

ARTICLE

Positive association of the oriented percolation cluster in randomly oriented graphs

François Bienvenu

Center for Interdisciplinary Research in Biology (CIRB), CNRS UMR 7241, Collège de France, Paris, France
Laboratoire de Probabilités, Statistique et Modélisation (LPSM), CNRS UMR 8001, Sorbonne Université, Paris, France
(Email: francois.bienvenu@normalesup.org)

(Received 28 November 2017; revised 6 May 2019; first published online 8 July 2019)

Abstract

Consider any fixed graph whose edges have been randomly and independently oriented, and write $\{S \rightsquigarrow i\}$ to indicate that there is an oriented path going from a vertex $s \in S$ to vertex i . Narayanan (2016) proved that for any set S and any two vertices i and j , $\{S \rightsquigarrow i\}$ and $\{S \rightsquigarrow j\}$ are positively correlated. His proof relies on the Ahlswede–Daykin inequality, a rather advanced tool of probabilistic combinatorics.

In this short note I give an elementary proof of the following, stronger result: writing V for the vertex set of the graph, for any source set S , the events $\{S \rightsquigarrow i\}$, $i \in V$, are positively associated, meaning that the expectation of the product of increasing functionals of the family $\{S \rightsquigarrow i\}$ for $i \in V$ is greater than the product of their expectations.

2010 MSC Codes: 60C05, 60K35, 82B43, 05C80

1. Introduction

Oriented percolation is the study of connectivity in a random oriented graph. In most settings, we start from a graph with a fixed orientation and then keep each edge with a given probability. Classical models of this type include the north-east lattice [3] and the hypercube [5].

Another broad and natural class of random oriented graphs is obtained by starting from a fixed graph and then orienting each edge, independent of the orientations of other edges. Note that, in the general case, the orientations of the edges need not be unbiased: some edges can be allowed to have a higher probability of pointing towards one of their ends than towards the other. Percolation on such *randomly oriented graphs* has been studied, for example in [7], and more recently in [8], which motivated the present work.

In [8], Narayanan showed that if the edges of any fixed graph are randomly and independently oriented, then writing $\{S \rightsquigarrow i\}$ to indicate that there is an oriented path going from a vertex $s \in S$ to vertex i , we have

$$\mathbb{P}(S \rightsquigarrow i, S \rightsquigarrow j) \geq \mathbb{P}(S \rightsquigarrow i) \mathbb{P}(S \rightsquigarrow j).$$

The aim of this note is to strengthen and simplify the proof of this result. More specifically, let V be the vertex set of the graph. We prove that the events $\{S \rightsquigarrow i\}$, $i \in V$, are positively associated, without resorting to advanced results such as the Ahlswede–Daykin inequality [1].

© Cambridge University Press 2019

1.1 Positive association and related notions

There are many ways to formalize the idea of a positive dependence between the random variables of a family $\mathbf{X} = (X_i)_{i \in I}$. A straightforward, weak one is to ask that these variables be pairwise positively correlated, that is,

$$\text{for all } i, j \in I, \quad \mathbb{E}(X_i X_j) \geq \mathbb{E}(X_i) \mathbb{E}(X_j).$$

A much stronger condition, due to [4], is known as positive association. In the following definition and throughout the rest of this note, we use bold letters to denote vectors, as in $\mathbf{X} = (X_i)_{i \in I}$, and we write $\mathbf{X} \leq \mathbf{X}'$ to say that $X_i \leq X'_i$ for all i . A function $f: \mathbb{R}^I \rightarrow \mathbb{R}$ is said to be *increasing* when $\mathbf{X} \leq \mathbf{X}' \implies f(\mathbf{X}) \leq f(\mathbf{X}')$.

Definition. The random vector $\mathbf{X} = (X_i)_{i \in I}$ is *positively associated* if, for all increasing functions f and g ,

$$\mathbb{E}(f(\mathbf{X})g(\mathbf{X})) \geq \mathbb{E}(f(\mathbf{X})) \mathbb{E}(g(\mathbf{X}))$$

whenever these expectations exist.

Without further mention, we only consider test functions f and g for which $\mathbb{E}(f(\mathbf{X}))$, $\mathbb{E}(g(\mathbf{X}))$ and $\mathbb{E}(f(\mathbf{X})g(\mathbf{X}))$ exist.

We say that the events $A_i, i \in I$, are positively associated when the corresponding vector of indicator variables $(\mathbb{1}_{A_i})_{i \in I}$ is positively associated. Similarly, a random subset R of the fixed set I can be seen as the vector

$$\mathbf{R} = (\mathbb{1}_{\{i \in R\}})_{i \in I},$$

so that R is said to be positively associated when the events $\{i \in R\}, i \in I$, are positively associated. This is equivalent to saying that for any increasing functions f and g from the power set of I to \mathbb{R} ,

$$\mathbb{E}(f(R)g(R)) \geq \mathbb{E}(f(R)) \mathbb{E}(g(R)),$$

where f being increasing is understood to mean that $r' \subset r \implies f(r') \leq f(r)$.

Positive association is famous for the FKG theorem, which states that it is implied by a lattice condition that can sometimes be very easy to check [6]. Another reason why it is so useful is that it implies weaker positive dependence notions that have to be checked in applications. One example of this is the existence of increasing couplings and the corresponding notion of *positive relation* used in the Stein–Chen method; see e.g. [2] and [9].

1.2 Notation

Let us fix some notation to be used throughout the rest of this note.

We study the simple graph $G = (V, E)$. Unless explicitly specified otherwise, V is assumed to be finite and we let $|V|$ denote its cardinality. The edges of G have a random orientation that is independent of the orientations of other edges and we write $\{i \rightarrow j\}$ to indicate that the edge $\{ij\}$ is oriented towards j . Formally, we are thus given a family of events $\{i \rightarrow j\}, \{ij\} \in E$, such that $\{i \rightarrow j\} = \{j \rightarrow i\}^c$ and for all $\{ij\}, \{i \rightarrow j\} \perp (\{k \rightarrow l\}, \{kl\} \neq \{ij\})$.

Finally, for every pair of vertices i and j , we write $\{i \rightsquigarrow j\}$ for the event that there exists an oriented path going from i to j . Similarly, for every source set S we let $\{S \rightsquigarrow i\} = \bigcup_{j \in S} \{j \rightsquigarrow i\}$ be the event that there is an oriented path from S to i , and for every target set T we let $\{i \rightsquigarrow T\} = \bigcup_{j \in T} \{i \rightsquigarrow j\}$ be the event that there is an oriented path from i to T . If there is an ambiguity regarding which graph is considered for these events, we will specify it with the notation $\{i \overset{G}{\rightsquigarrow} j\}$.

2. Positive association of the percolation cluster

2.1 Preliminary lemma

Lemma 2.1. *Let Γ be a finite set and let R be a positively associated random subset of Γ . Let X_i^r , $r \subset \Gamma$ and $i \in V$, be a family of events on the same probability space as R with the property that:*

- (i) $r' \subset r \implies X_i^{r'} \subset X_i^r$, for all $i \in V$,
- (ii) for all $r \subset \Gamma$, $(X_i^r)_{i \in V}$ is positively associated and independent of R .

For all $i \in V$, define X_i^R by

$$X_i^R := \bigcup_{r \subset \Gamma} \{R = r\} \cap X_i^r.$$

Then the events X_i^R , $i \in V$, are positively associated.

Proof. Let f and g be two increasing functions. We have

$$\begin{aligned} \mathbb{E}(f(\mathbf{X}^R)g(\mathbf{X}^R)) &= \sum_{r \subset \Gamma} \mathbb{E}(f(\mathbf{X}^r)g(\mathbf{X}^r)\mathbb{1}_{\{R=r\}}) \\ &= \sum_{r \subset \Gamma} \mathbb{E}(f(\mathbf{X}^r)g(\mathbf{X}^r)) \mathbb{P}(R = r) \\ &\geq \sum_{r \subset \Gamma} \mathbb{E}(f(\mathbf{X}^r)) \mathbb{E}(g(\mathbf{X}^r)) \mathbb{P}(R = r), \end{aligned}$$

because $\mathbf{X}^r \perp\!\!\!\perp R$ and \mathbf{X}^r is positively associated. Now, let $u: r \mapsto \mathbb{E}(f(\mathbf{X}^r))$ and $v: r \mapsto \mathbb{E}(g(\mathbf{X}^r))$, so that the last sum is $\mathbb{E}(u(R)v(R))$. Note that u and v are increasing, since f and g are increasing, and, by hypothesis, $r' \subset r \implies \mathbf{X}^{r'} \leq \mathbf{X}^r$. Therefore, by the positive association of R ,

$$\mathbb{E}(u(R)v(R)) \geq \mathbb{E}(u(R)) \mathbb{E}(v(R)).$$

Finally, again using the independence of \mathbf{X}^r and R , we have $\mathbb{E}(u(R)) = \mathbb{E}(f(\mathbf{X}^R))$ and $\mathbb{E}(v(R)) = \mathbb{E}(g(\mathbf{X}^R))$, which concludes the proof. □

2.2 Main result

Theorem 2.2. *Let G be a finite graph with vertex set V , whose edges have been randomly and independently oriented. Then, for any source set S , the events $\{S \rightsquigarrow i\}$, $i \in V$, are positively associated, that is, for all increasing functions f and g and writing $\mathbf{X} = (\mathbb{1}_{\{S \rightsquigarrow i\}})_{i \in V}$,*

$$\mathbb{E}(f(\mathbf{X})g(\mathbf{X})) \geq \mathbb{E}(f(\mathbf{X})) \mathbb{E}(g(\mathbf{X})).$$

Proof. Our proof uses the same induction on the number of vertices as Narayanan’s. The difference is that we use Lemma 2.1 rather than the Ahlswede–Daykin inequality to propagate the positive dependence.

The theorem is trivial for the graph consisting of a single vertex (a family of a single variable being always positively associated) so let us assume that it holds for every graph with strictly less than $|V|$ vertices.

Let Γ be the neighbourhood of S , that is,

$$\Gamma = \{v \in V \setminus S : \exists s \in S \text{ s.t. } \{vs\} \in E\}.$$

Then, let R be the random subset of Γ defined by

$$R = \{v \in \Gamma : \exists s \in S \text{ s.t. } s \rightarrow v\}.$$

Observe that the events $\{i \in R\}$, $i \in \Gamma$ are independent, so that the set R is positively associated.

Next, let H be the subgraph of G induced by $V \setminus S$. Note that, for all $i \in V \setminus S$,

$$\{S \overset{G}{\rightsquigarrow} i\} = \{R \overset{H}{\rightsquigarrow} i\}.$$

For every fixed $r \subset \Gamma$, the family $\{r \overset{H}{\rightsquigarrow} i\}$ for $i \in V \setminus S$ is independent of R because it depends only on the orientations of the edges of H , while R depends only on the orientations of the edges of G that go from S to Γ , and these two sets of edges are disjoint. Moreover, by the induction hypothesis, the events $\{r \overset{H}{\rightsquigarrow} i\}$, $i \in V \setminus S$, are positively associated. Since for fixed sets r and r' such that $r' \subset r$, $\{r' \rightsquigarrow i\} \implies \{r \rightsquigarrow i\}$ for all vertices, we can apply Lemma 2.1 to conclude that the events $\{R \rightsquigarrow i\}$, $i \in V \setminus S$, are positively associated.

To conclude the proof, note that the events $\{S \rightsquigarrow i\}$ are certain for $i \in S$ and that the union of a family of positively associated events and of a family of certain events is still positively related. \square

2.3 Corollaries

Corollary 2.3. *Let G be a finite graph with independently oriented edges. For any target set T , the events $\{i \rightsquigarrow T\}$, $i \in V$, are positively associated.*

Proof. Consider the randomly oriented graph H obtained by reversing the orientation of the edges of G , i.e. such that $\{i \overset{H}{\rightarrow} j\} = \{j \overset{G}{\rightarrow} i\}$. Then, for all $i \in V$,

$$\{i \overset{G}{\rightsquigarrow} T\} = \{T \overset{H}{\rightsquigarrow} i\},$$

and we already know from Theorem 2.2 that the events $\{T \overset{H}{\rightsquigarrow} i\}$, $i \in V$, are positively associated. \square

Corollary 2.4. *Let G be an infinite graph with independently oriented edges. Let f and g be increasing, non-negative functions on \mathbb{R}^V that depend only on a finite number of coordinates (i.e. such that there exists a finite set $U \subset V$ and $\tilde{f}: \mathbb{R}^U \rightarrow [0, +\infty[$ such that $f = \tilde{f} \circ \varphi$, where φ is the canonical surjection from \mathbb{R}^V to \mathbb{R}^U). Then, for any source set S , letting $\mathbf{X} = (\mathbb{1}_{\{S \rightsquigarrow i\}})_{i \in V}$,*

$$\mathbb{E}(f(\mathbf{X})g(\mathbf{X})) \geq \mathbb{E}(f(\mathbf{X})) \mathbb{E}(g(\mathbf{X})).$$

Proof. Let G_n be an increasing sequence of finite graphs such that $G = \bigcup_n G_n$, and for all $i \in V$, let

$$X_i^{(n)} = \{S \overset{G_n}{\rightsquigarrow} i\},$$

so that $X_i^{(n)} \subset X_i^{(n+1)}$ and $X_i = \bigcup_n X_i^{(n)}$. Since the functions f and g are increasing, so are the sequences $f(\mathbf{X}^{(n)})$ and $g(\mathbf{X}^{(n)})$. Thus, using Theorem 2.2 and monotone convergence,

$$\mathbb{E}(\lim_n f(\mathbf{X}^{(n)})g(\mathbf{X}^{(n)})) \geq \mathbb{E}(\lim_n f(\mathbf{X}^{(n)})) \mathbb{E}(\lim_n g(\mathbf{X}^{(n)})).$$

Finally, if f and g depend on a finite number of events X_i , then for every realization of \mathbf{X} we have $\lim_n f(\mathbf{X}^{(n)}) = f(\mathbf{X})$ and $\lim_n g(\mathbf{X}^{(n)}) = g(\mathbf{X})$. \square

Corollary 2.5. (Narayanan 2016). *For any (possibly infinite) graph with independently oriented edges, for any source set S and for any two vertices i and j ,*

$$\mathbb{P}(S \rightsquigarrow i, S \rightsquigarrow j) \geq \mathbb{P}(S \rightsquigarrow i) \mathbb{P}(S \rightsquigarrow j).$$

Proof. Take $f: (x_k)_{k \in V} \mapsto x_i$ and $g: (x_k)_{k \in V} \mapsto x_j$ in Corollary 2.4. \square

Corollary 2.6. Let G be a finite graph with independently oriented edges and vertex set V . For any source set S , let

$$N = \sum_{i \in V \setminus S} \mathbb{1}_{\{S \rightsquigarrow i\}}$$

denote the size of the oriented percolation cluster of G , and set $\lambda = \mathbb{E}(N)$. Then,

$$d_{TV}(N, \text{Poisson}(\lambda)) \leq \min(1, \lambda^{-1}) \left(\text{Var}(N) - \lambda + 2 \sum_{i \in V \setminus S} \mathbb{P}(S \rightsquigarrow i)^2 \right),$$

where d_{TV} denotes the total variation distance.

Proof. This is a direct application of the Stein–Chen method to the positively related variables $\mathbb{1}_{\{S \rightsquigarrow i\}}$, $i \in V \setminus S$: see, for example, Theorem 4.20 in [9]. \square

The gain of Corollary 2.6 is that we only need a suitable upper bound on $\text{Cov}(\mathbb{1}_{\{S \rightsquigarrow i\}}, \mathbb{1}_{\{S \rightsquigarrow j\}})$ to show that the size of the oriented percolation cluster is Poissonian.

Acknowledgements. I thank my adviser Amaury Lambert for a careful rereading of the first draft of this manuscript, and Jean-Jil Duchamps for helpful discussions. I also thank two anonymous reviewers.

References

- [1] Ahlswede, R. and Daykin, D. E. (1978) An inequality for the weights of two families of sets, their unions and intersections. *Probab. Theory Related Fields* **43** 183–185.
- [2] Barbour, A. D., Holst, L. and Janson, S. (1992) *Poisson Approximation*, Oxford Studies in Probability, Oxford University Press.
- [3] Durrett, R. (1984) Oriented percolation in two dimensions. *Ann. Probab.* **12** 999–1040.
- [4] Esary, J. D., Proschan, F. and Walkup, D. W. (1967) Association of random variables, with applications. *Ann. Math. Statist.* **38** 1466–1474.
- [5] Fill, J. A. and Pemantle, R. (1993) Percolation, first-passage percolation and covering times for Richardson’s model on the n -cube. *Ann. Appl. Probab.* **3** 593–629.
- [6] Fortuin, C. M., Kasteleyn, P. W. and Ginibre, J. (1971) Correlation inequalities on some partially ordered sets. *Commun. Math. Phys.* **22** 89–103.
- [7] Linusson, S. (2009) A note on correlations in randomly oriented graphs. [arXiv:0905.2881](https://arxiv.org/abs/0905.2881)
- [8] Narayanan, B. (2018) Connections in randomly oriented graphs. *Combin. Probab. Comput.* **27** 667–671.
- [9] Ross, N. (2011) Fundamentals of Stein’s method. *Probab. Surv.* **8** 201–293.