This is a ``preproof'' accepted article for *The Journal of Symbolic Logic*.
This version may be subject to change during the production process.
DOI: 10.1017/jsl.2023.90

EFFECTIVE CARDINALS AND Σ^0_4-DETERMINACY

J. P. AGUILERA

Abstract. By replacing the use of arbitrary bijections in the definition of "cardinal number" with that of suitably computable re-orderings, one arrives at the notion of an "effective cardinal." We use this notion to give a characterization of Σ^0_4-determinacy in the spirit of Reverse Mathematics.

1. Introduction

Assuming the Axiom of Choice, the infinite cardinal numbers can be constructed inductively in many ways, one of which is the following:

$$\aleph_0 = \mathbb{N}$$
$$\aleph_{\alpha+1} = \sup\{\gamma : \text{there is a wellordering of (a subset of) } \aleph_\alpha \text{ of length } \gamma\}$$
$$\aleph_\lambda = \sup\{\aleph_\alpha : \alpha < \lambda\}, \quad \text{at limit stages.}$$

An interesting concept is what results of this definition when one replaces the use of arbitrary wellorderings at each stage by only those which are appropriately computable. Let us be more precise. Recall Gödel's constructible hierarchy given by $L_0 = \varnothing$, $L_{\alpha+1}$ = all sets definable over L_α from finitely many elements of L_α, and $L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha$ at limit stages. A subset of an ordinal α is *α-recursively enumerable* (α-r.e.) if it is Σ_1-definable over L_α with parameters. Note that this definition makes sense for arbitrary α.

Definition 1. The *effective cardinal numbers* are defined inductively as follows:

$$\eta_0 = \mathbb{N}$$
$$\eta_{\alpha+1} = \sup\{\gamma : \text{there is an } \eta_\alpha\text{-r.e. wellordering of a subset of } \eta_\alpha \text{ of length } \gamma\}$$
$$\eta_\lambda = \sup\{\eta_\alpha : \alpha < \lambda\}, \quad \text{at limit stages.}$$

We have chosen to define effective cardinals in terms of recursive enumerability and not plain recursiveness (where a set is recursive if it is both r.e. and co-r.e.); the two notions will coincide in all cases of interest, but the computations involved will be simpler this way. Recall that an ordinal α is said to be *admissible* if L_α is a model of Kripke-Platek set theory, the result of removing from Zermelo-Fraenkel set theory the axioms of Powerset and Replacement and adding the axioms of Separation and Collection, both restricted to formulae in which only bounded quantifiers appear. For simplicity, we will assume that KP includes the schema of foundation for all formulae. For an ordinal α, we denote by α^+ the smallest admissible ordinal greater

Date: June 13, 2023.
2020 *Mathematics Subject Classification.* (2020) 03B30, 03F35, 03D60, 03E30 (primary), 03E10, 03E60 (secondary).
Key words and phrases. Reverse Mathematics, Admisible ordinal, Effective cardinal, Σ^0_4-determinacy, Gandy ordinal, Third-Order Arithmetic.

than α. According to our definition, all admissible ordinals are effective cardinals, though the converse is not true.

Definition 2. Let α, β be ordinals. η_α is said to be β-*Gandy* if $\alpha^+ \leq \eta_{\alpha+\beta}$.

Our definition is a generalization of that of a *Gandy ordinal* (which coincides with that of a 1-Gandy ordinal). These ordinals were first studied by H. Friedman (unpublished) and Gostanian [9] (and indirectly by Solovay [22]) and named by Abramson and Sacks [1].

A question of interest is that of the admissibility of η_α; another one is the related question of whether η_α is β-Gandy, for a given β. In this article, we shall present the main properties of effective cardinals and present an application to the reverse mathematics of determinacy. For context, we mention the following result, which is commonly known:

Theorem 3 (Aczel, Richter; Gostanian; Grilliot; Solovay)**.** *The following are equivalent over* KPl*:*

(1) There is an ordinal which is not 1*-Gandy.*
(2) Σ^0_2*-determinacy.*

Proof Sketch. Theorem 3 is obtained by combining several well-known results: first, by KPl, we have access to Shoenfield's absoluteness, so we may assume $V = L$ in both directions. By work of Gostanian [9], the existence of an ordinal which is not 1-Gandy is equivalent to the existence of an ordinal which is Π^+_1-reflecting (in the terminology of [3]). This is equivalent to Σ^1_1-reflection by Aczel-Richter [2]. By Aczel-Richter [2], this ordinal, if it exists, is equal to the closure ordinal of Σ^1_1-inductive definitions. By a theorem of Grilliot (unpublished, but see [4] for a proof), this ordinal, if it exists, is also the closure ordinal of *monotone* Σ^1_1-inductive definitions. A theorem of Solovay (unpublished, but see Kechris [12] or Moschovakis [17]) asserts that if this ordinal σ^1_1 exists, then all Σ^0_2 games won by Player I have a winning strategy in $L_{\sigma^1_1}$. As noted by Welch [24], the argument also shows that all other games are won by Player II, as witnessed by a strategy in any admissible set containing $L_{\sigma^1_1}$. (In fact, they appear in a strict initial segment of such an admissible set; see [6]). For the converse, if Σ^0_2-determinacy holds, then a theorem of Tanaka [23] asserts that all monotone Σ^1_1 inductive definitions reach a fixpoint. By Grilliot's theorem, all non-monotone Σ^1_1 inductive definitions have a fixpoint, so that there is a Σ^1_1-reflecting ordinal, by Aczel-Richter [2] and thus an ordinal which is not 1-Gandy, by Gostanian [9]. □

The ordinal of Theorem 3 has many equivalent characterizations, some of which we have mentioned; we refer the reader to [5] for a compilation of some others.

We shall prove the following analogue of Theorem 3:

Theorem 4. *The following are equivalent over* KPl*:*

(1) There is an ordinal which is not ω*-Gandy.*
(2) Σ^0_4*-determinacy.*

As an immediate consequence, we obtain the following boldface result in terms of a relativized form of Gandiness. Below, the notion of an ordinal ξ being α-*Gandy relative to* $x \in \mathbb{R}$ is defined just like before, except that Definitions 1 and 2 are modified to speak of the $L[x]$-hierarchy instead of the L-hierarchy.

Corollary 5. *The following are equivalent over* KPI*:*

(1) *For each $x \in \mathbb{R}$, there is an ordinal which is not ω-Gandy relative to x.*
(2) *$\mathbf{\Sigma}^0_4$-determinacy.*

We do not know if there is any kind of analogue of Theorem 3 and Theorem 4 for Σ^0_3 sets, though the existence of an ordinal which is not 2-Gandy implies the consistency of Σ^0_3-determinacy and indeed of Z_2. Let us finish this introduction by briefly recalling the history of Σ^0_4-determinacy.

The first proof of Σ^0_4-determinacy was obtained as a consequence of Martin's [14] proof of Σ^1_1-determinacy, which assumed the existence of a measurable cardinal. The first proof that did not require assumptions beyond ZFC was that of Paris [18]. Martin's [15] proof of Borel Determinacy showed that Σ^0_4-determinacy is indeed provable in Z_3 and further work of his (see [13]) showed this cannot be improved to Z_2. Martin's unprovability result built on work of Friedman [8] whereby Σ^0_5-determinacy is not provable in Z_2. A strengthening of Martin's unprovability result due to Montalbán and Shore [16] showed that Z_2 cannot even prove that all Boolean combinations of Σ^0_3 games are determined. However, Martin has shown (see [13]) that all wellfounded models of Z_2 satisfy Δ^0_4-determinacy. A previous characterization of Σ^0_4-determinacy in the spirit of Reverse Mathematics was carried out by Hachtman [10]; the proof of ours will rely on Hachtman's result.

2. Basic properties

We shall use basic properties of the constructible universe L without much detail. We refer the reader to Barwise [7], Jech [11], Simpson [21], or Sacks [20] for background. In talking about formulae which define sets over structures of the form L_α, it will be convenient to recall the existence of Gödel's pairing function, which allows multiple elements of L_α to be coded by tuples $\langle x, y, z\rangle$. L_α is closed under Gödel's pairing function whenever α is a multiplicatively indecomposable ordinal.

We have mentioned this before, but let us state it explicitly.

Lemma 6. *Suppose α is admissible. Then, α is an effective cardinal.*

Proof. This is simply because if $\prec$ is a wellorder (in the real world) which belongs to L_α, then L_α can recursively construct an isomorphism from $\prec$ to an ordinal, so if $\eta_\beta < \alpha$, then $\eta_{\beta+1} \leq \alpha$. □

Lemma 7. *Let α be a recursively inaccessible ordinal. Then $\alpha = \eta_\alpha$.*

Proof. This is immediate from the previous lemma. □

Below, we say that an ordinal γ is β-r.e. if there is a β-r.e. wellordering of β of length γ.

Lemma 8. *Let α, β, and γ be ordinals. Suppose that $\beta < \eta_\alpha$ and γ is β-r.e. Then, $\gamma < \eta_{\alpha+1}$.*

Proof. Let ϕ be the Σ_1 formula which defines over L_β a wellordering of length γ. Then, the formula

$$\phi^*(x, y, \beta) \leftrightarrow L_\beta \models \phi(x, y)$$

is Σ_0 (thus Σ_1) over L_{η_α} with parameters in L_{η_α}. □

Lemma 9. *Let α be an ordinal. Then, η_α is a limit ordinal and is closed under addition, multiplication, exponentiation with base ω, the Veblen functions, etc.*

https://doi.org/10.1017/jsl.2023.90 Published online by Cambridge University Press

Proof. This is proved by induction. The limit stages are immediate. Supposing η_α satisfies those closure properties and β, γ are η_α-r.e., we can use the fact that η_α is multiplicatively indecomposable (thus closed under coding of tuples) to combine the definitions of β, γ into η_α-r.e. wellorderings of length $\beta + \gamma$ and $\beta \cdot \gamma$. Closure under the Veblen functions is a bit tricker and will not be used below, but it can be done by lifting the proof that ω_1^{ck} is closed under the Veblen functions (see e.g., Rathjen-Weiermann [19]). $\square$

Lemma 9 will be very useful and used without explicit mention in the future. Its main consequence is that L_{η_α} is closed under the Cantor pairing function. Thus, we can replace mention of finite sequences of parameters in L_{η_α} by single parameters which can additionally be assumed to be ordinals. Many consequences of admissibility will hold in L_{η_α}, except possibly those that crucially depend on Σ_0-collection.

Lemma 10. *Let α be an ordinal. Then, every illfounded η_α-r.e. linear order has an infinite descending chain which is Π_2-definable over $L_{\eta_{\alpha+1}}$.*

Proof. Let $\prec$ be such a linear order. Using the closure properties provided by Lemma 9, we may construct inside of $L_{\eta_{\alpha+1}}$ an isomorphism

$$i_x : \prec\restriction x \to \beta_x$$

for some β_x, for each x in the wellfounded part of $\prec$. The construction of i_x is by transfinite recursion on $\prec\restriction x$ and requires β_x stages, so it belongs to $L_{\eta_\alpha+\beta_x}$. Since $L_{\eta_{\alpha+1}}$ is additively indecomposable, $i_x \in L_{\eta_{\alpha+1}}$. The function f given by $f(x) = \beta_x$ is partial Δ_1 over $L_{\eta_\alpha+1}$, with domain a subset of L_{η_α} which need not be Δ_1 over $L_{\eta_\alpha+1}$. Consider the set

$$A = \{x < \eta_\alpha : x \in \mathrm{dom}(f)\}.$$

Thus, A is Σ_1-definable over $L_{\eta_\alpha+1}$. A is precisely the wellfounded part of $\prec$, for if $x \in \mathrm{wfp}(\prec) \setminus A$, then $\prec\restriction x$ cannot be isomorphic to an ordinal smaller than $\eta_{\alpha+1}$ (otherwise $f(x)$ would be defined), so $\prec\restriction x$ is an η_α-r.e. wellorder of length at least $\eta_{\alpha+1}$, which contradicts the definition. Using A we can replicate the proof of König's lemma: by induction on $i \in \mathbb{N}$, define x_{i+1} to be the $<_L$-least $x \notin A$ such that $x \prec x_i$. This defines an infinite descending chain through $\prec$.

As for the complexity of this sequence s, we have $x_i < \eta_\alpha$ for each i, and the restriction of $<_L$ to L_{η_α} is Σ_1 over L_{η_α} and hence an element of $L_{\eta_\alpha+1}$. Therefore, s is the unique ω-sequence all of whose elements x_i have the following properties:

(1) $x_i \notin A$;
(2) $x_{i+1} \prec x_i$;
(3) for all $y \in \eta_\alpha$, if $y \prec x_i$ and $y \notin A$ then $x_{i+1} <_L y$.

The first condition is Π_1 over $L_{\eta_{\alpha+1}}$; the second is Δ_0; and the third is Π_2, since $x_{i+1} <_L y$ is equivalent to $L_{\eta_\alpha} \models x_{i+1} <_L y$ (we cannot conclude that the third condition is Σ_1, because $\eta_{\alpha+1}$ may not be admissible). $\square$

We remark the following consequence of the proof of the previous lemma:

Lemma 11. *Let α be an ordinal and let $\prec$ be an illfounded η_α-r.e. linear order. Then, either $\prec$ has an infinite descending sequence in $L_{\eta_\alpha+1}$, or the wellfounded part of $\prec$ is isomorphic to $\eta_{\alpha+1}$.*

https://doi.org/10.1017/jsl.2023.90 Published online by Cambridge University Press

Proof. This follows from the preceding argument, using the observation that if the range of f is bounded below $\eta_{\alpha+1}$, say $f[\eta_\alpha] \subset \beta < \eta_{\alpha+1}$. Then, A is Σ_1-definable over L_β. □

Below, an ordinal α is *locally countable* if every infinite $\beta < \alpha$ has a bijection with ω in L_α.

Lemma 12. *Let α be an ordinal. Suppose η_α is inadmissible. If η_α is locally countable, then $\eta_{\alpha+1}$ is admissible.*

Proof. Suppose first that η_α is a successor effective cardinal and let β be its predecessor. Since η_α is locally countable, there is a real $x \in L_{\eta_\alpha}$ coding a wellordering of $\mathbb{N}$ of length β. Let γ be least such that $x \in L_{\gamma+1}$. Then, for each ξ,

$$L_\xi \subset L_\xi[x] \subset L_{\gamma+1+\xi}.$$

Since η_α is additively indecomposable, we have

$$L_{\eta_\alpha} = L_{\eta_\alpha}[x].$$

The proof that ω_1^{ck} is the least admissible ordinal (see Barwise [7]) relativizes and shows that ω_1^x, the supremum of x-recursive ordinals, is least such that $L_{\omega_1^x}[x]$ is admissible. Thus,

$$L_{\eta_\alpha^+} = L_{\omega_1^x} = L_{\omega_1^x}[x].$$

Therefore, there are η_α-r.e. wellorderings of lengths cofinal below η_α^+ and the result follows.

The previous argument applies also to the case that η_α is a limit effective cardinal but there is a largest admissible smaller than η_α; the remaining case is that η_α is an inadmissible limit of admissibles. This case will not be necessary below, but it is good to note. To obtain it, follow the argument of Gostanian [9, Theorem 2.1] replacing KP by KPl throughout. This shows that if α is a locally countable limit of admissibles, then the α-recursive wellorders are cofinal below α^+ unless α is Σ^1_1-reflecting. This is enough to yield the result, since Σ^1_1-reflecting ordinals are all admissible. □

3. Σ^0_4-DETERMINACY

In this section we prove the main theorem.

Theorem 13. *The following are equivalent over* KP.

(1) Σ^0_4-determinacy;
(2) there is an ordinal which is not ω-Gandy.

To prove the corresponding result for Σ^0_2-determinacy (Theorem 3), one first uses Solovay's theorem on the complexity of winning strategies for Σ^0_2 games and positive Σ^1_1 induction, and then uses the results of Aczel-Richter, Gostanian, and Grilliot to relate this to 1-Gandy ordinals. The proof of Theorem 13 follows a similar outline: first we appeal to a theorem on the complexity of winning strategies for Σ^0_4 games, and then we relate that to ω-Gandy ordinals. The result we will require is due to Hachtman and we state it now. Here, recall that the *rank* of a wellfounded relation B is defined inductively as the strict supremum of the ranks $\rho(a)$ of its elements, where $\rho(a) = \sup\{\rho(b) + 1 : bBa\}$.

https://doi.org/10.1017/jsl.2023.90 Published online by Cambridge University Press

Theorem 14 (Hachtman [10])**.** *The least ordinal θ such that every Σ^0_4 game has a winning strategy definable over L_θ is the least ordinal such that L_θ satisfies "$\mathbb{R}$ exists and every wellfounded tree has a rank."*

Recall that by Gödel's condensation lemma, a structure of the form L_θ satisfies "$\mathbb{R}$ exists" if and only if it is not locally countable. Thus, we need to prove:

Theorem 15. *Assume* KP*. Then, the existence of an ordinal ζ which is not ω-Gandy is equivalent to the existence of an ordinal θ such that θ is not locally countable and $L_\theta \models$ "every wellfounded tree has a rank." Moreover, letting ζ and θ be the least such ordinals, then $\zeta = \eta_\zeta$ and $\theta = \eta_{\zeta+\omega}$.*

We prove this theorem in the remainder of the section. We would like to thank R. Lubarsky for bringing the following fact to our attention at some point in the past.

Lemma 16. *Let α be a multiplicatively indecomposable ordinal and let ρ_α be the supremum of ranks of wellfounded α-r.e. trees. Then ρ_α is the supremum of lengths of α-r.e. wellorderings of α.*

Proof. If T is a wellfounded α-r.e. tree, then its Kleene-Brouwer linearization $\prec_T$ is an α-r.e. wellorder. A simple induction shows that the length of $\prec_T$ is at least the rank of T. Conversely, suppose $\prec$ is an α-r.e. wellorder and let $T_\prec$ be the tree of all descending chains through $\prec$ ordered by end-extension. Then this is a wellfounded α-r.e. tree and – as before – a simple induction shows that its rank is at least the length of $\prec$. □

We will use the preceding lemma without mention, alternating between talking about trees and linear orders as we deem convenient. Lemma 17 below is already half of the proof of the theorem.

Lemma 17. *Suppose η_ξ is not ω-Gandy. Then, letting $\gamma = \eta_{\xi+\omega} < \xi^+$, γ is not locally countable and*

$$L_\gamma \models \text{"every wellfounded tree has a rank."}$$

Proof. It follows from Lemma 12 that for all $i \in \mathbb{N}$ with $i \neq 0$, $L_{\eta_{\xi+i}}$ is not locally countable. This implies in particular that for each i,

$$\mathcal{P}(\mathbb{N}) \cap L_\gamma = \mathcal{P}(\mathbb{N}) \cap L_{\eta_{\xi+i}} = \mathcal{P}(\mathbb{N}) \cap L_{\eta_\xi},$$

so L_γ is also not locally countable. It remains to show that:

$$L_\gamma \models \text{"every wellfounded tree has a rank."}$$

If $\prec$ is a linear ordering in L_γ, then $\prec$ is $\eta_{\xi+i}$-r.e. for some i, by Lemma 8. Thus, there are two possibilities: either $\prec$ is wellfounded (in the real world) and thus isomorphic to some ordinal $<\eta_{\xi+i+1}$. If so, then such an isomorphism can be constructed within $L_{\eta_{\xi+i+1}}$ and belongs to L_γ. Otherwise, $\prec$ is illfounded (in the real world) and thus has an infinite descending chain in $L_{\eta_{\xi+i}+1}$ (by Lemma 10) and thus in L_γ. □

For the rest of this section, we will denote by θ the least ordinal such that L_θ satisfies "$\mathbb{R}$ exists and every wellfounded tree has a rank," and we write

$$\zeta = \omega_1^{L_\theta}.$$

Observe that $\zeta = \eta_\zeta$.

https://doi.org/10.1017/jsl.2023.90 Published online by Cambridge University Press

Lemma 18. *$\theta = \eta_{\zeta+\omega}$. Moreover, $\eta_{\zeta+i}$ is inadmissible for all $i \in \mathbb{N}$.*

Proof. Clearly $\eta_{\zeta+\omega}$ is inadmissible. We prove that for each i, $\eta_{\zeta+i} < \theta$ and $\eta_{\zeta+i}$ is inadmissible. This implies that η_ζ is not ω-Gandy and that $\eta_{\zeta+\omega} \leq \theta$, so that $\eta_{\zeta+\omega} = \theta$ by Lemma 17 and the minimality of θ.

Inductively, suppose that $\eta_{\zeta+i} < \theta$ and that either $i = 0$ or $\eta_{\zeta+i}$ is inadmissible.

Claim 19. *Let $\prec$ be an illfounded $\eta_{\zeta+i}$-r.e. linear order. Then, there is an infinite descending chain through $\prec$ definable over $L_{\eta_{\zeta+i}}$.*

Proof. Suppose otherwise and let $\beta \neq \eta_{\zeta+i}$ be least such that there is an infinite descending chain through $\prec$ in $L_{\beta+1}$ and let d be such a chain. By the definition of θ, $\beta < \theta$. We have

$$\zeta = \omega_1^{L_\theta} = |\eta_{\zeta+i}|^{L_\theta},$$

as witnessed by a bijection $f : \eta_{\zeta+i} \to \zeta$. Using the hypothesis that each $\eta_{\zeta+j}$ is inadmissible for $0 < j \leq i$, we see that there is no admissible ordinal between ζ and $\eta_{\zeta+i}$, and thus such a bijection is definable over $L_{\eta_{\zeta+i}}$. Then, $f[d]$ is an ω-sequence of ordinals countable in L_θ which belongs to $L_{\beta+1}$ and not to $L_{\eta_{\zeta+i}+1}$. By Gödel's condensation lemma, ζ is no longer a cardinal in $L_{\beta+1}$, which is a contradiction. □

Claim 20. *$\eta_{\zeta+i+1}$ is inadmissible.*

Proof. This follows from the previous claim, since every $\eta_{\zeta+i}$-r.e. linear order can be mapped in a $\Delta_1^{L_{\eta_{\zeta+i+1}}}$ way to an infinite descending chain or to its order-type, and these order-types are cofinal in $\eta_{\zeta+i+1}$. □

It follows from the first claim that the set of indices of $\eta_{\zeta+i}$-r.e. wellorders is definable over $L_{\eta_{\zeta+i}+1}$. Thus, there is a linear order in $L_{\eta_{\zeta+i}+2}$ (and thus in L_θ) which is longer than all $\eta_{\zeta+i}$-r.e. wellorders (namely, the sum of all $\eta_{\zeta+i}$-r.e. wellorders) and thus has length at least $\eta_{\zeta+i+1}$. Therefore, $\eta_{\zeta+i+1} < \theta$. □

We have seen that $\theta = \eta_{\zeta+\omega}$ is a supremum of inadmissible effective cardinals. Thus, ζ is not ω-Gandy. This completes the proof of the theorem.

4. Closing Remarks

We have exhibited a characterization of Σ^0_4-determinacy in the spirit of Reverse Mathematics which is analogous to the Aczel-Richter-Gostanian-Grilliot-Solovay-Tanaka characterization of Σ^0_2-determinacy. Some questions are left open by this, however. Tanaka [23] characterized Σ^0_2-determinacy in terms of a theory of monotone inductions. Is there an interesting characterization for Σ^0_4-determinacy in terms of inductive definability? Is there a characterization for Σ^0_3-determinacy or Σ^0_5-determinacy in terms of Gandy ordinals?

We believe that the notion of an effective cardinal is a natural one and foreshadow that there is more to be said about them.

Acknowledgements. This work was partially supported by FWO grant 3E017319 and FWF grant I-4513.

References

[1] F. G. Abramson and G. E. Sacks. Uncountable Gandy Ordinals. *J. London Math. Soc.*, 14(2):387–392, 1976.

[2] P. Aczel and W. Richter. Inductive Definitions and Reflecting Properties of Admissible Ordinals. In J. E. Fenstad and P. G. Hinman, editors, *Generalized Recursion Theory*, pages 301–381. 1974.

[3] J. P. Aguilera. F_σ Games and Reflection in $L(\mathbb{R})$. *J. Symbolic Logic*, 85:1102–1123, 2020.

[4] J. P. Aguilera and F. V. Kerchove. A proof of Grilliot's theorem. Preprint.

[5] J. P. Aguilera and R. Lubarsky. Feedback Hyperjump. To appear.

[6] J. P. Aguilera and R. Lubarsky. Winning strategies for F_σ Games. Preprint.

[7] J. Barwise. *Admissible Sets and Structures*. Perspectives in Mathematical Logic. Springer-Verlag, Berlin, 1975.

[8] H. M. Friedman. Higher set theory and mathematical practice. *Ann. Math. Logic*, 2(3):325 – 357, 1971.

[9] R. Gostanian. The Next Admissible Ordinal. *Ann. Math. Logic*, 17:171–203, 1979.

[10] S. Hachtman. Calibrating Determinacy Strength in Levels of the Borel Hierarchy. *J. Symbolic Logic*, 82:510–548, 2017.

[11] T. J. Jech. *Set Theory*. Springer Monographs in Mathematics. Springer, 2003.

[12] A. S. Kechris. On Spector Classes. In A. S. Kechris, B. Löwe, and J. R. Steel, editors, *Ordinal definability and recursion theory, The Cabal Seminar, Volume III*. Cambridge University Press, 2016.

[13] D. A. Martin. Untitled Book on Determinacy. Unpublished.

[14] D. A. Martin. Measurable cardinals and analytic games. *Fundamenta Mathematicae*, 66:287–291, 1970.

[15] D. A. Martin. Borel Determinacy. *Ann. Math.*, 102(2):363–371, 1975.

[16] A. Montalbán and R. A. Shore. The Limits of Determinacy in Second-Order Arithmetic. *Proc. London Math. Soc.*, 104:223–252, 2011.

[17] Y. N. Moschovakis. *Descriptive set theory, second edition*, volume 155 of *Mathematical Surveys and Monographs*. AMS, 2009.

[18] J. Paris. ZF $\vdash \Sigma^0_4$ determinateness. *J. Symbolic Logic*, 37(4):661–667, 1972.

[19] M. Rathjen and A. Weiermann. Reverse Mathematics and Well-Ordering Principles. In S. Cooper and A. Sorbi, editors, *Computability in Context: Computation and Logic in the Real World*. 2011.

[20] G. E. Sacks. *Higher Recursion Theory*. Lecture Notes in Logic. Springer-Verlag, Berlin, New York, 1990.

[21] S. Simpson. *Subsystems of Second-Order Arithmetic*. 1999.

[22] R. M. Solovay. Hyperarithmetically Encodable Sets. *Trans. Amer. Math. Soc.*, 239, 1978.

[23] K. Tanaka. Weak Axioms of Determinacy and Subsystems of Analysis, II: Σ^0_2 Games. *Ann. Pure Appl. Logic*, 52:181–193, 1991.

[24] P. D. Welch. Weak Systems of Determinacy and Arithmetical Quasi-Inductive Definitions. *J. Symbolic Logic*, 76:418–436, 2011.

Department of Mathematics, Ghent University, Krijgslaan 281-S8, B9000 Ghent, Belgium

Institute of Discrete Mathematics and Geometry, Vienna University of Technology, Wiedner Hauptstrasse 8–10, 1040 Vienna, Austria.

Email address: aguilera@logic.at