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QUASI-FLOWS*

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The purpose of this paper is to investigate a quasi-flow which is a one-parameter group of non-singular measurable point transformations on a measure space. If, in particular, the transformations are all measure preserving (i.e. a flow is given), the ergodicity together with the mixing property, the spectral or metrical type, increasing partitions of the space and the entropy of the flow are our main interests. Those methods used in the study of a flow are frequently useful for our approach. For example, the concept of a special flow introduced by W. Ambrose [3] plays an important role and the representation of a given flow in terms of a special flow is a powerful tool in the study of flows. L.M. Abramov [1] calculates the entropy of a flow with the help of the representation. As another example we give attention to the work of G. Maruyama [10] and H. Totoki [15] where they discuss a general time-change of flows the basic idea of which was originated by E. Hopf [8]. They discuss the invariant measure of a general time-changed flow and prove that the ergodicity is inherited and the entropy is kept invariant by the time-change. In the study of quasi-flows, we shall use both the repesentation in terms of a special quasi-flow and a time-change. Besides a quasi-flow requires its own methods in the investigation and it gives us some further problems such as the existence of an invariant measure (c.f. [4], [5], [6] and [7]) and related topics.

We are much interested in a study of two flows (quasi-flows) which are linked by a particular kind of commutation relation. Ya. G. Sinai ([13], [14]) has introduced the concept of transversal fields of a flow on a Riemannian manifold and he has obtained the results on the ergodicity of the flow. As one of generalizations, he has dealt with an admissible continuous one-parameter group of non-singular point transformations which is a transversal

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field of the flow. Further generalization of his result to quasi-flow will be discussed in this paper.

In Section 2 we shall define a quasi-flow, and we shall introduce some related concepts and state their simple properties. In Section 3 we shall first introduce a concept of an S-quasi-flow (Definition 3. 1) which is analogous to a special flow, and then we shall proceed to the representation theorem which asserts that we can form an S-quasi-flow equivalent to a given quasi-flow. Section 4 will be devoted to the discussion of time-change of a quasi-flow based on a positive function (Definition 4. 1 and Theorem 4. 1)

The results in § 3-4 lead us to state the following remarks. We shall be able to give an example which shows how important the representation theorem is in study of quasi-flows. The representation theorem gives us a negative answer to the question proposed by Sinai [14];" Is every quasi-flow admissible?". In fact, one of the conditions of the admissibility which requires the boundedness of the density of the conditional measure dose not hold in general. The other conditions for the admissibility are satisfied if the quasi-flow has no fixed points. Further we shall find that his results proved by using an admissible quasi-flow can be obtained similarly dropping a condition which requires a bounded density of the conditional measure (c.f. §6 and §7). We shall further show that our representation theorem and the theory of time-change enable us to give those conditions under which a quasi-flow is metrically transitive or conservative, and to find if a quasi-flow has a σ-finite invariant measure.

In Section 5 we shall consider a TQ-system. It is defined in the following manner. Let $\{Z_t\}$ be a quasi-flow and T be an automorphism. Suppose that $\lambda(w)$ be a positive measurable function. If a quasi-flow $\{TZ_tT^{-1}\}$ is a time-changed quasi-flow based on $\lambda(w)$, then we call the tripple $[\{Z_t\}, \lambda(w); T]$ a TQ-system, and $\{Z_t\}$ is called a transversal quasi-flow of the automorphism T with the coefficient of expansion $\lambda(w)$. Sinai has shown that if in particular $\{Z_t\}$ is a flow, $\lambda(w)$ is $\mathfrak{B}(\nu_{\{Z_t\}})$-measurable. Concerning his result we shall show the following. Given a TQ-system $[\{Z_t\}, \lambda(w); T]$. *If* $\log \lambda(w)$ *is quasi-integralble and* $E[\log \lambda | \nu_T](w) \neq 0$ *holds a.e.* (dP), *then* $\{Z_t\}$ *is a flow if and only if* $\lambda(w)$ *is* $\mathfrak{B}(\nu_{\{Z_t\}})$-*measurable.* Under the same conditions ($\{Z_t\}$ is necessarily a flow) *the entropy* $h\{Z_t\}$ *of* $\{Z_t\}$ *is either* 0 *or* ∞.

We shall apply our results to the investigation of the ergodic property

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of an automorphism and a flow in Section 6. There we shall assume some what weaker conditions than those in Sinai [14] to obtain the same results.

In Section 7 we shall consider an increasing partition ζ and the conditional entropy $H(T\zeta|\zeta)$ of an automorphism T in connection with a TQ-system. In particular, if $\lambda(w)$ in TQ-system is $\mathfrak{B}(\nu_{\{Z_t\}})$-measurable and if $E[\log\lambda|\nu_T](w) > 0$ holds *a.e.* (dP) then there exists an increasing partition ζ and we obtain $H(T\zeta|\zeta) = E[\log\lambda]$ whenever $\{Z_t\}$ has not fixed points.

In the final section we shall deal with three examples. Although they are well-known or rather simple, however each example tells us important remarks and its own specific suggestions. Example 1 is a so-called Bernoulli automorphism. It is an example of an automorphism which has no transversal flow but has a transversal quasi-flow. This quasi-flow is conservative and metrically transitive. It can not be a flow by any kind of time-change; in other words it has no invariant measure*. In example 2, we shall form the flow induced by the Ornstein-Uhlenbeck's Brownian motion and the flow of Brownian motion in such a way that they form a TQ-system. It is noted that the flow of Brownian motion appearing as a transversal flow in this example has infinite entropy. As is shown in Theorem 5. 2, the entropy of the transversal flow is either 0 or ∞. While many examples of transversal flows of entropy 0 are known, for example transversal flows of the group automorphism on the two dimensional torus, the geodesic flow on the manifold of constant negative curvature, the infinite shift and so forth, so far as the author knows no example of a transversal flow of entropy ∞ has not been given. Example 3 is considered as an example to show that we can discuss even multidimensional transversal field in our set up by introducing a quasi-flow with a general unimodular group as the parameter space.

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* This example for a Bernoulli automorphism suggests us a new approach to the investigation of Bernoulli automorphisms as will be prescribled in Example 1.

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2. Preliminaries.

Throughout this paper the probability space denoted by $(\Omega, \mathfrak{B}, P)$ is a Lebesgue space in the sense of V. A. Rohlin [12]*. We denote by R the real line with the ordinary Lebesgue measure $dm(u) = du$, and denote by $(R, \mathfrak{R}, m)$ the ordinary Lebesgue measure space. A bimeasurable, one-to-one, measure preserving transformation of Ω onto Ω' is called an *isomorphism* of Ω onto Ω'. An isomorphism of Ω onto itself is an *automorphism* of Ω. A one-parameter group of automorphisms $\{T_t; -\infty<t<\infty\}$ is called a *flow* if the mapping $(t, w) \to T_t w$ is $\overline{\mathfrak{R}\times\mathfrak{B}}$-measurable**. Now we consider bimeasurable transformations which are not necessarily measure preserving.

Definition 2. 1. We call a one-to-one bimeasurable transformation S of Ω onto itself a *quasi-automorphism* of Ω if it is *non-singular*, that is, both $P(SA)$ and $P(S^{-1}A)$ vanish whenever $P(A)$ vanishes. A one-parameter group of quasi-automorphisms $\{Z_t; -\infty < t < \infty\}$ is a *quasi-flow* if the mapping $(t, w) \to Z_t w$ is $\overline{\mathfrak{R}\times\mathfrak{B}}$-measurable.

Given a quasi-flow $\{Z_t\}$, then by the non-singularity of $\{Z_t\}$ there exists a collection of positive integrable functions $\{\alpha_t(w); -\infty < t < \infty\}$ such that

$$P(Z_t B) = \int_B \alpha_t(w) dP \qquad \text{for any} \qquad B \in \mathfrak{B}, \tag{2. 1}$$

or equivalently,

$$\int f(Z_t^{-1} w) dP = \int f(w) \alpha_t(w) dP \tag{2. 2}$$

holds for any bounded measurable function $f(w)$. By the group property of $\{Z_t\}$, it holds that

$$P(Z_{t+s}B) = P(Z_t(Z_s B)) = \int_{Z_s B} \alpha_t(w) dP = \int_B \alpha_t(Z_s w) \alpha_s(w) dP$$

for $B \in \mathfrak{B}$. So we have

$$\alpha_{t+s}(w) = \alpha_t(Z_s w) \alpha_s(w) \qquad a.e. \ (dP) \tag{2. 3}$$

for any fixed t and s. We call the (t, w) function $\alpha_t(w)$ defined by (2, 1) the *multiplicative density function* of $\{Z_t\}$.

Remark 2. 1. By virtue of Theorem 3. 1, we can prove that there exists

* We shall use terminologies in [11, 12] throughout this paper.

** We denote by $\overline{\mathfrak{R}\times\mathfrak{B}}$ the completion of the product σ-field $\mathfrak{R}\times\mathfrak{B}$.

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a version of $\alpha_t(w)$ which is $\mathfrak{R}\times\mathfrak{B}$-measurable and equality (2, 3) holds for every t and s with probability 1.

A set B is called *$\{Z_t\}$-invariant* if $Z_tB=B$ for all t. A set B is called *$\{Z_t\}$-invariant (mod 0)*, if $P(Z_tB\ominus B)=0$ holds for each t*. It is easily seen that for any $\{Z_t\}$-invariant (mod 0) set B, there exists a $\{Z_t\}$-invariant set B' such that $P(B\ominus B')=0$. Two quasi-flows $\{Z_t\}$ on Ω and $\{Z'_t\}$ on Ω' are said to be *isomorphic* if there exist two invariant null sets $N\subset\Omega$ and $N'\subset\Omega'$, and an isomorphism T of $\Omega-N$ onto $\Omega'-N'$ such that $Z_tw=T^{-1}Z'_tTw$ for all $w\in\Omega-N$ and t.

Let ζ be a partition of Ω. We denote by $\mathfrak{B}(\zeta)$ the completion of the σ-fleld $\{B\in\mathfrak{B};\ B$ is a ζ-set$\}$. The *factor measure space* $(\Omega_{/\zeta},\mathfrak{B}_\zeta,P_\zeta)$ is a Lebesgue space if ζ is measurable. We denote by $C=C_\zeta$ the element of ζ and denote by $C(w)=C_\zeta(w)$ the element of ζ which contains the point w. For any measurable partition ζ, there exists a *canonical system of measures* $\{(C,\mathfrak{B}_C,P(\cdot\,|\,\zeta;\ C)),\ C\in\zeta\}$. We denote simply by $P(A\,|\,C_\zeta)=P(A\cap C_\zeta\,|\,\zeta;C_\zeta)$. Then the conditional expectation of measurable function $f(w)$ with respect to $\mathfrak{B}(\zeta)$ is given by

$$(2.\ 4)\qquad E[f\,|\,\zeta](w)=E[f\,|\,\mathfrak{B}(\zeta)](w)=\int_{C(w)}f\,dP(\cdot\,|\,\zeta;\ C(w))\quad \text{a.e. } (dP).$$

Let T be an automorphism of Ω and ζ be a T-invariant measurable partition of Ω. Then the transformation T_ζ: $T_\zeta C=\{Tw;\ w\in C\}\in\zeta$ is an automorphism of $(\Omega_{/\zeta},\mathfrak{B}_\zeta,P_\zeta)$. We denote by T^C the restriction of T to C. Then the transformation T^C is an isomorphism of $(C,\mathfrak{B}_C,P(\cdot\,|\,\zeta;\ C))$ onto $(T_\zeta C,\mathfrak{B}_{T_\zeta C},P(\cdot\,|\,\zeta;\ T_\zeta C))$ for almost every $C\in\zeta$ (dP_ζ).

We denote by ν the trivial partition of Ω and, by ε the partition into individual points of Ω. Let $\{Z_t\}$ be a quasi-flow on Ω. We denote by $\nu_{\{Z_t\}}$ the measurable covering of the partition into the trajectories of $\{Z_t\}$: $\{Z_tw;-\infty<t<\infty\}$. The partition ν_S is defined from a quasi-automorphism S in similar way. A quasi-flow (or a quasi-automorephism) is called *metrically transitive* if $\nu_{\{Z_t\}}=\nu$ (resp. $\nu_S=\nu$).

Remark 2. 2. A quasi-flow $\{Z_t\}$ is metrically transitive if and only if it has only trivial invariant sets, i.e. either $P(B)=0$ or 1 whenever B is a measurable invariant set.

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* We denote by $A\ominus B$ the symmetric difference of A and B.

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Let $\{Z_t\}$ be a quasi-flow and let $g(w)$ be a measurable function on Ω. Then $g(t, w) = g(Z_t w)$ is a $\overline{\mathfrak{R} \times \mathfrak{B}}$-measurable function. Hence the integral

$$\int_a^b g(Z_t w) dt, \qquad -\infty \leqq a < b \leqq \infty$$

is well defined for almost all w if $g(w) \geq 0$. A quasi-flow $\{Z_t\}$ is called *conservative* if, for any positive measurable function $g(w)$ i.e. $g(w) > 0$ for all w, the equality

$$\int_0^\infty g(Z_t w) dt = \int_{-\infty}^0 g(Z_t w) dt = \infty, \tag{2. 5}$$

holds for almost all $w(dP)$. The definition of conservative automorphism is similar. We should note that any flow $\{T_t\}$ (automorphism T) is conservative because of the finiteness of the invariant measure P (c.f. [8]).

3. Representation of quasi-flows.

In this section, we shall represent quasi-flows by means of a special quasi-flow which is an analogue of [3, 4]. Our formulation is necessary for the proofs of theorems in the later sections.*

Let S be a quasi-automorphism of a Lebesgue space $(X, \mathfrak{A}, \mu)$, and $f(x)$ be a positive measurable function defined on X satisfying the following equality

$$\sum_{k=1}^{\infty} f(S^k x) = \sum_{k=-1}^{-\infty} f(S^k x) = \infty \tag{3. 1}$$

for every $x \in X$. Set $\tilde{\Omega} = \{(x, u);\ 0 \leqq u < f(x),\ x \in X\}$ and let $\tilde{\mathfrak{B}}$ be the restriction of $\overline{\mathfrak{A} \times \mathfrak{R}}$ to $\tilde{\Omega}$. Let $p(x, u)$ be a positive $\tilde{\mathfrak{B}}$-measurable function on $\tilde{\Omega}$ such that

$$\int_X \int_0^{f(x)} p(x, u) du\, d\mu(x) = 1. \tag{3. 2}$$

Setting $d\tilde{P}(x, u) = p(x, u) du\, d\mu(x)$, the measure space $(\tilde{\Omega}, \tilde{\mathfrak{B}}, \tilde{P})$ becomes a Lebesgue space. We define a quasi-flow on $\tilde{\Omega}$ by

$$Z_t(x, u) = (S^n x, u + t - f_n(x)) \quad \text{for } f(S^n x) > u + t - f_n(x) \geq 0, \tag{3. 3}$$

where

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* Recently, the author was privately informed by H. Totoki that U. Krengel [9] has studied representations of one-parameter semigroups of non-singular measurable transformations under more general settings than ours.

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$$(3.4)\qquad f_n(x)=\begin{cases}\sum_{k=0}^{n-1} f(S^k x) & n\geqq 1\\ 0 & n=0\\ -\sum_{k=1}^{-n} f(S^{-k}x) & n\leqq -1.\end{cases}$$

The multiplicative density function of $\{Z_t\}$ is given by the form

$$(3.5)\qquad \alpha_t(x,u)=\frac{\bar{p}(x,u+t)}{\bar{p}(x,u)}\quad (x,u)\in\tilde{\Omega},$$

where $\bar{p}(x,u)$ is a function on $X\times R$ defined by

$$(3.6)\qquad \bar{p}(x,u)=p(S^n x, u-f_n(x))\times\begin{cases}\rho(x)\cdots\rho(S^{n-1}x) & n\geqq 1\\ 1 & n=0\\ [\rho(S^{-1}x)\cdots\rho(S^n x)]^{-1} & n\leqq -1\end{cases}$$

for $f_n(x)\leqq u<f_{n+1}(x)$, with $\rho(x)=\dfrac{d\mu(Sx)}{d\mu}$.

DEFINITION 3.1. The quasi-flow $\{Z_t\}$ defined by (3.3) is called an *S-quasi-flow*. We say that $\{Z_t\}$ is built up by $(X,\mathfrak{A},\mu,f(x),p(x,u),S)$.

DEFINITION 3.2. Let $\{Z_t\}$ be a quasi-flow. An S-quasi-flow $\{\tilde{Z}_t\}$ is called an *S-representation* of $\{Z_t\}$, if $\{\tilde{Z}_t\}$ is isomorphic to $\{Z_t\}$.

With these definitions, we can now state the theorem,

THEOREM 3.1. *A quasi-flow without fixed points on a Lebesgue space has an S-representation.*

First we prepare two lemmas to prove the theorem.

LEMMA 3.1. *Let $(X,\mathfrak{A},\mu)$ be a Lebesgue space, S be a bimeasurable one-to-one transformation of X onto itself and $f(x)$ be a positive measurable function satisfying* (3.1). *Set $\tilde{\Omega}=\{(x,u);\ 0\leqq u<f(x),\ x\in X\}$ and let ξ be the partition of $\tilde{\Omega}$ into the vertical lines. Suppose that $(\tilde{\Omega},\mathfrak{B},P)$ is a Lebesgue space with a certain measure P such that the partition ξ is measurable and the measure on the factor measure space $\tilde{\Omega}_{/\xi}$ is given by $dP_\xi(C(x))=d\mu(x)$, where $C(x)=\{(x,u);\ 0\leqq u<f(x)\}$. If the one-parameter group of transformations defined by* (3.3) *is a quasi-flow on the space $(\tilde{\Omega},\mathfrak{B},P)$, then S is a quasi-automorphism of $(X,\mathfrak{A},\mu)$ and the measure P is given by the multiple integral*

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$$(3.7)\qquad P(B)=\int_X\int_0^{f(x)}\chi_B(x,u)p(x,u)du\,d\mu(x)$$

for some positive function $p(x,u)$*. *Further, if the functions* $f(x,u)=f(x)$ *and* $g(x,u)=u$ *are both* $\mathfrak{B}$*-measurable, then* $\mathfrak{B}=\tilde{\mathfrak{B}}$ *holds and* $p(x,u)$ *is* $\tilde{\mathfrak{B}}$*-measurable.*

Proof. Since ξ is a measurable partition of $\tilde{\Omega}$, there exists the canonical system of measures: $\{(C,\mathfrak{B}_C,P(\cdot\,|\,\xi;\,C));\,C\in\xi\}$. By the measurability of $\{Z_t\}$, we can easily show that any segment; $\{(x,u);\,a\leqq u<b\}$ of $C(x)$ is $\mathfrak{B}_C$-measurable. Put $P(du\,|\,C(x))=p(x,du)$. Then the multiplicative density function $\alpha_t(x,u)$ satisfies

$$(3.8)\qquad \alpha_t(x,u)p(x,du)d\mu(x)$$
$$=\sum\chi_{\{f(S^nx)>t+u-f_n(x)\geq 0\}}(x,u)p(S^nx,d(u+t-f_n(x)))d\mu(x).$$

Therefore $p(x,du)\alpha_t(x,u)=p(x,d(u+t))$ holds a.e. (dP) for $0\leqq u,\ u+t<f(x)$. Hence $p(x,du)$ is equivalent to the ordinary Lebesgue measure on $[0,f(x))$. We denote by $p(x,u)$ the density $\frac{p(x,du)}{du}$. Obviously we have, $p(x,u)>0$ a.e. (dP). So (3. 7) holds. Noting (3. 8) again, we have

$$\alpha_t(x,u)p(x,u)d\mu(x)=p(Sx,u)d\mu(Sx)\quad\text{for}\quad f(Sx)>u+t-f(x)\geqq 0,$$

that is, $d\mu(Sx)$ is equivalent to $d\mu(x)$. Hence S is a quasi-automorphism of X. If $f(x,u)$ and $g(x,u)$ are both $\mathfrak{B}$-measurable, we can easily prove that $\mathfrak{B}$ coincides with $\tilde{\mathfrak{B}}$ and that $p(x,u)$ is $\tilde{\mathfrak{B}}$-measurable by the formula (3. 7).

By virtue of Wiener's ergodic theorem [16], we have the following lemma.

LEMMA 3. 2. *Let* $\{Z_t\}$ *be a quasi-flow,* $g(w)$ *be a bounded measurable function on* Ω *whose absolute value is dominated by* K*, and let* N *be the null set of* w *out side of which* $g(Z_tw)$ *is* $\mathfrak{R}$*-measurable. If we put*

$$(3.9)\qquad g_a(w)=\begin{cases}\dfrac{1}{a}\displaystyle\int_0^a g(Z_uw)du & w\notin N\\[2ex] 0 & w\in N\end{cases}$$

then we have

$$\text{(i)}\qquad g_a(w)\to g(w)\qquad a\to 0\qquad \text{a.e.,}$$

* We denote by $\chi_B(x,u)$ the characteristic function of B.

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(ii) $|g_a(w)| \leqq K$

(iii) $|g_a(Z_t w) - g_a(Z_s w)| \leqq \frac{2K}{|a|} |t - s|$.

Moreover $C(\{Z_t\})$ *is dense both in* $L^2(\Omega, \mathfrak{B}, P)$ *and in* $L^1(\Omega, \mathfrak{B}, P)$. *Here* $C(\{Z_t\}) = C(\{Z_t\}, \Omega, \mathfrak{B}, P)$ *is the totality of all the bounded measurable functions each of which satisfies* $|h(Z_t w) - h(Z_s w)| \leqq M|t - s|$ *with a suitable constant* M.

Proof of Theorem 3.1. Since $\{Z_t\}$ has no fixed points, there exists a measurable set $B \in \mathfrak{B}$ and a positive number t_0 such that $P(B^c \cap Z_{t_0}B) > 0$. By Lemma 3.2, $\psi_a(w) = \frac{1}{a}\int_0^a \chi_B(Z_u w)du \to \chi_B(w)$ $(a \to 0)$ a.e. Hence there exists a positive number a such that $P(B_1 \ominus B^c) < \frac{1}{2} P(B^c \cap Z_{t_0}B)$ and $P(Z_{t_0}B_2 \ominus Z_{t_0}B) < \frac{1}{2} P(B^c \cap Z_{t_0}B)$ hold for $B_1 = \left\{w;\ \psi_a(w) < \frac{1}{4}\right\}$ and $B_2 = \left\{w;\ \psi_a(w) > \frac{3}{4}\right\}$. Fix such a, then $P(B_1 \cap Z_{t_0}B_2) > 0$ and $|\psi_a(Z_t w) - \psi_a(Z_s w)| \leqq \frac{2|t-s|}{a}$ by Lemma 3.2. Hence the functions $\bar{\varphi}$ and $\underline{\varphi}$ defined bellow are measurable.

$$\bar{\varphi}(w) = \begin{cases} \sup\ \{u;\ Z_u w \in B_1 \cap Z_{t_0}B_2\} \\ -\infty \text{ if the above set is empty,} \end{cases}$$

(3.10)

$$\underline{\varphi}(w) = \begin{cases} \inf\ \{u;\ Z_u w \in B_1 \cap Z_{t_0}B_2\} \\ \infty \text{ if the above set is empty.} \end{cases}$$

Set $\Omega_1 = \{w;\ \bar{\varphi}(w) = \infty,\ \underline{\varphi}(w) = -\infty\}$, $\Omega_2 = \{w;\ \bar{\varphi}(w) = \infty,\ \underline{\varphi}(w) > -\infty\}$, $\Omega_3 = \{w;\ -\infty < \bar{\varphi}(w) < \infty\}$ and $\Omega_4 = \{w;\ \bar{\varphi}(w) = -\infty\}$. Each Ω_j; $j = 1, 2, 3, 4$ is a $\{Z_t\}$-invariant measurable set and $\Omega_i \cap \Omega_j = \phi$ $i \neq j$, $\Omega = \Omega_1 + \Omega_2 + \Omega_3 + \Omega_4$ hold. Further we have $P(\Omega_1 + \Omega_2 + \Omega_3) \geqq P(B_1 \cap Z_{t_0}B_2) > 0$.

Let $\mathfrak{B}_1$ and P_1 be restrictions of $\mathfrak{B}$ and P to Ω_1, respectively. We shall construct the S-representation on the set Ω_1. Set $\Omega_1^* = \{w;\ \psi_a(w) = 1/2,\ \psi_a(Z_t w) > 1/2,\ \text{for } 0 < t < a/8\}$, $f(w^*) = \inf\ \{t > 0;\ Z_t w^* \in \Omega_1^*\}$ for $w^* \in \Omega_1^*$ and $Sw^* = Z_{f(w^*)}w^*$. Then Ω_1 is divided into the segments of trajectories of $\{Z_t\}$: $C(w^*) = \{Z_t w^*;\ 0 \leqq t < f(w^*)\}$. We denote by ξ such partition. Set $f(w) = f(w^*)$ if $w \in C(w^*)$ and $g(w) = u$ if $Z_{-u}w = w^*$, $w \in C(w^*)$. Then we can easily see that $f(w)$ and $g(w)$ are both measurable and the partition ξ is measurable. Observing the one-to-one mapping H of $\tilde{\Omega}_1 = \{(w^*, u);\ 0 \leqq u - f(w^*),\ w^* \in \Omega_1^*\}$ onto Ω_1 defined by $H(w^*, u) = Z_u w^*$, we can prove

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that $\{Z_t\}$ has an S-representation on Ω_1 by Lemma 3.1, similarly to the proof of Theorem 2 in [3].

Let us consider the set Ω_2. Put $\Omega_2^{(n)} = \{w;\ n+1 > \underline{\varphi}(w) \geqq n\}$ $n = 0, \pm 1, \pm 2, \cdots$. Then $\Omega_2^{(n)}$ is measurable and $\Omega_2 = \sum_n \Omega_2^{(n)}$. Put $\Omega_2^{*(n)} = \{w;\ \underline{\varphi}(w) = n\}$, then $Z_1\Omega_2^{*(n)} = \Omega_2^{*(n-1)}$. Considering the partition ξ_2 of Ω_2: $C_{\xi_2}(w^*) = \{Z_u w^*;\ 0 \leqq u < 1\}$, $w^* \in \Omega_2^* = \sum_n \Omega_2^{*(n)}$, we have the assertion of the theorem on Ω_2 by a discussion similar to that of previous paragraph.

For Ω_3, we can also prove the assertion similarly Ω_2. For the set Ω_4, let us repeat the same discussion above, and we have the assertion of the theorem for some subset of Ω_4 with positive measure. Performing this procedure successively and using transfinite induction, we can conclude the proof.

Corollary 3.1. (I. M. Ambrose) *If $\{Z_t\}$ is a flow, there exists an S-flow $\{\tilde{Z}_t\}$ built up by $(X, \mathfrak{A}, \mu, f(x), 1, S)$ where S is an automorphism of finite measure space X, and $\{\tilde{Z}_t\}$ is isomorphic to $\{Z_t\}$.*

Remark 3.1. By the proof of Theorem 3, we can easily see the following.

Let $\{Z_t\}$ be a quasi-flow and let $F\{Z_t\}$ be the set of all fixed points of $\{Z_t\}$. Let $\{Z_t'\}$ be the restriction of $\{Z_t\}$ to Ω-$F\{Z_t\}$. Then there exists an S-quasi-flow $\{\tilde{Z}_t\}$ built up by some $(X, \mathfrak{A}, \mu, f(x), p(x,u), S)$ which satisfies the following conditions (i) there exist S-invariant sets X_n; $n = 1, 2, 3, \cdots$ such that $X_n \cap X_m = \phi$ if $n \neq m$ and $X = \sum_n X_n$, (ii) there exists a positive constant θ_n such that $f(x) \geqq \theta_n$ for $x \in X_n$ for each n. (iii) $\{\tilde{Z}_t\}$ is isomorphic to $\{Z_t'\}$.

By this representation theorem of a quasi-flow, we have the following.

Corollary 3.2. *Let $\{Z_t\}$ be a quasi-flow and let $\alpha_t(w)$ be the multiplicative density function of $\{Z_t\}$. Let $a_n(w)$ be a sequence of $\mathfrak{B}(\nu_{\{Z_t\}})$-measurable functions which converges to* 0 *as $n \to \infty$* a.e. (dP). *Then it holds that $\alpha_{a_n(w)}(w)$ converges to* 1 *in* $L^1(\Omega, \mathfrak{B}, P)$.

Especially, $\alpha_t(w)$ converges to 1 in $L^1(\Omega, \mathfrak{B}, P)$ as $t \to 0$.

Proof. Since $\alpha_t(w) = 1$ on $F\{Z_t\}$ and $F\{Z_t\}$ is a measurable set, it is sufficients to prove the assertion in the case of an S-quasi-flow which satisfies the conditions in Remark 3.1. We may assume that there exists an S-invariant function $b_n(x)$ such that $b_n(x) = a_n(x,u)$ a.e. $(du\, d\mu(x))$. By Remark 3.1, for any $\varepsilon > 0$, there exists a natural number $N_1(\varepsilon)$ such that $P\{(x,u);$

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$0 \leqq u < f(x),\ x \in \sum_{N_1} X_n\} < \frac{\varepsilon}{4}$ and $f_n(x) \geqq \varepsilon$ for $x \in Y_1(\varepsilon) = \sum_{N_1} X_n$. By the assumption, there exist a natural number $N_2 = N_2(\varepsilon)$ and an S-invariant set $Y_2(\varepsilon)$ such that $P(\{(x, u);\ 0 \leqq u < f(x),\ x \in Y_2\}) < \frac{\varepsilon}{4}$ and $a_n(x) < \varepsilon$ for $x \in Y_2(\varepsilon)$. Then $\{\bar{p}(x, u + a_n(x));\ n \geqq N_2\}$ is uniformly integrable on $\Omega_\varepsilon = \{(x, u);\ 0 \leqq u < f(x),\ x \in X - Y_1(\varepsilon) - Y_2(\varepsilon)\}$ for $n \geqq N_2(\varepsilon)$. Therefore we have

$$\lim_{n \to \infty} \int_{X - Y_1 - Y_2} \int_0^{f(x)} |\bar{p}(x, u + a_n(x)) - \bar{p}(x, u)|\, du\, d\mu(x) = 0. \tag{3. 11}$$

It is obvious that

$$\int_{Y_1 \cup Y_2} \int_0^{f(x)} \left| \frac{\bar{p}(x, u + a_n(x))}{\bar{p}(x, u)} - 1 \right| p(x, u) du\, d\mu(x) \leqq \varepsilon \tag{3. 12}$$

by the invariantness of $\Omega - \Omega_\varepsilon$ under $\{Z_t\}$ and by $P(\Omega - \Omega_\varepsilon) < \frac{\varepsilon}{2}$. From (3. 5), (3. 11) and (3. 12), it follows that

$$\overline{\lim_{n \to \infty}} \int_X \int_0^{f(x)} |\alpha_{a_n(x)}(x, u) - 1|\, du\, d\mu(x) \leqq \varepsilon \text{ for any } \varepsilon > 0.$$

Hence $\{\alpha_{a_n(x)}(x, u)\}$ converges to 1 in $L^1(\Omega, \mathfrak{B}, P)$ as $n \to \infty$.

Now we remark the connections between the properties of a quasi-flow $\{Z_t\}$ and the properties of the basic quasi-automorphism which appears in the S-representation of $\{Z_t\}$.

Remark 3. 2. Let $\{Z_t\}$ be an S-quasi-flow built up by $(X, \mathfrak{A}, \mu, f(x), p(x, u), S)$. Then the following propositions hold;

(i) $\{Z_t\}$ is metrically transitive if and only if S is metrically transitive,

(ii) $\{Z_t\}$ is conservative if and only if S is conservative,

(iii) $\{Z_t\}$ has a σ-finite invariant measure equivalent to P if and only if S has a σ-finite invariant measure equivalent to μ.

Remark 3. 3. Ya. G. Sinai [14] has called a quasi-flow $\{Z_t\}$ admissible if

(i) there exists a regular partition ξ of Ω for $\{Z_t\}$, that is, almost every element of the partition ξ is a segment of a trajectory of $\{Z_t\}$ and the time length of each element is a measurable function on Ω,

(ii) for any regular partition ξ of Ω, there exist two constants a and b; $0 < a < b < \infty$, such that the conditional measure on an element $C = C_\xi(w) = \{Z_u w;\ s(w) \leqq u < r(w)\}$ is expressed by a density $p(u | C(w))$ in the form

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(3. 13) $$P(A|C) = \int_{A\cap C} p(u|C)du$$

and $p(u|C)$ satisfies

(3. 14) $$a \leqq p(u|C)(r-s) \leqq b.$$

By virtue of Theorem 3. 1 and Lemma 3. 1, we have that any quasi-flow without fixed points satisfies the conditions (i) and (ii) except the inequality (3. 14).

4. Time-changes of quasi-flows.

In this section, we introduce the time-change of a quasi-flow. The method of time-changes is useful to study the geometrical structure of trajectories of the quasi-flow. But it seems to be difficult to study the time-changes of quasi-flows through general additive functionals, such as G. Maruyama [10] and H. Totoki [15] did in flows. Perhaps the general time-change is not useful for the investigation of quasi-flows. So we shall discuss only classical time-changes induced by positive measurable functions.

Let $\{Z_t\}$ be a quasi-flow and $\lambda(w)$ be a positive measurable function on Ω. Let $\lambda(w)$ be integrable along the trajectories of $\{Z_t\}$, that is, $\lambda(Z_u w)$ is a locally integrable u-function for every $w \in \Omega$. If we put

(4. 1) $$\varphi(t, w) = \int_0^t \lambda(Z_u w)du,$$

then it holds that

(4. 2) $$\varphi(t+s, w) = \varphi(t, w) + \varphi(s, Z_t w)$$

by the group property of $\{Z_t\}$. We say that the function $\varphi(t, w)$ defined by (4. 1) is the additive functional of $\{Z_t\}$ which is induced by $\lambda(w)$, if $\lambda(w)$ is integrable along the trajectories of $\{Z_t\}$ and if

(4. 3) $$\lim_{t\to\pm\infty} \varphi(t, w) = \pm\infty.$$

We denote by $\tau(t, w)$ the inverse function of $\varphi(t, w)$ for each w, that is,

(4. 4) $$\tau(t, w) = u \text{ if and only if } \varphi(u, w) = t$$

Define a system of point transformations $\{\hat{Z}_t\}$ by

(4. 5) $$\hat{Z}_t w = Z_{\tau(t, w)} w,$$

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then we have, by (4. 2),

$$\hat{Z}_{t+s}w = \hat{Z}_t\hat{Z}_s w, \tag{4. 6}$$

THEOREM 4. 1. *Let* $\{Z_t\}$ *be a quasi-flow on* $(\Omega, \mathfrak{B}, P)$ *and* $\varphi(t, w)$ *be the additive functional of* $\{Z_t\}$ *which is induced by* $\lambda(w)$. *Then the system* $\{\hat{Z}_t\}$ *defined by* (4. 5) *is again a quasi-flow.*

Further let $\alpha_t(w)$ *and* $\hat{\alpha}_t(w)$ *be multiplicative density functions of* $\{Z_t\}$ *and* $\{\hat{Z}_t\}$, *respectively. Then it holds that*

$$\hat{\alpha}_t(w) = \frac{\lambda(w)}{\lambda(Z_{\tau(t,w)}\,w)}\,\alpha_{\tau(t,w)}(w) \quad \text{a.e.}. \tag{4. 7}$$

We can easily prove the theorem similarly to the proof of Theorem 4. 2 in [16]. Now we define,

DEFINITION 4. 1. The quasi-flow $\{\hat{Z}_t\}$, defined by (4. 5), is called the *time-change* of $\{Z_t\}$ by $\lambda(w)$.

The following proposition is easily seen,

PROPOSITION 4. 1. *Let* $\{Z_t\}$ *be a quasi-flow and* $\{\hat{Z}_t\}$ *be a time-changed quasi-flow of* $\{Z_t\}$ *by* $\lambda(w)$. *Then we have,*

(i) $\{Z_t\}$ *is a time-changed quasi-flow of* $\{\hat{Z}_t\}$ *by* $1/\lambda(w)$,

(ii) $\nu_{\{Z_t\}} = \nu_{\{\hat{Z}_t\}}$,

especially $\{Z_t\}$ *is metrically transitive if and only if* $\{\hat{Z}_t\}$ *is metrically transitive,*

(iii) $\{Z_t\}$ *is conservative if and only if* $\{\hat{Z}_t\}$ *is conservative.*

Remark 4. 1. The equality (4. 7) implies that if $\{Z_t\}$ is a flow, then its time-changed quasi-flow $\{\hat{Z}_t\}$ by $\lambda(w)$ has a σ-finite invariant measure $\hat{P}$: $d\hat{P} = \lambda(w)dP$.

Remark 4. 2. Let $\{Z_t\}$ be a quasi-flow on a Lebesgue space $(\Omega, \mathfrak{B}, P)$. There arises a question whether $\{Z_t\}$ has a σ-finite invariant measure Q equivalent to P. Such a problem has been discussed by many authors in case of discrete parameters. In the continuous parameter cases, we have some results by virtue of Remark 3. 1, Theorem 4. 1 and Propsoition 4. 1. The following conditions are equivalent for a conservative quasi-flow $\{Z_t\}$,

(i) $\{Z_t\}$ has a σ-finite invariant measure equivalent to P,

(ii) $\{Z_t\}$ becomes a flow through time-change by some $\lambda(w)$,

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(iii) there exists a positive measurable function $\lambda(w)$ which is integrable along the trajectories of $\{Z_t\}$ such that there exists the non trivial limit

$$\lim_{t\to\infty}\frac{\int_0^t g(Z_u w)du}{\int_0^t \lambda(Z_u w)du} \qquad \text{a.e.} \quad (dP) \tag{4. 8}$$

for any λ-bounded, non-negative measurable function $g(w)$. (Here we say that a function $g(w)$ is λ-bounded if there exists a constant K such that $|g(w)| \leqq K\lambda(w)$.) [c.f. 5].

(iv) For any $\varepsilon > 0$, there exists a countable partition $\{\Omega_n\}$ of Ω such that

$$\frac{1}{1+\varepsilon} < \alpha_t(w) < 1+\varepsilon \quad \text{if} \quad w, Z_t w \in \Omega_n \tag{4. 9}$$

[c.f. 2],

(v) for any $\{Z_t\}$-invariant set A, there exists a measurable set $B \subset A$ with positive measure such that the Hopf's compressibility measure of B is positive [c.f. 7],

(vi) $\{Z_t\}$ is σ-bounded [c.f. 6].

5. *TQ*-systems.

In this section, we shall study some properties of quasi-flows which have a special commutation relation with other automorphisms. The commutation relation is important in the study of automorphisms and flows (c.f. §6 and 7).

Definition 5. 1. A system $[\{Z_t\}, \lambda(w); T]$ of a quasi-flow $\{Z_t\}$ and a positive measurable function $\lambda(w)$ and an automorphism T is called a *TQ-system*, if

$$TZ_tT^{-1} = \hat{Z}_t \tag{5. 1}$$

holds, where $\{\hat{Z}_t\}$ is the time-changed quasi-flow of $\{Z_t\}$ by $\lambda(w)$. We say that a *TQ-system* $[\{Z_t\}, \lambda(w); T]$ is a *TF-system* if $\{Z_t\}$ is a flow.

The geometrical meaning of *TQ-system* is as follows. An automorphism T transforms the trajectories of a quasi-flow $\{Z_t\}$ onto themselves so that any segment of a trajectory is transformed again to another segment of another one. We can easily see the following proposition.

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PROPOSITION 5. 1. *Let* $[\{Z_t\}, \lambda(w); T]$ *be a TQ-system and let* $\{\tilde{Z}_t\}$ *be a time-changed quasi-flow of* $\{Z_t\}$ *by some* $\gamma(w)$. *Then we have,*

(i) *the system* $[\{\tilde{Z}_t\}, \lambda(w)\gamma(T^{-1}w)/\gamma(w); T]$ *is a TQ-system,*

(ii) *the system* $[\{Z_t\}, \lambda^{(k)}(w); T^k]$ *is a TQ-system, where* $\lambda^{(k)}(w)$ *is defined by*

$$\lambda^{(k)}(w) = \begin{cases} \lambda(w)\lambda(T^{-1}w)\cdots\lambda(T^{-k+1}w) & k \geqq 1 \\ 1 & k = 0 \\ [\lambda(Tw)\lambda(T^2w)\cdots\lambda(T^{-k}w)]^{-1} & k \leqq -1. \end{cases} \tag{5. 2}$$

Let $[\{Z_t\}, \lambda(w); T]$ be a *TQ-system* and let $\{\hat{Z}_t\}$ be the time-changed quasi-flow of $\{Z_t\}$ by $\lambda(w)$: $\hat{Z}_t w = Z_{\tau(t,w)}w = TZ_tT^{-1}w$. Let $\alpha_t(w)$ be the multiplicative density function of $\{Z_t\}$. Since it holds that $P(\hat{Z}_tB) = P(TZ_tT^{-1}B) = = P(Z_tT^{-1}B) = \int_{T^{-1}B}\alpha_t(w)dP = \int_B \alpha_t(T^{-1}w)dP$, we have the following proposition by (4. 7).

PROPOSITION 5. 2. *Let* $[\{Z_t\}, \lambda(w); T]$ *be a TQ-system, then it holds that*

$$\alpha_{\tau(t,w)}(w) = \frac{\lambda(\hat{Z}_tw)}{\lambda(w)}\,\alpha_t(T^{-1}w) \qquad \text{a.e. } (dP). \tag{5. 3}$$

COROLLARY 5. 1. (Ya. G. Sinai) *If* $\{Z_t\}$ *is a flow,* $\lambda(w)$ *is* $\mathfrak{B}(\nu_{\{Z_t\}})$*-measurable.*

Proof. Since $\{Z_t\}$ is a flow, $\alpha_t(w) = 1$ holds. From (5. 3), it follows that $\lambda(\hat{Z}_tw)/\lambda(w) = 1$ a.e. (dP). Thus we conclude that $\lambda(w)$ is $\mathfrak{B}(\nu_{\{\hat{Z}_t\}})$-measurable. Hence $\lambda(w)$ is $\mathfrak{B}(\nu_{\{Z_t\}})$-measurable by Propostiion 4. 1 (ii).

Now, we are interested converse problem i.e. whether $\mathfrak{B}(\nu_{\{Z_t\}})$-measurability of $\lambda(w)$ implies that $\alpha_t(w) = 1$ a.e. (dP). For this purpose, we shall prepare the following simple lemma. We say that a measurable function $h(w)$ is *quasi-integrable*, if either the positive part or the negative part of $h(w)$ is integrable. For any quasi-integrable function $h(w)$, the conditional expectation $E[h(w)|\zeta]$ with respect to a σ-field $\mathfrak{B}(\zeta)$ for any partition ζ is well defined. And we have that

$$E[h|\nu_T](w) = \begin{cases} \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} h(T^kw) & \text{a.e. } (dP) \\ \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} h(T^{-k}w) & \text{a.e. } (dP) \end{cases} \tag{5. 4}$$

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by Birkhoff's ergodic theorem.

LEMMA 5.1. *Let* $[\{Z_t\}, \lambda(w); T]$ *be a TQ-system and let* $h(w)$ *be quasi-integrable. If* $h(w)$ *is* $\mathfrak{B}(\nu_{\{Z_t\}})$*-measurable, then the conditional expectation* $E[h|\nu_T]$ *is* T*- and* $\{Z_t\}$*-invariant (mod 0).*

Proof. Noting that $\{T^k Z_t T^{-k}\}$ is a time-changed quasi-flow of $\{Z_t\}$, we have together with Proposition 4.1 (ii)

$$E[h|\nu_T](Z_t w) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} h(T^k Z_t T^{-k} T^k w)$$

$$= \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} h(T^k w) = E[h|\nu_T](w) \qquad \text{a.e.} \quad (dP).$$

DEFINITION 5.2. We say that a TQ-system $[\{Z_t\}, \lambda(w); T]$ has the *property* (A), if $\log\lambda(w)$ is quasi-integrable and satisfies

$$E[\log\lambda|\nu_T] \neq 0 \qquad \text{a.e.} \quad (dP). \tag{5.5}$$

We say that a TQ-system has the *property* (AC) (or (AD)), if $\log\lambda(w)$ is quasi-integrable and satisfies

$$E[\log\lambda|\nu_T] > 0 \ \ (\text{resp.} < 0) \ \ \text{a.e.} \ (dP). \tag{5.6}$$

THEOREM 5.1. *Let* $[\{Z_t\}, \lambda(w); T]$ *be a TQ-system with the property* (A). *Then* $\{Z_t\}$ *is a flow if and only if* $\lambda(w)$ *is* $\mathfrak{B}(\nu_{\{Z_t\}})$*-measurable.*

Proof. Assume that $\lambda(w)$ is $\mathfrak{B}(\nu_{\{Z_t\}})$-measurable. Then by Lemma 5.1, there exist T- and $\{Z_t\}$-invariant sets Ω^+ and Ω^- such that

$$E[\log\lambda|\nu_T] \begin{cases} > 0 & \text{a.e. } (dP) \text{ on } \Omega^+ \\ < 0 & \text{a.e. } (dP) \text{ on } \Omega^-, \end{cases} \tag{5.7}$$

and $P(\Omega^+) + P(\Omega^-) = 1$. On the other hand, the set of all fixed points of $\{Z_t\}$ denoted by $F\{Z_t\}$ is also T- and $\{Z_t\}$-invariant. In fact, for $w \in F\{Z_t\}$, $Z_t T^{-1} w = T^{-1}\hat{Z}_t w = T^{-1} w$ holds, and hence $T^{-1}w \in F\{Z_t\}$. Therefore $\alpha_t(w) = 1$ on $F\{Z_t\}$. Hence we may assume that $E[\log\lambda|\nu_T] > 0$ (or < 0) a.e. (dP) and $\{Z_t\}$ has no fixed points. Let us consider the case $E[\log\lambda|\nu_T] > 0$, it follows that

$$\lim_{n\to\infty} 1/\lambda^{(n)}(w) = \lim_{n\to\infty} \frac{1}{\lambda(w)\,\lambda(T^{-1}w)\cdots\lambda(T^{-n+1}w)} = 0 \qquad \text{a.e.,} \tag{5.8}$$

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because $\lim_{n\to\infty}[\lambda(w)\lambda(T^{-1}w)\cdots\lambda(T^{-n+1}w)]^{1/n}=\exp\{E[\log\lambda(w)|\nu_T]\}>1$ a.e. . By the $\mathfrak{B}(\nu_{\{Z_t\}})$-measurability of $\lambda(w)$, it follows that

$$(5.\ 9)\qquad \tau(t,w)=\int_0^t\frac{du}{\lambda(\hat{Z}_u w)}=\frac{t}{\lambda(w)}\qquad \text{a.e. for each } t.$$

By Proposition 5. 1 (ii) and Proposition 5. 2, it follows that

$$(5.\ 10)\qquad \alpha_t(T^{-k}w)=\alpha_{t/\lambda^{(k)}(w)}(w)\quad \text{a.e.}\quad \text{for each } t.$$

Since the sequence $\{\alpha_{t/\lambda^{(k)}(w)}(w)\}$ converges to 1 in $L^1(\Omega,\mathfrak{B},P)$ by Corollary 3. 2, the sequence $\{g(\alpha_t(T^{-n}w))\}$ converges to $g(1)$ in $L^1(\Omega,\mathfrak{B},P)$ for any bounded continuous function g on R. Therefore it holds that

$$E[g(\alpha_t)|\nu_T](w)=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}g(\alpha_t(T^{-k}w))$$
$$=\lim_{n\to\infty}g(\alpha_t(T^{-n}w))=g(1)\quad\text{in}\quad L^1(\Omega,\mathfrak{B},P).$$

Thus our proof is complete if $E[\log\lambda(w)|\nu_T]>0$. For the case $E[\log\lambda(w)|\nu_T]<0$, we can prove similarly.

We can easily see the following corollary by Theorem 5. 1 and Proposition 5. 1.

COROLLARY 5. 2. *Let* $[\{Z_t\},\lambda(w);T]$ *be a TQ-system with the property* (A). *Then* $\{Z_t\}$ *becomes a flow via time-change by* $\gamma(w)$, *if and only if* $\lambda(w)\gamma(T^{-1}w)/\gamma(w)$ *is* $\mathfrak{B}(\nu_{\{Z_t\}})$*-measurable.*

We shall next point out another property, concerned with the entropy, of a flow in a TF-system. First we prove the following lemma for an automorphism.

LEMMA 5. 2. *Suppose we are given an automorphism* T *and invariant measurable set* B. *Let* $h(w)$ *and* $\lambda(w)>0$ *be measurable functions such that* $\log\lambda(w)$ *is quasi-integrable and satisfies* (5. 5) *on the set* B, *and*

$$(5.\ 11)\qquad h(w)=\lambda(w)h(T^{-1}w)\qquad \text{a.e.}\quad (dP)\quad \text{on } B.$$

Then the value of $|h(w)|$ *is zero or infinity for almost all* $w\in B$.

Proof. By the equality (5. 11), there exist T-invariant sets B_+ and B_- such that $0<h(w)<+\infty$ on B_+ and $-\infty<h(w)<0$ on B_- and $h(w)=0$ or infinity a.e. (dP) on $B-B_+-B_-$. Assume that $P(B_+)>0$. Then there exist two positive constants a and b such that $P(B_{a,b})>0$, where $B_{a,b}=\{w;a<$

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$h(w) < b\} \cap B_+$. Since, (5. 4) holds for $\log\lambda(w)$ by Birkhoff's ergodic theorem, there exist a point $w_0 \in B_{a,b}$ and an increasing subsequence of natural numbers $\{n_k\}$ such that $T^{n_k}w_0 \in B_{a,b}$ $k = 1, 2, \cdots$ and

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \log\lambda(T^k w_0) \neq 0 \tag{5. 12}$$

hold. While it follows from (5. 11) that

$$0 \neq |\lim_{k\to\infty} \frac{1}{n_k} \sum_{j=1}^{n_k} \log\lambda(T^j w_0)| = |\lim_{k\to\infty} \frac{1}{n_k} \sum_{j=1}^{n_k} (\log h(T^j w_0) - \log h(T^{j-1} w_0))|$$

$$= |\lim_{k\to\infty} \frac{1}{n_k} (\log h(T^{n_k} w_0) - \log h(w_0))| \leqq \lim_{k\to\infty} \frac{\log b - \log a}{n_k} = 0.$$

This is a contradiction, so we have $P(B_+) = 0$. By a similar way, $P(B_-)=0$ is proved.

THEOREM 5. 2. *Let* $\{Z_t\}$ *be a flow and* $[\{Z_t\}, \lambda(w); T]$ *be a TQ-system with property* (A). *Then the entropy of the flow* $\{Z_t\}$ *is zero or infinity.*

Proof. Let us consider the restriction $\{Z_t^C\}$ of the flow $\{Z_t\}$ onto $C \in \nu_{\{Z_t\}}$. Then the entropy $h(\{Z_t\})$ of $\{Z_t\}$ is given by the form,

$$h(\{Z_t\}) = \int_{\Omega/\nu_{\{Z_t\}}} \bar{h}(C) dP_{\nu_{\{Z_t\}}}(C), \tag{5. 13}$$

where the function $\bar{h}(C) = h(\{Z_t^C\})$ defined on $\Omega/\nu_{\{Z_t\}}$ is equal to the entropy of the flow $\{Z_t^C\}$ on C[c.f. 12]. Hence it is sufficient to prove that $\bar{h}(C)=0$ or ∞ (a.e. $P_{\nu_{\{Z_t\}}}$).

Since $\lambda(w)$ is $\mathfrak{B}(\nu_{\{Z_t\}})$-measurable by Corollary 5. 1, we can define a function on $\Omega/\nu_{\{Z_t\}}$ by $\bar{\lambda}(C) = \lambda(w)$ $w \in C$. Then it holds that

$$TZ_tT^{-1}w = Z_{t/\bar{\lambda}(C)}w \text{ a.e. } w \in C, \text{ for a.e. } C \in \Omega/\nu_{\{Z_t\}}. \tag{5. 14}$$

But $\nu_{\{Z_t\}}$ is T-invariant because $Z_tT^{-1}B = T^{-1}\hat{Z}_tB = T^{-1}B$ for any $\{Z_t\}$-invariant set. Hence (5. 14) can be written as

$$T^{C'}Z_t^{C'}(T^{C'})^{-1} = Z_{t/\bar{\lambda}(C)}^{C} \quad (\text{mod } 0),\ C' = T^{-1}C.$$

Clearly $T^{C'}$ is an isomorphism of $T^{-1}C$ to C, and hence $\{Z_t^{T^{-1}C}\}$ is isomorphic to $\{Z_{t/\bar{\lambda}(C)}^C\}$. Therefore we have

$$\bar{h}(T^{-1}C) = h(\{Z_{t/\bar{\lambda}(C)}^C\}) = \frac{h(\{Z_t^C\})}{\bar{\lambda}(C)} = \frac{\bar{h}(C)}{\bar{\lambda}(C)} \quad \text{a.e. } C. \tag{5. 15}$$

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From (5. 15) and Lemma 5. 2, it follows that $\bar{h}(C)=0$ or ∞ a.e. $(dP_{\nu_{\{Z_t\}}})$. Now proof is complete.

Now we consider a commutation relation of a quasi-flow $\{Z_t\}$ and a flow $\{T_t\}$, similarly to the above discussions.

DEFINITION 5. 3. A system $[\{Z_t\}, \kappa(w); \{T_t\}]$ is called a *TQ-system*, if $\kappa(w)$ is integrable along the trajectories of $\{T_t\}$ and if $\left[\{Z_t\}, \exp\left(\int_0^s \kappa(T_{-u}w)du\right); T_s\right]$ is a TQ-system for each s.

We say that a TQ-system has *property* (A) (resp. (AC) or (AD)), if

$$E[\kappa(w)\,|\,\nu_{\{T_t\}}] \neq 0 \ \ (\text{resp.} > 0 \text{ or } < 0) \ \text{ a.e. } (dP) \tag{5. 16}$$

holds. We call a TQ-system $[\{Z_t\}, \kappa(w); \{T_t\}]$ a *TF-system* if $\{Z_t\}$ is a flow. Then we have,

THEOREM 5. 1′. *Let $[\{Z_t\}, \kappa(w); \{T_t\}]$ be a TQ-system with property (A). Then $\{Z_t\}$ is a flow if and only if $\kappa(w)$ is $\mathfrak{B}(\nu_{\{Z_t\}})$-measurable.*

THEOREM 5. 2′. *Under the same assumption of Theorem 5. 1′, if $\{Z_t\}$ is a flow, then its entropy is zero or infinity.*

6. *TQ*-systems and the ergodicity of automorphisms and flows.

In this section, we discuss the ergodicity of flows on abstract measure space in connection with TQ-systems (c.f. EX. 1 and 2 in §8).

We may call a quasi-flow $\{Z_t\}$ a *transversal quasi-flow* of an automorphism T (or a flow $\{T_t\}$) with a *coefficient of expansion* $\lambda(w)$ (resp. $\kappa(w)$), if $[\{Z_t\}, \lambda(w); T]$ (resp. $[\{Z_t\}, \kappa(w); \{T_t\}]$) is a TQ-system. A transversal quasi-flow is called a *transversal flow* if $\{Z_t\}$ is a flow. This definition of transversal quasi-flows is a generalization of one dimensional transversal fields, and moreover, of transversal admissible one-parameter group of transformations in [14].

THEOREM 6. 1. *Let $[\{Z_t\}, \lambda(w); T]$ be a TQ-system with property (A). If either $\lambda(w) \geqq 1$ for all $w \in \Omega$ or $0 < \lambda(w) \geqq 1$ for all $w \in \Omega$ holds, then it holds that*

$$\nu_T \leqq \nu_{\{Z_t\}}. \tag{6. 1}$$

Proof. By Lemma 3. 2, it is easily seen that $\{E[h\,|\,\nu_T];\ h \in C(\{Z_t\})\}$ is dense in $L^1(\Omega, \mathfrak{B}(\nu_T), P)$ and (5. 4) holds. Hence it is sufficient to prove our

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assertion that for $g(w) \in C(\{Z_t\})$, $E[g|\nu_T](w)$ is $\{Z_t\}$-invariant (mod 0). By Proposition 5. 1 (ii), $T^k Z_t T^{-k} w = Z_{\tau^k(t,w)} w$ holds, where

$$\tau^k(t, w) = \int_0^t \frac{du}{\lambda^{(k)}(T^k Z_u T^{-k} w)} = \int_0^t \lambda^{(-k)}(Z_u T^{-k} w) du. \tag{6. 2}$$

Therefore we have

$$E[g|\nu_T](Z_t w) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^n g(T^k Z_t w) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^n g(Z_{\tau^k(t, T^k w)} T^k w)$$

for almost all w. Now, if $\lambda(w) \geqq 1$, the sequence $1/\lambda^{(k)}(T^k w) = \lambda^{(-k)}(w) = [\lambda(Tw) \cdots \lambda(T^k w)]^{-1}$ converges to 0 boundedly for almost all w, by the same reasoning as (5. 8). From (6. 2), it follows that

$$\begin{aligned} &|E[g|\nu_T](Z_t w) - E[g|\nu_T](w)| \\ &= \left| \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^n \{g(Z_{\tau^k(t, T^k w)} T^k w) - g(T^k w)\} \right| \\ &\leqq \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^n K|\tau^k(t, T^k w)| = \lim_{n\to\infty} K|\tau^n(t, T^n w)| \\ &= \lim_{n\to\infty} K\left|\int_0^t \frac{du}{\lambda^{(n)}(T^n Z_u w)}\right| = 0 \quad \text{a.e.} \quad (dP) \end{aligned} \tag{6. 3}$$

for some constant K. Therefore $E[g|\nu_T]$ is $\{Z_t\}$-invariant (mod 0). We can similarly prove the assertion for $\lambda(w) \leqq 1$.

THEOREM 6. 2. *If a TF-system* $[\{Z_t\}, \lambda(w); T]$ *has property* (A), *then* (6. 1) *holds.*

Proof. With the same reasoning as the proof of Theorem 5. 1, we may assume that $[\{Z_t\}, \lambda(w); T]$ has property (AC) or (AD). If it has property (AC), the sequence $\lambda^{(-n)}(w)$ converges to 0 by (5. 8). Since it holds that

$$\tau^k(t, T^k w) = t\lambda^{(-k)}(w) \quad \text{a.e.} \quad (dP) \tag{6. 4}$$

by Theorem 5. 1, it follows that $E[g|\nu_T]$ is $\{Z_t\}$-invariant (mod 0) by (6. 3). If property (AD) holds, we can similarly prove our assertion.

COROLLARY 6. 1. *Let a TQ-system* $[\{Z_t\}, \lambda(w); T]$ *have property* (A). *If there exists a positive measurable function* $\gamma(w)$, *integrable along the trajectories of* $\{Z_t\}$, *such that* $\tilde{\lambda}(w) = \lambda(w)\gamma(T^{-1}w)/\gamma(w)$ *is* $\mathfrak{B}(\nu_{\{Z_t\}})$*-measurable and* $\log \tilde{\lambda}(w)$ *is quasi-integrable* (or $\tilde{\lambda}(w) \geqq 1$ *for all* w *or* $\tilde{\lambda}(w) \leqq 1$ *for all* w) *then* (6. 1) *holds.*

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Proof. Let $\{\tilde{Z}_t\}$ be the time-changed quasi-flow of $\{Z_t\}$ by $\gamma(w)$. By Proposition 5. 1 (i), $[\{\tilde{Z}_t\}, \tilde{\lambda}(w); T]$ is a TQ-system with property (A). Since $\lambda(w)$ is $\mathfrak{B}(\nu_{\{Z_t\}})$-measurable by the assumption, $\{Z_t\}$ is a flow by Theorem 5. 1. Hence we have our assertion by Theorem 6. 2 (or Theorem 6. 1).

The following theorems are proved with the similar methods to the Theorem 6. 1 and 6. 2.

THEOREM 6. 1′. *Let $[\{Z_t\}, \kappa(w); \{T_t\}]$ be a TQ-system with property (A). If either $\kappa(w) \geqq 0$ for all $w \in \Omega$ or $\kappa(w) \leqq 0$ for all $w \in \Omega$ holds, then we have*

$$\nu_{\{T_t\}} \lesssim \nu_{\{Z_t\}} \quad (\text{mod } 0). \tag{6. 5}$$

THEOREM 6. 2′. *Let $[\{Z_t\}, \kappa(w); \{T_t\}]$ be a TF-system with property (A). Then* (6. 5) *holds.*

Remark 6. 1. Relation (6. 1) (resp. (6. 5)) means that if $\{Z_t\}$ is metrically transitive then T(resp. $\{T_t\}$) is ergodic.

7. *TQ*-systems and increasing partitions of automorphisms.

In this section, we shall study increasing partitions with respect to automorphisms in connection with TQ-system. These results are generalizations of the results in [14].

Let ξ and ζ be measurable partitions. The conditional entropy $H(\xi|\zeta)$ of ξ with respect to ζ is defined by

$$H(\xi|\zeta;\ w) = -\log P(D \cap C|\zeta;\ C),\quad w \in D \cap C,\ D \in \xi,\ C \in \zeta, \tag{7. 1}$$

$$H(\xi|C_\zeta) = H(\xi|\zeta;\ C_\zeta) = \int_{C_\zeta} H(\xi|\zeta;\ w)dP(\cdot|C_\zeta),\quad C_\zeta \in \zeta, \tag{7. 2}$$

$$H(\xi|\zeta) = E[H(\xi|\zeta;\ w)]. \tag{7. 3}$$

THEOREM 7. 1. *Let $[\{Z_t\}, \lambda(w); T]$ be a TF-system with property (AC). Then, there exists a partition ζ of Ω, such that almost every element of ζ is a segment of a trajectory of $\{Z_t\}$ and that*

(i) $T\zeta \geqq \zeta$ (mod 0),

(ii) $\bigvee_k T^k\zeta = \varepsilon$ (mod 0),

(iii) $\bigwedge_k T^k\zeta = \nu_{\{Z_t\}}$ (mod 0),

(iv) $H(T\zeta|\zeta) = \int_{\Omega - F\{Z_t\}} \log\lambda(w)dP.$

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Proof. By the proof of Theorem 5. 1, the set $F\{Z_t\}$ of all fixed points of $\{Z_t\}$ is T- and $\{Z_t\}$-invariant. For any T-invariant set B, there exists a T- and $\{Z_t\}$-invariant set B' such that $P(B\ominus B')=0$, by Theorem 6. 1. If there exists a denumerable partition $\{\Omega_n\}$ of $\Omega-F\{Z_t\}$ such that (1) each Ω_n is T- and $\{Z_t\}$-invariant subset of $\Omega-F\{Z_t\}$, (2) there exists a partition ζ_n of Ω_n which satisfies the conditions (i)$\sim$(iii) on Ω_n and

$$\text{(iv)}' \qquad \int_{\Omega_n}\log P(C_{T\zeta_n}(w)\,|\,C_{\zeta_n}(w))dP=\int_{\Omega_n}\log\lambda(w)dP.$$

Then the partition ζ of Ω, which is equal to ζ_n on Ω_n and is equal to the partition of $F\{Z_t\}$ into individual points on $F\{Z_t\}$, is a desirable one. The conditions (i)$\sim$(iii) are obviously fulfilled and it holds that

$$H(T\zeta\,|\,\zeta)=E[\log P(C_{T\zeta}(w)\,|\,C_\zeta(w)]=\sum_{n=1}^{\infty}\int_{\Omega_n}\log P(C_{T\zeta_n}(w)\,|\,C_{\zeta_n}(w))dP$$

$$=\sum_n\int_{\Omega_n}\log\lambda(w)\,dP=\int_{\Omega-F\{Z_t\}}\log\lambda(w)dP.$$

Hence it is sufficient to prove the assertion that there exists a T- and $\{Z_t\}$-invariant subset Ω_0 of $\Omega-F\{Z_t\}$ with positive measure and exists a partition ζ of Ω_0 which satisfies the conditions (i), (ii), (iii) and (iv)$'$.

Since we may assume that the flow $\{Z_t\}$ has not fixed points, we can suppose by Corollary 3. 1 that $\{Z_t\}$ is an S-flow built up by some $(X,\mathfrak{A},\mu, f(x), 1, S)$, where S is an automorphism of X. We may assume that $\lambda(w)=\lambda(x,u)=\lambda(S^kx,v)$ for $0\leqq v<f(S^kx)$, by Theorem 5. 1. Let ξ be the partition of $\Omega=\{(x,u);\ 0\leqq u<f(x),\ x\in X\}$ into the vertical lines: $C_\xi(x)=\{(x,v);\ 0\leqq v<f(x)\}$. Put $V_\gamma=\{w;\ E[\log\lambda\,|\,\nu_T](w)>\gamma\}$, then $P(V_\gamma)>0$ holds for sufficiently small γ (>0). Since V_γ is $\mathfrak{B}(\nu_{\{Z_t\}})$-measurable, by Theorem 6. 2, there exists an S-invariant set A such that $P(V_\gamma\ominus\{(x,u);\ 0\leqq u<f(x),\ x\in A\})=0$. Let b be a positive constant such that $\mu(\{x;\ f(x)>b\}\cap A)>0$. Put

$$\text{(7. 4)} \qquad V=\{(x,u);\ f(x)>b,\ w\in A,\ 0\leqq u<b\},$$

and

$$\text{(7. 5)} \qquad \Omega_0=\bigcup_{k=-\infty}^{\infty}T^kV.$$

Let η be the partition of V into the vertical lines of it i.e. the restriction

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of ξ to V, and let $\tilde{\eta}$ be the partition of Ω_0 which is equal to η on V and degenerated on $\Omega_0 - V$. We shall show in the following that the partition $\zeta = \bigvee_{k \leqq 0} T^k \tilde{\eta}$ is a desirable one.

For any $\varepsilon > 0$, there exist a $\mathfrak{B}(\nu_{\{Z_t\}})$-measurable set G_ε with $P(G_\varepsilon) > 1 - \varepsilon$ and a natural number $n(\varepsilon)$ such that for any $w \in G_\varepsilon$ and $n > n(\varepsilon)$,

$$(7.6) \qquad \exp[nE[\log \lambda(w) \mid \nu_T] + n\varepsilon] > \lambda(Tw) \cdots \lambda(T^n w) > \exp[nE[\log \lambda(w) \mid \nu_T] - n\varepsilon]$$

holds. From $\mathfrak{B}(\nu_{\{Z_t\}})$-measurability of $\lambda(w)$ and (6.4), it follows that the time length of the segment $C_{T^n\eta}(T^n w)$ is $b[\lambda(Tw) \cdots \lambda(T^n w)]^{-1}$ and it is less than $b \times \exp[-nE[\log \lambda(w) \mid \nu_T] + n\varepsilon] < be^{-n(\gamma - \varepsilon)}$ for $w \in G_\varepsilon$ and $n > n(\varepsilon)$.

Setting $K_n = \{w \in V;\ H(\eta \mid C_{T^n\tilde{\eta}}(T^n w)) > 0\}$, we have

$$(7.7) \qquad \begin{aligned} \sum P(K_n \cap G_\varepsilon) &= \sum P(T^n K_n \cap T^n G_\varepsilon) \\ &\leqq \sum E \exp[-nE[\log \lambda \mid \nu_T] + n\varepsilon] \leqq b \sum e^{(-\gamma + \varepsilon)n} < \infty \end{aligned}$$

for any ε with $\gamma > \varepsilon > 0$. From (7.7) and Borel-Cantelli's lemma, it follows that $P(\overline{\lim_n} K_n \cap G_\varepsilon) = 0$ for small ε. Hence we have $P(\overline{\lim_n} K_n) = 0$, that is, almost every w belongs to only finite numbers of K_n's. Hence ζ divides almost every $C_{\tilde{\eta}} \subset V$ into almost countable segments, that is, almost every element of ζ which is contained in V is a semi-open segment of a trajectory of $\{Z_t\}$. For $k > 0$, put

$$W_k = T^{-k}V - \bigcup_{m=1}^{k-1} T^{-m}V,$$

then $P(\Omega_0 - \sum_{k=1}^{\infty} W_k) = 0$ holds. Since $C_{T^k\zeta}(w) = T^{-k}C_\zeta(T^k w)$ for $w \in W_k$, almost every element of ζ is a semi-open segment of a trajectory of $\{Z_t\}$.

Let us show that ζ satisfies the condition (i)$\sim$(iii) and (iv)$'$. Let $h_n(w)$ be the time length of the segment $C_{T^n\zeta}(w)$ of a trajectory of $\{Z_t\}$. Then $\{h_n(w)\}$ is a non-increasing sequense of finite valued positive measurable functions, because ζ is an increasing partition with respect to T. On the other hand, from the commutation relation of T and $\{Z_t\}$, it follows that $h_n(Tw) = h_{n-1}(w)/\lambda(Tw)$, $w \in \Omega_0$. If we put $\lim_{n\to\infty} h_n(w) = h(w)$, then it holds that

$$(7.8) \qquad \lambda(Tw)h(Tw) = h(w) \qquad \text{a.e. on } \Omega_0.$$

From (7.8) and Lemma 5.2, it follows that $h(w) = 0$ a.e. on Ω_0. This fact means that the condition (ii) is fulfilled. The condition (iii) can be shown similarly, that is, $\lim_{n\to\infty} h_n^*(w) = h^*(w)$ (this limit exists for almost all $w \in \Omega_0$

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admitting infinity), satisfies the equality (7. 8), where $h_n^*(w)$ is the time distance between w and the boundary of $C_\zeta(w)$, so we have $h^*(w)=\infty$ a.e. on Ω_0.

Now we see easily that

$$(7.\ 9)\qquad P(C_{T\zeta}(w)\,|\,C_\zeta(w)) = h_1(w)/h_0(w) = h_0(T^{-1}w)\,[h_0(w)\lambda(w)]^{-1} \text{ for } w\in\Omega_0.$$

The positive part of the logalism of the left hand side of (7. 9) is integrable, and $\log\lambda(w)$ is quasi-integrable and $E[\log\lambda\,|\,\nu_T]>0$ a.e. on Ω_0. Let B be the T-invariant set such that either $\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\log\lambda(T^k w)<\infty$ or $-\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\log P(C_{T\zeta}(T^k w)\,|\,C_\zeta(T^k w))<\infty$ holds for almost every $w\in B$. Then we have by Lemma 5. 2, $E[\log(\lambda(w)P(C_{T\zeta}(w))\,|\,C_\zeta(w))\,|\,\nu_T]=0$ a.e. (dP) on B. On the other hand $\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\log\lambda(T^k w)=-\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\log P(C_{T\zeta}(T^k w)\,|\,C_\zeta(T^k w))=\infty$ a.e. (dP) on Ω_0-B. Therefore we have

$$(7.\ 10)\qquad -\int_{\Omega_0}E[\log P(C_{T\zeta}(w)\,|\,C_\zeta(w))\,|\,\nu_T]dP$$

$$=\int_{\Omega_0}E[\log\lambda(w)\,|\,\nu_T]dP=\int_{\Omega_0}\log\lambda(w)dP.$$

So our assertion was proved.

For TQ-systems, we have similar theorem,

THEOREM 7. 2. *Let* $[\{Z_t\},\lambda(w);\,T]$ *be a TQ-system with property* (*AC*). *If* $\lambda(w)\geqq 1$ *holds, there exists a partition* ζ *of* Ω, *such that almost every element of which is a segment of a trajectory of* $\{Z_t\}$ *and* ζ *satisfies the same conditions* (i)$\sim$(iv) *in Theorem* 7. 1.

Proof. By the same reason in the proof of Theorem 7. 1, it is sufficient to prove the assertion that there exists a T- and $\{Z_t\}$-invariant subset Ω_0 of $\Omega-F\{Z_t\}$ with positive measure and exists a partition ζ satisfying the conditions (i)$\sim$(iii) and (iv)$'$ in the proof of Theorem 7. 1.

Since we may assume that $\{Z_t\}$ has not fixed points, we can suppose that $\{Z_t\}$ is an S-quasi-flow built up by some $(X,\mathfrak{A},\mu,f(x),p(x,u),S)$ by Theorem 3. 1. Let ξ be the partition of $\Omega=\{w=(x,u);\,0\leqq u<f(x),\,x\in X\}$ into the vertical lines: $C_\xi(w)=C_\xi(x,u)=\{(x,v);\,0\leqq v<f(x)\}$. Then the conditional measure on $C_\xi(x,u)$ is given in the form

$$(7.\ 11)\qquad dP(\cdot\,|\,C_\xi(x,u))=\frac{p(x,v)dv}{\int_0^{f(x)}p(x,v)dv}$$

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Put

(7. 12)
$$V_{\alpha'\beta'}=\left\{w=(x,u);\ \frac{\int_0^t p(x,u+v)\,dv}{t\int_0^{f(x)}p(x,v)\,dv}<\alpha',\ \text{for } 0<|t|<\beta',\right.$$
$$\left.\beta'<u,\ \text{and}\ 2\beta'<f(x)\right\},$$

Since $\lim_{t\to 0}\frac{1}{t}\int_0^t p(x,u+v)dv=p(x,u)$ a.e. $(dud\mu(x))$, there exist two constants α' and β' such that $P(V_{\alpha'\beta'})>0$. Further there exists a measurable set $A\subset X$ and positive constants b and β $(<\beta')$ such that (x,b), $(x,b+\beta)\in V_{\alpha'\beta'}$ for any $x\in A$ and $\mu(A)>0$ holds. Fix such constants b, α, β', β and the set A. Set $V'=\{w=(x,u);\ b\leqq u<b+\beta,\ x\in A\}$ and let η' be the partition of V' into the vertical lines of it. Let us fix positive constants γ and γ' such that $P(V')>3\sqrt{\gamma'}$ and

(7. 13)
$$P(\{w;\ P(D|C_{\nu_T}(w))>3\gamma\})>1-\gamma',$$

where
$$D=\{w;\ \lambda(w)>1\}\cap\left\{w;\ \lim_{t\to 0}\frac{1}{t}\int_0^t\frac{du}{\lambda(Z_u w)}=\frac{1}{\lambda(w)}\right\}.$$

The existence of such γ and γ' is assured by the hypothesis of that $E[\log\lambda(w)]>0$ a.e. and by that $P(\{w;\ \lambda(w)>1\}-D)=0$ holds by Lemma 3. 2 (i). Put

(7. 14)
$$A_{\alpha\beta}=\left\{w;\ \frac{1}{t}\int_0^t\frac{du}{\lambda(Z_u w)}<\alpha,\quad 0<t\leqq\beta\right\},$$

for $0<\alpha<1$. Since,
$$\bigcup_{0<\alpha<1}A_{\alpha\beta}=\left\{w;\ \frac{1}{t}\int_0^t\frac{du}{\lambda(Z_u w)}<\alpha,\quad 0<t\leqq\beta\right\}\supset D$$

holds, we have
$$\lim_{\alpha\to 1-0}P(A_{\alpha\beta}|C_{\nu_T}(w))\geqq P(D|C_{\nu_T}(w))\quad \text{a.e.}.$$

From (7. 13), it follows that there exists a constant α (<1) such that

(7. 15)
$$P(A_{\alpha\beta\gamma})>1-2\gamma',\quad A_{\alpha\beta\gamma}=\{w;\ P(A_{\alpha\beta}|C_{\nu_T}(w))>2\gamma\}.$$

By Birkhoff's ergodic theorem, there exists a measurable set G with $P(G)>1-\gamma'$ and a natural number n_0 such that

(7. 16)
$$|s_n(w)-nP(A_{\alpha\beta}|C_{\nu_T}(w))|<n\gamma$$

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hold for any $w \in G$ and $n > n_0$, where $s_n(w)$ is the numbers of i $(0 \leqq i \leqq n)$ with $T^i w \in A_{\alpha\beta}$. Put

$$V = \{w \in V';\ P(G \cap A_{\alpha\beta\gamma} | C_{\eta'}(w)) > 1 - \sqrt{\gamma'}\ \}. \tag{7. 17}$$

From (7. 13) and (7. 15), it follows from Tschebyscheff's inequality that

$$P(V' - V) \leqq \frac{1}{\sqrt{\gamma'}} \int_{V'-V} [1 - P(G \cap A_{\alpha\beta\gamma} | C_{\eta'}(w))] dP$$

$$\leqq \frac{1}{\sqrt{\gamma'}} (1 - P(G \cap A_{\alpha\beta\gamma})) < \frac{3\gamma'}{\sqrt{\gamma'}} = 3\sqrt{\gamma'}\ .$$

Hence we have $P(V) > 0$. Let η be the partition of V into the vertical lines of it and $\tilde{\eta}$ be the partition of $\Omega_0 = \bigcup_k T^k V$ which is equal to η on V and degenerated on $\Omega_0 - V$. Define a partition ζ of Ω_0 by

$$\zeta = \bigvee_{k \leqq 0} T^k \tilde{\eta}. \tag{7. 18}$$

It can be proved by the same method as the proof of Theorem 7. 1, that almost every element of ζ is a semi-open segment of a trajectory of $\{Z_t\}$ and the conditions (i)$\sim$(iv)$'$ are fulfilled.

8. Examples.

We shall give three TQ-systems on Lebesgue spaces. First, let us give examples of TQ-systems such that the quasi-flows consisting of the systems have not σ-finite invariant measures.

Example 1. Bernoulli automorphisms. Let $X = [0, 1)$ and $(\mathfrak{A}, \mu)$ be the ordinary Lebesgue measure on X. Let $p_j = 1, 2, \cdots, N$ be positive numbers with $p_1 + p_2 + \cdots + p_N = 1$ and let S be the quasi-automorphism (mod 0) of X defined by

$$Sx = \left(\frac{p_1}{p_N}\right)^m \left(\frac{p_{k+1}}{p_k}(x - q_{k-1}) + q_k\right) \tag{8. 1}$$

for $1 - p_N^m + p_N^m q_{k-1} \leqq x < 1 - p_N^m + p_N^m q_k$, $k = 1, 2, \cdots N, m = 0, 1, 2, \cdots$, where $q_k = p_1 + p_2 + \cdots + p_k, q_0 = 0$. let $\{Z_t\}$ be the S-quasi-flow of $\Omega = \{(x, u);\ 0 \leqq u < 1,\ 0 \leqq x < 1\}$ built up by $(X, \mathfrak{A}, \mu, 1, 1, S)$. The automorphism T of Ω defined by

$$T(x, u) = (x', u'), \tag{8. 2}$$

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$$(8.3)\qquad \begin{cases} x' = \dfrac{1}{p_k}(x - q_{k-1}) \\ u' = p_k u + q_{k-1} \end{cases} \quad \text{for } q_{k-1} \leqq x < q_k,$$

is isomorphic to a Bernoulli automorphism. Further the system $[\{Z_t\}, \lambda(x,u); T]$ is a TQ-system, where $\lambda(x,u) = 1/p_k$ for $q_{k-1} \leqq x < q_k$. Now the partition ζ of Ω into vertical lines is a partition given by Theorem 7.1 and the conditional entropy $H(T\zeta|\zeta) = -\sum_j p_j \log p_j = E[\log \lambda(x,u)]$ is equal to the entropy of the automorphism T.

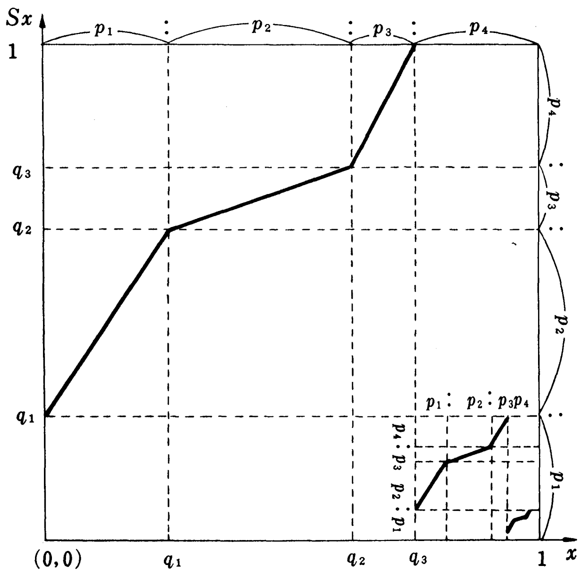

Fig. 1. N=4. $p_1 = p_4 = \frac{1}{4}$ $p_2 = \frac{3}{8}$ $p_3 = \frac{1}{8}$

The quasi-flow $\{Z_t\}$ is a flow if and only if S is an automorphism, that is, if and only if $p_1 = p_2 = \cdots = p_N = \dfrac{1}{N}$ holds. In the other cases, $\{Z_t\}$ and S have no σ-finite invariant measures equivalent to the Lebesgue measure. Especially, the case $N = 3$ and $p_1 = p_3 = \dfrac{1}{4}$, $p_2 = \dfrac{1}{2}$ is such an example, which is given by A. Brunel. The case $N = 2$ and $p_2/p_1 = \alpha$ is the example given by L. K. Arnold (see [2]).

In the case of $p_j = \dfrac{1}{N}$, $j = 1, 2, \cdots, N$, the flow $\{Z_t\}$ is ergodic and has pure point spectrum $\left\{2\pi i \dfrac{k}{N^m}\,;\ k \text{ and } m \text{ are integers}\right\}$. Hence the entropy of $\{Z_t\}$ is 0.

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EXAMPLE 2. Let Ω be the space of continuous functions on real line R and let $(\Omega, \mathfrak{B}, P)$ be the measure space of Brownian motion. Then the sift $\{\theta_t\}$ acting on the w: $(\theta_t w)(s) = w(s-t)$, is a flow on $(\Omega, \mathfrak{B}, P)$. Define a oneparameter group of transformations $\{T_t\}$ of Ω by

$$(8.4) \qquad (T_t w)(s) = e^{-at} w(e^{2at} s).$$

We can easily see that $\{T_t\}$ is a flow on Ω which is induced from 2-dimensional Ornstein-Uhlenbeck's Brownian motion. We have

$$(8.5) \qquad T_s \theta_t T_{-s} = \theta_{t \cdot \exp(2as)},$$

that is, $[\{\theta_t\}, -2a; \{T_t\}]$ is a TF-system. This is an example of a transversal flow which has infinite valued entropy. The relation (8. 5) satisfied by the transformation groups θ_t and T_t on the functional space comes from the commutation relation for the shift and the multiplication acting on reals.

EXAMPLE 3. We can generalize our formulations to multi-dimensional transversal fields as follows. Let G be a connected unimodular Lie group and $\{Z_g;\ g \in G\}$ be a group of non-singular transformations of a Lebesgue space $(\Omega, \mathfrak{B}, P)$ such that

(i) $\quad Z_{g_1} Z_{g_2} w = Z_{g_1 g_2} w$ for $g_1, g_2 \in G$ and $w \in \Omega$,

(ii) the mapping Z_g: $(g, w) \to Z_g w$ is $\overline{\mathfrak{G} \times \mathfrak{B}}$-measurable, where $\mathfrak{G}$ is the topological Borel field on G. We call a mapping $\varphi(g, w)$ of $G \times \Omega$ onto G to be a multiplicative mapping for $\{Z_g;\ g \in G\}$ if it is $\mathfrak{G} \times \mathfrak{B}$-measurable and $\varphi(g, w)$ is a one-to-one onto mapping of G for each w and satisfies

$$(8.6) \qquad \varphi(g_1 g_2, w) = \varphi(g_1, Z_{g_2} w) \cdot \varphi(g_2, w) \quad \text{for each } w.$$

Let $\tau(g, w)$ be the inverse mapping of $\varphi(g, w)$ for each w. Then the system $\{\hat{Z}_g;\ g \in G\}$ defined by

$$(8.7) \qquad \hat{Z}_g w = Z_{\tau(g, w)} w$$

is again a group of bimeasurable point transformations of Ω. We say that $\{\hat{Z}_g;\ g \in G\}$ is the time-change of $\{Z_g;\ g \in G\}$ by $\varphi(g, w)$. If the measure P is invariant under $\{Z_g;\ g \in G\}$, $\{\hat{Z}_g;\ g \in G\}$ has an invariant measure. In fact, if the mapping $\varphi(g, w)$ of G (for fixed w) is non-singular with respect to the invariant measure dg of G, then the density is given in the form

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$\lambda(Z_g w)$ with some $\mathfrak{B}$-measurable function $\lambda(w)$ and the measure $dQ = \lambda(w)dP$ is invariant under $\{\hat{Z}_g;\ g \in G\}$.

Under the suitable hypothesis, we can perform the similar discucssions to §3~§7. For an example, let T be an $n\times n$ matrix with integral coefficients with determinant ± 1. Then T can be considered as an automorphism of n-dimensional torus $\boldsymbol{T}^n$. The Jordan's canonical form of T is given by a regular real matrix $C = (c_{ij})$ in the form

(8. 8) $$C^{-1}TC = A = \begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_k \end{pmatrix} = (a_{ij}),\quad A_j = \begin{pmatrix} a_jI & & & \\ & a_jI & & \\ & & \ddots & I \\ & & & a_j \end{pmatrix},$$

where either a_j is a 1×1 matrix and I is the idddentity 1×1 matrix or a_j is 2×2-matrix in the form

(8. 8′) $$a_j = \begin{pmatrix} \alpha_j \cos\theta_j & -\alpha_j \sin\theta_j \\ \alpha_j \cos\theta_j & \alpha_j \cos\theta_j \end{pmatrix} \quad \alpha_j > 0.$$

and I is the identity 2×2 matrix.

We suppose that $\alpha_1, \alpha_2, \cdots, \alpha_m < 1$. We define $\{Z_t;\ t = (t^1, t^2, \cdots, t^N) \in R^N\}$ (where N is the total dimensions of $A_1, \cdots, A_m$) by

(8. 9) $$Z_t x = x + Ct \quad (Ct)_i = \sum_{j=1}^{N} c_{ij}t^j.$$

It is easily seen that $\{Z_t;\ t \in R^N\}$ is a group of measure preserving transformations and satisfies

(8. 10) $$TZ_tT^{-1} = Z_{\bar{A}t},\quad \bar{A} = (a_{ij})_{i,j=1,2,\cdots,N}.$$

Then we can construct an increasing partition ζ with respect to T such that ζ satisfies the condition (i) ~ (iii) in Theorem 7. 1 and $H(T\zeta|\zeta) = -\sum_{j=1}^{m} \log|\det A_j|$. It is well known that $H(T\zeta|\zeta)$ is equal to the entropy of T(c.f. [11]).

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