J. Fluid Mech. (2001), *vol.* 431, *pp.* 427–432. Printed in the United Kingdom

Bounds on the convective heat transport in containers

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(Received 5 November 2000 and in revised form 15 December 2000)

Using the Hopf–Doering–Constantin decomposition, we derive upper bounds on the vertical heat flux in closed containers. It is found that the original bound of Doering & Constantin (1996) for Nusselt number as a function of Rayleigh number, $Nu \leqslant \sqrt{R}/4$, holds, at the very least, asymptotically as $R \to \infty$ under reasonably diverse experimental settings.

1. Introduction

Arguing away the influence of boundaries in fluid dynamics is often a precarious step, if comparison with experiment is a goal. Ignoring lateral boundaries when studying rotating convection in cylinders, for example, can hide the real onset of motion from theoretical view (Ecke, Zhong & Knobloch 1992 and the references therein). Even in the highest Rayleigh number experiments, the vertical heat flux continues to depend on container geometry (Wu & Libchaber 1992; Xu, Bajaj & Ahlers 2000). Does this dependence survive at even larger R?

In earlier work, bounds on the heat transported by convection were obtained using, in part, either suppositions or boundary conditions applicable only to physically extended systems (Howard 1963; Busse 1969; Doering & Constantin 1996, hereafter referred to as DC). Here convection in finite rectangular boxes and right circular cylinders is analysed with various lateral boundary conditions. If the lateral boundaries Ω are kept nearly insulated in an environment having constant ambient temperature $T_\star$ (assumed within the extremes of the experiment), then in dimensionless form

$$-\hat{\boldsymbol{n}} \cdot \nabla T = \epsilon(T - T_\star) \quad (\text{on } \Omega) \tag{1.1}$$

(where T is the pointwise fluid temperature) is appropriate with $\epsilon > 0$. For either geometry we shall find that

$$Nu + 1 \leqslant \tfrac{3}{4}(\tfrac{S}{A})\epsilon + \tfrac{1}{4}R^{1/2}, \tag{1.2}$$

where Nu is Nusselt number and aspect ratio A/S is the base surface area over the lateral surface area. Note that if R is large, or the aspect ratio is large, or the lateral boundaries are well insulated ($\epsilon \to 0$), the relationship approaches

$$Nu + 1 \leqslant \tfrac{1}{4}R^{1/2}, \tag{1.3}$$

that is, the result derived in DC for the periodically extended domain. While the prefactor was further refined in DC and elsewhere (Gebhardt *et al.* 1995; Nicodemus *et al.* 1997), ultimately with a reduction of about 8 (Busse 1969; Kerswell 1997), it is not pursued here.

While it seems that (1.1) closely mimics laboratory conditions in most reported experiments, the issue of the robustness of the bound to slightly different or imprecisely maintained lateral boundary conditions comes to mind. Perhaps reassuringly, regardless of the specifics of the lateral boundary conditions, (1.3) holds *asymptotically* provided the mean-square wall flux remains subdominant to $R^{1/4}$, i.e.

$$\left[\frac{1}{S}\iint_{\Omega} |\hat{\boldsymbol{n}}\cdot\nabla T|^2\,\mathrm{d}S\right]^{1/2} \equiv \|\hat{\boldsymbol{n}}\cdot\nabla T\| \sim o(R^{1/4}),$$

where the notation $\iint_{\Omega}\mathrm{d}S$ means long-time-averaged surface integral over the lateral boundary Ω.

2. The formulation

A fluid having expansion coefficient α is subjected to a downward gravitational field $-g\hat{\boldsymbol{k}}$ while contained in a right circular cylinder of height h and radius ρh. The bottom face is maintained hotter than the top by a temperature difference ΔT. The scaled Boussinesq equations are

$$\boldsymbol{u}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p = \sigma\nabla^2\boldsymbol{u} + \sigma R T\hat{\boldsymbol{k}},$$

$$T_t + \boldsymbol{u}\cdot\nabla T = \nabla^2 T,$$

and

$$\nabla\cdot\boldsymbol{u} = 0,$$

where $R = \alpha g\Delta T d^3/\kappa\nu$ is the Rayleigh number and $\sigma = \nu/\kappa$ is the Prandtl number. As scaled, $T = 1$ at $z = 0$ and $T = 0$ at $z = 1$ and the fluid is assumed to 'stick' to all walls, that is $\boldsymbol{u} = \boldsymbol{0}$ on the cylinder.

The Hopf–Doering–Constantin decomposition sets $T(r,\phi,z,t) = \tau(z) + \theta(r,\phi,z,t)$ where θ vanishes at $z = 0, 1$, and in the simplest analysis τ eventually has the form

$$\tau(z) = \begin{cases} 1-(\delta^{-1}-1)z, & 0\leqslant z\leqslant\delta \\ z, & \delta\leqslant z\leqslant 1-\delta \\ (\delta^{-1}-1)(1-z), & 1-\delta\leqslant z\leqslant 1, \end{cases} \tag{2.1}$$

where $\delta(R)\leqslant 1/2$, and various consequences of (2.1) such as $0\leqslant\tau^2\leqslant\tau\leqslant 1$, $\int_0^1\tau(z)\,\mathrm{d}z = 1/2$, or $\int_0^1\int_0^z\tau(z_\star)\,\mathrm{d}z_\star\,\mathrm{d}z\geqslant 0$, will frequently be used in subsequent analysis to simplify and to shorten the overall presentation, though sometimes at the expense of numerical prefactors. By use of the averaging operators

$$\overline{(\cdot)} = \lim_{t_\star\to\infty}\frac{1}{t_\star}\int_0^{t_\star}\frac{1}{A}\int_0^{2\pi}\int_0^{\rho}(\cdot)\,r\,\mathrm{d}r\,\mathrm{d}\phi\,\mathrm{d}t \quad\text{and}\quad \langle(\cdot)\rangle = \int_0^1(\cdot)\,\mathrm{d}z$$

on various algebraic derivatives of the Boussinesq equations one obtains

$$Nu = \tfrac{1}{2} + \tfrac{1}{2}\langle{\tau_z}^2\rangle + G - H = \delta^{-1} - 1 + G - H, \tag{2.2}$$

$$G = \frac{1}{2}\int_0^1\int_0^z\overline{\nabla_H^2 T}(z_\star)\,\mathrm{d}z_\star\,\mathrm{d}z + \frac{1}{2A}\iint_{\Omega}(\theta-\tau)\hat{\boldsymbol{n}}\cdot\nabla\theta\,\mathrm{d}S, \tag{2.3}$$

and

$$H = \tfrac{1}{2}\langle|\nabla\theta|^2\rangle + \langle\theta w(\tau_z - 1)\rangle + \frac{1}{2R}\langle|\nabla\boldsymbol{u}|^2\rangle, \tag{2.4}$$

where the Nusselt number $Nu \equiv \langle \hat{\boldsymbol{k}} \cdot [T\boldsymbol{u} - \nabla T] \rangle$ is the long-time, volumetrically averaged, vertical heat flux and ∇_H^2 denotes the horizontal Laplacian.

The choice (2.1) for τ allowed DC to bound the middle term of H and arrive at

$$H \geqslant \left(\frac{1}{2} - \frac{\delta}{4c} \right) \langle |\nabla \theta|^2 \rangle + \left(\frac{1}{2R} - \frac{\delta c}{16} \right) \langle |\nabla \boldsymbol{u}|^2 \rangle. \tag{2.5}$$

In lieu of a detailed proof that the above bound on H holds equally well for the closed containers considered in this paper, the reader may prefer to note that with the exception of the two inequalities $\langle \theta_z^2 \rangle \leqslant \langle |\nabla \theta|^2 \rangle$ and $4\langle w_z^2 \rangle \leqslant \langle |\nabla \boldsymbol{u}|^2 \rangle$† the analysis leading to (2.5) is concentrated in the z-direction, requiring no reference to horizontal dependence beyond an outermost average.

G comprises the novel terms associated with the presence of lateral boundaries; in its absence, DC equated the coefficients in the above inequality to zero, resulting in $\delta(R) = 4/\sqrt{R}$, $H \geqslant 0$, and $Nu + 1 \leqslant \sqrt{R}/4$, provided $R \geqslant 64$.

3. Insulated or nearly insulated sides

Here we suppose that

$$-\hat{\boldsymbol{n}} \cdot \nabla T = \epsilon (T - T_\star) \quad (\epsilon \geqslant 0) \tag{3.1}$$

holds on the lateral wall (modelling a nearly insulated experiment) and work towards bounding G from above. Thus

$$\begin{aligned}
\frac{1}{2} \int_0^1 \int_0^z \overline{\nabla_H^2 T} \, \mathrm{d}z_\star \, \mathrm{d}z &= \lim_{t_\star \to \infty} \frac{1}{t_\star} \int_0^{t_\star} \frac{1}{2A} \int_0^1 \int_0^z \int_0^{2\pi} \int_0^\rho \left[(rT_r)_r + \frac{1}{r} T_{\theta\theta} \right] \mathrm{d}r \, \mathrm{d}\phi \, \mathrm{d}z_\star \, \mathrm{d}z \, \mathrm{d}t \\
&= \lim_{t_\star \to \infty} \frac{1}{t_\star} \int_0^{t_\star} \frac{\rho}{2A} \int_0^1 \int_0^z \int_0^{2\pi} T_r|_{r=\rho} \, \mathrm{d}\phi \, \mathrm{d}z_\star \, \mathrm{d}z \, \mathrm{d}t \\
&= \lim_{t_\star \to \infty} \frac{1}{t_\star} \int_0^{t_\star} \frac{\epsilon\rho}{2A} \int_0^1 \int_0^z \int_0^{2\pi} (T_\star - \tau - \theta) \, \mathrm{d}\phi \, \mathrm{d}z_\star \, \mathrm{d}z \, \mathrm{d}t \\
&\leqslant \frac{\epsilon S T_\star}{4A} + \frac{\epsilon}{2A} \int\!\!\int_\Omega |\theta| \, \mathrm{d}S.
\end{aligned}$$

In moving to the inequality, the $T_\star$ term is integrated, the negative-definite τ term is simply dropped, and the inner z-integration is extended to $z = 1$ on the positively modified θ integral (which now takes the form of a surface integral). The remainder of G can also be bounded producing similar terms, i.e.

$$\begin{aligned}
\frac{1}{2A} \int\!\!\int_\Omega (\theta - \tau) \hat{\boldsymbol{n}} \cdot \nabla \theta \, \mathrm{d}S &= \frac{\epsilon}{2A} \int\!\!\int_\Omega (\tau^2 - T_\star \tau + T_\star \theta - \theta^2) \, \mathrm{d}S \\
&\leqslant \frac{\epsilon S (1 - T_\star)}{4A} + \frac{\epsilon T_\star}{2A} \int\!\!\int_\Omega |\theta| \, \mathrm{d}S - \frac{\epsilon}{2A} \int\!\!\int_\Omega \theta^2 \, \mathrm{d}S,
\end{aligned}$$

† The first of these is obviously independent of the domain and the derivation of the second inequality requires, besides $\nabla \cdot \boldsymbol{u} = \boldsymbol{0}$, only integration by parts and produces boundary terms which always contain a component of velocity and therefore drop out, as they do in a periodic domain (cf. DC). Evidently these inequalities hold in more general domains than considered in this article, provided the fluid sticks to the boundaries.

where the last inequality is achieved in part by replacing τ^2 with τ and noting $\iint_\Omega \tau \, \mathrm{d}S = S/2$. Combining these results with $|\theta| \leqslant c + \theta^2/4c$ (for any $c > 0$) yields

$$G \leqslant \frac{\epsilon S}{4A}[1 + 2c(1+T_\star)] + \frac{\epsilon}{2A}\left(\frac{1+T_\star}{4c} - 1\right)\iint_\Omega \theta^2 \, \mathrm{d}S.$$

While θ vanishes at $z = 0, 1$ it remains unknown elsewhere on Ω. However, the above boundary term can be avoided through the choice $c = (1+T_\star)/4$ which yields

$$G \leqslant \frac{\epsilon S}{4A}\left[1 + \frac{(1+T_\star)^2}{2}\right] \quad (0 \leqslant T_\star \leqslant 1).$$

The analysis of DC carries us to the bound (1.2). Note, without details, that a similar analysis for a rectangular box gives identical results.

4. Unknown side conditions

If lateral boundary conditions are not precisely known or maintained the following analysis of such effects on bounds may provide helpful guidelines concerning the asymptotic robustness of data relative to the *intended* experiment. Whatever the actual boundary condition, let us use the notation

$$\hat{\boldsymbol{n}} \cdot \nabla T|_{r=\rho} = F(\phi, z, t),$$

and investigate how F affects the bounds. We return to (2.3) and bound G from above by observing that

$$\frac{1}{2}\int_0^1 \int_0^z \overline{\nabla_H^2 T} \, \mathrm{d}z_\star \, \mathrm{d}z = \lim_{t_\star \to \infty} \frac{1}{t_\star}\int_0^{t_\star} \frac{\rho}{2A}\int_0^1 \int_0^z \int_0^{2\pi} F(\phi, z_\star, t) \, \mathrm{d}\phi \, \mathrm{d}z_\star \, \mathrm{d}z \, \mathrm{d}t$$
$$\leqslant \frac{S}{A2\sqrt{2}}\|F\|,$$

and

$$\frac{1}{2A}\iint_\Omega (\theta - \tau)\hat{\boldsymbol{n}} \cdot \nabla\theta \, \mathrm{d}S \leqslant \frac{S}{A2\sqrt{2}}\|F\| + \frac{1}{2A}\iint_\Omega \theta F \, \mathrm{d}S,$$

giving

$$G \leqslant \frac{S}{A\sqrt{2}}\|F\| + \frac{1}{2A}\iint_\Omega |\theta F| \, \mathrm{d}S. \tag{4.1}$$

This time the lateral surface integral involving θ (which is unknown on Ω) is dealt with directly using a lemma designed to extend it into the volume from which the 'z' boundary conditions on θ can then be used to derive gradient estimates – eventually folded into H.

LEMMA. *If $f(r)$ is differentiable on $0 \leqslant r \leqslant \rho$ then*

$$|f(\rho)| \leqslant \frac{2}{\rho^2}\int_0^\rho r|f(r)| \, \mathrm{d}r + \frac{\sqrt{2\pi}}{4}\left(\int_0^\rho r f'(r)^2 \, \mathrm{d}r\right)^{1/2}.$$

Proof. Starting with $f(\rho) = f(r) + \int_r^\rho f'(r_\star) \, \mathrm{d}r_\star$, multiplying by r, integrating over the domain, and using the Schwartz inequality on the inner integral yields

$$\frac{\rho^2}{2}f(\rho) \leqslant \int_0^\rho r f(r) \, \mathrm{d}r + \int_0^\rho r\left(\int_r^\rho r_\star^{-1} \, \mathrm{d}r_\star\right)^{1/2}\left(\int_r^\rho r_\star f'(r_\star)^2 \, \mathrm{d}r_\star\right)^{1/2} \mathrm{d}r.$$

Doing the left inner integral and changing the lower limit to 0 on the right gives

$$f(\rho) \leqslant \frac{2}{\rho^2}\int_0^\rho r|f(r)|\,\mathrm{d}r + \frac{2}{\rho^2}\left(\int_0^\rho r\sqrt{\ln\left(\frac{\rho}{r}\right)}\,\mathrm{d}r\right)\left(\int_0^\rho r f'^2\,\mathrm{d}r\right)^{1/2}.$$

Integrating and also considering f replaced by $-f$ provides the result.

Returning to the second term in (4.1) and applying the above lemma gives

$$\frac{1}{2A}\iint_\Omega |\theta F|\,\mathrm{d}S \leqslant \frac{1}{2A}\iint_\Omega \left[\frac{2}{\rho^2}\int_0^\rho r|\theta F|\,\mathrm{d}r + \frac{\sqrt{2\pi}}{4}\left(\int_0^\rho r\left(\frac{\partial}{\partial r}\theta F\right)^2\mathrm{d}r\right)^{1/2}\right]\mathrm{d}S$$

$$\leqslant \lim_{t_\star\to\infty}\frac{1}{t_\star}\int_0^{t_\star}\int_0^1\int_0^{2\pi}\left[\frac{1}{A\rho^2}|F|\int_0^\rho r|\theta|\,\mathrm{d}r + \frac{\sqrt{2\pi}}{8A}|F|\left(\int_0^\rho r\theta_r^2\,\mathrm{d}r\right)^{1/2}\right]\rho\,\mathrm{d}\phi\,\mathrm{d}z\,\mathrm{d}t.$$

Various Schwartz inequalities and a Poincaré inequality (on θ in direction z) gives

$$\frac{1}{2A}\iint_\Omega |\theta F|\,\mathrm{d}S \leqslant \frac{\|F\|}{\sqrt{A}}\left[\frac{1}{\sqrt{\pi}}\langle\theta_z^2\rangle^{1/2} + \frac{S}{8}\langle\theta_r^2\rangle^{1/2}\right].$$

Using $b \leqslant |c|/2 + b^2/(2|c|)$ on these two terms and balancing them for a clean application of $\langle\theta_z^2+\theta_r^2\rangle \leqslant \langle|\nabla\theta|^2\rangle$ gives

$$\frac{1}{2A}\iint_\Omega |\theta F|\,\mathrm{d}S \leqslant \frac{\|F\|^2}{\lambda}\left(\frac{1}{S^2}+\frac{\pi}{64}\right) + \lambda\langle|\nabla\theta|^2\rangle \quad (\text{for any } \lambda > 0).$$

Returning to (2.2) with the above bound gives

$$Nu + 1 \leqslant \delta^{-1} + \frac{S}{A\sqrt{2}}|F\| + \frac{\|F\|^2}{\lambda}\left(\frac{1}{S^2}+\frac{\pi}{64}\right) - \tilde{H},$$

where

$$\tilde{H} = (\tfrac{1}{2}-\lambda)\langle|\nabla\theta|^2\rangle + \langle\theta w(\tau_z - 1)\rangle + \frac{1}{2R}\langle|\nabla \boldsymbol{u}|^2\rangle$$

$$\geqslant \left(\frac{1}{2} - \frac{\delta}{4c} - \lambda\right)\langle|\nabla\theta|^2\rangle + \left(\frac{1}{2R} - \frac{\delta c}{16}\right)\langle|\nabla\boldsymbol{u}|^2\rangle.$$

Finally setting the above coefficients to zero gives $\tilde{H} \geqslant 0$ and

$$Nu + 1 \leqslant \frac{\sqrt{R}}{4}(1-2\lambda)^{-1/2} + \frac{S}{A\sqrt{2}}\|F\| + \left(\frac{1}{S^2}+\frac{\pi}{64}\right)\|F\|^2\lambda^{-1},$$

where $0 < \lambda < 1/2$ is a free parameter. If the goal is to consider only large $\|F\|$ that still preserves the dominant $R^{1/2}$ scaling in the bound, a look at the subdominant terms above suggests the optimal scaling $\lambda = \|F\|R^{-1/4}$ giving

$$Nu + 1 \leqslant \frac{\sqrt{R}}{4}(1-2\|F\|R^{-1/4})^{-1/2} + \frac{S}{A\sqrt{2}}\|F\| + \left(\frac{1}{S^2}+\frac{\pi}{64}\right)\|F\|R^{1/4},$$

which to leading order is $Nu + 1 \leqslant \sqrt{R}/4$ provided $\|F\| \sim o(R^{1/4})$.

5. Discussion and prospects

Bound (1.2) was derived to address measurements in experimental containers; however, the additive aspect-ratio term bears no relation to the aspect-ratio dependence observed in *Nu* at high *R*. Xu *et al.* (2000) plausibly argue, based on their experiments, that the aspect ratio slightly alters the multiplicative prefactor (analogous to our $1/4$) in the *R* power-law term. The best bound for the infinite slab (Busse 1969) is about a factor of 8 better than the one given here and has been reproduced under the DC umbrella and shown to be sharp by Kerswell (1997). And while an exponent of 0.3 rather than 0.5 is closer to what has been observed experimentally (Niemela *et al.* 2000), it would still be interesting to know if a more refined treatment of the container problem would produce an aspect-ratio-dependent prefactor. One can, by handling the G term as in this paper, concentrate on optimizing the choice of τ and refining the $H \geqslant 0$ criterion using Kerswell's Euler–Lagrange approach. Especially ripe for this is the perfectly insulated container whose simple lateral boundary condition $\hat{\boldsymbol{n}}\cdot\nabla\theta = 0$ is passed on to close Kerswell's Euler–Lagrange equations. Would a bounded version of Busse's multi-wavenumber asymptotics emerge? Equally important (to this author), the connection between the infinite and finite domain made here relieves most inhibitions associated with continued efforts, in the (simpler) infinite domain, to reduce bounds through the inclusion of more information from the governing PDEs (Ierley & Worthing 2000).

I thank C. Doering for discussing his method, G. Ierley for providing helpful comments, and S. Grossmann for urging me to increase the scope of section 3 which was originally limited to experiments in 'cold' environments, i.e. having $T_\star = 0$.

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