Glasgow Math. J. **47** (2005) 363–372. © 2005 Glasgow Mathematical Journal Trust.
doi:10.1017/S0017089505002594. Printed in the United Kingdom

LIPSCHITZIAN ELEMENTS OVER p-ADIC FIELDS

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Abstract. Let p be a prime number, $\mathbf{Q}_p$ the field of p-adic numbers, K a finite field extension of $\mathbf{Q}_p$, $\bar{K}$ a fixed algebraic closure of K, and $\mathbf{C}_p$ the completion of $\bar{K}$ with respect to the p-adic valuation. We discuss some properties of Lipschitzian elements, which are elements T of $\mathbf{C}_p$ defined by a certain metric condition that allows one to integrate Lipschitzian functions along the Galois orbit of T over K with respect to the Haar distribution.

2000 *Mathematics Subject Classification.* 11S99.

1. Introduction. Let p be a prime number, $\mathbf{Q}_p$ the field of p-adic numbers, K a fixed finite field extension of $\mathbf{Q}_p$, $\bar{K}$ a fixed algebraic closure of K, and $\mathbf{C}_p$ the completion of $\bar{K}$ with respect to the p-adic valuation. We denote by $|.|$ the p-adic absolute value on $\mathbf{C}_p$, normalized by $|p| = \frac{1}{p}$. We also denote by $O_{\mathbf{C}_p}$ the ring of integers of $\mathbf{C}_p$ and by G_K the group of continuous automorphisms of $\mathbf{C}_p$ over K. In what follows, by a Galois orbit over K we mean a set of the form $C_K(T) := \{\sigma(T) : \sigma \in G_K\}$, with T in $\mathbf{C}_p$. Associated to each such Galois orbit $C_K(T)$ we have a Haar distribution (in the sense of Mazur and Swinnerton-Dyer [**7**]) on $C_K(T)$, call it $\pi_{K,T}$, which is the unique distribution on $C_K(T)$ with values in $\mathbf{Q}_p$, normalized by $\pi_{K,T}(C_K(T)) = 1$, which is G_K-invariant, in the sense that for any ball B in $C_K(T)$ and any $\sigma \in G_K$ one has $\pi_{K,T}(\sigma(B)) = \pi_{K,T}(B)$. For any element α of $\bar{K}$, and any subfield L of $\bar{K}$ containing α, which is a finite extension of K, consider the ratio

$$Tr_K(\alpha) = \frac{Tr_{L/K}(\alpha)}{[L : K]}.$$

This element of K depends on α and K only, but not on the choice of the field L. The significance of $Tr_K(\alpha)$ is that of the arithmetic mean of the conjugates of α over K. A generalization of this notion of trace is obtained if one replaces the above ratio by an appropriate integral over the corresponding Galois orbit with respect to the Haar distribution. Following [**3**] one may define the trace of an element T of $\mathbf{C}_p$ over K by the formula

$$Tr_K(T) = \int_{C_K(T)} x \, d\pi_{K,T}(x),$$

provided that the integral on the right side is well defined. This certainly is the case when $T = \alpha$ is algebraic over K, and in such case this notion of trace coincides with the one defined above. In this case the Haar distribution $\pi_{K,T} = \pi_{K,\alpha}$ coincides with the arithmetic mean of Dirac measures supported at the conjugates of α over K. The trace

$Tr_K(T)$ is also well defined for any element T of $\mathbf{C}_p$ for which the Haar distribution $\pi_{K,T}$ is bounded, that is, for which $\pi_{K,T}$ is a measure. In such case any continuous function $f : C_K(T) \to \mathbf{C}_p$ is integrable with respect to $\pi_{K,T}$. For such an element T, it follows from the general theory of p-adic Cauchy transform (see the work of Barsky [**5**]) that the trace function $F_K(T, z)$ defined by

$$F_K(T, z) = \int_{C_K(T)} \frac{1}{1 - zx} d\pi_{K,T}(x),$$

is well defined and rigid analytic on almost the entire $\mathbf{C}_p$. This is an analytic object that embodies a significant amount of algebraic data. For instance, recall that by Galois theory in $\mathbf{C}_p$, as developed by Tate [**9**], Sen [**8**] and Ax [**4**], closed subgroups of the Galois group G_K are in one-to-one correspondence with the closed subfields of $\mathbf{C}_p$ which contain K. If E is a closed subfield of $\mathbf{C}_p$, containing K, on which the trace map Tr_K is defined and it is continuous, and if T is a generating element of E over K (see [**6**], [**1**], [**2**]), then the trace map Tr_K on the entire field E is determined by its values at $1, T, \dots, T^n, \dots$. All these values are in turn determined by the trace function $F_K(T, z)$, as they appear as coefficients in the Taylor series expansion of $F_K(T, z)$ about $z = 0$,

$$F_K(T, z) = \sum_{n=0}^{\infty} Tr_K(T^n) z^n.$$

A larger class of elements T was defined in [**3**], in terms of a metric condition. For any real number $\epsilon > 0$, the Galois orbit $C_K(T)$ of T can be written as a finite disjoint union of open balls of radius ϵ. We denote by $N_{K,T,\epsilon}$ the number of these balls. We say that T is Lipschitzian over K provided that

$$\lim_{\epsilon \to 0} \frac{\epsilon}{|N_{K,T,\epsilon}|} = 0,$$

where the absolute value in the denominator on the left side is the p-adic absolute value. As explained in [**3**], this condition is very useful in integration theory along Galois orbits in $\mathbf{C}_p$. Any Lipschitzian function $f : C_K(T) \to \mathbf{C}_p$ is integrable with respect to the Haar distribution $\pi_{K,T}$ provided that T is Lipschitzian over K, even if $\pi_{K,T}$ is not bounded. In particular, any element T of $\mathbf{C}_p$ which is Lipschitzian over K has a trace $Tr_K(T)$ and a trace function $F_K(T, z)$ over K. In light of these nice properties of Lipschitzian elements, it would be valuable to have a general theory that studies such classes of elements of $\mathbf{C}_p$. In the present paper we establish some properties of Lipschitzian elements. More specifically, we discuss the following three basic questions. Firstly, if we have a Lipschitzian element T over K, how can we find other elements U in the field generated by T over K, or in its topological closure, which are also Lipschitzian over K? Secondly, with T as above, if we find such a new Lipschitzian element U over K, how can we relate the integrals of Lipschitzian functions along $C_K(U)$ to integrals along $C_K(T)$? And thirdly, with T as above, do these Lipschitzian elements (or families of elements) U have any natural algebraic structure, besides the metric structure which is built into the definition of Lipschitzian elements? In order to state our results, we first introduce some notation. Let T be an element of $\mathbf{C}_p$ and let E be the topological closure of the field $K(T)$ in $\mathbf{C}_p$. For any element U of E one has a canonical map $h_{K,T,U} : C_K(T) \to C_K(U)$, given by

$$h_{K,T,U}(\sigma(T)) = \sigma(U),$$

for any $\sigma \in G_K$. We set

$$A_{K,T} = \big\{U \in O_{\mathbf{C}_p} \cap E : |\sigma_1(U) - \sigma_2(U)| \leq |\sigma_1(T) - \sigma_2(T)|, \\ \text{for any automorphisms } \sigma_1, \sigma_2 \in \mathrm{G_K}\big\}.$$

We also set

$$Lip_{K,T} = \{U \in E : h_{K,T,U} \text{ is a Lipschitzian function}\}.$$

Then we prove the following results.

THEOREM 1. *Let T be an element of $\mathbf{C}_p$ which is Lipschitzian over K. Then any element of $Lip_{K,T}$ is Lipschitzian over K.*

As a consequence of Theorem 1 above, if T is Lipschitzian over K and if U is an element of $Lip_{K,T}$, then one can integrate Lipschitzian functions defined on either one of the two Galois orbits $C_K(T)$ and $C_K(U)$. The next theorem provides a transformation formula for such integrals.

THEOREM 2. *Let T be an element of $\mathbf{C}_p$ which is Lipschitzian over K and let U be an element of $Lip_{K,T}$. Then for any Lipschitzian function $f : C_K(U) \to \mathbf{C}_p$,*

$$\int_{C_K(U)} f\, d\pi_{K,U} = \int_{C_K(T)} f \circ h_{K,T,U}\, d\pi_{K,T}. \tag{1}$$

In particular, from Theorem 2 we see that one can recover the trace of U over K by integrating the canonical map $h_{K,T,U}$ along the Galois orbit $C_K(T)$.

COROLLARY 1. *Let T be an element of $\mathbf{C}_p$ which is Lipschitzian over K. Then, for any element U of $Lip_{K,T}$,*

$$Tr_K(U) = \int_{C_K(T)} h_{K,T,U}\, d\pi_{K,T}. \tag{2}$$

The next theorem is concerned with the algebraic structure of the sets $Lip_{K,T}$ and $A_{K,T}$.

THEOREM 3. *Let T be an element of $\mathbf{C}_p$ which is Lipschitzian over K and let E denote the topological closure of $K(T)$ in $\mathbf{C}_p$. Then*

(i) *$Lip_{K,T}$ is a dense subfield of E which contains $K(T)$, and*
(ii) *$A_{K,T}$ is a closed subring of $\mathbf{C}_p$, with field of fractions $Lip_{K,T}$.*

2. Proofs of the results. Let p be a prime number, and let $\mathbf{Q}_p$, K, $\bar{K}$, $\mathbf{C}_p$, and G_K be as in the introduction. For any element T of $\mathbf{C}_p$ we denote as above the Galois orbit of T over K by $C_K(T)$. Choose such an element T and denote by E the topological closure of $K(T)$ in $\mathbf{C}_p$. Then E is a closed subfield of $\mathbf{C}_p$ and T is a so called generating element of E. It is known (see [**6**], [**1**], [**2**]) that, conversely, any closed subfield of $\mathbf{C}_p$ is of this type, that is, it has a generating element. Let now T be an element of $\mathbf{C}_p$, let

E be the topological closure of $K(T)$ in $\mathbf{C}_p$, and let U be any element of E. One has a canonical map $h_{K,T,U} : C_K(T) \to C_K(U)$, given by

$$h_{K,T,U}(\sigma(T)) = \sigma(U),$$

for any $\sigma \in G_K$. The map $h_{K,T,U}$ is well defined, since U belongs to E. Indeed, if σ_1 and σ_2 are elements of G_K such that $\sigma_1(T) = \sigma_2(T)$, then $\sigma_2^{-1}\sigma_1$ is a continuous automorphism of $\mathbf{C}_p$ over K for which $\sigma_2^{-1}\sigma_1(T) = T$, thus any element of the closed subfield E is fixed by this automorphism. In particular $\sigma_2^{-1}\sigma_1(U) = U$, and hence $\sigma_1(U) = \sigma_2(U)$. This shows that the map $h_{K,T,U}$ is well defined. Next, we define the sets $A_{K,T}$ and $Lip_{K,T}$ as above,

$$A_{K,T} = \{U \in O_{\mathbf{C}_p} \cap E : |\sigma_1(U) - \sigma_2(U)| \leq |\sigma_1(T) - \sigma_2(T)|, \sigma_1, \sigma_2 \in G_K\},$$

and

$$Lip_{K,T} = \{U \in E : h_{K,T,U} \text{ is Lipschitzian}\}.$$

We will use the following terminology. We will say that a subset B of a Galois orbit $C_K(T)$ is an open ball of radius δ in $C_K(T)$, provided that B is a subset of the form

$$B = B(y, \delta) = \{z \in C_K(T) : |z - y| < \delta\},$$

for some y in $C_K(T)$. Let us remark that the same subset B of $C_K(T)$ may be an open ball of radius δ_1 of $C_K(T)$, and at the same time it may be an open ball of radius δ_2 of $C_K(T)$, for some strictly positive real numbers $\delta_1 \neq \delta_2$. In other words, the above definition of an open ball B of radius δ does not force δ to coincide with the diameter of the set B, defined as usual by

$$\text{diameter (B)} = \sup\{|\text{x} - \text{y}| : \text{x}, \text{y} \in \text{B}\}.$$

For any element T of $\mathbf{C}_p$ and any element U of $Lip_{K,T}$ there exist real numbers $M > 0$ such that

$$|\sigma_1(U) - \sigma_2(U)| \leq M|\sigma_1(T) - \sigma_2(T)| \tag{3}$$

for any automorphisms $\sigma_1, \sigma_2 \in G_K$. Let us denote by $M_{K,T,U}$ the infimum of the set of those real numbers M for which the inequality (3) holds for any $\sigma_1, \sigma_2 \in G_K$. Evidently we will then have

$$|\sigma_1(U) - \sigma_2(U)| \leq M_{K,T,U}|\sigma_1(T) - \sigma_2(T)|, \tag{4}$$

for any $\sigma_1, \sigma_2 \in G_K$.

We first prove the following result.

LEMMA 1. *Let T be an arbitrary element of $\mathbf{C}_p$, and let U be an element of $Lip_{K,T}$. Then, for any $\epsilon > 0$ there exists a positive integer number $N_{K,T,U,\epsilon}$ with the following property. For any open ball B of radius ϵ in $C_K(U)$, $h_{K,T,U}^{-1}(B)$ is the disjoint union of exactly $N_{K,T,U,\epsilon}$ open balls of radius $\frac{\epsilon}{M_{K,T,U}}$ in $C_K(T)$.*

Proof. Let T and U be as in the statement of the lemma. Fix a real number $\epsilon > 0$ and choose an open ball B of radius ϵ in $C_K(U)$. Let us fix an element y of $h_{K,T,U}^{-1}(B)$. For

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any element z of $C_K(T)$ satisfying the inequality $|z-y| < \frac{\epsilon}{M_{K,T,U}}$, relation (4) applied for some automorphisms σ_1 and σ_2 for which $\sigma_1(T) = y$ and $\sigma_2(T) = z$, shows that

$$|h_{K,T,U}(z) - h_{K,T,U}(y)| \leq M_{K,T,U}|z-y| < \epsilon.$$

Since $h_{K,T,U}(y)$ belongs to B, which is an open ball of radius ϵ in $C_K(U)$, it follows that $h_{K,T,U}(z)$ belongs to B, and hence z belongs to $h^{-1}_{K,T,U}(B)$. We conclude that the open ball of radius $\frac{\epsilon}{M_{K,T,U}}$ in $C_K(T)$ which contains an arbitrary element y of $h^{-1}_{K,T,U}(B)$, is entirely contained in $h^{-1}_{K,T,U}(B)$. This means that $h^{-1}_{K,T,U}(B)$ is a union of balls of radius $\frac{\epsilon}{M_{K,T,U}}$ in $C_K(T)$.

Next, $h_{K,T,U}$ being continuous, and B being closed in $C_K(U)$, it follows that $h^{-1}_{K,T,U}(B)$ is a closed subset of $C_K(T)$, and hence it is compact. Therefore $h^{-1}_{K,T,U}(B)$ is compact covered by open balls of radius $\frac{\epsilon}{M_{K,T,U}}$. This shows that one has a finite covering with such balls. Let us denote the number of open balls of radius $\frac{\epsilon}{M_{K,T,U}}$ in $C_K(T)$ whose disjoint union coincides with $h^{-1}_{K,T,U}(B)$, by $N_{K,T,U,\epsilon}(B)$.

In order to finish the proof of the lemma, it remains to show that for any two open balls B_1 and B_2 of radius ϵ in $C_K(U)$, one has $N_{K,T,U,\epsilon}(B_1) = N_{K,T,U,\epsilon}(B_2)$. Let B_1, B_2 be such balls. There exists an automorphism $\sigma_0 \in G_K$ such that $\sigma_0(B_1) = B_2$. Let us choose an arbitrary element z of $h^{-1}_{K,T,U}(B_1)$. There exists an automorphism $\sigma \in G_K$ such that $\sigma(T) = z$. Thus $h_{K,T,U}(z) = \sigma(U)$. On the other hand $\sigma_0(z) = \sigma_0\sigma(T)$, and therefore

$$h_{K,T,U}(\sigma_0(z)) = h_{K,T,U}(\sigma_0\sigma(T)) = \sigma_0\sigma(U) = \sigma_0(h_{K,T,U}(z)).$$

Since $h_{K,T,U}(z)$ belongs to B_1, it follows that

$$h_{K,T,U}(\sigma_0(z)) = \sigma_0(h_{K,T,U}(z)) \in \sigma_0(B_1) = B_2,$$

which means that $\sigma_0(z)$ belongs to $h^{-1}_{K,T,U}(B_2)$. Hence we have the inclusion

$$\sigma_0\big(h^{-1}_{K,T,U}(B_1)\big) \subseteq h^{-1}_{K,T,U}(B_2).$$

By a similar reasoning, the equality $B_1 = \sigma_0^{-1}(B_2)$ implies the inclusion

$$\sigma_0^{-1}\big(h^{-1}_{K,T,U}(B_2)\big) \subseteq h^{-1}_{K,T,U}(B_1),$$

which is equivalent to the inclusion

$$h^{-1}_{K,T,U}(B_2) \subseteq \sigma_0\big(h^{-1}_{K,T,U}(B_1)\big).$$

By the above inclusions we obtain the equality

$$h^{-1}_{K,T,U}(B_2) = \sigma_0\big(h^{-1}_{K,T,U}(B_1)\big).$$

Now we know that $C_K(T)$ can be written as a disjoint union of $N_{K,T,\delta}$ open balls of radius δ, where we let $\delta = \frac{\epsilon}{M_{K,T,U}}$, and that σ_0 produces a permutation of these balls. Hence the same number of balls occur in $h^{-1}_{K,T,U}(B_1)$ as in $\sigma_0(h^{-1}_{K,T,U}(B_1))$, which coincides with $h^{-1}_{K,T,U}(B_2)$. This completes the proof of the lemma.

https://doi.org/10.1017/S0017089505002594 Published online by Cambridge University Press

As a consequence of the above lemma let us remark that each of the $N_{K,U,\epsilon}$ open balls of radius ϵ in $C_K(U)$ produces $N_{K,T,U,\epsilon}$ open balls of radius $\frac{\epsilon}{M_{K,T,U}}$ in $C_K(T)$. Therefore $C_K(T)$ is a disjoint union of $N_{K,U,\epsilon}N_{K,T,U,\epsilon}$ open balls of radius $\frac{\epsilon}{M_{K,T,U}}$. We thus obtain the following corollary.

COROLLARY 2. *For any element T of C_p, any element U of $Lip_{K,T}$, and any $\epsilon > 0$,*

$$N_{K,U,\epsilon}N_{K,T,U,\epsilon} = N_{K,T,\frac{\epsilon}{M_{K,T,U}}}. \tag{5}$$

COROLLARY 3. *For any element T of C_p, any element U of $Lip_{K,T}$, and any open ball B in $C_K(U)$,*

$$\pi_{K,U}(B) = \pi_{K,T}\big(h^{-1}_{K,T,U}(B)\big). \tag{6}$$

Proof. Let T and U be as in the statement of the corollary. If B is an open ball of radius $\epsilon > 0$ in $C_K(U)$, then by the definition of the Haar distribution $\pi_{K,U}$ we have

$$\pi_{K,U}(B) = \frac{1}{N_{K,U,\epsilon}}. \tag{7}$$

On the other hand we know that $h^{-1}_{K,T,U}(B)$ is a disjoint union of exactly $N_{K,T,U,\epsilon}$ open balls of radius $\frac{\epsilon}{M_{K,T,U}}$ in $C_K(T)$, and the Haar measure of each of these balls equals $1/N_{K,T,\frac{\epsilon}{M_{K,T,U}}}$. This implies that

$$\pi_{K,T}\big(h^{-1}_{K,T,U}(B)\big) = \frac{N_{K,T,U,\epsilon}}{N_{K,T,\frac{\epsilon}{M_{K,T,U}}}}. \tag{8}$$

The equality (6) now follows from (7), (8) and (5).

We are now ready to prove the results stated in the introduction.

Proof of Theorem 1. Let T be a Lipschitzian element of $\mathbf{C}_p$, and let U be an element of $Lip_{K,T}$. By the equality (5) we deduce that for any $\epsilon > 0$ we have

$$\frac{\epsilon}{\left|N_{K,U,\epsilon}\right|} = \frac{\epsilon\left|N_{K,T,U,\epsilon}\right|}{\left|N_{K,T,\frac{\epsilon}{M_{K,T,U}}}\right|} \leq \frac{\epsilon}{\left|N_{K,T,\frac{\epsilon}{M_{K,T,U}}}\right|}, \tag{9}$$

where the absolute value in each of these expressions is the normalized p-adic absolute value. Since T is Lipschitzian, we know that

$$\lim_{\epsilon\to 0} \frac{\frac{\epsilon}{M_{K,T,U}}}{\left|N_{K,T,\frac{\epsilon}{M_{K,T,U}}}\right|} = 0. \tag{10}$$

By (9) and (10) it follows that

$$\lim_{\epsilon\to 0} \frac{\epsilon}{\left|N_{K,U,\epsilon}\right|} = 0, \tag{11}$$

and hence U is Lipschitzian. This completes the proof of Theorem 1.

Proof of Theorem 2. Let T be a Lipschitzian element of $\mathbf{C}_p$, let U be an element of $Lip_{K,T}$, and let $f : C_K(U) \to \mathbf{C}_p$, f Lipschitzian. By Theorem 1 we know that U is Lipschitzian, and therefore the integral on the left hand side of equality (1) is well defined. Moreover, f and $h_{K,T,U}$ being Lipschitzian functions, their composition $f \circ h_{K,T,U}$ will be Lipschitzian. Thus the integral on the right hand side of (1) is also well defined, since T is Lipschitzian. Now that we know that the two integrals are well defined, in order to show that they are equal it would be enough to find sequences of Riemann sums associated to the two integrals, which are close enough to each other. Therefore what we do is the following. We choose an $\epsilon > 0$, and we break the Galois orbit $C_K(U)$ as a disjoint union of open balls of radius ϵ, denote them by B_j, $1 \leq j \leq N_{K,U,\epsilon}$. Inside each of these balls B_j we choose a point x_j, and consider the corresponding Riemann sum

$$S = \sum_{1\leq j\leq N_{K,U,\epsilon}} f(x_j)\pi_{K,U}(B_j) = \frac{1}{N_{K,U,\epsilon}} \sum_{1\leq j\leq N_{K,U,\epsilon}} f(x_j). \tag{12}$$

We know that S will be as close to the integral $\int_{C_K(U)} f \, d\pi_{K,U}$ as we please, for ϵ small enough. In order to finish the proof of the theorem it remains to show that for ϵ sufficiently small, S is as close as we please to the integral $\int_{C_K(T)} f \circ h_{K,T,U} \, d\pi_{K,T}$. Next, in order to prove this, we break the Galois orbit $C_K(T)$ as a disjoint union of open balls of radius $\frac{\epsilon}{M_{K,T,U}}$. We know by Lemma 1 that these balls can be arranged in sets consisting of $N_{K,T,U,\epsilon}$ balls each, to form the open subsets $h^{-1}_{K,T,U}(B_j)$. Let us denote these $N_{K,T,\frac{\epsilon}{M_{K,T,U}}}$ open balls of radius $\frac{\epsilon}{M_{K,T,U}}$ by $B_{j,m}$, $1 \leq j \leq N_{K,U,\epsilon}$, $1 \leq m \leq N_{K,T,U,\epsilon}$, so that we have

$$h^{-1}_{K,T,U}(B_j) = \cup_{1\leq m\leq N_{K,T,U,\epsilon}} B_{j,m}, \tag{13}$$

for $j = 1, 2, \ldots, N_{K,U,\epsilon}$. Next, inside each ball $B_{j,m}$ we choose an element $y_{j,m}$, and form the corresponding Riemann sum

$$\begin{aligned} S^* &= \sum_{\substack{1\leq j\leq N_{K,U,\epsilon}\\ 1\leq m\leq N_{K,T,U,\epsilon}}} (f \circ h_{K,T,U})(y_{j,m})\pi_{K,T}(B_{j,m}) \\ &= \frac{1}{N_{K,T,\frac{\epsilon}{M_{K,T,U}}}} \sum_{\substack{1\leq j\leq N_{K,U,\epsilon}\\ 1\leq m\leq N_{K,T,U,\epsilon}}} f(h_{K,T,U}(y_{j,m})). \end{aligned} \tag{14}$$

We know that for ϵ small enough, S^* is as close as we please to the integral from the right hand side of equality (1). It remains to show that for ϵ small, $|S - S^*|$ is small. By the relations (12), (14) and (5) we see that $S - S^*$ can be written in the form

$$\begin{aligned} &\frac{1}{N_{K,T,\frac{\epsilon}{M_{K,T,U}}}} \left(N_{K,T,U,\epsilon} \sum_{1\leq j\leq N_{K,U,\epsilon}} f(x_j) - \sum_{\substack{1\leq j\leq N_{K,U,\epsilon}\\ 1\leq m\leq N_{K,T,U,\epsilon}}} f(h_{K,T,U}(y_{j,m})) \right) \\ &= \frac{1}{N_{K,T,\frac{\epsilon}{M_{K,T,U}}}} \sum_{1\leq j\leq N_{K,U,\epsilon}} \sum_{1\leq m\leq N_{K,T,U,\epsilon}} (f(x_j) - f(h_{K,T,U}(y_{j,m}))). \end{aligned} \tag{15}$$

It follows that

$$|S - S^*| \leq \frac{1}{\left|N_{K,T,\frac{\epsilon}{M_{K,T,U}}}\right|} \max_{\substack{1\leq j\leq N_{K,U,\epsilon}\\ 1\leq m\leq N_{K,T,U,\epsilon}}} |f(x_j) - f(h_{K,T,U}(y_{j,m}))|. \tag{16}$$

Now f being Lipschitzian, there exists a real number $M_f > 0$ with the property that

$$|f(x) - f(x')| \leq M_f |x - x'|, \tag{17}$$

for any elements x and x' of $C_K(U)$. By the inequalities (16) and (17) we deduce that

$$|S - S^*| \leq \frac{M_f}{\left|N_{K,T,\frac{\epsilon}{M_{K,T,U}}}\right|} \max_{\substack{1\leq j\leq N_{K,U,\epsilon}\\ 1\leq m\leq N_{K,T,U,\epsilon}}} |x_j - h_{K,T,U}(y_{j,m})|. \tag{18}$$

For any j and any m with $1 \leq j \leq N_{K,U,\epsilon}$ and $1 \leq m \leq N_{K,T,U,\epsilon}$, x_j belongs to B_j and $h_{K,T,U}(y_{j,m})$ belongs to $h_{K,T,U}(B_{j,m})$, which is contained in B_j, hence

$$|x_j - h_{K,T,U}(y_{j,m})| < \epsilon. \tag{19}$$

By (18) and (19) we obtain

$$|S - S^*| < \frac{\epsilon M_f}{\left|N_{K,T,\frac{\epsilon}{M_{K,T,U}}}\right|}. \tag{20}$$

This finishes the proof of the theorem, since in (20) the right hand side tends to zero as $\epsilon \to 0$, T being Lipschitzian.

Proof of Theorem 3. Let T be a Lipschitzian element of $\mathbf{C}_p$.
We first prove (i). Let U and V be two elements of $Lip_{K,T}$. We need to show that $U - V$ and UV belong to $Lip_{K,T}$, and, if V is nonzero, then $\frac{U}{V}$ also belongs to $Lip_{K,T}$. We start by observing that for any automorphisms $\sigma_1, \sigma_2 \in G_K$ we have

$$\begin{aligned} |h_{K,T,U-V}(\sigma_1(T)) - h_{K,T,U-V}(\sigma_2(T))| &= |\sigma_1(U - V) - \sigma_2(U - V)| \\ &= |\sigma_1(U) - \sigma_2(U) - \sigma_1(V) + \sigma_2(V)| \\ &\leq \max\{|\sigma_1(U) - \sigma_2(U)|\,, |\sigma_1(V) - \sigma_2(V)|\} \\ &\leq \max\{M_{K,T,U}, M_{K,T,V}\} \cdot |\sigma_1(T) - \sigma_2(T)|\,, \end{aligned}$$

by (4). This shows that $U - V$ belongs to $Lip_{K,T}$, and that moreover we have

$$M_{K,T,U-V} \leq \max\{M_{K,T,U}, M_{K,T,V}\}. \tag{21}$$

We further have

$$\begin{aligned} |h_{K,T,UV}(\sigma_1(T)) - h_{K,T,UV}(\sigma_2(T))| &= |\sigma_1(UV) - \sigma_2(UV)| \\ &= |\sigma_1(U)\sigma_1(V) - \sigma_1(U)\sigma_2(V) + \sigma_1(U)\sigma_2(V) - \sigma_2(U)\sigma_2(V)| \\ &\leq \max\{|\sigma_1(U)| \cdot |\sigma_1(V) - \sigma_2(V)|\,, |\sigma_2(V)| \cdot |\sigma_1(V) - \sigma_2(V)|\} \\ &\leq \max\{|U| \cdot M_{K,T,V}, |V| \cdot M_{K,T,U}\} \cdot |\sigma_1(T) - \sigma_2(T)|\,. \end{aligned}$$

Hence UV belongs to $Lip_{K,T}$, and moreover

$$M_{K,T,UV} \leq \max\{|V| M_{K,T,U}, |U| M_{K,T,V}\}. \tag{22}$$

Let now $V \neq 0$. It will be sufficient to show that $\frac{1}{V}$ belongs to $Lip_{K,T}$, because then $\frac{U}{V}$ will also be an element of $Lip_{K,T}$. We have

$$\begin{aligned}\left|h_{K,T,\frac{1}{V}}(\sigma_1(T)) - h_{K,T,\frac{1}{V}}(\sigma_2(T))\right| &= \left|\sigma_1\left(\frac{1}{V}\right) - \sigma_2\left(\frac{1}{V}\right)\right| \\ &= \left|\frac{\sigma_2(V) - \sigma_1(V)}{\sigma_1(V)\sigma_2(V)}\right| \\ &= \frac{1}{|V|^2}\left|\sigma_2(V) - \sigma_1(V)\right| \\ &\leq \frac{M_{K,T,V}}{|V|^2}\left|\sigma_2(T) - \sigma_1(T)\right|.\end{aligned}$$

Therefore $\frac{1}{V}$ belongs to $Lip_{K,T}$, and

$$M_{K,T,\frac{1}{V}} \leq \frac{M_{K,T,V}}{|V|^2}. \tag{23}$$

Note that if we apply the inequality (23) with V replaced by $\frac{1}{V}$, then we obtain

$$M_{K,T,V} \leq |V|^2 M_{K,T,\frac{1}{V}}. \tag{24}$$

By combining the inequalities (23) and (24), we find that

$$M_{K,T,\frac{1}{V}} = \frac{M_{K,T,V}}{|V|^2}. \tag{25}$$

We conclude that $Lip_{K,T}$ is a field. Evidently T belongs to $Lip_{K,T}$, and

$$M_{K,T,T} = 1. \tag{26}$$

It is also clear that we have the inclusion $K \subseteq Lip_{K,T}$, and that

$$M_{K,T,U} = 0, \tag{27}$$

for any element U of K. Therefore $K(T) \subseteq Lip_{K,T}$, which completes the proof of part (i).

We now proceed to prove part (ii). We clearly have the inclusion $A_{K,T} \subseteq Lip_{K,T}$. It is easy to see that

$$A_{K,T} = \left\{U \in Lip_{K,T} \cap O_{\mathbf{C}_p} : M_{K,T,U} \leq 1\right\}. \tag{28}$$

By combining (21), (22), (27) and (28) it follows that $A_{K,T}$ is a subring of $Lip_{K,T}$ which contains O_K. Let us also observe that for any element U of $Lip_{K,T}$ and for any integer number m we have

$$M_{K,T,p^m U} = \left|p^m\right| M_{K,T,U}. \tag{29}$$

In fact, for any element a of K and any element U of $Lip_{K,T}$ we have

$$M_{K,T,aU} = |a|\, M_{K,T,U}. \tag{30}$$

https://doi.org/10.1017/S0017089505002594 Published online by Cambridge University Press

Therefore, for a fixed element U of $Lip_{K,T}$ we indeed have $p^m U \in A_{K,T}$ for all large enough natural numbers m. This shows that the field of fractions of $A_{K,T}$ coincides with $Lip_{K,T}$. It remains to show that $A_{K,T}$ is topologically complete. Let U be an element of $\mathbf{C}_p$ and let $(U_m)_{m\in\mathbf{N}}$ be a sequence of elements from $A_{K,T}$ with the property that

$$U = \lim_{m\to\infty} U_m. \tag{31}$$

This already implies that U belongs to $O_{\mathbf{C}_p}$. It remains to show that U belongs to $Lip_{K,T}$, and that $M_{K,T,U} \leq 1$. Let E denote the topological closure of $K(T)$ in $\mathbf{C}_p$. Let us note in the first place that since U_m belongs to E for any m, we also have that U belongs to E. Thus the map $h_{K,T,U}$ is well defined. Next, let us fix arbitrary automorphisms $\sigma_1, \sigma_2 \in G_K$. Then, choose an arbitrary $\epsilon > 0$. For all m large enough we have $|U_m - U| \leq \epsilon$. This in turn implies that $|\sigma_1(U_m) - \sigma_1(U)| \leq \epsilon$ and $|\sigma_2(U_m) - \sigma_2(U)| \leq \epsilon$. It follows that

$$\begin{aligned} &|h_{K,T,U}(\sigma_1(T)) - h_{K,T,U}(\sigma_2(T))| = |\sigma_1(U) - \sigma_2(U)| \\ &\leq \max\{|\sigma_1(U) - \sigma_1(U_m)|\,,\ |\sigma_1(U_m) - \sigma_2(U_m)|\,,\ |\sigma_2(U_m) - \sigma_2(U)|\} \\ &\leq \max\{\epsilon,\ M_{K,T,U_m} \cdot |\sigma_1(T) - \sigma_2(T)|\} \leq \max\{\epsilon,\ |\sigma_1(T) - \sigma_2(T)|\}, \end{aligned}$$

where we have used the fact that $M_{K,T,U_m} \leq 1$ for all m. Since the above inequality holds for any $\epsilon > 0$, we obtain

$$|h_{K,T,U}(\sigma_1(T)) - h_{K,T,U}(\sigma_2(T))| \leq |\sigma_1(T) - \sigma_2(T)|\,.$$

Since this last inequality holds for any automorphisms $\sigma_1, \sigma_2 \in G_K$, it follows that $M_{K,T,U} \leq 1$. In conclusion, U belongs to $A_{K,T}$, and this completes the proof of the theorem.

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