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$\mathbb{Z}^d$ group shifts and Bernoulli factors

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Abstract. In this paper, a *group shift* is an expansive action of $\mathbb{Z}^d$ on a compact metrizable zero-dimensional group by continuous automorphisms. All group shifts factor topologically onto equal-entropy Bernoulli shifts; abelian group shifts factor by continuous group homomorphisms onto canonical equal-entropy Bernoulli group shifts; and completely positive entropy abelian group shifts are weakly algebraically equivalent to these Bernoulli factors. A completely positive entropy group (even vector) shift need not be topologically conjugate to a Bernoulli shift, and the Pinsker factor of a vector shift need not split topologically.

1. *Introduction*

By an *algebraic* $\mathbb{Z}^d$ *action* we will mean an action α of $\mathbb{Z}^d$ by continuous automorphisms $\alpha^{\mathbf{v}}$ ($\mathbf{v} \in \mathbb{Z}^d$) on a compact metrizable group X. Since the inaugural paper of Kitchens and Schmidt [**12**], a remarkably rich and thorough theory has been developed for these actions, closely related to the theory of modules over Laurent polynomial rings [**19**]. The case that X is expansive and zero-dimensional is one fundamental part of the study. In this paper, we focus on aspects of this case related to Bernoullicity and entropy.

In this paper, we define a *group shift* to be an expansive action of $\mathbb{Z}^d$ on a compact metrizable zero-dimensional group by continuous automorphisms. (In some works treating more general situations, 'group shift' has a more general meaning, as in [**6**].) We show in §3 that all group shifts factor topologically onto equal-entropy Bernoulli shifts; this is our only result for possibly non-abelian groups. This result is another example of the relative civility of group shifts in contrast to general $\mathbb{Z}^d$ shifts of finite type: for d, N positive integers greater than one, there exist $\mathbb{Z}^d$ shifts of finite type of entropy $\log N$ which do not factor topologically onto a $\mathbb{Z}^d$ Bernoulli shift on N symbols [**3**].

In §6, we prove that a finite-entropy abelian zero-dimensional algebraic $\mathbb{Z}^d$ action has as an algebraic factor a canonical Bernoulli group shift of equal entropy. Motivated by the work of Einsiedler and Schmidt [**4**], in §7 we prove that an abelian group shift is weakly

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algebraically equivalent to this Bernoulli factor if and only if it has completely positive entropy. This allows in the zero-dimensional case a more simple proof (see Remark 7.3) for the difficult Bernoullicity theorem of Rudolph and Schmidt [**17**].

We consider possible generalizations for $\mathbb{Z}^d$ group shifts of the Kitchens theorem which classified $\mathbb{Z}$ group shifts up to topological conjugacy. In §4, we recall Kitchens' theorem and find two generalizations. In particular for $d > 1$ the Pinsker factor of a $\mathbb{Z}^d$ group shift splits topologically if the closure of the homoclinic group is algebraically conjugate to a Bernoulli group shift. Mostly the generalized statements of Kitchens' theorem are simply false, and in §5 we give a variety of counterexamples. For example, a completely positive entropy $\mathbb{Z}^d$ group shift need not be topologically conjugate to a Bernoulli shift; we give an alternative argument for this unpublished result of Kitchens [**11**]. Also, the Pinsker factor of a $\mathbb{Z}^d$ group shift need not split topologically, even if it has entropy log 2 (and is thus a vector shift as in [**10, 11**]).

For the most part our proofs make little use of the Noetherian module theory which is central to the study of algebraic $\mathbb{Z}^d$ actions. It seems to us that there is some value where possible to seeing alternative arguments, and in some cases we provide them. Here a critical tool for us is the homoclinic group, as studied in the abelian case by Einsiedler, Lind and Schmidt [**4, 13**]. The alternative proof we provide in Proposition 2.5 extends some of this to the zero-dimensional non-abelian case, which we have not found in the literature. Moreover §2 contains general background and particularly background for the homoclinic group.

2. *The Pinsker factor and the homoclinic group*

Again, by an *algebraic $\mathbb{Z}^d$ action* we will mean an action α of $\mathbb{Z}^d$ by continuous automorphisms $\alpha^{\mathbf{v}}$ ($\mathbf{v} \in \mathbb{Z}^d$) on a compact metrizable group X. In general there are two fundamental cases, one where X is connected and the other where X is totally disconnected (zero-dimensional). Our focus in this paper is the expansive zero-dimensional case. An algebraic $\mathbb{Z}^d$ action (X, α) is *expansive* if there exists an open neighborhood $\mathcal{U} \subset X$ of the identity $0 \in X$ such that $\bigcap_{\mathbf{v}\in\mathbb{Z}^d} \alpha^{\mathbf{v}}(\mathcal{U}) = \{0\}$. To shorten notation, we will often use X to represent (X, α) or α.

A topological factor map from a topological dynamical system (X, α) to another such system (Y, β) is a surjective continuous map from X onto Y that intertwines the actions α and β. A topological conjugacy is a bijective factor map. An *algebraic factor map* between algebraic $\mathbb{Z}^d$ actions is a topological factor map defined by a group epimorphism and an *algebraic conjugacy* is an algebraic factor map which is a topological conjugacy.

Let G be any finite group. The product $G^{\mathbb{Z}^d}$ with coordinatewise addition is a compact zero-dimensional metrizable topological group. For $x \in G^{\mathbb{Z}^d}$ and $\mathbf{v} \in \mathbb{Z}^d$, we may write $x(\mathbf{v})$ as $x_{\mathbf{v}}$. Together with the usual d-dimensional shift action σ, given by $(\sigma^{\mathbf{v}}(x))_{\mathbf{w}} = x_{\mathbf{v}+\mathbf{w}}$, it defines the Bernoulli $\mathbb{Z}^d$ group shift $(B(G), \sigma)$. We say that an algebraic $\mathbb{Z}^d$ action is *algebraically Bernoulli* if it is algebraically conjugate to some Bernoulli group shift. In this paper, by a $\mathbb{Z}^d$ *group shift* we will mean an expansive algebraic action on a zero-dimensional group. Any closed shift-invariant subgroup of a Bernoulli group shift $B(G)$, together with the $\mathbb{Z}^d$ action given by the restriction of σ, is a group shift. We may assume our group shifts have this form (up to algebraic conjugacy, they must).

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The *alphabet group* of a group shift X is the group $\{x_{\mathbf{v}} : x \in X,\ \mathbf{v} \in \mathbb{Z}^d\} \leq G$ of symbols that actually occur in elements of X. When we say a group is a p-group, we mean that p is a rational prime and every element of the group has order a power of p. An abelian group shift X is algebraically conjugate to the product (over finitely many rational primes p) of p-group shifts $X_{(p)}$.

A fundamental result of Kitchens and Schmidt [**12**] yields that an algebraic action α of $\mathbb{Z}^d$ on a zero-dimensional group is expansive if and only if it satisfies the *descending chain condition*, i.e. any nested sequence of closed α-invariant subgroups stabilizes after finitely many steps. As the descending chain condition is equivalent to the existence of an algebraic conjugacy to some closed shift-invariant subgroup of $(G^{\mathbb{Z}^d}, \sigma)$ with G some (possibly finite) compact Lie group [**12,** Theorem 3.2], every expansive $\mathbb{Z}^d$ action on a compact zero-dimensional group is algebraically isomorphic to a closed shift-invariant subgroup of some Bernoulli shift. Using the descending chain condition one can easily show that a $\mathbb{Z}^d$ group shift in fact must be a $\mathbb{Z}^d$ shift of finite type (SFT) [**12, 19**].

Moreover [**12,** Theorem 3.16], a general (perhaps non-expansive) zero-dimensional algebraic $\mathbb{Z}^d$ action is algebraically conjugate to an inverse limit $X = (X_1 \leftarrow X_2 \leftarrow \cdots)$ of group shifts $X_k, k \in \mathbb{N}$ (here the maps $X_k \leftarrow X_{k+1}$ are algebraic factor maps). Any group shift has entropy $\log N$ for some $N \in \mathbb{N}$ (see the remarks before Theorem 3.2); so, if X has finite entropy, then for some $N \in \mathbb{N}$ we have $h(X) = \log N = h(X_k)$ for all but finitely many k. We assume familiarity with entropy, and refer to [**14, 19**] for thorough background. However, for definiteness recall that the $\mathbb{Z}^d$ (topological) entropy of a $\mathbb{Z}^d$ group shift X is $\lim_n (1/|C_n|) \log |\{x|_{C_n} : x \in X\}|$, where C_n denotes the cube $\{\mathbf{v} \in \mathbb{Z}^d : 0 \leq v_i < n\}$. We might denote this entropy as $h_d(\alpha)$, or just $h_d(X)$ or $h(X)$ if context makes the simpler notation clear. The measure-theoretic entropy $h(X, \mu)$ of α with respect to an α-invariant Borel probability μ is generalized analogously from the $\mathbb{Z}$ case.

We will use the following result repeatedly.

THEOREM 2.1. (Addition formula) [**14**] *If $\phi : X \to Y$ is an algebraic factor map of algebraic $\mathbb{Z}^d$ actions, then*

$$h(X) = h(Y) + h(\ker(\phi)). \tag{2.2}$$

The addition formula (2.2) is a special case of more general results [**14,** Appendix B], which follow from work of Yuzvinskii as explained in [**14**]. The formula (2.2) for topological entropy also holds for entropy with respect to Haar measure: for an algebraic $\mathbb{Z}^d$ action α on X, the Haar measure λ on X is a measure of maximal entropy for α, i.e. λ is an α-invariant Borel probability such that $h(X, \lambda) = h(X)$.

An algebraic $\mathbb{Z}^d$ action (X, α) has a *Pinsker factor* $\mathcal{P}(X)$. This is the maximal zero-entropy algebraic factor of (X, α); it is also the maximal zero-entropy continuous (not necessarily algebraic) factor of (X, α); and with its Haar measure, it is the maximal zero-entropy measurable factor of (X, α, λ). We say a measurable/topological dynamical system has *completely positive entropy (c.p.e.)* if its only zero-entropy factor system is the system containing just one point. An algebraic system (X, α) is c.p.e. (topologically or with respect to Haar measure) if and only if the Pinsker factor group is the trivial group. An algebraic system of finite entropy is c.p.e. if and only if the Haar measure is its unique measure of maximal entropy [**19**].

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Let $\|\mathbf{v}\| = \|\mathbf{v}\|_\infty = \max\{|v_i| : 1 \le i \le d\}$ denote the maximum norm on $\mathbb{Z}^d$. We now define a very useful tool for the sequel.

Definition 2.3. For an algebraic $\mathbb{Z}^d$ action (X, α), $x \in X$ is called a *homoclinic point* if for every sequence of $\mathbb{Z}^d$-vectors $\mathbf{v}_n$ with $\lim_n \|\mathbf{v}_n\| \to \infty$ the sequence $\alpha^{\mathbf{v}_n}(x)$ converges to the identity in X. The set of homoclinic points forms a subgroup in X, which is called the *homoclinic group* of (X, α) and is denoted by Δ_X.

When X is a group shift, Δ_X has a very simple definition; in this case,

$$\Delta_X = \{x \in X \mid x_{\mathbf{v}} = e \text{ for all but finitely many } \mathbf{v} \in \mathbb{Z}^d\},$$

where e above denotes the identity element in the alphabet group of X.

Lind and Schmidt [**13**] used the homoclinic group to significantly clarify the nature of c.p.e. and the Pinsker factor for expansive algebraic actions, as follows. (In the statement, $\overline{\Delta_X}$ denotes the (topological) closure of Δ_X in X.)

THEOREM 2.4. [**13**] *For an expansive $\mathbb{Z}^d$ action by continuous automorphisms on a compact abelian group X, the following hold:*
(1) *X has positive entropy if and only if Δ_X is non-trivial;*
(2) *$h(X) = h(\overline{\Delta_X})$;*
(3) *the Pinsker factor map can be presented as the map $\pi : X \to X/\overline{\Delta_X}$; and*
(4) *X has completely positive entropy if and only if Δ_X is dense.*

So, $\overline{\Delta_X}$ is the maximal closed invariant subgroup of X on which the restricted action has c.p.e.. Lind and Schmidt used Fourier analysis for part of their proof. In the zero-dimensional (group shift) case, we can avoid the abelian hypothesis and the Fourier analysis, as follows.

PROPOSITION 2.5. *Let X be a group shift. Then the following hold:*
(1) *X has positive entropy if and only if Δ_X is non-trivial;*
(2) *$h(X) = h(\overline{\Delta_X})$;*
(3) *the Pinsker factor map can be presented as the map $\pi : X \to X/\overline{\Delta_X}$;*
(4) *X has completely positive entropy if and only if Δ_X is dense; and*
(5) *$X/\overline{\Delta_X}$ is algebraically conjugate to a group shift.*

Proof. First suppose that x is a non-trivial homoclinic point in X. Pick $R > 0$ such that $x_{\mathbf{v}} = e$ if $\|\mathbf{v}\| \ge R$. By composing appropriate translates of x, we see that there are at least 2^{n^d} distinct configurations in $\{x|_{\{0,1,\ldots,nR-1\}^d} : x \in X\}$, and therefore $h_d(X) \ge (1/R)^d \log 2 > 0$.

To prove the converse direction of (1), for n, r in $\mathbb{N}$ with $r < n$ define

$$E_n = \{\mathbf{v} \in \mathbb{Z}^d \mid \|\mathbf{v}\| \le n\},$$
$$B_{n,r} = \{\mathbf{v} \in \mathbb{Z}^d \mid n - r \le \|\mathbf{v}\| \le n\}.$$

It is routine to check that, for every positive integer r,

$$h_d(X) = \lim_n \frac{1}{n^d} \log \max_{x \in X} \operatorname{card}\{y|_{E_n} \mid y \in X \text{ and } y|_{B_{n,r}} = x|_{B_{n,r}}\}. \tag{2.6}$$

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Therefore the assumption $h_d(X) > 0$ implies that for any r we can find $n \in \mathbb{N}$ and points x, y in X such that x and y agree on $B_{n,r}$ and are not equal on E_{n-r}. Define a configuration z by setting

$$z_{\mathbf{v}} = \begin{cases} (xy^{-1})_{\mathbf{v}} & \text{if } \mathbf{v} \in E_n, \\ e & \text{otherwise.} \end{cases}$$

Recalling that the group shift X is a shift of finite type, we see that, if r was chosen large enough, then $z \in X$. Clearly z is a non-trivial homoclinic point. This finishes the proof of (1). We also see that the limit in (2.6) can be achieved using homoclinic points for x in the formula of (2.6). This proves (2).

It follows from (2) and the addition formula for entropy (2.2) that $h_d(X/\overline{\Delta_X}) = 0$, so the kernel of π contains the kernel of the Pinsker factor map. It follows from (1) that the kernel of the Pinsker factor map must contain $\overline{\Delta_X}$, which is the kernel of π. Thus (3) is true, and (4) follows immediately from (3). Finally, (5) follows from Proposition 7.10. □

The following consequence will be convenient for us.

PROPOSITION 2.7. *Suppose $\phi : X \to Y$ is an algebraic factor map of group shifts. Then ϕ maps $\overline{\Delta_X}$ onto $\overline{\Delta_Y}$.*

Proof. Clearly $\phi(\overline{\Delta_X}) \subset \overline{\Delta_Y}$. Also, we have a factor map $X/\overline{\Delta_X} \to Y/\phi(\overline{\Delta_X})$, with $h(X/\overline{\Delta_X}) = 0$ and $Y/\phi(\overline{\Delta_X}) \supset \overline{\Delta_Y}/\phi(\overline{\Delta_X})$. It follows that $\overline{\Delta_Y}/\phi(\overline{\Delta_X})$ is trivial, since $\overline{\Delta_Y}$ has completely positive entropy. □

Notation 2.8. Given $n \in \mathbb{N}$, we will use $F(n)$ to denote the Bernoulli shift $B(G)$ such that the alphabet group is $\{g \in \mathbb{R}/\mathbb{Z} : ng = 0\}$. For example, the group shifts $B(\mathbb{Z}/2)$ and $F(2)$ are algebraically conjugate, and they are topologically conjugate to $(1/2)F(2)$, which is a subshift of the group shift $F(4)$ but is not a subgroup.

Notation 2.9. In what follows $\mathfrak{R}_d = \mathbb{Z}[u_1^{\pm 1}, \ldots, u_d^{\pm 1}]$ denotes the Laurent polynomial ring in d variables with coefficients in $\mathbb{Z}$, and $\mathfrak{R}_d^{(p)} = \mathbb{F}_p[u_1^{\pm 1}, \ldots, u_d^{\pm 1}]$, the Laurent polynomial ring in d variables with coefficients in $\mathbb{F}_p$, where p is a rational prime and $\mathbb{F}_p$ the finite field of p elements.

3. *Group shifts factor topologically onto equal-entropy Bernoulli shifts*

In this section, we will show (Theorem 3.2) that every group shift has an equal-entropy topological Bernoulli factor.

Definition 3.1. Let G be a finite group, with identity element e. Let X be a group subshift of $G^{\mathbb{Z}^d}$. Let $\prec$ denote lexicographic order on $\mathbb{Z}^d$. Given $x \in X$, the *follower set of* x in X is defined to be

$$\mathcal{F}(x) = \{g \in G : \exists\, y \in X \text{ such that } y_{\mathbf{0}} = g \text{ and } y_{\mathbf{v}} = x_{\mathbf{v}} \text{ for all } \mathbf{v} \prec \mathbf{0}\}.$$

Abusing notation, we let $\mathcal{F}(e)$ denote the follower set of the identity in X.

It is easily seen [**9**] that $\mathcal{F}(e)$ is a normal subgroup of G, and, for any x, the set $\mathcal{F}(x)$ is the coset of $\mathcal{F}(e)$ which contains $x_{\mathbf{0}}$. It is well known from work of Yuzvinskii and Conze

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that $h(X) = \log|\mathcal{F}(e)|$ (see the proof of [**14,** Theorem 4.5]); for completeness an argument is included in the proof of the next result.

THEOREM 3.2. *Let X be a $\mathbb{Z}^d$ group shift with alphabet group G. Then there is a topological factor map ϕ from X onto a Bernoulli shift B of equal entropy, namely $B = \mathcal{F}(e)^{\mathbb{Z}^d}$.*

Proof. For any x in X, the set $\mathcal{F}(x)$ is the coset of $\mathcal{F}(e)$ which contains $x_{\mathbf{0}}$. For each such coset, choose a bijection b to $\mathcal{F}(e)$. Define a one-block code ϕ from X to $B = (\mathcal{F}(e))^{\mathbb{Z}^d}$ by the rule $(\phi x)_{\mathbf{0}} = b(x_{\mathbf{0}})$.

Every finite B-word occurs in the image of ϕ, and therefore ϕ is surjective. Moreover, because X is SFT [**12,** Theorem 5.2], there must exist $R > 0$ such that, for all n, for any fixed configuration c on the R neighborhood in $\mathbb{Z}^d$ of the cube $C_n = \{\mathbf{v} \in \mathbb{Z}^d : 0 \le v_i < n, 1 \le i \le d\}$, there are at most $|\mathcal{F}(e)|^{n^d}$ configurations on C_n compatible with c. This implies that the entropy of X is at most $\log|\mathcal{F}(e)|$. Since X factors onto B and $h(B) = \log|\mathcal{F}(e)|$, it follows that $h(X) = h(B)$. □

Remark 3.3. We note some useful and well-known facts for a group shift X which is a p-group. Here $\mathcal{F}(e)$ must be a p-group, so $h(X) = \log|\mathcal{F}(e)|$ implies that $h(X)$ is an integer multiple of $\log p$. In particular, because $F(p)$ has no proper subsystem of equal entropy, the only group shift in $F(p)$ with positive entropy is $F(p)$ itself.

Remark 3.4. We will show below (Theorem 6.5) that for abelian group shifts it is possible to construct an *algebraic* factor map onto a Bernoulli shift of equal entropy. However, this cannot in general be done with the construction used in the proof of Theorem 3.2, even for $\mathbb{Z}$ group shifts. In that construction, if there is a choice of bijections b which makes ϕ a group homomorphism, then there is a homomorphism of the alphabet groups $G \to \mathcal{F}(e)$ which restricts to a bijection $\mathcal{F}(e) \to \mathcal{F}(e)$. Here is a $\mathbb{Z}$ group Markov shift example (a modification of the Kitchens example (7.9)) for which no such homomorphism exists. The alphabet group G is $\mathbb{Z}/4 \oplus \mathbb{Z}/4$. The allowed transitions are, for a, b in $\mathbb{Z}/4$,

$$(a, b) \to (0, 0), (0, 2), (2, 0), (2, 2), (1, 1), (1, 3), (3, 1), (3, 3) \quad \text{if } 2b = 0 \in \mathbb{Z}/4,$$
$$(a, b) \to (1, 0), (1, 2), (3, 0), (3, 2), (2, 1), (2, 3), (0, 1), (0, 3) \quad \text{if } 2b \neq 0 \in \mathbb{Z}/4.$$

No homomorphism $\psi : G \to \mathcal{F}(e)$ can restrict to a bijection $\mathcal{F}(e) \to \mathcal{F}(e)$, because $\mathcal{F}(e)$ has index two in G; $\ker(\psi)$ would have to contain an element of order two in the complement of $\mathcal{F}(e)$, and such an element does not exist.

4. *Two extensions of Kitchens' theorem*

In his influential early paper on $\mathbb{Z}$ group shifts, Kitchens gave the following decisive topological classification theorem for $\mathbb{Z}$ group shifts.

THEOREM 4.1. [**9**] *Let X be a $\mathbb{Z}$ group shift. Then X is topologically conjugate to the product of a Bernoulli shift and an automorphism of a finite group.*

We will consider possible generalizations of Theorem 4.1 to $\mathbb{Z}^d$ group shifts with $d > 1$. (In [**11**], Kitchens himself addressed some of these issues for vector shifts and thus in

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some cases group shifts, as we will see below.) There are severe limitations, which we will indicate by examples in the next section. In this section we record the two extensions of Theorem 4.1 which we do know, and state an open problem.

Let X be a group shift. Recall that Δ_X is the homoclinic group of X, $\overline{\Delta_X}$ is the group shift which is the kernel of the Pinsker factor map, and the image $\mathcal{P}(X)$ of that factor map is the group shift which is the maximal zero-entropy continuous factor of X.

PROPOSITION 4.2. *Suppose that $n \in \mathbb{N}$ and X is a group subshift of $F(n)$. Then $\overline{\Delta_X}$ is algebraically Bernoulli, and X is topologically conjugate to the product of the group shifts $\mathcal{P}(X)$ and $\overline{\Delta_X}$.*

Proof. Let $n = \prod_p p^{k_p}$ be the factorization of n as a product of powers of the primes p which divide n. Then $F(n)$ is algebraically conjugate to the product $\prod_p F(p^{k_p})$, and X is algebraically conjugate to a product of group shifts $X_{(p)}$, where $X_{(p)} \leq F(p^{k_p})$. So without loss of generality we may assume for a prime p that $X = X_{(p)} \leq F(p^k)$, and also $h(X) > 0$.

Let j be the smallest integer such that $h(p^j X) = 0$. It follows from the addition formula (2.2) that the kernel K of the multiplication-by-p map $p^{j-1}X \to p^j X$ has positive entropy. Because $K \leq F(p)$, it follows that $K = F(p)$ (see Remark 3.3).

Write an element x of X in the form $x = p^{-k}x_k + \cdots + p^{-1}x_1$, where $x_i \in \{0, 1, \ldots, p-1\}^{\mathbb{Z}^d}$. Because $K = F(p)$, we see that, for every element a_j of $\{0, 1, \ldots, p-1\}^{\mathbb{Z}^d}$, there exists x in X such that $x_j = a_j$ and $x_t = 0$ when $t > j$. Then, if $1 \leq i \leq j$ and $a_i \in \{0, 1, \ldots, p-1\}^{\mathbb{Z}^d}$, there exists x in X such that $x_i = a_i$ and $x_t = 0$ when $t > i$. By considering sums of such elements, we see that $F(p^j)$ is a group subshift of X. Since the kernel of the multiplication-by-p^j map $\gamma : X \to p^j X$ is contained in $F(p^j)$, we conclude that $\ker(\gamma) = F(p^j)$. Because $F(p^j)$ has completely positive entropy, it is contained in the kernel of the Pinsker factor map; thus the zero-entropy factor $p^j X$ of X must be the maximal zero-entropy algebraic factor. Therefore γ is the map $X \to \mathcal{P}(X)$, and $\overline{\Delta_X}$ is algebraically Bernoulli because it is the kernel of γ which equals $F(p^j)$.

Moreover, if $p^{-k}x_k + \cdots + p^{-1}x_1 \in X$, then by subtracting the element $p^{-j}x_j + \cdots + p^{-1}x_1$ of $F(p^j)$, we see that $p^{-k}x_k + \cdots + p^{-(j+1)}x_{j+1} \in X$. We conclude that the map

$$p^{-k}x_k + \cdots + p^{-1}x_1 \mapsto (p^{-k+j}x_k + \cdots + p^{-1}x_{j+1}, p^{-j}x_j + \cdots + p^{-1}x_1)$$

defines a topological (not necessarily algebraic) conjugacy of group shifts $X \to \mathcal{P}(X) \times F(p^j)$. □

Recall that an integer is *squarefree* if it is not divisible by the square of any prime. Below, we use Notations 2.8 and 2.9.

THEOREM 4.3. *Suppose that X is an abelian group shift and $\overline{\Delta_X}$ is algebraically Bernoulli. Then the following are true:*

(1) *X is topologically conjugate to the product of $\overline{\Delta_X}$ and $\mathcal{P}(X)$; and*
(2) *if $nX = 0$ for a squarefree integer n, then X is algebraically conjugate to the product of $\overline{\Delta_X}$ and $\mathcal{P}(X)$.*

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Proof. After considering the presentations of $F(n)$ and X as direct sums of p-groups for primes p, we assume without loss of generality that $h(X) > 0$ and there are positive integers k, M such that X is a group subshift of $(F(p^k))^M$. Let Y denote $\overline{\Delta_X}$ and set $D = \{y \in Y : py = 0\}$. Because Y is algebraically Bernoulli, it follows that D is algebraically Bernoulli; also, $h(Y) > 0$ implies that $h(D) > 0$. Note that $D \leq (F(p))^M$.

As in [**10**] and [**19**], we may view D and $(F(p))^M$ as metrizable $\mathfrak{R}_d^{(p)}$-modules (where each u_i must act by a continuous group automorphism) and use the associated duality theory. This is Pontryagin duality of locally compact abelian groups, with additional structure. The action of u_i on $(F(p))^M$ is given by the ith coordinate shift map, $u_i(x) = \sigma^{\mathbf{e_i}}(x)$. The dual $\mathfrak{R}_d^{(p)}$-module of a module $\mathfrak{M}$, denoted $\widehat{\mathfrak{M}}$, is the group of continuous homomorphisms $\mathfrak{M} \to \mathbb{F}_p$. The action of $\mathfrak{R}_d^{(p)}$ on an $\mathfrak{R}_d^{(p)}$-module $\widehat{\mathfrak{M}}$ is dual to the action on $\mathfrak{M}$; for $\chi \in \widehat{\mathfrak{M}}$, $(u_i(\chi))(m) = \chi(u_i(m))$. Of course, $\mathfrak{M}$ is discrete if and only if its dual is compact.

Because $pD = 0$ and D is algebraically Bernoulli with positive entropy, there is a positive integer j such that D is algebraically conjugate to $(F(p))^j$, the product of j copies of $F(p)$. The $\mathfrak{R}_d^{(p)}$-module $\widehat{D}$ is then a free $\mathfrak{R}_d^{(p)}$-module, isomorphic to a direct sum of j copies of $\mathfrak{R}_d^{(p)}$. Since $D \leq (F(p))^M$, we have a dual module epimorphism $\pi : \widehat{(F(p))^M} \to \widehat{D}$. By freeness of $\widehat{D}$, this epimorphism π splits, and its kernel $D^\perp$ has a complementary internal direct summand, $\mathcal{N}$. Define the group shift

$$C = \mathcal{N}^\perp = \{x \in (F(p))^M : \chi(x) = 0 \text{ for all } \chi \in \mathcal{N}\}.$$

We conclude that $(F(p))^M$ is the internal direct sum of C and D, i.e. the map $C \times D \to (F(p))^M$ given by $(c, d) \mapsto c + d$ is an algebraic conjugacy of group shifts.

Write an element x in $(F(p^k))^M$ in the form $x = p^{-k}x_k + \cdots + p^{-1}x_1$, where $x_i \in (\{0, 1, \ldots, p-1\}^{\mathbb{Z}^d})^M$. Let $p^{-1}x_1 = c_1(x) + d_1(x)$ be the internal direct sum representation with C and D. Let γ be the epimorphism of group shifts $X \to X/D$. Now $\ker(\gamma) \subset \overline{\Delta_X}$ and by Proposition 2.7 $\gamma(\overline{\Delta_X}) = \overline{\Delta_{\gamma X}}$, so we have $\gamma^{-1}(\overline{\Delta_{\gamma X}}) = \overline{\Delta_X}$.

Note that, given $x = p^{-k}x_k + \cdots + p^{-2}x_2 + c_1(x) + d_1(x) \in X$, we have $x - d_1(x) = p^{-k}x_k + \cdots + p^{-2}x_2 + c_1(x) \in X$. Let W be the subset of X consisting of all points of the form $p^{-k}x_k + \cdots + p^{-2}x_2 + c_1(x)$. In general, W need not be a subgroup of X (W might contain an element w such that $0 \neq pw \in D$), but it is closed and shift-invariant, i.e. a subshift of X. The map $W \times D \to X$ given by the rule $(w, d) \to w + d$ is continuous, shift-commuting, surjective and injective; so, X is topologically conjugate to the product of subshifts $W \times D$. Also, the restriction of γ to W is bijective, and thus a topological conjugacy from W to the group shift γX. It follows that X is topologically conjugate to the product of group shifts $\gamma X \times D$.

If γX has positive entropy, then we can repeat this operation until we reach a quotient group shift of zero entropy. At that point we have X topologically conjugate to a product $B \times Z$, where B is a product of copies of $F(p)$, and Z is a zero-entropy group shift which is the image of X under a homomorphism whose kernel is $\overline{\Delta_X}$, i.e. Z is the Pinsker factor of X. The shifts B and $\overline{\Delta_X}$ are topologically conjugate because all Bernoulli shifts of equal entropy are topologically conjugate.

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In the case $pX = 0$, X is the internal direct sum of those subgroups C and D; we have $D = \overline{\Delta_X}$ and $h(\gamma X) = 0$; the subshift W is C; the restriction of γ to C defines an algebraic conjugacy $C \to \gamma X$; and X is algebraically isomorphic to $D \times C$. □

Remark 4.4. We mention some related results of Kitchens. Given a finite field $\mathbb{F}$, a $\mathbb{Z}^d$ vector shift is a closed shift-invariant vector subspace of some full-$\mathbb{F}^m$ shift, considered as a vector space over $\mathbb{F}$. Kitchens showed [**11,** Theorem 3.4] that any vector shift X admits a finite sequence of vector shift epimorphisms

$$X = X_0 \to X_1 \to X_2 \to \cdots \to X_{r-1} \to X_r = \{0\},$$

where $\ker(X_i \to X_{i+1})$ either has zero entropy or is a full-$\mathbb{F}^{k_i}$ shift. This is an application by duality of the primary decomposition theorem for Noetherian modules. This setting generalizes the case of group shifts X for which $pX = 0$ with p prime (in which case, X is a vector shift over the field of p elements).

Kitchens also explained the cocycle structure of a vector shift epimorphism $\phi : X \to X/Y$. Here, he chose a vector shift presentation X_1 of X/Y and chose a continuous map $c : \mathbb{Z}^d \times X_1 \to X$ such that, for all $\mathbf{v} \in \mathbb{Z}^d$, the map $(\sigma_X)^{\mathbf{v}}$ on X is topologically conjugate to the map on $X_1 \times Y \to X_1 \times Y$ defined by the rule

$$(z, y) \mapsto ((\sigma_{X_1})^{\mathbf{v}}(z), (\sigma_Y)^{\mathbf{v}}(y)) + (0, c(\mathbf{v}, z)),$$

and consequently the map ϕ is presented as a coordinate projection.

Theorem 4.1 contains the result that all c.p.e. abelian $\mathbb{Z}$ group shifts are topologically conjugate to Bernoulli shifts, even though such group shifts need not be algebraically conjugate to Bernoulli shifts. For $d > 1$, if X is a c.p.e. $\mathbb{Z}^d$ abelian group shift which is not algebraically Bernoulli, it is possible for X to be topologically conjugate (5.9) or not topologically conjugate (5.3) to a Bernoulli shift.

Problem 4.5. Classify c.p.e. $\mathbb{Z}^d$ group shifts up to topological conjugacy. In particular, when is a c.p.e. abelian group shift X topologically conjugate to a Bernoulli shift?

5. *Counterexamples around Kitchens' theorem*

In this section we will give examples which rule out various generalizations of the Theorem 4.1 of Kitchens. Some must be well known and are provided for completeness. We use Notations 2.8 and 2.9.

The following example shows that the topological conjugacy of Kitchens' Theorem 4.1 cannot in general be chosen to be a group isomorphism.

Example 5.1. Let X be the $\mathbb{Z}$ group shift generated by $F(2)$ and the point $\{1/4\}^{\mathbb{Z}}$. Here, X is topologically conjugate to the group shift Y which is the product of $F(2)$ and the identity automorphism on $\mathbb{Z}/2$. However, X is not algebraically conjugate to Y, because $2Y$ is trivial and $2X$ is non-trivial.

The phenomenon above likewise occurs in $\mathbb{Z}^d$ group shifts for $d > 1$, as in the next example.

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Example 5.2. Let Z be any non-trivial zero-entropy $\mathbb{Z}^2$ group subshift of $F(2)$, such as the Ledrappier 'three-dot' shift. Consider the group subshift of $\mathbb{T}^{\mathbb{Z}^2}$ which is

$$X = F(2) + \tfrac{1}{2}Z \leq F(4).$$

Here the closure of the homoclinic group Δ_X is $F(2)$; the Pinsker factor $\mathcal{P}(X)$ is $X/F(2)$, which is algebraically conjugate to Z; and X is topologically conjugate to the group shift $Y = F(2) \times Z$ by the one-block map $x \mapsto ((1/2)\lfloor 2x \rfloor, 2x)$. However, the topological conjugacy cannot be achieved by a group isomorphism, because $2Y = 0$ but $2X \neq 0$.

The next result is due to Kitchens [**11**]. We give a different argument for the completely positive entropy step.

PROPOSITION 5.3. *The example* [**10,** Example 3.2] *of Kitchens is a* $\mathbb{Z}^2$ *abelian group shift* S *of completely positive entropy such that* $2S = 0$, $h(S) = \log 2$ *and thus* S *is measurably conjugate to a Bernoulli shift on two symbols, but* S *is not topologically conjugate to any full shift.*

Proof. Following [**10**], we write a coordinate entry of an element s of the Bernoulli group shift $B = B(\mathbb{F}_2 \oplus \mathbb{F}_2)$ in the form $s_{(i,j)} = (s^1_{(i,j)}, s^2_{(i,j)})$. The example [**10,** Example 3.2] is the subshift S consisting of all s in B such that

$$s^1_{(i,j)} + s^1_{(i,j+1)} + s^2_{(i,j)} + s^2_{(i+1,j)} = 0, \tag{5.4}$$

for all $(i, j) \in \mathbb{Z}^2$. The entropy of S is log 2, because $|\mathcal{F}(e)| = |\{(0, 0), (0, 1)\}| = 2$.

Stated in dynamical language, the purpose of the example S in [**10**] was to exhibit a c.p.e. group shift that is not algebraically conjugate to a Bernoulli shift, which was shown by investigating algebraic properties of its dual module. The same subshift reappears as [**11,** Example 4.11] where Kitchens observed that S is not even topologically conjugate to a Bernoulli shift. This is because all four fixed points of B are contained in S, but a Bernoulli $\mathbb{Z}^2$ shift with entropy log 2 has only two fixed points.

To finish the proof, it remains to see that S has completely positive entropy and thus is measurably conjugate to some Bernoulli shift. Kitchens [**10**] proved this by showing that the dual $\mathfrak{R}^{(2)}_2$-module of S is torsion free but not free. Appealing to Proposition 2.5, we shall give an alternative proof by showing that the homoclinic group is dense in S.

Suppose that $n > 1$ and $(a_{(i,j)})$, $1 \leq i, j \leq n$ is a square configuration occurring in a point of S. It suffices to show that there is a point s in S such that

$$s_{(i,j)} = \begin{cases} a_{(i,j)} & \text{if } (i, j) \in \{1, 2, \ldots, n\}^2, \\ (0, 0) & \text{if } (i, j) \notin \{0, 1, 2, \ldots, n+1\}^2. \end{cases}$$

To use a picture, we will give the (perfectly general) argument for the case $n = 4$. The 8×8 array below has coordinate set $\{-1, 0, \ldots, 6\}^2$, with $(-1, -1)$ at the lower left corner. The inner 4×4 square is the given configuration on $\{1, 2, 3, 4\}^2$; the (boldface) boundary entries are all $(0, 0)$; and we must specify the remaining entries. The entries marked + are determined by the inner 4×4 configuration and the requirement (5.4) for (i, j) in $\{1, 2, 3, 4\}^2$. The entries marked $\cdot$ are yet to be determined. We have chosen some

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of the remaining entries to be 0; those choices in particular guarantee that (5.4) holds at (i, j) if $i = -1$ or $j = -1$ and at $(i, j) = (0, 0)$.

$$\begin{array}{cccccccc}
(\mathbf{0},\mathbf{0}) & (\mathbf{0},\mathbf{0}) & (\mathbf{0},\mathbf{0}) & (\mathbf{0},\mathbf{0}) & (\mathbf{0},\mathbf{0}) & (\mathbf{0},\mathbf{0}) & (\mathbf{0},\mathbf{0}) & (\mathbf{0},\mathbf{0}) \\
(\mathbf{0},\mathbf{0}) & (\cdot, 0) & (+,\cdot) & (+,\cdot) & (+,\cdot) & (A, B) & (C, D) & (\mathbf{0},\mathbf{0}) \\
(\mathbf{0},\mathbf{0}) & (\cdot, 0) & (\square,\square) & (\square,\square) & (\square,\square) & (\mathbf{E},\mathbf{F}) & (G, H) & (\mathbf{0},\mathbf{0}) \\
(\mathbf{0},\mathbf{0}) & (\cdot, 0) & (\square,\square) & (\square,\square) & (\square,\square) & (\square,\square) & (\cdot, +) & (\mathbf{0},\mathbf{0}) \\
(\mathbf{0},\mathbf{0}) & (\cdot, 0) & (\square,\square) & (\square,\square) & (\square,\square) & (\square,\square) & (\cdot, +) & (\mathbf{0},\mathbf{0}) \\
(\mathbf{0},\mathbf{0}) & (0, 0) & (\square,\square) & (\square,\square) & (\square,\square) & (\square,\square) & (\cdot, +) & (\mathbf{0},\mathbf{0}) \\
(\mathbf{0},\mathbf{0}) & (0, 0) & (0, 0) & (0,\cdot) & (0,\cdot) & (0,\cdot) & (0,\cdot) & (\mathbf{0},\mathbf{0}) \\
(\mathbf{0},\mathbf{0}) & (\mathbf{0},\mathbf{0}) & (\mathbf{0},\mathbf{0}) & (\mathbf{0},\mathbf{0}) & (\mathbf{0},\mathbf{0}) & (\mathbf{0},\mathbf{0}) & (\mathbf{0},\mathbf{0}) & (\mathbf{0},\mathbf{0})
\end{array}$$

Working up column 0, and then across row 5, we see there are unique choices for the corresponding $\cdot$ entries, and then B, such that (5.4) holds on column 0 and at $(i, 5)$ for $i = 1, 2, 3$. Similarly, working right in row 0 and then up in column 5, we see there are unique choices for the corresponding $\cdot$ entries, and then G, such that (5.4) holds on row 0 and at $(5, j)$ for $j = 1, 2, 3$.

At this point, condition (5.4) is satisfied at all coordinates (i, j) except perhaps the four coordinates of the square $S'' = \{4, 5\}^2$, whose group entries are named A, B, C, D, E, F, G, H; we have defined B and G; and the remaining entries must be specified to satisfy the required four equations. We set $A = 0$; $H = E + F$; $C = G + E + F$; and $D = B$. The four equations will then be satisfied if $C = D$, i.e. if in $\mathbb{Z}/2$

$$B + E + F + G = 0. \tag{5.5}$$

If we sum the left-hand side of (5.4) over all (i, j) not in S'', then each entry to the left or below S'' appears in this sum exactly twice and $B + E + F + G$ occurs once. The sum is zero and this proves (5.5). □

PROPOSITION 5.6. *There is a $\mathbb{Z}^2$ group shift T such that $2T = 0$, $h(T) = \log 2$, T is not c.p.e. and T is not topologically conjugate to the product of a c.p.e. system and a zero-entropy system.*

Proof. We will construct T as a group subshift of $B = B(\mathbb{F}_2 \oplus \mathbb{F}_2)$. Define the point x in B by setting

$$x_{(i,j)} = \begin{cases} (0, 1) & \text{if } i \text{ is even,} \\ (1, 0) & \text{if } i \text{ is odd,} \end{cases}$$

and then set $y = \sigma^{(1,0)}x$ and $z = x + y$. Let H denote the four-element, shift-invariant subgroup $\{0, x, y, z\}$ of B. Define $T = S + H$, where S is the example of Kitchens studied in Proposition 5.3. Then $x \notin S$, and $x + S = y + S$ because $x - y = z \in S$. Then T is the disjoint union of the subshifts S and $x + S$. The closure of the homoclinic group of T is S. The Pinsker factor $\mathcal{P}(T)$ is the identity map on a two-element group. The Pinsker factor of a group shift T is the maximal continuous zero-entropy factor of T viewed as a topological dynamical system (forgetting the algebraic structure). Therefore, if T is topologically conjugate to the product of a c.p.e. system T' and a zero-entropy system Z', then Z' must be a system consisting of two fixed points, and T must be topologically conjugate to the

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disjoint union of two c.p.e. systems of full entropy which are topologically conjugate to each other. One of these two must be S, which is c.p.e. of full entropy; and then the other must be the subshift $x + S$. However, these two are not topologically conjugate, because S has all four fixed points of B and $x + S$ has none. □

In the proof of Proposition 5.6, $\overline{\Delta_T}$ is not algebraically Bernoulli. By Theorem 4.3 (or in this case by earlier results [**11**] of Kitchens described in Remark 4.4), this is unavoidable.

Remark 5.7. Of course, Theorem 4.1 fails badly without the algebraic hypothesis, even for $\mathbb{Z}$ SFTs. For example, if p is prime then the $\mathbb{Z}$ SFT X defined by the matrix

$$\begin{pmatrix} 0 & p \\ 1 & 0 \end{pmatrix},$$

has as a maximal continuous zero-entropy factor (which is also the Pinsker factor for the measure of maximal entropy) the system Z which is the transposition of two points; but there is no system W such that X is topologically conjugate to $Z \times W$, because the full shift on p symbols has no square root [**2**]. If $1 < n \in \mathbb{N}$, and Y is the $\mathbb{Z}$ SFT defined by the matrix

$$\begin{pmatrix} n & 1 \\ 0 & 1 \end{pmatrix},$$

then the measure of maximal entropy does not have full support; there is a continuous map onto a maximal zero-entropy topological factor of Y, but this zero-dimensional factor is not a subshift.

Let p be a rational prime. In the case that a c.p.e. abelian $\mathbb{Z}$ group shift X satisfies $pX = 0$, Kitchens proved that X is algebraically Bernoulli ([**9**], also see [**19,** Examples 10.11]). (In the $\mathbb{Z}^2$ case, this fails by Proposition 5.3.) Kitchens [**9**] also gave an example of an abelian $\mathbb{Z}$ group shift X such that $4X = 0$ and X is topologically but not algebraically Bernoulli. Fagnani [**5**] classified up to algebraic conjugacy the c.p.e. Bernoulli $\mathbb{Z}$ group shifts X such that $p^2X = 0$. In particular, for $k = 2$ and every rational prime p, Fagnani (correcting a finiteness claim in [**9**]) constructed an infinite family of group shifts of entropy $k \log p$, pairwise not algebraically conjugate, but all topologically conjugate to a Bernoulli shift. Schmidt [**19,** Examples 10.11] extended this construction to every $k \geq 2$.

We will finish this section by remarking that a simple construction used in [**7, 8**] produces $\mathbb{Z}^d$ examples from these $\mathbb{Z}$ examples.

Definition 5.8. Suppose that X is a $\mathbb{Z}^d$ group shift with alphabet group G. The *full* $\mathbb{Z}^{d+c}$ *extension* of X is defined to be the $\mathbb{Z}^{d+c}$ group shift $X^{(d\to d+c)}$ with domain

$$\{x = x_{(\mathbf{u},\mathbf{v})} \in G^{\mathbb{Z}^d \times \mathbb{Z}^c} : x_{(.,\mathbf{v})} \in X \text{ for all } \mathbf{v} \in \mathbb{Z}^c\},$$

and the usual definition of addition coordinatewise.

Remark 5.9. Suppose that X and Y are $\mathbb{Z}^d$ group shifts. We remark that: $h_{d+c}(X^{(d\to d+c)}) = h_d(X)$; $X^{(d\to d+c)}$ is topologically/algebraically Bernoulli if and only if X is; and $X^{(d\to d+c)}$ is topologically/algebraically conjugate to $Y^{(d\to d+c)}$ if and only if X is topologically/algebraically conjugate to Y.

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To deduce conjugacy (in either sense) of X and Y from conjugacy of $X^{(d\to d+c)}$ and $Y^{(d\to d+c)}$, let X_1 denote the subgroup of points in $X^{(d\to d+c)}$ fixed by every element of $\{0\} \times \mathbb{Z}^c$. A conjugacy $X^{(d\to d+c)} \to Y^{(d\to d+c)}$ must induce a conjugacy of the associated actions of $\mathbb{Z}^d \times \{0\}$ on X_1 and Y_1. But the action of $\mathbb{Z}^d \times \{0\}$ on X_1 is conjugate to X. We leave the verification of the remaining claims as an exercise (see also [**8**]).

Thus from any of the infinite Fagnani/Schmidt families of $\mathbb{Z}$ group shifts X which are topologically isomorphic to the same Bernoulli but pairwise not algebraically isomorphic, we get a family of $\mathbb{Z}^d$ group shifts $X^{(1\to d)}$ with the same properties.

6. *Abelian group shifts factor algebraically onto equal-entropy Bernoulli group shifts*

Recall that for a group shift X its homoclinic subgroup is denoted Δ_X. We begin with the following simple lemma. In the abelian (and not necessarily zero-dimensional) case, the essence of the lemma already appears in [**4,** Lemma 4.5].

LEMMA 6.1. *Suppose that $\gamma : X \to Y$ is an algebraic factor map between group shifts. Then the following are equivalent:*
(1) $h(\ker(\gamma)) = 0$;
(2) $h(X) = h(Y)$;
(3) *for every closed shift-invariant subgroup W of X, $h(\gamma W) = h(W)$; and*
(4) $\Delta_X \cap \ker(\gamma)$ *is the trivial group.*

Proof.
(1) $\iff$ (2) This follows from the addition formula for entropy (2.2).
(1) $\iff$ (4) This follows from Proposition 2.5.
(3) $\implies$ (2) This is trivial.
(4) $\implies$ (3) Let W be a closed shift-invariant subgroup of X. Because $\ker(\gamma|_W) \subset \ker(\gamma)$ and $\Delta_W \subset \Delta_X$, it follows from (4) that $\Delta_W \cap \ker(\gamma|_W)$ is trivial. As before this implies $h(\gamma W) = h(W)$. □

We review a little algebra. For $\mathfrak{R}_d$ as in Notation 2.9 and $\mathbf{n} = (n_1, \dots, n_d) \in \mathbb{Z}^d$, we use the notation

$$u^{\mathbf{n}} = (u_1)^{n_1} \cdots (u_d)^{n_d}$$

and we write an element f of $\mathfrak{R}_d$ as

$$f = \sum_{\mathbf{n}\in\mathbb{Z}^d} c_f(\mathbf{n}) u^{\mathbf{n}},$$

where each $c_f(\mathbf{n})$ is in $\mathbb{Z}$ and $c_f(\mathbf{n}) \neq 0$ for only finitely many $\mathbf{n}$. Such an f defines a character χ_f on $\mathbb{T}^{\mathbb{Z}^d}$ by the rule

$$\chi_f : x \mapsto \sum_{\mathbf{n}\in\mathbb{Z}^d} c_f(\mathbf{n}) x_{\mathbf{n}}$$

and any character on a subgroup of $\mathbb{T}^{\mathbb{Z}^d}$ is the restriction of some χ_f. For f in $(\mathfrak{R}_d)^J$ we use the notation $f = (f_1, \dots, f_J)$. From such an f we obtain a character χ_f on $(\mathbb{T}^{\mathbb{Z}^d})^J$ by the rule

$$x = (x_1, \dots, x_J) \mapsto f_1(x_1) + \cdots + f_J(x_J)$$

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and all characters on $(\mathbb{T}^{\mathbb{Z}^d})^J$ arise in this way. Given a character χ_f on $\mathbb{T}^{\mathbb{Z}^d}$ we define a homomorphism $\pi_f : \mathbb{T}^{\mathbb{Z}^d} \to \mathbb{T}^{\mathbb{Z}^d}$ by the rule $(\pi_f x)_{\mathbf{n}} = \chi_f(\sigma^{\mathbf{n}} x)$.

We will usually restrict the domain of χ_f and π_f, and will indicate the restricted domain where needed. When we say that f is zero we mean that it is identically zero, and likewise a group is zero if it contains just one element.

LEMMA 6.2. *Suppose that p is a rational prime, $k \in \mathbb{N}$, $f \in \mathfrak{R}_d$ and f is not zero* mod p. *Let γ denote the restriction of π_f to $X = F(p^k)$. Then the following hold:*
(1) *$h(\gamma Y) = h(Y)$, for every closed shift-invariant subgroup Y of $F(p^k)$; and*
(2) *$\gamma X = X$.*

Proof. To prove (1), it suffices by Lemma 6.1 to show, given a non-trivial homoclinic point x in X, that $\gamma(x) \neq 0$. Let $\prec$ denote lexicographic order on $\mathbb{Z}^d$, and let $\equiv$ denote congruence modulo p. Define vectors $\mathbf{v}$, $\mathbf{w}$ in $\mathbb{Z}^d$ by

$$x_{\mathbf{v}} \not\equiv 0 \quad \text{and} \quad \mathbf{n} \prec \mathbf{v} \implies x_{\mathbf{n}} \equiv 0;$$
$$c_f(\mathbf{w}) \not\equiv 0 \quad \text{and} \quad \mathbf{w} \prec \mathbf{n} \implies c_f(\mathbf{n}) \equiv 0.$$

Because $\ker(\gamma)$ is shift invariant, we may suppose that $\mathbf{v} = \mathbf{w} = \mathbf{0}$. Then

$$(\gamma x)_{\mathbf{0}} = \sum_{\mathbf{n} \in \mathbb{Z}^d} c_f(\mathbf{n}) x_{\mathbf{n}}$$
$$\equiv c_f(\mathbf{0}) x_{\mathbf{0}} \not\equiv 0.$$

This proves (1). Now (2) follows because X has no proper subshift of full entropy. □

LEMMA 6.3. *Let p be a rational prime. Suppose X is an abelian algebraic subshift of a Bernoulli group shift $B = B_1 \times \cdots \times B_J$ where $B_j = F(p^{k_j})$, and $X \neq B$. Then there is a continuous group homomorphism $\phi : B \to B'$, where B' is the Bernoulli group shift $B' = B'_1 \times \cdots \times B'_J$ such that the following hold:*
(1) *there is an index i such that $B'_i = F(p^{k_i - 1})$ and $B'_j = B_j$ for $j \neq i$; and*
(2) *$h(\phi X) = h(X)$.*

Proof. By Pontryagin duality, since $X \leq B$ and $X \neq B$, there is a character χ which annihilates X but does not annihilate B. Pick $f = (f_1, \ldots, f_J) \in (\mathfrak{R}_d)^J$ so that $\chi = \chi_f$. Without loss of generality choose $f_j = 0$ if χ_{f_j} annihilates B_j. Let $g = (g_1, \ldots, g_J)$ be the unique element of $(\mathfrak{R}_d)^J$ such that $f = p^r g$ where r is a non-negative integer and g is not identically zero modulo p. Let $\mathcal{J} = \{j : f_j \neq 0\}$. If $j \in \mathcal{J}$, then $\chi_{f_j}(B_j) \neq 0$, and therefore $r < k_j$, for every j in $\mathcal{J}$. Define the integer $K = \max\{k_j : j \in \mathcal{J}\}$. Without loss of generality, if necessary after permuting the coordinate groups B_j, we may assume that $1 \in \mathcal{J}$ and $k_1 = K$ and g_1 is not zero modulo p. Because $r < K$ and $p^r \chi_g$ annihilates X, we have

$$\pi_g(X) \leq F(p^r) = p^{K-r} F(p^K) \leq p F(p^K). \tag{6.4}$$

Define $\phi : B \to B$ by the rule

$$(\phi x)_i = x_i \quad \text{if } i > 1,$$
$$(\phi x)_1 = p x_1 - \pi_g(x).$$

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It follows from (6.4) that $\phi X \leq pB_1 \times B_2 \times \cdots \times B_J$. Because $F(p^{K-1}) = pB_1$, the lemma will follow if we show that $h(\phi X) = h(X)$. By the addition formula (2.2), it suffices to show that the kernel of ϕ has zero entropy. Clearly,

$$\ker(\phi) = \{x = (x_1, 0, \ldots, 0) \in X \mid px_1 - \pi_{g_1}(x_1) = 0\}.$$

Define an element ℓ of $\mathfrak{R}_d$ by $\ell = pu^{\mathbf{0}} - g_1$, so the restriction γ of π_ℓ to $F(p^K)$ sends a point y to the point $py - \pi_{g_1}(y)$. Since g_1 modulo p is not zero, it follows from Lemma 6.2 that $\ker(\gamma)$ has entropy zero. Because the algebraic shift $\ker(\phi)$ is isomorphic to the shift $\ker(\gamma)$, it follows that $\ker(\phi)$ has entropy zero. □

THEOREM 6.5. *Suppose α is a finite-entropy action of $\mathbb{Z}^d$ by continuous automorphisms on a compact metrizable zero-dimensional abelian group. Then there is an algebraic factor map $\phi : X \to B$ where B is a $\mathbb{Z}^d$ Bernoulli group shift such that $h(X) = h(B)$. This Bernoulli group shift is unique up to isomorphism of its alphabet group.*

Remark 6.6. Above, the group shift B is canonically associated to α, but we are not constructing a canonical factor map to B.

Proof of Theorem 6.5. Without loss of generality we assume that α has positive entropy. First we prove the existence of ϕ. Given α, there exists an algebraic factor map onto a group shift of equal entropy. So, we can assume that X is a group shift. There is a continuous group isomorphism which sends X to a group which is the product of finitely many groups $X_{(p)}$, where $X_{(p)}$ is a subgroup of a group of the form $B = F(p^{k_1}) \times \cdots \times F(p^{k_J})$, and $X_{(p)}$ is invariant under the $\mathbb{Z}^d$ shift. For each $X_{(p)}$, iteration of Lemma 6.3 produces a continuous group epimorphism $\phi_{(p)}$ onto a Bernoulli shift of entropy $h(X_{(p)})$. Define $\phi = \bigoplus_p \phi_{(p)}$. This proves the existence of the Bernoulli shift B.

For uniqueness, let $\phi : X \to B$ be an algebraic factor map with B Bernoulli. First suppose that X is a group shift. As usual, without loss of generality we assume that X is a p-group. We can assume $B = \bigoplus_{k=1}^K F(p^k)^{d_k}$ with $d_K > 0$ and $d_k \geq 0$ for $1 \leq k < K$, and clearly those numbers d_k determine B up to isomorphism of the alphabet group. We observe that

$$\begin{aligned} h(p^j B) &= 0 \quad \text{if } j \geq K, \\ h(p^{K-1} B) &= d_K \log p, \\ h(p^{K-2} B) &= (d_{K-1} + 2d_K) \log p, \\ h(p^j B) &= (d_{j+1} + 2d_{j+2} + \cdots + (K-j)d_K) \log p \quad \text{if } j \leq K-2. \end{aligned}$$

For every non-negative integer j, ϕ maps $p^j X$ onto $p^j B$, and then by Lemma 6.1 we have $h(p^j X) = h(p^j B)$. Thus the entropies $h(p^j X)$ and the displayed equations determine the numbers d_k recursively. This proves the uniqueness claim in the case that X is a group shift.

In general, up to algebraic conjugacy, we may assume that X is an inverse limit by algebraic factor maps of group shifts, $X = (X_1 \leftarrow X_2 \leftarrow X_3 \leftarrow \cdots)$, with $x \in X$ written as $x = (x_1, x_2, \ldots)$. By finite entropy and zero dimension, we have $h(X_n) = h(X)$ for all large n; by uniform continuity and zero dimension, for all large n there are algebraic

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factor maps $\phi_n : X_n \to B$ such that $\phi(x) = \phi_n(x_n)$. Now for all large n and all non-negative integers j we have $h(p^j X_n) = h(p^j X_{n+1}) = h(p^j X)$. The uniqueness claim then follows from the group shift case. □

Remark 6.7. For an action α of $\mathbb{Z}^d$ on a compact metrizable group, there is a canonical maximal algebraic factor map onto a zero-dimensional algebraic action, which is the homomorphism whose kernel is the connected component of the identity [**14**]. Thus it follows from Theorem 6.5 that α has as a quotient a canonical Bernoulli group shift of maximal entropy whenever this maximal zero-dimensional factor has finite entropy; in particular, whenever α itself has finite entropy.

7. *Classification of c.p.e. abelian group shifts up to algebraic weak equivalence*

Following Einsiedler and Schmidt [**4**], we say that two actions α, α' of $\mathbb{Z}^d$ by continuous automorphisms on compact groups X, X' are *algebraically weakly equivalent* if there are continuous group epimorphisms $X \to X'$ and $X' \to X$ which intertwine the actions. In this section we classify c.p.e. abelian group shifts up to algebraic weak equivalence, as follows.

THEOREM 7.1. *Suppose that X is an abelian group shift of completely positive entropy. Then X is weakly algebraically equivalent to a Bernoulli group shift. Two Bernoulli group shifts are weakly algebraically equivalent if and only if they have isomorphic alphabet groups.*

Remark 7.2. There are some earlier examples in this direction from [**11**]. There, Kitchens constructed several explicit vector shifts as algebraic quotients of full shifts, and in some cases constructed an explicit (non-obvious) algebraic conjugacy back to the full shift.

Remark 7.3. Before continuing to the proof of Theorem 7.1 and related matters, we note that Theorem 7.1 gives an alternative proof in the zero-dimensional case of a difficult theorem of Rudolph and Schmidt ([**17**] or [**19,** Theorem 23.1]): a c.p.e. $\mathbb{Z}^d$ action by continuous automorphisms on a compact abelian group is measurably (with respect to Haar measure) isomorphic to a $\mathbb{Z}^d$ Bernoulli shift. In the zero-dimensional case, such an action is an inverse limit of abelian group shifts, which by Theorem 7.1 are factors of Bernoulli group shifts. Thus the zero-dimensional case of the theorem by Rudolph and Schmidt follows immediately given two results of the general Bernoulli theory for amenable group actions: a factor of Bernoulli is measurably Bernoulli [**15**] and an inverse limit of Bernoulli's is measurably Bernoulli (Dan Rudolph pointed out to us that this follows from [**18**]).

Let X be a compact metrizable abelian group with expansive $\mathbb{Z}^d$ action α by continuous automorphisms; then the homoclinic group Δ_X is a countable group, and becomes an $\mathfrak{R}_d$ module (recall Notation 2.9) via the $\mathbb{Z}^d$ action by restriction of α. Einsiedler and Schmidt [**4**] defined the *adjoint action* α^* to be the $\mathbb{Z}^d$ action dual to this module. They showed among other results that: α^* is expansive and c.p.e.; α^* is weakly algebraically equivalent to α if α is c.p.e.; and α^{***} is algebraically isomorphic to α^*. When X is an abelian group shift, so is X^*.

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Proof of Theorem 7.1. By Theorem 6.5 there is a continuous algebraic homomorphism ϕ from X onto a Bernoulli group shift B of equal entropy. Theorem 7.1 will then follow from the uniqueness statement of Theorem 6.5 if we can find an algebraic factor map from B onto X. For this, we will give two proofs.

First, because $h(X) = h(B)$, it follows from Lemma 6.1 that $\phi : X \to B$ induces an embedding of homoclinic modules $\Delta_X \to \Delta_B$. Duality gives an algebraic epimorphism $B^* \to X^*$. The general result of Einsiedler and Schmidt [**4**] mentioned above gives an algebraic factor map $X^* \to X$. The composition $B^* \to X^* \to X$ gives an algebraic factor map $B^* \to X$. Finally, B^* and B are algebraically isomorphic, so we have the required map $B \to X$.

Second, for a self-contained proof in the spirit of seeing the group shift case directly, we shall give an elementary construction of an algebraic factor map $B \to X$. The basic idea is that any map from the natural generating set of Δ_B into Δ_X extends uniquely to an algebraic map $B \to X$, and with a little care we can guarantee that this map is surjective.

Without loss of generality, suppose that $h(X) > 0$, p is prime and $B = \bigoplus_{k=1}^{K} (F(p^k))^{d_k}$, where $d_K > 0$ and $d_k \geq 0$ for $1 \leq k < K$. Let $\{b_i^k : d_k > 0,\ 1 \leq i \leq d_k,\ 1 \leq k \leq K\}$ be the natural corresponding set of generating homoclinic points in B: here $b_i^k(\mathbf{v}) = 0$ for every $\mathbf{v} \in \mathbb{Z}^d$ except $\mathbf{v} = \mathbf{0}$, and $b_i^k(\mathbf{0})$ is 1 in the coordinate for the ith copy of $F(p^k)$, and 0 in other coordinates. For each b_i^k we will pick an image homoclinic point x_i^k of order p^k in X. Such a choice determines a map $\psi : B \to X$ by the rule

$$\psi : \sum_{k=1}^{K} \sum_{i=1}^{d_k} \sum_{j=0}^{k-1} \sum_{\mathbf{v} \in \mathbb{Z}^d} c_{i,j,\mathbf{v}}^{(k)} p^j \sigma^{\mathbf{v}} b_i^{(k)} \mapsto \sum_{k=1}^{K} \sum_{i=1}^{d_k} \sum_{j=0}^{k-1} \sum_{\mathbf{v} \in \mathbb{Z}^d} c_{i,j,\mathbf{v}}^{(k)} p^j \sigma^{\mathbf{v}} x_i^{(k)},$$

where the $c_{i,j,\mathbf{v}}^{(k)}$ are arbitrary in $\{0, 1, \ldots, p-1\}$. Every point of B has a unique expression of the input form above; when the $x_i^{(k)}$ are homoclinic, the output expressions are well defined and ψ is a continuous shift commuting group homomorphism.

For a group shift Y, let $E_k(Y) = \{y \in Y : p^k y = 0\}$ and let $D_k(Y)$ be the closure of the homoclinic subgroup of $E_k(Y)$. We claim that, for $1 \leq k \leq K$,

$$h(p^{k-1}(D_k(X))) = (d_K + \cdots + d_k) \log p. \tag{7.4}$$

To prove the claim, first note that by the entropy addition formula (2.2) we have

$$h(X) = h(p^k X) + h(E_k(X)),$$
$$h(B) = h(p^k B) + h(E_k(B)).$$

We also have $h(X) = h(B)$ and by Lemma 6.1(3) we have $h(p^k X) = h(p^k B)$, and therefore $h(E_k(X)) = h(E_k(B))$. Then again by the addition formula (2.2), we have

$$h(E_k(X)/E_{k-1}(X)) = h(E_k(B)/E_{k-1}(B)) = (d_K + \cdots + d_k) \log p.$$

Because $h(D_k(X)) = h(E_k(X))$ and $D_{j-1}(X) \leq D_j(X)$, again using the addition formula (2.2) we have

$$h(D_k(X)/D_{k-1}(X)) = h(E_k(X)/E_{k-1}(X)) = (d_K + \cdots + d_k) \log p.$$

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This proves the claim (7.4), because the group shifts $D_k(X)/D_{k-1}(X)$ and $p^{k-1}D_k(X)$ are isomorphic.

We will need a little notation for our recursive choice of the homoclinic points $x_i^{(k)}$ and some related objects. For $1 \le k \le K$, let W_0^k be the group subshift generated by the set

$$\mathcal{S}_k = \{p^{t-1}x_i^{(t)} \mid t \ge k,\, d_t > 0,\, 1 \le i \le d_t\}.$$

(So, $W_0^{K+1} = 0$.) For k with $d_k > 0$, we will make our choices of the points $x_i^{(k)}$ using the inductive hypothesis that the points $x_i^{(t)}$ have been chosen for $t > k$ and that

$$h(W_0^{k+1}) = (d_K + \cdots + d_{k+1}) \log p, \tag{7.5}$$

where for $k = K$ we interpret the right side of (7.5) to be zero.

So, assume that $d_k > 0$ and the inductive hypothesis is satisfied. For $1 \le i \le d_k$, we will inductively choose points $x_i^{(k)}$ and y_i, and group shifts W_i. Here $W_0 = W_0^{K+1}$, and for $1 \le i \le d_k$ the group shift generated by W_{i-1} and y_i is W_i. At step i we will then choose points x_i and y_i satisfying the following conditions, assuming if $i > 1$ that they have been satisfied at steps $t < i$:

(1) $y_i \in p^{k-1}D_k$ and $y_{i-1} \notin W_{i-1}$;

(2) $y_i = p^{k-1}x_i^{(k)}$, where $x_i^{(k)}$ is a homoclinic point in D_k;

(3) $x_i^{(k)}$ has order p^k; and

(4) $h(W_i) = (d_K + \cdots + d_{k+1} + i) \log p$.

(Note that the last item for $i = d_k$ establishes the inductive hypothesis (7.5) for the next stage.) Now we explain why we can make the choices at stage i satisfying the listed conditions.

(1) We have $W_0 \le p^{k-1}D_k$ and $y_t \in p^{k-1}D_k$ for $t < i$, so $W_{i-1} \le p^{k-1}D_k$. By (7.4) we have $h(W_{i-1}) < (d_K + \cdots + d_k) \log p = h(p^{k-1}D_k)$. Therefore we can choose y_i as required.

(2) The point y_i is in the complement of the closed subset W_{i-1} of $p^{k-1}D_k$, and the homoclinic points of D_k are dense in D_k. So perhaps after redefining y_i, we can choose a homoclinic point $x_i^{(k)}$ of D_k such that $y_i = p^{k-1}x_i^{(k)}$.

(3) We have $p^{k-1}x_i^{(k)} \ne 0$ because $p^{k-1}x_i^{(k)} \notin W_{i-1}$. We have $p^k x_i^{(k)} = 0$ because $p^k D_k = 0$. So, $x_i^{(k)}$ has order p^k.

(4) Because $pW_{i-1} = 0 = pW_i$, recalling Remark 3.3 we see that the entropies of W_{i-1} and W_i are integer multiples of $\log p$. Because W_i is generated as a group shift by the group shift W_{i-1} and the single point y_i of order p, W_i is an algebraic factor of $W_{i-1} \times F(p)$, so either $h(W_i) = h(W_{i-1})$ or $h(W_i) = h(W_{i-1}) + \log p$. Because y_i is a non-zero homoclinic point not in W_i, we have $h(W_i) > h(W_{i-1})$, so $h(W_i) = h(W_{i-1}) + \log p = (d_K + \cdots + d_{k+1} + i) \log p$ as required.

This finishes our choice of the homoclinic points $x_i^{(k)}$ of order p^k. It remains to check that the algebraic map $B \to X$ determined by the rule $b_i^{(k)} \mapsto x_i^{(k)}$ is surjective. Let U denote the group shift W_1 above, generated by the $D = d_K + \cdots + d_1$ points $p^{k-1}x_i^{(k)}$ of $\mathcal{S}_1$. Now $h(U) = D \log p$ and $pU = 0$.

There is an algebraic factor map γ mapping $F(p)^D = Y$ onto U, determined by sending the D natural generating homoclinic points of Y to the points $p^{k-1}x_i^{(k)}$.

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Because $h(Y) = h(U)$, the restriction of γ to Δ_Y is injective, and thus an isomorphism onto the group generated by $\mathcal{S}'$. Consequently, given any choices of terms $c^{(k)}_{i,k-1,\mathbf{v}}$ from $\{0, 1, \ldots, p-1\}$ with only finitely many of them non-zero, we have

$$\sum_{k=1}^{K}\sum_{i=1}^{d_k}\sum_{\mathbf{v}\in\mathbb{Z}^d} c^{(k)}_{i,k-1,\mathbf{v}} p^{k-1}\sigma^{\mathbf{v}} x^{(k)}_i = 0 \quad\Longrightarrow\quad \text{every } c^{(k)}_{i,k-1,\mathbf{v}} = 0. \tag{7.6}$$

Now we consider our map $\psi : B \to X$ determined by the choices $b^{(k)}_i \mapsto x^{(k)}_i$. Because B and X are c.p.e. with equal entropy, ψ will be surjective if its restriction to Δ_B is injective. For this, given an arbitrary choice of terms $c^{(k)}_{i,j,\mathbf{v}}$ from $\{0, 1, \ldots, p-1\}$ with all but finitely many non-zero, and given

$$z := \sum_{k=1}^{K}\sum_{i=1}^{d_k}\sum_{j=1}^{k-1}\sum_{\mathbf{v}\in\mathbb{Z}^d} c^{(k)}_{i,j,\mathbf{v}} p^{j}\sigma^{\mathbf{v}} x^{(k)}_i = 0, \tag{7.7}$$

it suffices to show that all the terms $c^{(k)}_{i,j,\mathbf{v}}$ are zero. So, suppose not. A non-zero term $c^{(k)}_{i,j,\mathbf{v}} p^{j}\sigma^{\mathbf{v}} x^{(k)}_i$ in the sum for z in (7.7) is a point of order $p^{k-j} > 0$. Let p^t be the maximum such order and consider the presentation of $0 = p^{t-1}z$ as the sum of the non-zero points $c^{(k)}_{i,j,\mathbf{v}} p^{j+t-1}\sigma^{\mathbf{v}} x^{(k)}_i$. The sum of these non-zero points is a sum of the form on the left side of (7.6). This contradicts (7.6), and finishes the proof. □

Remark 7.8. If $\phi : X \to Y$ is an algebraic factor map of equal-entropy group shifts, then ϕ restricts to an injection $\gamma : \Delta_X \to \Delta_Y$ and ϕ maps $\overline{\Delta_X}$ onto $\overline{\Delta_Y}$. This by no means assures that the map $\Delta_X \to \Delta_Y$ will be bijective. For example, suppose X is c.p.e. abelian, $Y = B$ a Bernoulli group shift and γ is surjective. Then $\gamma^{-1} : \Delta_B \to \Delta_X$ extends to an algebraic factor map ψ from B onto X. In this case $\phi\psi$ is the identity on $\overline{\Delta_B} = B$ and it follows that ϕ is an algebraic conjugacy. In particular, if X is c.p.e. and not algebraically Bernoulli, then the homomorphism $\phi : X \to B$ constructed in Theorem 6.5 can never restrict to a bijection $\Delta_X \to \Delta_B$.

Example 7.9. In this example, X is a c.p.e. $\mathbb{Z}$ group shift such that X^* is not algebraically Bernoulli. Inductively set $X_{\langle 0\rangle} = X$ and $X_{\langle n+1\rangle} = X^*_{\langle n\rangle}$. Then for all $n > 0$, the group shift $X_{\langle n\rangle}$ is also not algebraically Bernoulli.

Here X comes from [**9,** Example 3] of Kitchens; X is the Markov subgroup of the Bernoulli shift with alphabet $\mathbb{Z}/4 \oplus \mathbb{Z}/2$ with transitions

$$(0, 0), (2, 0), (1, 1), (3, 1) \to \mathbb{Z}/4 \oplus \{0\},$$
$$(1, 0), (3, 0), (2, 1), (0, 1) \to \mathbb{Z}/4 \oplus \{1\}.$$

The homoclinic group Δ_X is generated under the shift by two points of the form $x = (0, 0)^\infty.(2, 0)(0, 0)^\infty$ and $x' = (0, 0)^\infty(1, 0).(1, 1)(0, 0)^\infty$. We note that as an $\mathfrak{R}_1$-module, Δ_X is not cyclic. For if Δ_X were generated by a single homoclinic point $z = (z_i)_{i\in\mathbb{Z}} = (z^1_i, z^2_i)_{i\in\mathbb{Z}}$, then x' would be a sum of shifted multiples of z and so $z^2_i = 1$ for at least one $i \in \mathbb{Z}$. Obviously having another $z^2_j = 1$ would force an infinite number of coefficients $a_n \in \mathbb{Z}/4$ in $x' = \sum_n a_n\sigma^n(z)$ to be non-zero. Without loss of generality assume that $z^2_0 = 1$ and $z^2_j = 0$ for all $j \neq 0$. Now all a_n $(n \neq 0)$ have to be even but a_0 has to

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be odd. This implies that z_0^1 and z_{-1}^1 are odd and z_i^1 are even for all $i \notin \{0, -1\}$ and therefore $2z = (0, 0)^\infty(2, 0).(2, 0)(0, 0)^\infty$. As $x_i^2 = 0$ for all $i \in \mathbb{Z}$, the equation $x = \sum_n b_n \sigma^n(z)$ is only valid if every b_n is even, but then again the two odd components z_{-1}^1 and z_0^1 force infinitely many b_n to be non-zero.

The system X^* contains elements of order four and has entropy log 4. If X^* were algebraically Bernoulli, then its dual module Δ_X would have to be cyclic with a generator of order 4; but Δ_X is not cyclic. Therefore X^* is not Bernoulli. By [**4,** Theorem 4.7], $X_{\langle n \rangle}$ is algebraically conjugate to $X_{\langle n+2 \rangle}$ for all $n \geq 1$. If Y is algebraically Bernoulli, then Y is algebraically conjugate to Y^*. It follows that no $X_{\langle n \rangle}$ is algebraically Bernoulli.

To continue the concrete example, note that projection onto the $\mathbb{Z}/4$ coordinate defines an algebraic factor map ϕ from X onto the Bernoulli group shift B with alphabet $\mathbb{Z}/4$. As guaranteed by Remark 7.8, the induced injection of homoclinic groups is not surjective; $\phi(\Delta_X)$ is the set of homoclinic points y in B such that $\sum_n y_n$ is zero mod 2, and $\phi(\Delta_X)$ has index 2 in Δ_B. The Laurent modules Δ_B and Δ_X are of course not isomorphic, because Δ_B is cyclic and Δ_X is not.

We do not know if the abelian hypothesis in Theorem 7.1 is necessary. On the other hand, an algebraic factor of a group shift must be algebraically conjugate to a group shift, for the following reason.

PROPOSITION 7.10. *Suppose that* (X, α) *and* (X', α') *are algebraic* $\mathbb{Z}^d$ *actions, and* α' *is an algebraic factor of* α. *If* X *is zero-dimensional, then so is* X'. *If* α *is expansive, then so is* α'.

Proof. First, suppose that X is zero-dimensional, and u, v are points in X with distinct images under the assumed algebraic factor map $X \to X'$, with kernel K. For sufficiently small clopen neighborhoods C_u, C_v of u, v the sets $K + C_u$, $K + C_v$ are disjoint and clopen. Therefore X' is zero-dimensional.

For the other claim, we use equivalence of expansiveness and the descending chain condition ([**12**] or [**19,** Corollary 4.7]). Suppose that α' is not expansive. Then there is an infinite decreasing chain of distinct α' invariant subgroups, and their preimages give a chain of the same sort for α. Therefore α is not expansive. □

For the case $\mathbb{Z}^d = \mathbb{Z}$ and X' a non-expansive inverse limit of equal-entropy Bernoulli shifts, it follows from [**1,** Theorem 2.10] that there cannot even exist a shift of finite type X which admits a topological factor map from X onto X'.

Dedication. This paper is dedicated in memory of Bill Parry. Bill made seminal contributions to ergodic theory and symbolic dynamics; for example, the current paper studies certain $\mathbb{Z}^d$ shifts of finite type, and Bill's paper [**16**] introduced the $\mathbb{Z}$ shifts of finite type (topological Markov shifts), the most fundamental objects of symbolic dynamics. The older author of the current paper is one of the mathematicians who benefited hugely from Bill's mathematics and kindness, and feels his absence.

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