a conjecture on the asymptotics of the threshold as n goes to infinity.

AMS 2010 *Mathematics subject classification*: Primary 05C35
Secondary 05C70, 05C72, 05D40

1. Introduction

Many results in graph theory study the relation between the minimum degree of a given graph and its spanning subgraphs. For example, Dirac's theorem [4] asserts that a graph on n vertices with minimum degree at least $\lceil \frac{n}{2} \rceil$ contains a Hamilton cycle. Hajnal and

† Supported in part by NSF CAREER grant DMS-0745185, UIUC Campus Research Board grants 09072 and 11067, OTKA grant K76099, and TAMOP-4.2.1/B-09/1/KONV-2010-0005 project. The work was partly done while at the Department of Mathematics, University of California, San Diego La Jolla, CA 92093, USA.
‡ Supported in part by a Samsung Scholarship.

Szemerédi [5] proved that every graph on $n \in r\mathbb{Z}$ vertices with minimum degree at least $(1-\frac{1}{r})n$ contains a spanning subgraph consisting of $\frac{n}{r}$ vertex-disjoint copies of K_r (we call such a subgraph a K_r-factor).

In this note we propose investigating this relation in edge-weighted graphs. As a concrete problem, we study the particular case when the spanning subgraph is the graph formed by vertex-disjoint copies of K_r (in other words, we would like to extend the Hajnal–Szemerédi theorem to edge-weighted graphs). Suppose we equip the complete graph K_n with edge weights $w : E(K_n) \to [0,1]$. For a given weighted graph and vertex v we let $\deg_w(v)$ denote the weighted degree of the vertex v. Let $\delta_w(K_n)$ be the minimum weighted degree. The main question can be formulated as follows: How large must $\delta_w(K_n)$ be to guarantee that there exists a K_r-factor such that every K_r in the factor has total edge weight at least $t\binom{r}{2}$ for some given $t \in [0,1]$?

More formally, for $n \in r\mathbb{Z}$ let $\mathcal{W}(r,t,n)$ be the collection of edge weightings on K_n such that every K_r-factor has a clique with weight strictly smaller than $t\binom{r}{2}$. We then define

$$\delta(r,t,n) = \sup_{w\in\mathcal{W}(r,t,n)} \delta_w(K_n) \quad \text{and} \quad \delta(r,t) = \limsup_{n\to\infty} \frac{\delta(r,t,n)}{n}.$$

The main open question that we raise is the following.

Question 1.1. Determine the value of $\delta(r,t)$ for all r and t.

Let $\mathcal{W}^*(r,t,n)$ be the collection of edge weightings of K_n such that every K_r-factor has a clique with weight at most $t\binom{r}{2}$ (instead of strictly smaller than $t\binom{r}{2}$), and define the functions $\delta^*(r,t,n)$ and $\delta^*(r,t)$ accordingly. The compactness of the space $\mathcal{W}^*(r,t,n)$ gives the following result (whose proof we provide in the arXiv version of our paper [1]).

Proposition 1.2. *For all r, t, and n, $\delta(r,t,n) = \delta^*(r,t,n)$. Therefore, $\delta(r,t) = \delta^*(r,t)$.*

The proposition above shows that if an edge-weighting of K_n has minimum degree greater than $\delta(r,t,n)$, then there exists a K_r-factor such that every copy of K_r has weight greater than $t\binom{r}{2}$. Therefore, the Hajnal–Szemerédi theorem is in fact a special case of our problem when $t = \left(\binom{r}{2}-1\right)/\binom{r}{2}$ where we only consider the integer weights $\{0,1\}$. Thus we believe that the following special case is an important and interesting instance of the problem corresponding to the Hajnal–Szemerédi theorem for $r = 3$ (which was first proved by Corrádi and Hajnal [2]).

Question 1.3. What is the value of $\delta(3, \frac{2}{3})$?

2. Lower bound

It is not too difficult to deduce the bound $\delta(r,t) \geqslant (1-1/r)t$ from the graph showing the sharpness of the Hajnal–Szemerédi theorem. Our first proposition provides a better lower bound to this function.

https://doi.org/10.1017/S0963548313000059 Published online by Cambridge University Press

Proposition 2.1. *The following holds for every integer $r \geqslant 2$ and real $t \in (0,1]$:*

$$\delta(r,t) \geqslant \frac{1}{r} + \left(1 - \frac{1}{r}\right)t.$$

Proof. The following construction implies the bound. Let $n \in r\mathbb{Z}$ with $n > r$ and let $k = \frac{n}{r}$. Let A be an arbitrary set of $k-1$ vertices and let B be the remaining $k(r-1)+1$ vertices. Consider the weight function w that assigns weight t to edges whose endpoints are both in B, and weight 1 to all other edges. □

Proposition 2.1 illustrates the fundamental difference between the minimum degree threshold for containing a K_r-factor in graphs and edge-weighted graphs. For example, when $r = 3$ we see that $\delta(3, 2/3) \geqslant 7/9$, while the corresponding function for graphs has value $2/3$ by Hajnal–Szemerédi theorem.

3. Upper bound

Next, we provide some upper bounds on $\delta(r,t)$. For some of the bounds, in order to avoid distraction arising from technical issues, we omit the proofs and refer the reader to the arXiv version of our note [1] for details.

The following observation establishes the correct value of the function $\delta(2,t)$.

Observation 3.1. *For every $t \in (0,1]$ we have $\delta(2,t) = \frac{1+t}{2}$.*

Proof. The lower bound on $\delta(2,t)$ follows from Proposition 2.1, and thus it suffices to establish the upper bound. Let w be a weight function such that $\delta_w(K_n) \geqslant \frac{1+t}{2}n$. Let G be the subgraph of K_n consisting of the edges of weight at least t. In this graph, the degree d of any vertex v satisfies $\deg_w(v) < (n-1-d)t + d \cdot 1$. Then the minimum weighted degree condition implies that $d \geqslant \frac{n}{2}$, and so by Dirac's theorem there is a K_2-factor in G. The corresponding K_2-factor in the weighted graph establishes the bound $\delta(2,t) \leqslant \frac{1+t}{2}$. □

For general r, by using a similar argument to that of Observation 3.1, one can consider hypergraphs and try to use Daykin and Häggkvist's theorem [3] (or a stronger conjecture given in [6]) which relates the minimum degree of a hypergraph with the existence of a perfect matching. To a weighted graph we assign an r-uniform hypergraph: the r-edges of the hypergraph formed by the sufficiently heavy K_r of the weighted graph. However, the bound obtained in this way turns out to be weaker than the following two bounds: see [1] for details.

The first bound below determines the correct value of the function $\delta(r,t)$ for small values of t.

Theorem 3.2. *For every $r \geqslant 3$, there exists a positive real t_r such that for every $t \in (0, t_r)$ we have $\delta(r,t) = \frac{1}{r} + \left(1 - \frac{1}{r}\right)t$.*

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Our proof gives an explicit bound on t_r. As an instance, it shows that $t_3 \geqslant \frac{1}{12}$. The second bound is based on a simple reduction scheme.

Theorem 3.3. *For every $r \geqslant 3$ and $t \in (0,1]$, $\delta(r,t) \leqslant \frac{1}{2} + \frac{t}{2}$.*

Proof. Let $\delta' = \max\{\delta(r-1,t), \frac{1}{2} + \frac{t}{2}\}$. We prove that $\delta(r,t) \leqslant \delta'$. Let ε be an arbitrary fixed positive real, and assume that n_0 is large enough so that $\delta(r-1,t,n) \leqslant (\delta(r-1,t) + \frac{\varepsilon}{2})n$ for all $n \geqslant n_0$. Assume that we are given a weight function for the complete graph on $n \geqslant 2n_0$ vertices such that the minimum weighted degree is at least $(\delta' + \varepsilon)n$. We partition randomly the vertices into a set A of size $\frac{r-1}{r}n$ and a set B of size $\frac{1}{r}n = k$. By the Chernoff–Hoeffding inequalities, for large enough n, there is such a partition which additionally satisfies that for every vertex the weighted degree into A is at least $(\delta' + \frac{\varepsilon}{2})\frac{r-1}{r}n$ and into B is at least $\delta'\frac{1}{r}n$. By the assumption on δ' and n_0, we can find a K_{r-1}-factor $\mathcal{K}_A$ on A with minimum average weight t.

Using $\mathcal{K}_A$ we construct a complete weighted bipartite graph H, where the vertices on one side are associated with cliques in $\mathcal{K}_A$ and the vertices on the other side are associated with vertices in B. For a clique $K \in \mathcal{K}_A$ and a vertex $v \in B$, we assign as weight of the edge (K, v) the average of the weights of the edges between v and the vertices in K. Notice that the minimum weighted degree of H is at least $\delta' k \geqslant (\frac{1}{2} + \frac{t}{2})k$. Let $H(t)$ be the unweighted subgraph of H consisting of edges with weight at least t. A computation similar to that in Observation 3.1 shows that the minimum degree in $H(t)$ is at least $\frac{k}{2}$. Thus, by Hall's theorem, there is a perfect matching $\mathcal{M}$ in $H(t)$.

Now notice that $\mathcal{K}_A$ and $\mathcal{M}$ lift to a K_r-factor of K_n with minimum weight $t\binom{r-1}{2} + t(r-1) = t\binom{r}{2}$. Consequently, $\delta(r,t,n) \leqslant (\delta' + \varepsilon)n$. Since ε can be arbitrarily small, we have $\delta(r,t) \leqslant \delta' = \max\{\delta(r-1,t), \frac{1}{2} + \frac{t}{2}\}$. Thus our conclusion follows from Observation 3.1, which asserts that $\delta(2,t) = \frac{1}{2} + \frac{t}{2}$. □

For the special case related to triangle factors discussed earlier, we have $\frac{7}{9} \leqslant \delta(3, \frac{2}{3}) \leqslant \frac{5}{6}$.

In this article we have proposed the study of the function $\delta(r,t)$. As seen in Section 2, this function shows different behaviour from that of its non-weighted counterpart (which is related to the Hajnal–Szemerédi theorem). Based on the evidence given by Proposition 2.1 and Theorem 3.2, we make the following conjecture.

Conjecture 3.4. *For every $r \geqslant 2$ and $t \in (0,1]$,*

$$\delta(r,t) = \frac{1}{r} + \left(1 - \frac{1}{r}\right)t.$$

We refer the reader to the arXiv version of our paper [1] for further discussion.

Acknowledgements

We thank Wojciech Samotij for his valuable comments. We thank an anonymous referee for helpful suggestions.

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