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REPORT ON THE FINITENESS OF SILTING OBJECTS

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Abstract We discuss the finiteness of (two-term) silting objects. First, we investigate new triangulated categories without silting object. Second, we study two classes of τ-tilting-finite algebras and give the numbers of their two-term silting objects. Finally, we explore when τ-tilting-finiteness implies representation-finiteness and obtain several classes of algebras in which a τ-tilting-finite algebra is representation-finite.

Keywords: silting object; support τ-tilting module; τ-tilting-finite

2020 *Mathematics subject classification:* Primary 16G20; 16G60

1. Introduction

In this paper, we discuss three subjects on the finiteness of (two-term) silting objects.

The case we should first consider is when the number of silting objects is zero; that is, the question is which triangulated categories have no silting object. For example, the bounded derived category $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,\Lambda)$ over a finite-dimensional algebra Λ has no non-zero silting object if and only if Λ has infinite global dimension. When Λ is non-semisimple self-injective, its stable module category $\underline{\mathsf{mod}}\,\Lambda$ admits no silting object. (See [5].) Inspired by these two cases, we ask if the singularity category $\mathsf{D}_{\mathsf{sg}}(\Lambda)$ of Λ has no non-zero silting object. Here is the first main theorem of this paper.

Theorem 1 (Theorem 2.2 and Corollary 2.4). $\mathsf{D}_{\mathsf{sg}}(\Lambda)$ *admits no non-zero silting object if* Λ *has finite right self-injective dimension. In particular, the stable category of the Cohen–Macaulay category over an Iwanaga–Gorenstein algebra has no non-zero silting object.*

Next, we restrict our viewpoint from silting objects to two-term silting ones. Thanks to Adachi–Iyama–Reiten, there is a one-to-one correspondence between two-term silting objects and support τ-tilting modules [3]. We call an algebra τ-*tilting-finite* provided there are only finitely many support τ-tilting modules. For example, we know the following τ-tilting-finite algebras: representation-finite algebras, preprojective algebras of Dynkin type [26] and algebras of dihedral, semi-dihedral and quaternion type [17] and so on.

The second subject of this paper is to give two new classes of τ-tilting-finite algebras. One is the class of weakly symmetric algebras of tubular type with non-singular Cartan matrix [12, 14]. The other is the class of non-standard self-injective algebras which are socle equivalent to self-injective algebras of tubular type [13, 14]. Here is the second main theorem of this paper. (See Figures 1 and 2 for the notation of A_i's and Λ_i's.)

Theorem 2 (Theorems 3.1 and 3.3). *(1) Any weakly symmetric algebra of tubular type with non-singular Cartan matrix is* τ*-tilting-finite. In particular, we have the number of support* τ*-tilting modules:*

$A_1(\lambda)$	$A_2(\lambda)$	A_3	A_4	A_5	A_6	A_7	A_8
24	6	192	132	8	8	108	100

A_9	A_{10}	A_{11}	A_{12}	A_{13}	A_{14}	A_{15}	A_{16}
108	116	100	32	28	32	30	30

(2) Any non-standard self-injective algebra which is socle equivalent to a self-injective algebra of tubular type is τ*-tilting-finite. In particular, we have the number of support* τ*-tilting modules:*

Λ_1	Λ_2	$\Lambda_3(\lambda)$	Λ_4	Λ_5	Λ_6	Λ_7	Λ_8	Λ_9	Λ_{10}
8	8	6	32	28	32	30	30	192	≥ 500

(3) Every algebra as in (1) and (2) is tilting-discrete.

The last subject is on 'representation-finiteness vs. τ-tilting-finiteness". Evidently, a representation-finite algebra is τ-tilting-finite, but the converse does not necessarily hold. Thus, we naturally ask when τ-tilting-finiteness implies representation-finiteness. A typical example is the hereditary case; that is, τ-tilting-finite hereditary algebras are representation-finite. For more examples, it was proved that τ-tilting-finite cycle-finite algebras are representation-finite [24]. Recently, the gentle case was verified; τ-tilting-finite gentle algebras are representation-finite [28]. Now, we give new classes of algebras which satisfy this property.

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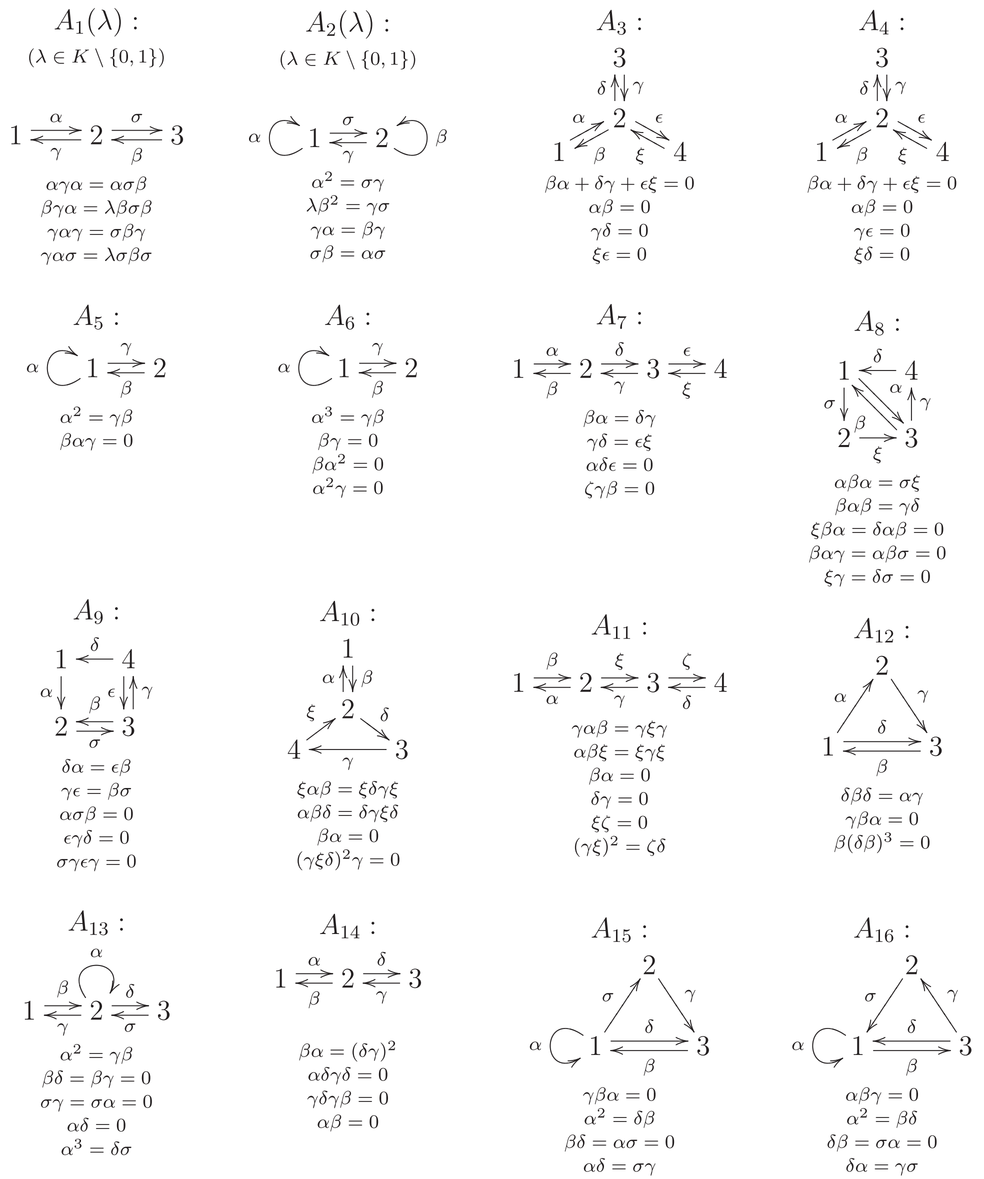

Figure 1. List of weakly symmetric algebras of tubular type.

Theorem 3 (Corollaries 4.2 and 4.3, Theorems 4.8 and 4.11). *The following algebras are representation-finite if they are τ-tilting-finite:*

(1) quasitilted algebras;

(2) algebras satisfying the separation condition;

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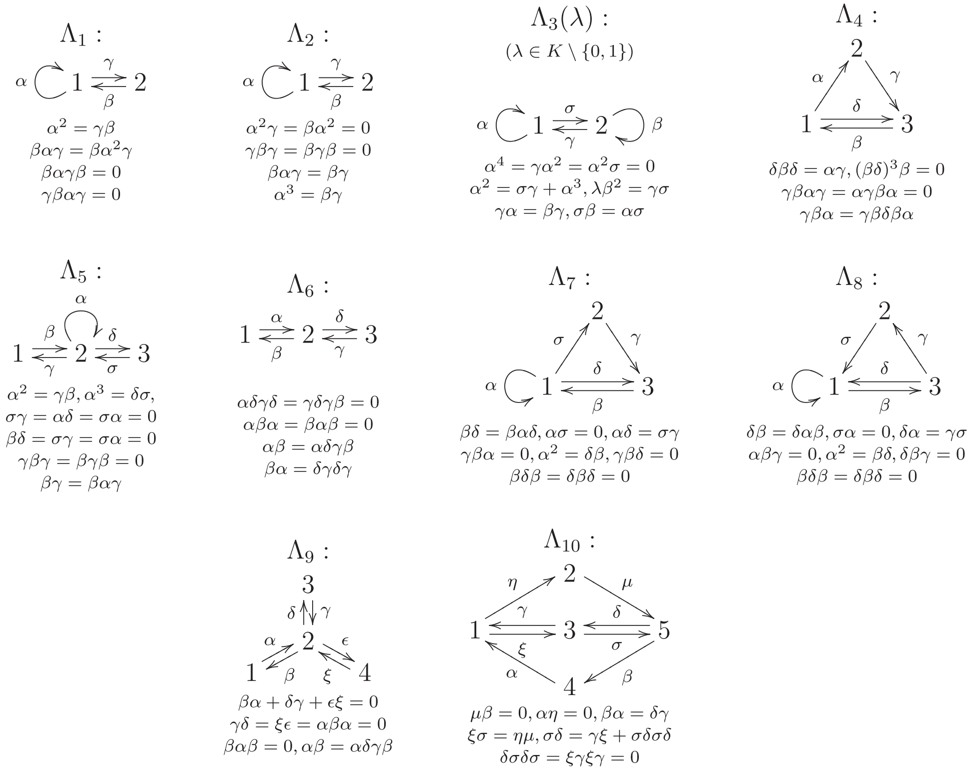

Figure 2. List of non-standard self-injective algebras which are socle equivalent to self-injective algebras of tubular type.

(3) the trivial extensions of tree quiver algebras with radical square zero;

(4) locally hereditary algebras;

Notation. Throughout this paper, algebras are always assumed to be basic, indecomposable and finite-dimensional over an algebraically closed field K. Modules are finitely generated and right. For an algebra Λ, we denote by $\mathsf{mod}\,\Lambda(\mathsf{proj}\,\Lambda,\ \mathsf{inj}\,\Lambda)$ the category of (projective, injective) modules over Λ.

2. The existence of silting objects

Let $\mathcal{T}$ be a Krull–Schmidt triangulated category which is K-linear and Hom-finite. For example, we consider the bounded derived category $\mathsf{D}^{\mathsf{b}}(\mathsf{mod}\,\Lambda)$ and the perfect derived category $\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\,\Lambda)$ over an algebra Λ. In this section, we explore when a triangulated category has no silting object. Let us recall the definition of silting objects.

Definition 2.1. An object T of $\mathcal{T}$ is said to be *presilting* (*pretilting*) if it satisfies $\mathrm{Hom}_{\mathcal{T}}(T,\ T[i]) = 0$ for any $i > 0$ $(i \neq 0)$. It is called *silting* (*tilting*) if in addition $\mathcal{T} =$

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$\mathsf{thick}\,T$. Here, $\mathsf{thick}\,T$ stands for the smallest thick subcategory of $\mathcal{T}$ containing T. We denote by $\mathsf{silt}\,\mathcal{T}$ the set of isomorphism classes of basic silting objects of $\mathcal{T}$.

A typical example of silting objects is the stalk complex Λ (and its shifts) in $\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\,\Lambda)$. If we can find even one silting object, silting mutation produces infinitely many ones [5]. However, we know triangulated categories with no silting object [5, Example 2.5].

Let Λ be an algebra. We denote by $\mathsf{D}_{\mathsf{sg}}(\Lambda)$ the singularity category of Λ; that is, it is the Verdier quotient of $\mathsf{D}^{\mathsf{b}}(\mathsf{mod}\,\Lambda)$ by $\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\,\Lambda)$. Here is the main result of this section.

Theorem 2.2. $\mathsf{D}_{\mathsf{sg}}(\Lambda)$ *has no non-zero silting object if* $\operatorname{inj.dim}\Lambda_\Lambda < \infty$.

To prove this theorem, silting reduction [5, 21] plays a crucial role.

In the rest, fix a presilting object T of $\mathcal{T}$ and define a subset $\mathsf{silt}_T\,\mathcal{T}$ of $\mathsf{silt}\,\mathcal{T}$ by

$$\mathsf{silt}_T\,\mathcal{T} := \{P \in \mathsf{silt}\,\mathcal{T} \mid T \text{ is a direct summand of } P\}.$$

Moreover, one puts $\mathcal{S} := \mathsf{thick}\,T$. The Verdier quotient of $\mathcal{T}$ by $\mathcal{S}$ is denoted by $\mathcal{T}/\mathcal{S}$.

Then, silting reduction [21, Theorem 3.7] says:

Theorem 2.3. *The canonical functor* $\mathcal{T} \to \mathcal{T}/\mathcal{S}$ *induces a bijection* $\mathsf{silt}_T\,\mathcal{T} \to \mathsf{silt}\,\mathcal{T}/\mathcal{S}$ *if any object* X *of* $\mathcal{T}$ *satisfies* $\operatorname{Hom}_{\mathcal{T}}(T, X[\ell]) = 0 = \operatorname{Hom}_{\mathcal{T}}(X, T[\ell])$ *for* $\ell \gg 0$.

For example, this is the case where $\mathcal{T}$ has a silting object [5, Proposition 2.4].

Now, we are ready to show our main theorem of this section.

Proof of Theorem 2.2. We will apply silting reduction to $\mathcal{T} = \mathsf{D}^{\mathsf{b}}(\mathsf{mod}\,\Lambda)$ and $T = \Lambda$; in this setting, $\mathcal{S} = \mathsf{thick}\,\Lambda = \mathsf{K}^{\mathsf{b}}(\mathsf{proj}\,\Lambda)$ and $\mathcal{T}/\mathcal{S} = \mathsf{D}_{\mathsf{sg}}(\Lambda)$. To do that, we check that the conditions $\operatorname{Hom}_{\mathcal{T}}(\Lambda, X[\ell]) = 0 = \operatorname{Hom}_{\mathcal{T}}(X, \Lambda[\ell])$ are satisfied for any object X and $\ell \gg 0$. The first equality holds evidently. Let us show that the second equality holds true. Since Λ has finite right self-injective dimension, it can be regarded as a complex in $\mathsf{K}^{\mathsf{b}}(\mathsf{inj}\,\Lambda)$, which is obtained by applying the Nakayama functor $\nu := - \otimes^{\mathbf{L}}_{\Lambda} D\Lambda$ to some complex P in $\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\,\Lambda)$. Then we get isomorphisms

$$\operatorname{Hom}_{\mathcal{T}}(X, \Lambda[\ell]) \simeq \operatorname{Hom}_{\mathsf{K}^{\mathsf{b}}(\mathsf{mod}\,\Lambda)}(X, \nu P[\ell]) \simeq D\operatorname{Hom}_{\mathsf{K}^{\mathsf{b}}(\mathsf{mod}\,\Lambda)}(P[\ell], X).$$

As the complex X is bounded, the last above is zero for sufficiently large ℓ. Thus, silting reduction brings us a bijection $\mathsf{silt}_{\Lambda}\,\mathsf{D}^{\mathsf{b}}(\mathsf{mod}\,\Lambda) \to \mathsf{silt}\,\mathsf{D}_{\mathsf{sg}}(\Lambda)$. It follows from [5, Example 2.5(1)] that the LHS of the bijection is $\{\Lambda\}$ if Λ has finite global dimension, and is otherwise empty. Hence, we conclude that $\mathsf{D}_{\mathsf{sg}}(\Lambda)$ admits no non-zero silting object. $\square$

An algebra Λ is said to be *Iwanaga–Gorenstein* if it has finite right and left self-injective dimension. In that case, the singularity category $\mathsf{D}_{\mathsf{sg}}(\Lambda)$ is triangle equivalent to the stable category $\underline{\mathsf{CM}}\,\Lambda$ of the full subcategory of $\mathsf{mod}\,\Lambda$ consisting of Cohen–Macaulay modules M; i.e. $\operatorname{Ext}^i_\Lambda(M, \Lambda) = 0$ for $i > 0$. So, we immediately obtain the following corollary.

Corollary 2.4. $\underline{\mathsf{CM}}\,\Lambda$ *has no non-zero silting object if* Λ *is Iwanaga–Gorenstein.*

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3. The finiteness of support τ-tilting modules

Let Λ be an algebra. We say that an object X of $\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\,\Lambda)$ is *two-term* if the ith term of X is zero unless $i = 0, -1$. A *support τ-tilting* module is defined to be the 0th cohomology of a two-term silting object of $\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\,\Lambda)$. (See [3] for details.) Denote by $\mathsf{s}\tau\text{-}\mathsf{tilt}\,\Lambda$ the set of isomorphism classes of basic support τ-tilting modules. We call Λ *τ-tilting-finite* if $\mathsf{s}\tau\text{-}\mathsf{tilt}\,\Lambda$ is a finite set.

In this section, we discuss τ-tilting-finiteness of weakly symmetric algebras of tubular type with non-singular Cartan matrix, which were completely classified up to Morita equivalence by [12] as follows (Figure 1).

The main theorem of this section is the following.

Theorem 3.1. *Any weakly symmetric algebra of tubular type with non-singular Cartan matrix is τ-tilting-finite.*

In Appendix, we will see the numbers of support τ-tilting modules of A_i's.

Proof. Note that A_i is symmetric for all i but $i = 3$ [12, Theorem 2]. Observe that the Cartan matrix of A_i has positive definite. We then apply [17, Theorem 13] to deduce the conclusion that A_i is τ-tilting-finite for all i but $i = 3$. The algebra A_3 is just the preprojective algebra of Dynkin type $\mathbb{D}_4$, and so it is τ-tilting-finite by [26, Theorem 2.21]. □

A self-injective algebra is said to be *tilting-discrete* if for any $n > 0$, there are only finitely many tilting objects of length n. Here is a corollary of Theorem 3.1.

Corollary 3.2. *Any weakly symmetric algebra of tubular type with non-singular Cartan matrix is tilting-discrete.*

Proof. A weakly symmetric algebra of tubular type with non-singular Cartan matrix is derived equivalent to one of A_i's [14], which is τ-tilting-finite by Theorem 3.1. It follows from [6, Corollary 2.11] that the algebra is tilting-discrete. □

Thanks to Białkowski–Skowroński [13], we also have a complete list of Morita equivalence classes of self-injective algebras which are socle equivalent to self-injective algebras of tubular type. We focus on such algebras which are not of tubular type. The following classes of algebras coincide [32]:

(i) self-injective algebras which are socle equivalent to self-injective algebras of tubular type but not of tubular type;

(ii) non-standard self-injective algebras which are socle equivalent to self-injective algebras of tubular type;

(iii) non-standard non-domestic self-injective algebras of polynomial growth;

(iv) algebras presented by the quivers and relations as in Figure 2.

Then we have a similar result as Theorem 3.1.

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Theorem 3.3. *All non-standard self-injective algebras which are socle equivalent to self-injective algebras of tubular type are τ-tilting-finite. Moreover, they are tilting-discrete.*

The proof of this theorem is by direct calculation, and we will give the numbers of support τ-tilting modules of Λ_i in Appendix. So, we now leave here.

4. Representation-finiteness vs. τ-tilting-finiteness

The aim of this section is to provide several classes of algebras whose τ-tilting-finiteness implies representation-finiteness.

Let C be a connected component of the Auslander–Reiten quiver of an algebra. We say that C is *preprojective* if it has no oriented cycle, and any module in C is of the form $\tau^{-n}P$ for some non-negative integer n and some indecomposable projective module P. Dually, define *preinjective* components.

We start with the following proposition, which was given in [27] as a remark; see also [1].

Proposition 4.1. *A τ-tilting-finite algebra with preprojective or preinjective component is representation-finite.*

We give several classes of algebras as in Proposition 4.1.

A *quasitilted* algebra is defined to be the endomorphism algebra of a tilting object T over a hereditary abelian K-category $\mathcal{H}$. When $\mathcal{H} = \mathsf{mod}\, K\Delta$ for some acyclic quiver Δ, the algebra is called *tilted* of type Δ. If in addition, T is preprojective, then the algebra is said to be *concealed*. We know from [15] that every quasitilted algebra admits a preprojective component. This leads to the following corollary, which is a slight generalization of Zito's result [34, Theorem 3.1].

Corollary 4.2. *A τ-tilting-finite quasitilted algebra is representation-finite.*

Let Λ be an algebra associated with an acyclic quiver Q and i a vertex of Q. We write the full subquiver of Q generated by the non-predecessors of i by $Q(i)$. An algebra Λ is said to *satisfy the separation condition* if for any vertex i of Q, all distinct indecomposable summands of rad P_i have supports lying in different connected components of $Q(i)$. Here, P_i denotes the indecomposable projective module corresponding to i. In the case, Λ admits a preprojective component [9, IX, Theorem 4.5]. So, we get the following corollary.

Corollary 4.3. *A τ-tilting-finite algebra satisfying the separation condition is representation-finite.*

Since every tree quiver algebra satisfies the separation condition [9, IX, Lemma 4.3], the following is also obtained.

Corollary 4.4. *A τ-tilting-finite tree quiver algebra is representation-finite.*

We study the nice (isomorphism) class $\mathcal{C}$ of algebras which are representation-finite or have a tame concealed algebra as a factor. Such a class contains the classes of algebras with a preprojective component [30, XIV, Theorem 3.1], cycle-finite algebras [24] and loop-finite algebras [31, Theorem 4.5]. Here is a generalization of Proposition 4.1.

Proposition 4.5. *A τ-tilting-finite algebra in $\mathcal{C}$ is representation-finite.*

Proof. Combine Corollary 4.2 and [16, Theorem 5.12(d)]. □

A *commutative ladder* of degree n is an algebra presented by the quiver

$$\begin{array}{ccccccc} 1 & \longrightarrow & 2 & \longrightarrow & \cdots & \longrightarrow & n \\ \downarrow & & \downarrow & & & & \downarrow \\ 1' & \longrightarrow & 2' & \longrightarrow & \cdots & \longrightarrow & n' \end{array}$$

with all possible commutative relations, which is isomorphic to $K\overrightarrow{\mathbb{A}_2} \otimes K\overrightarrow{\mathbb{A}_n}$. Here, $\overrightarrow{\mathbb{A}_n}$ stands for the linearly oriented quiver of type $\mathbb{A}_n$. By [18, Theorem 3], a commutative ladder of degree n is representation-finite if and only if $n \leq 4$. We derive a corollary from Proposition 4.5.

Corollary 4.6. *A τ-tilting-finite commutative ladder is representation-finite.*

Proof. Let Λ be a commutative ladder of degree 5. As the Happel–Vossieck list [20] (see also [29]), the factor algebra of Λ by the idempotents corresponding to the vertices 1 and $5'$ is a tame concealed algebra of type $\widetilde{\mathbb{E}_7}$. Observe that a commutative ladder of degree ≥ 5 has Λ as a factor. Thus, the class of commutative ladders is contained in $\mathcal{C}$. □

We can also deduce Corollary 4.6 from Corollary 4.3; this is because a commutative ladder satisfies the separation condition, since all indecomposable projectives have indecomposable radicals.

Remark 4.7. Inspired by this work, the fourth named author of this paper showed that any τ-tilting-finite strongly simply connected algebra is representation-finite [33, Theorem 2.6], which generalizes Corollaries 4.4 and 4.6.

Let us discuss algebras with radical square zero. To do that, we first recall the definition of separated quivers.

For a quiver Q, we construct a new quiver Q^s as follows:

(1) the vertices of Q^s are those of Q and their copies; we denote by i' the copy of a vertex i of Q.

(2) an arrow $a \to b$ of Q^s are drawn whenever a is a vertex i of Q, b is the copy of a vertex j of Q, and Q has an arrow $i \to j$.

We call the acyclic quiver Q^s the *separated quiver* of Q. As is well known, a radical square zero algebra presented by a quiver Q is stable equivalent to the hereditary algebra KQ^s [10, X, Theorem 2.4]. A full subquiver of a separated quiver Q^s is said to be *single*

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if it has at most one of vertices i and i' for each vertex i of Q. Then an algebra given by a quiver Q with radical square zero is τ-tilting-finite if and only if every single subquiver of Q^s is a disjoint union of Dynkin quivers [1, Theorem 3.1].

Thanks to these results, we show the following result.

Theorem 4.8. *Let Λ be an algebra presented by a tree quiver with radical square zero.*

(1) If Λ is τ-tilting-finite, then it is representation-finite.

(2) If the trivial extension of Λ is τ-tilting-finite, then it is representation-finite.

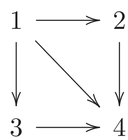
Proof. (1) This is due to Corollary 4.4, but we give another proof here, in which we use combinatorial discussion.

As the quiver of Λ is tree, we observe that every connected component R of the separated quiver has no same latter i and i'. Then we can apply [1, Theorem 3.1] for R to deduce the fact that R is of Dynkin type, since Λ is τ-tilting-finite. Hence, it follows from [10, X, Theorem 2.6] that Λ is representation-finite.

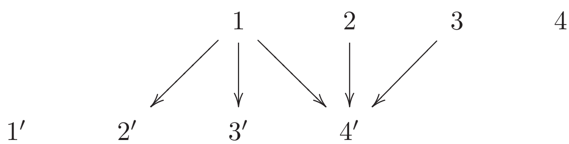
(2) If the trivial extension $T(\Lambda)$ of Λ is τ-tilting-finite, then so is Λ by [16, Theorem 5.12(d)], and hence Λ is representation-finite by (1). We observe that Λ is simply-connected and has the quadratic form of positive definite, which implies that it is an iterated tilted algebra of Dynkin type [7, Proposition 5.1]. (See also [19].) It follows from [8, Theorem 3.1] that $T(\Lambda)$ is representation-finite. □

Theorem 4.8 does not necessarily hold if Λ is given by a non-tree acyclic quiver.

Example 4.9. (1) Let Λ be an algebra presented by the quiver



with radical square zero. Then the separated quiver is the following:



Observe that it contains the extended Dynkin diagram $\widetilde{\mathbb{D}}_5$ as an underlying graph, whence Λ is τ-tilting-finite by [1] but not representation-finite by [10].

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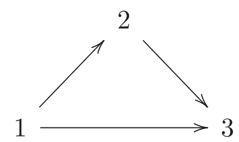
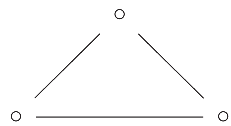
(2) Let us consider the algebra presented by the quiver



with radical square zero. Then the trivial extension is the Brauer graph algebra given by the Brauer graph



This is τ-tilting-finite by [4, Theorem 6.7] but not representation-finite.

Let Q be a quiver. The *double quiver* of Q, denoted by Q^d, is constructed from Q by adding the inverse arrow of every arrow in Q. Here is an easy observation.

Proposition 4.10. *Let Q be a tree quiver and I an admissible ideal of KQ^d. Put $\Lambda := KQ^d/I$. If Λ is τ-tilting-finite, then Q is of Dynkin type.*

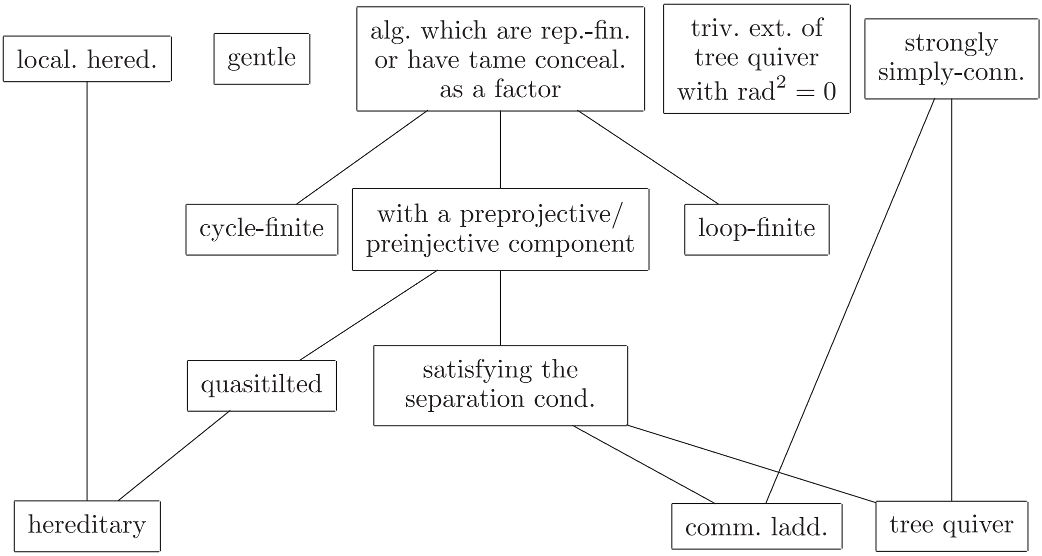
Proof. By assumption, it follows from [16] that $\Lambda/\operatorname{rad}^2\Lambda$ is τ-tilting-finite. We observe that the separated quiver of Q^d is the disjoint union of two quivers R_1 and R_2 which satisfy $i \in R_j \Leftrightarrow i' \notin R_j$ $(j = 1, 2)$ and whose underlying graphs coincide with that of Q. We apply [1] to deduce the fact that R_1, R_2, and hence Q, are of Dynkin type. □

Let us discuss the locally hereditary case. An algebra is said to be *locally hereditary* provided every homomorphism between indecomposable projective modules is a monomorphism or zero; see [11, 23, 25]. We know that such an algebra is presented by an acyclic quiver and the relations contain no monomials. We show the following theorem.

Theorem 4.11. *A τ-tilting-finite locally hereditary algebra is representation-finite.*

Proof. Let Λ be a τ-tilting-finite locally hereditary algebra. As is easy to see, the local hereditariness yields that Λ has no monomial relation and the quiver Q is acyclic. The τ-tilting-finiteness implies that Q does not contain a subquiver of extended Dynkin type, whence Λ admits all possible commutative relations. Then, we figure out that Λ is strongly simply-connected; see [22] for example. The assertion follows from [33, Theorem 2.6]. □

We close this section by giving an interesting observation. Denote by $\mathcal{A}$ the class of algebras in which τ-tilting-finiteness implies representation-finiteness; we put a hierarchy of classes contained in $\mathcal{A}$:



Proposition 4.12. *The class $\mathcal{A}$ is closed under taking factors by ideals contained in the centre and the radical.*

Proof. Let Λ be in $\mathcal{A}$ and put $\Gamma := \Lambda/I$, where I is an ideal of Λ contained in the centre and the radical. By [17, Theorem 11] (in Appendix), these algebras have the same poset of support τ-tilting modules. Therefore, if Γ is τ-tilting-finite, then so is Λ. By assumption, it turns out that Λ is representation-finite, so is Γ. □

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Appendix. The numbers of support τ-tilting modules over weakly symmetric algebras of tubular type

In this Appendix, we give the numbers of support τ-tilting modules of A_i and Λ_i as in § 3. (See the introduction for the tables of the numbers.)

The following theorem plays an important role.

Theorem A.1 (Eisele et al. [17, Theorem 11]). *Let I be a two-sided ideal of Λ which is contained in the centre and the radical of Λ. Then we have an isomorphism of posets* $\mathsf{s}\tau\text{-}\mathsf{tilt}\,\Lambda$ *and* $\mathsf{s}\tau\text{-}\mathsf{tilt}\,\Lambda/I$.

For our algebra Λ, the strategy is the following.

(i) Find central elements which are in the radical.

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(ii) Construct an ideal I generated by the elements as in (i).

(iii) Consider the factor algebra Λ/I. By Theorem A.1, we have an isomorphism of posets $\mathsf{s}\tau\text{-}\mathsf{tilt}\,\Lambda$ and $\mathsf{s}\tau\text{-}\mathsf{tilt}\,\Lambda/I$. Then, one counts the number or draws the Hasse quiver of $\mathsf{s}\tau\text{-}\mathsf{tilt}\,\Lambda/I$. If possible, we may find a nice algebra whose factor algebra is isomorphic to Λ/I and which admits a well-known Hasse quiver of support τ-tilting modules.

The number of $\mathsf{s}\tau$-tilt A_i

First, let us discuss for A_i's. In any case, we can easily check that the following elements belong to the centre.

$i = 1$: $\alpha\gamma + \gamma\alpha$ and $\beta\sigma + \sigma\beta$;	$i = 2$: $\alpha + \beta$;
$i = 3$: –;	$i = 4$: $\beta\alpha - \gamma\delta - \xi\varepsilon$ and $\alpha\delta\gamma\beta$;
$i = 5$: $\alpha\beta + \beta\alpha$;	$i = 6$: α^2 and $\beta\alpha\gamma$;
$i = 7$: $\alpha\beta + \beta\alpha + \gamma\delta + \xi\varepsilon$;	$i = 8$: $\alpha\beta + \beta\alpha$;
$i = 9$: $\beta\sigma + \varepsilon\gamma + \sigma\beta$;	$i = 10$: $\alpha\beta + \gamma\xi\delta + \xi\delta\gamma$;
$i = 11$: $\alpha\beta + \gamma\xi$;	$i = 12$: $\alpha\gamma\beta + \beta\alpha\gamma$ and $\gamma\beta\delta\beta\alpha$;
$i = 13$: α^2, $\sigma\delta$ and $\beta\alpha\gamma$;	$i = 14$: $\alpha\delta\gamma\beta$, $\delta\gamma\beta\alpha$, $\gamma\beta\alpha\delta$ and $\beta\alpha + \gamma\delta\gamma\delta$;
$i = 15$: α^2, $\beta\alpha\delta$ and $\gamma\beta\sigma$;	$i = 16$: α^2, $\delta\alpha\beta$ and $\sigma\alpha\beta$.

Let I_i be the ideal of A_i generated by the elements above and the socle, and $\overline{A_i} := A_i/I_i$. In the following, we feel free to utilize Theorem A.1 and refer to [26] for support τ-tilting modules over preprojective algebras of Dynkin type.

$i = 1$. It is seen that $\overline{A_1}$ is isomorphic to the factor algebra of the preprojective algebra of Dynkin type $\mathbb{A}_3$ by the intersection of the centre and the radical. This implies that A_1 has 24 support τ-tilting modules.

$i = 2$. Observe that $\overline{A_2}$ is the Nakayama algebra presented by the quiver $\bullet \underset{y}{\overset{x}{\rightleftarrows}} \bullet$ with relations $xy = 0 = yx$, whence there are six support τ-tilting modules of A_2.

$i = 3$. A_3 is the preprojective algebra of type $\mathbb{D}_4$, which has 192 support τ-tilting modules.

$i = 5, 6$. It is obvious that $\overline{A_5}$ and $\overline{A_6}$ are isomorphic, which are furthermore isomorphic to $R(2AB)$ in Table 2 of [17]. Hence, A_5 and A_6 have 8 support τ-tilting modules.

$i = 7$. By Theorem A.1, we have an isomorphism of posets $\mathsf{s}\tau\text{-}\mathsf{tilt}\,A_7 \simeq \mathsf{s}\tau\text{-}\mathsf{tilt}\,\overline{A_7}$. Moreover, one observes that $\overline{A_7}$ is isomorphic to the factor algebra of the preprojective algebra Γ of type $\mathbb{A}_4$ by the central elements in the radical, and the socle. However, the socle of Γ is not contained in the centre, and so we can not apply Theorem A.1 to obtain the Hasse quiver of support τ-tilting modules.

Now, let us apply Adachi's method [2]. We fix the numbering of the vertices of $\mathbb{A}_4$ by $1 \;—\; 2 \;—\; 3 \;—\; 4$ and let $\overline{\Gamma}$ be the factor algebra of Γ by the central elements in the radical. We can still apply Theorem A.1 to get an isomorphism $\mathsf{s}\tau\text{-}\mathsf{tilt}\,\Gamma \simeq \mathsf{s}\tau\text{-}\mathsf{tilt}\,\overline{\Gamma}$. Let P be the indecomposable projective module of $\overline{\Gamma}$ corresponding to the vertex 1 and

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define a subset $\mathcal{N}$ of $\mathsf{s}\tau\text{-}\mathsf{tilt}\,\overline{\Gamma}$ by

$$\mathcal{N} := \{N \in \mathsf{s}\tau\text{-}\mathsf{tilt}\,\big(\overline{\Gamma}/\operatorname{soc} P\big) \mid P/\operatorname{soc} P \in \mathsf{add}\, N \text{ and } \operatorname{Hom}_{\overline{\Gamma}}(N, P) = 0\}.$$

Here, $\operatorname{soc} P$ stands for the socle of P. We see that $\mathcal{N}$ has six elements; see [26] for example. It follows from [2, Theorem 3.3(1)] that the Hasse quiver of $\mathsf{s}\tau\text{-}\mathsf{tilt}\,\overline{\Gamma}$ can be constructed by $\mathsf{s}\tau\text{-}\mathsf{tilt}(\overline{\Gamma}/\operatorname{soc} P)$ and the copy of $\mathcal{N}$. A similar argument works for the indecomposable projective module P' of $\overline{\Gamma}$ at the vertex 4 instead of P. As $\overline{A_7}$ is isomorphic to the factor algebra of $\overline{\Gamma}$ by the socle of P and P', it turns out that $\overline{A_7}$ has precisely 12 support τ-tilting modules fewer than $\overline{\Gamma}$, so than Γ. Consequently, we obtain that A_7 has 108 support τ-tilting modules.

$i = 8, 9, 11$. We can use 'String Applet' (https://www.math.uni.-bielefeld.de/jgeuenich/string-applet/); apply it to $\overline{A_i}$.

Remark A.2. The applet can be also run for A_7.

$i = 4, 10$. We count the number of τ-tilting modules over the factor algebra by each idempotent. Let $\{e_1, \dots, e_n\}$ be a complete set of primitive orthogonal idempotents of an algebra Λ and I be a subset of $\{1, \dots, n\}$ (possibly, $I = \emptyset$). We denote by t_I the number of τ-tilting modules of $\Lambda/(e)$, where $e = \sum_{i\in I} e_i$. Here, $t_\emptyset$ means the number of τ-tilting modules of Λ. Note that the number of support τ-tilting modules over Λ is equal to $\sum_I t_I$.

We demonstrate the way of counting for $i = 4$; it similarly works for $i = 10$. Putting $\Lambda := \overline{A_4}$, e_i denotes the primitive idempotent corresponding to the vertex i.

(i) We observe that $\Lambda/(e_1)$ is the factor algebra of the Brauer tree algebra of the Brauer tree $\circ$ —— $\circ$ —— $\circ$ —— $\circ$ by some socles, and so one easily obtains $t_{\{1\}} = 9$.

(ii) When I has the vertex 2, $\Lambda/(e)$ is semisimple, so $t_I = 1$; there are eight cases.

(iii) In the cases that $I = \{3\}$ and $\{4\}$, $\Lambda/(e)$ is the preprojective algebra of type $\mathbb{A}_3$, so $t_I = 13$; see [26] for example.

(iv) For $I = \{1, 3\}, \{1, 4\}, \{3, 4\}$, see the case of $i = 2$; $t_I = 3$.

(v) We easily get $t_{\{1,3,4\}} = 1$.

There remains to count the number of τ-tilting modules of Λ. To do that, we use the GAP-package QPA; Λ is representation-finite, and so all indecomposable τ-rigid modules can be got on QPA. Then, we obtain $t_\emptyset = 79$. Consequently, one sees that there are 132 support τ-tilting modules of Λ, so of A_4.

We only put the table for A_{10}.

I	4 points 3 points	{1,2} {1,3} {1,4} {2}	{2,3} {2,4}	$\{3,4\}$	$\{1\}$	{3} {4}	$\emptyset$	total
t_I	1 (5 cases)	2	1	3	10	8	72	116

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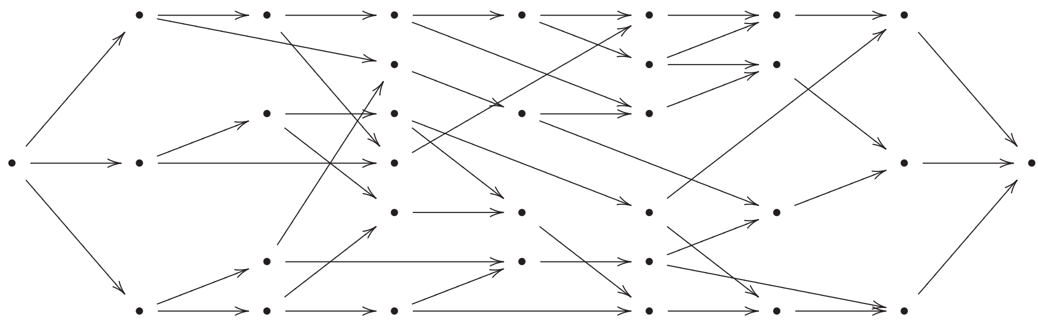
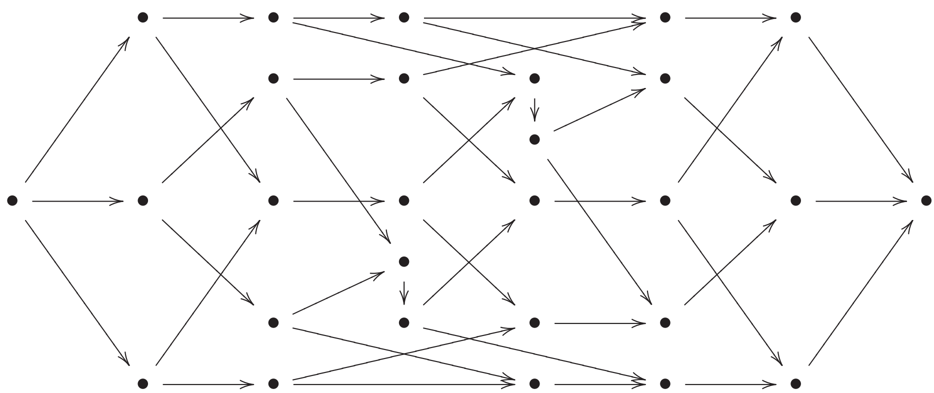
Remark A.3. It is not difficult to draw the Hasse quivers directly, but they are too large.

$i = 12 - 15$. We directly construct the Hasse quiver of $\mathsf{s\tau\text{-}tilt}\, A_i$ as follows.

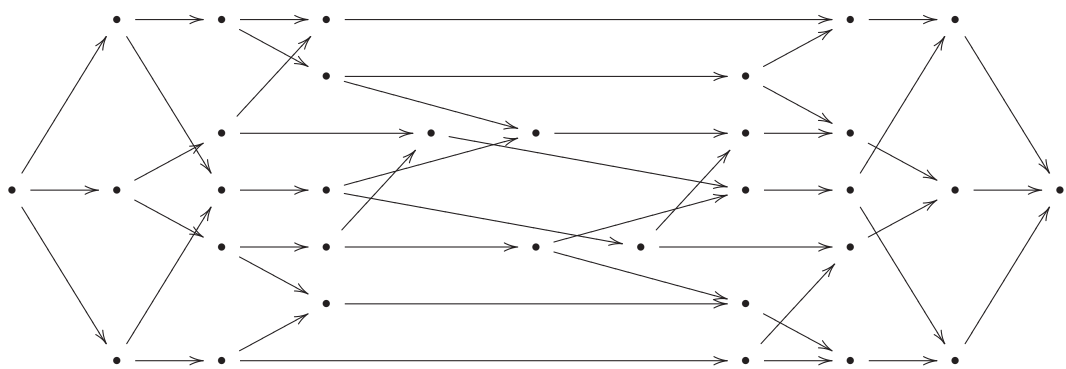
(1) The Hasse quiver of $\mathsf{s\tau\text{-}tilt}\, A_{12}$:



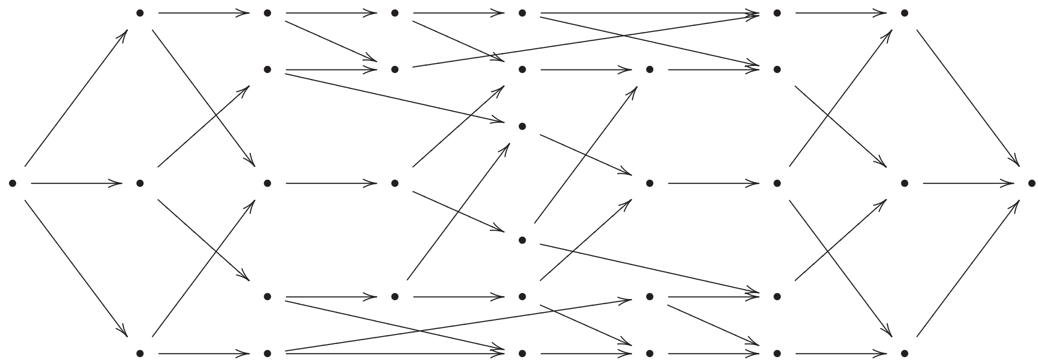
(2) The Hasse quiver of $\mathsf{s\tau\text{-}tilt}\, A_{13}$:



(3) The Hasse quiver of $\mathsf{s\tau\text{-}tilt}\, A_{14}$:



(4) The Hasse quiver of sτ-tilt A_{15}:



Remark A.4. Note that A_{16} is the opposite algebra of A_{15}. So, it follows from [3, Theorem 2.14] that there is a bijection between sτ-tilt A_{15} and sτ-tilt A_{16}. We also remark that $\overline{A_{15}}$ is not representation-finite.

The number of sτ-tilt Λ_i

Next, we discuss for Λ_i's. One gets central elements.

$i = 1$: $\alpha^2 + \beta\gamma$ and $\beta\alpha\gamma$;	$i = 2$: α^2, $\beta\gamma$ and $\gamma\beta$;
$i = 3$: $\alpha + \beta$, $\sigma\gamma$ and $\gamma\sigma$;	$i = 4$: $\gamma\beta\alpha$ and $\beta\alpha\gamma + \alpha\gamma\beta$;
$i = 5$: $\beta\gamma$, $\gamma\beta$, $\delta\sigma$ and $\sigma\delta$;	$i = 6$: $\alpha\beta$ and $\beta\alpha + \gamma\delta\gamma\delta$;
$i = 7$: $\beta\delta$, $\delta\beta$ and $\gamma\beta\sigma$;	$i = 8$: –;
$i = 9$: $\alpha\beta$, $\gamma\beta\alpha\delta$, $\delta\gamma\beta\alpha$ and $\xi\delta\gamma\epsilon$;	$i = 10$: $\gamma\xi - \sigma\delta$.

Let I_i be the ideal of Λ_i generated by the elements above and the socle. Putting $\overline{\Lambda_i} := \Lambda_i/I_i$, we observe isomorphisms as follows.

	$\overline{\Lambda_1}$	$\overline{\Lambda_2}$	$\overline{\Lambda_3}$	$\overline{\Lambda_4}$	$\overline{\Lambda_5}$	$\overline{\Lambda_6}$	$\overline{\Lambda_7}$	Λ_8	$\overline{\Lambda_9}$
$\simeq$	$\overline{A_5}$	$\overline{A_5}$	$\overline{A_2}$	$\overline{A_{12}}$	$\overline{A_{13}}$	$\overline{A_{14}}$	$\overline{A_{15}}$	Λ_7^{op}	$\overline{A_3}$

Here, Λ^{op} stands for the opposite algebra of an algebra Λ. Thus it turns out that Λ_i for every i except $i = 10$ is τ-tilting-finite by Theorem 3.1. Moreover, we have the number of support τ-tilting modules of Λ_i as in the introduction.

The left case Λ_{10} will be dealt with in a forthcoming paper by the first and second named authors.

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