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ALGEBRAIC STRUCTURE OF THE RANGE OF A TRIGONOMETRIC POLYNOMIAL

LEONID V. KOVALEV and XUERUI YANG

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Abstract

The range of a trigonometric polynomial with complex coefficients can be interpreted as the image of the unit circle under a Laurent polynomial. We show that this range is contained in a real algebraic subset of the complex plane. Although the containment may be proper, the difference between the two sets is finite, except for polynomials with a certain symmetry.

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1. Introduction

In 1976, Quine [6, Theorem 1] proved that the image of the unit circle $\mathbb{T}$ under an algebraic polynomial p of degree n is contained in a real algebraic set, that is, a set $V = \{(x, y) \in \mathbb{R}^2 : q(x, y) = 0\}$, where q is a polynomial of degree $2n$. In general, $p(\mathbb{T})$ is a proper subset of V, but we will show that $V \setminus p(\mathbb{T})$ is finite and that $V = p(\mathbb{T})$ whenever V is connected.

Consider a trigonometric polynomial $P(t) = \sum_{k=-m}^{n} a_k e^{ikt}$, $t \in \mathbb{R}$, with complex coefficients a_k. It is natural to require $a_{-m} a_n \neq 0$. The range of P is precisely the image of the unit circle $\mathbb{T}$ under the Laurent polynomial $p(z) = \sum_{k=-m}^{n} a_k z^k$. This motivates our investigation of $p(\mathbb{T})$ for Laurent polynomials. Our main result, Theorem 2.1, asserts that $p(\mathbb{T})$ is contained in the zero set V of a polynomial of degree $2 \max(m, n)$. This matches Quine's theorem in the case when p is an algebraic polynomial, that is, $m = 0$. The difference $V \setminus p(\mathbb{T})$ is finite when $m \neq n$, but may be infinite when $m = n$.

In Section 4, we investigate the exceptional case when $V \setminus p(\mathbb{T})$ is infinite and we relate it to the properties of the zero set of a certain harmonic rational function. The structure of zero sets of such functions is a topic of current interest with applications to gravitational lensing [1, 2].

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Finally, in Section 5, we use the algebraic nature of the polynomial images of $\mathbb{T}$ to estimate the number of intersections of two such images, that is, the number of shared values of two trigonometric polynomials.

2. Algebraic nature of polynomial images of circles

A real algebraic subset of $\mathbb{R}^2$ is a set of the form $\{(x, y) \in \mathbb{R}^2 : q(x, y) = 0\}$, where $q \in \mathbb{R}[x, y]$ is a polynomial in x, y. Consider a Laurent polynomial

$$p(z) = \sum_{k=-m}^{n} a_k z^k \quad \text{for all } z \in \mathbb{C} \setminus \{0\}, \tag{2.1}$$

where $m \geq 0$, $n \geq 1$ and $a_{-m} a_n \neq 0$. This includes the case of algebraic polynomials ($m = 0$) because the condition $a_0 \neq 0$ can be ensured by adding a constant to p, which does not affect the algebraic nature of $p(\mathbb{T})$. Since we are interested in the image of the unit circle, which is invariant under the substitution of z^{-1} for z, it suffices to consider the case $m \leq n$.

Theorem 2.1. *Let p be the Laurent polynomial (2.1) with $m \leq n$.*

(a) *The image of $\mathbb{T}$ under p is contained in the zero set V of some polynomial $h \in \mathbb{R}[x, y]$ of degree $2n$.*
(b) *If h is expressed as a polynomial $h_{\mathbb{C}} \in \mathbb{C}[w, \overline{w}]$ via the substitution $w = x + iy$, the degree of $h_{\mathbb{C}}$ in each of the variables w and $\overline{w}$ separately is $m + n$.*
(c) *If $m < n$, then the set $V \setminus p(\mathbb{T})$ is finite.*
(d) *In the case $m = n$ the set $V \setminus p(\mathbb{T})$ is finite if and only if V is bounded.*

The proof of Theorem 2.1 involves two polynomials

$$g(z) = z^m (p(z) - w) \quad \text{and} \quad g^*(z) = z^{n+m} \overline{g(1/\bar{z})} = z^n \overline{(p(1/\bar{z}) - w)} \tag{2.2}$$

which are the subject of the following lemma.

Lemma 2.2. *The resultant $h_{\mathbb{C}} = \operatorname{res}(g, g^*)$ of the polynomials (2.2) is a polynomial in $\mathbb{C}[w, \overline{w}]$ of degree $2n$. Moreover, $h_{\mathbb{C}}$ has degree $m + n$ in each of the variables w and $\overline{w}$ separately. Finally, $h(x, y) := h_{\mathbb{C}}(x + iy, x - iy)$ is a polynomial of degree $2n$ in $\mathbb{R}[x, y]$.*

Proof. Both g and g^* are polynomials of degree $m + n$ in z, except for the case $m = 0$ and $w = a_0$, which we ignore in this proof because considering a generic w is enough. By definition, the resultant of g and g^* is the determinant of the following matrix of size $2(m + n)$.

$$R = \begin{pmatrix} a_{-m} & \cdots & \cdots & a_0 - w & \cdots & a_n & 0 & 0 \\ 0 & \ddots & & & \ddots & & \ddots & 0 \\ 0 & 0 & a_{-m} & \cdots & \cdots & a_0 - w & \cdots & a_n \\ \overline{a_n} & \cdots & \overline{a_0} - \overline{w} & \cdots & \cdots & \overline{a_{-m}} & 0 & 0 \\ 0 & \ddots & & \ddots & & & \ddots & 0 \\ 0 & 0 & \overline{a_n} & \cdots & \overline{a_0} - \overline{w} & \cdots & \cdots & \overline{a_{-m}} \end{pmatrix}.$$

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All appearances of w or $\overline{w}$ in R are in the columns numbered $m+1$ to $m+2n$, which are the middle $2n$ columns of the matrix R. Therefore $h_{\mathbb{C}}$ is a polynomial of degree at most $2n$.

First, let us prove that $h_{\mathbb{C}}$ has degree $n+m$ in each variable separately. It obviously cannot be greater than $n+m$, since each of w and $\overline{w}$ appears $n+m$ times in the matrix. The position of $a_0 - w$ in the top half of the matrix shows that the Leibniz formula for $\det R$ contains the term $\pm\overline{a_n}^{\,m}\overline{a_{-m}}^{\,n}(a_0 - w)^{n+m}$ and no other terms with the monomial w^{n+m}. Therefore the coefficient of w^{n+m} in h is $\pm\overline{a_n}^{\,m}\overline{a_{-m}}^{\,n} \neq 0$. Similarly, the coefficient of $\overline{w}^{2n}$ in h is $\pm a_{-m}^{n} a_n^{m} \neq 0$. This proves that $h_{\mathbb{C}}$ has degree $n+m$ in w and $\overline{w}$ separately.

When $m = n$, the preceding paragraph shows that $h_{\mathbb{C}}$ has degree $2n$ in w and $\overline{w}$ separately, which implies that $\deg h = 2n$.

We proceed to prove that $\deg h_{\mathbb{C}} = 2n$ in the case $m < n$. Let R_1 be the matrix obtained from R by replacing all constant entries in the columns $m+1, \ldots, m+2n$ by 0. Since the cofactor of any of the entries we replaced is a polynomial of degree less than $2n$, the difference $\det R - \det R_1$ has degree less than $2n$. Thus, it suffices to show that $\det R_1$ has degree $2n$. When deriving a formula for $\det R_1$ we may assume that $w \neq a_0$. Let us focus on the columns of R_1 numbered $m+1, \ldots, 2m$. The only nonzero entries in these columns are:

- $a_0 - w$ at $(j-m, j)$ for $m+1 \leq j \leq 2m$; and
- $\overline{a_0} - \overline{w}$ at $(j+m, j)$ for $n+1 \leq j \leq 2m$.

We can use column operations to eliminate all nonzero entries in the upper-left $m \times m$ submatrix of R_1. Since this submatrix is upper-triangular, the process only involves adding some multiples of the jth column with $m+1 \leq j \leq 2m$ to columns numbered k, where $j - m \leq k \leq m$. Such a column operation also affects the bottom half of the matrix, where we add a multiple of the entry $(j+m, j)$ to the entry $(j+m, k)$. Since $(j+m) - k \leq j + m - (j-m) = 2m < n+m$, the affected entries of the bottom half are strictly above the diagonal $\{(n+m+j, j) \colon 1 \leq j \leq m\}$, which is filled with the value a_n. In conclusion, these column operations do not substantially affect the upper-triangular submatrix formed by the entries (i, j) with $n+m+1 \leq i \leq n+2m$, $1 \leq j \leq m$, in the sense that the submatrix remains upper-triangular and its diagonal entries remain equal to a_n.

Similar column operations on the right-hand side of the matrix eliminate all nonzero entries in the bottom right $m \times m$ submatrix of R_1. Let R_2 be the resulting matrix: that is,

$$R_2 = \begin{pmatrix} 0 & \cdots & a_0 - w & \cdots & \cdots & 0 & 0 & 0 \\ 0 & \ddots & & \ddots & & & \ddots & 0 \\ 0 & 0 & 0 & \cdots & a_0 - w & \cdots & \cdots & a_n \\ \overline{a_n} & \cdots & \cdots & \overline{a_0} - \overline{w} & \cdots & 0 & 0 & 0 \\ 0 & \ddots & & & \ddots & & \ddots & 0 \\ 0 & 0 & 0 & \cdots & \cdots & \overline{a_0} - \overline{w} & \cdots & 0 \end{pmatrix}.$$

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We claim that $\det R_2 = \pm|a_n|^{2m}|a_0 - w|^{2n}$. Indeed, the first m columns of R_2 contain only an upper-triangular submatrix with $\overline{a_n}$ on the diagonal; the last m columns contain only a lower-triangular matrix with a_n on the diagonal. After these are accounted for, we are left with a $2n \times 2n$ submatrix in which every row has exactly one nonzero element, either $a_0 - w$ or its conjugate. This completes the proof of $\deg h_{\mathbb{C}} = 2n$.

Define $h(x, y) = h_{\mathbb{C}}(x + iy, x - iy)$ for real x, y. We claim that h is real valued, and thus has real coefficients. Recall (for example, [4, page 11]) that the resultant can be expressed in terms of the roots of the polynomials g, g^*. Let $z_1, \dots, z_{n+m}$ be the roots of g listed with multiplicity. To simplify notation, we separate the cases $m > 0$ and $m = 0$.

Case 1: $m > 0$. We have $\prod_{k=1}^{m+n} z_k = (-1)^{n+m} a_{-m}/a_n$; in particular, $z_k \neq 0$ for all k. It follows from (2.2) that g^* has roots $1/\overline{z_k}$ for $k = 1, \dots, n + m$. The leading terms of g and g^* are a_n and $\overline{a_{-m}}$, respectively. Thus,

$$\begin{aligned}
\operatorname{res}(g, g^*) &= (\overline{a_{-m}} a_n)^{m+n} \prod_{i,j=1}^{n+m} (z_i - 1/\overline{z_j}) = (\overline{a_{-m}} a_n)^{n+m} \prod_{i,j=1}^{n+m} \frac{z_i \overline{z_j} - 1}{\overline{z_j}} \\
&= (\overline{a_{-m}} a_n)^{n+m} \Big(\prod_{j=1}^{m+n} \overline{z_j} \Big)^{-(n+m)} \prod_{i,j=1}^{n+m} (z_i \overline{z_j} - 1) \\
&= (-1)^{n+m} (\overline{a_{-m}} a_n)^{m+n} (\overline{a_n / a_{-m}})^{n+m} \prod_{i,j=1}^{n+m} (z_i \overline{z_j} - 1) \\
&= (-1)^{n+m} |a_n|^{2(m+n)} \prod_{i,j=1}^{n+m} (z_i \overline{z_j} - 1). \qquad (2.3)
\end{aligned}$$

The latter product is evidently real.

Case 2: $m = 0$. We have $\prod_{k=1}^{m+n} z_k = (-1)^n (a_0 - w)/a_n$; in particular, $z_k \neq 0$ for all k provided that $w \neq a_0$. The rest of the proof goes as in case $m > 0$, with a_{-m} replaced by $a_0 - w$ throughout. Since a_{-m} cancels out at the end of (2.3), the conclusion that h is real valued still holds. □

The following description of the local structure of the zero set of a complex-valued harmonic function is due to Sheil–Small (unpublished) and appears in [10].

Theorem 2.3 [10, Theorem 3]. *Let $\Omega \subset \mathbb{C}$ be a domain and let $f : \Omega \to \mathbb{C}$ be a harmonic function. Suppose that the points $\{z_k\}_{k=1}^{\infty}$ are distinct zeros of f which converge to a point $z^* \in \Omega$. Then z^* is an interior point of a simple analytic arc γ which is contained in $f^{-1}(0)$ and contains infinitely many of the points z_k.*

The fact that $z_k \in \gamma$ for infinitely many k is not stated in [10, Theorem 3] but is a consequence of the proof.

Proof of Theorem 2.1. (a)–(b) Suppose that $w \in p(\mathbb{T})$. Then the rational functions $p(z) - w$ and $\overline{p(1/\overline{z}) - w}$ have a common zero, namely, any preimage of w that lies on $\mathbb{T}$. Consequently, the polynomials (2.2) have a common zero, which implies that their

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resultant $h_{\mathbb{C}} = \operatorname{res}(g, g^*)$ vanishes at w. Claims (a) and (b) follow from Lemma 2.2. For future reference, note that the zero set of h can be written as

$$V = h^{-1}(0) = p(E) \quad \text{where } E = \{z \in \mathbb{C} \setminus \{0\} : p(z) = p(1/\bar{z})\}. \tag{2.4}$$

(c) In view of (2.4), to prove that $V \setminus p(\mathbb{T})$ is finite it suffices to show that $E \setminus \mathbb{T}$ is finite. Let $q(z) = p(z) - p(1/\bar{z})$, which is a harmonic Laurent polynomial. Since $m < n$, it follows that $q(z) = p(z) + O(|z|^m) = a_n z^n + O(|z|^{n-1})$ as $|z| \to \infty$. Thus E is a bounded set. By symmetry, E is also bounded away from zero.

Suppose that $E \setminus \mathbb{T}$ is infinite. Then it contains a convergent sequence of distinct points $z_k \to z^* \neq 0$. By Theorem 2.3, there exists a simple analytic arc Γ such that $g_{|\Gamma} = 0$ and z^* is an interior point of Γ. In the case $z^* \in \mathbb{T}$, the arc Γ is not a subarc of $\mathbb{T}$ because it contains infinitely many of the points z_k which are not on $\mathbb{T}$. By virtue of its analyticity, γ has finite intersection with $\mathbb{T}$. By shrinking γ we can achieve that $\gamma \cap \mathbb{T} = \{z^*\}$ if $z^* \in \mathbb{T}$, and $\gamma \cap \mathbb{T} = \emptyset$ otherwise.

Since the endpoints of γ lie in $E \setminus \mathbb{T}$, the process described above can be iterated to extend γ further in both directions. This continuation process can be repeated indefinitely. Since E is bounded, we conclude that E contains a simple closed analytic curve Γ, as in the proof of [10, Theorem 4].

If Γ does not surround zero, then the maximum principle yields $q \equiv 0$ in the domain enclosed by Γ, which is impossible since q is nonconstant. If Γ surrounds zero, then the complement of $\Gamma \cup \mathbb{T}$ has a connected component G such that $0 \notin G$. The maximum principle yields $q \equiv 0$ in G, which is a contradiction. The proof of (c) is complete.

(d) The proof of (c) used the assumption that $m < n$ only to establish that the set E in (2.4) is bounded. Thus the conclusion still holds if $m = n$ and E is a bounded set. Recalling that $V = p(E)$ and $|p(z)| \to \infty$ as $|z| \to \infty$, we find that E is bounded whenever V is bounded.

Finally, if V is an unbounded set, then $V \setminus p(\mathbb{T})$ must be infinite because $p(\mathbb{T})$ is bounded. □

Since a real algebraic set has finitely many connected components [9, Theorem 3], it follows from Theorem 2.1 that when $V \setminus p(\mathbb{T})$ is finite, the set $p(\mathbb{T})$ coincides with one of the connected components of V and the other components of V are singletons. The number of singleton components of V can be arbitrarily large, even when p is an algebraic polynomial.

Remark 2.4. For every integer N, there exists a polynomial p such that the set $V \setminus p(\mathbb{T})$ described in Theorem 2.1 contains at least N points.

Proof. Let $a_1, \dots, a_N$ be distinct complex numbers with $0 < |a_k| < 1$ for $k = 1, \dots, N$. Using Lagrange interpolation, we get a polynomial q of degree $2N - 1$ such that $q(a_k) = q(1/\bar{a}_k) = k$ for $k = 1, \dots, N$. Let r be a polynomial of degree $2N$ with zeros at the points a_k and $1/\bar{a}_k$, $k = 1, \dots, N$. Since $\inf_{\mathbb{T}} |r| > 0$, for sufficiently large constant M the polynomial $p = q + Mr$ satisfies $q(a_k) = q(1/\bar{a}_k) = k$ for $k = 1, \dots, N$ as well as $|p(z)| > N$ for $z \in \mathbb{T}$. It follows that the algebraic set V, as described by (2.4), contains the points $1, \dots, N$, none of which lie on the curve $p(\mathbb{T})$. □

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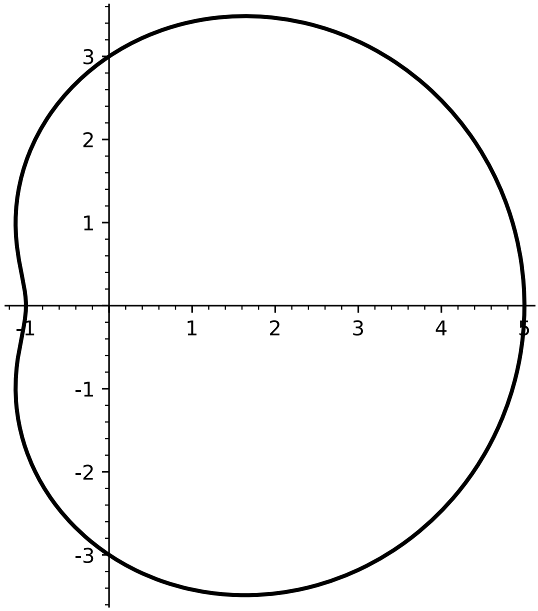

FIGURE 1. Nonalgebraic image of the circle.

3. Examples

First, we observe that $p(\mathbb{T})$ need not be a real algebraic set, even for a quadratic polynomial p.

EXAMPLE 3.1. Let $p(z) = z^2 + 3z + 1$. Then $p(\mathbb{T})$ is not a real algebraic set.

PROOF. Direct computation of the polynomial h in Theorem 2.1 yields

$$\begin{aligned} h(x,y) &= \det\begin{pmatrix} 1-w & 3 & 1 & 0 \\ 0 & 1-w & 3 & 1 \\ 1 & 3 & 1-\overline{w} & 0 \\ 0 & 1 & 3 & 1-\overline{w} \end{pmatrix} \\ &= x^4 + 2x^2y^2 + y^4 - 4x^3 - 4xy^2 - 5x^2 - 9y^2, \end{aligned}$$

where $w = x + iy$. By Theorem 2.1, the set $h^{-1}(0)$ contains $p(\mathbb{T})$. Since $p \neq 0$ on $\mathbb{T}$, we have $0 \in h^{-1}(0) \setminus p(\mathbb{T})$. If $p(\mathbb{T})$ was an algebraic set, then V would be reducible. However, h is an irreducible polynomial. Indeed, the fact that the zero set of h is bounded implies that any nontrivial factorisation $h = fg$ would have $\deg f = \deg g = 2$. This means that V is the union of two conic sections, which it evidently is not, as $p(\mathbb{T})$ is not an ellipse (see Figure 1). □

According to Theorem 2.1, the set $p(\mathbb{T})$ can be completed to a real algebraic set by adding finitely many points, provided that p is either an algebraic polynomial or a

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Laurent polynomial with $m < n$. The following example shows that the case $m = n$ is indeed exceptional.

EXAMPLE 3.2. Let $p(z) = z + z^{-1}$. Then $p(\mathbb{T})$ is the line segment $[-2, 2]$. The smallest real algebraic set containing $p(\mathbb{T})$ is the real line $\mathbb{R}$.

The claimed properties of Example 3.2 are straightforward to verify. In addition, the polynomial h from Theorem 2.1 can be computed as $h(x, y) = -4y^2$, which shows that h is not necessarily irreducible.

4. Zero set of harmonic Laurent polynomials

The relation (2.4) highlights the importance of the zero set of the harmonic Laurent polynomial $P(z) = p(z) - p(1/\bar{z})$, where p is a Laurent polynomial. It is not a trivial task to determine whether a given harmonic Laurent polynomial has unbounded zero set: for example, Khavinson and Neumann [2] remarked on the varied nature of zero sets for rational harmonic functions in general. In this section, we develop a necessary condition, in terms of the coefficients of p, for the function P to have an unbounded zero set.

Suppose that p is a Laurent polynomial (2.1) such that the associated function $P(z) = p(z) - p(1/\bar{z})$ has unbounded zero set. Consider the algebraic part of P, namely,

$$q(z) = \sum_{k=1}^{n} a_k z^k - \sum_{k=1}^{m} a_{-k}\bar{z}^k. \tag{4.1}$$

Then q is a harmonic polynomial such that $\liminf_{z\to\infty} |q(z)|$ is finite. In other words, q is not a proper map of the complex plane.

One necessary condition is immediate: if $m < n$, then $|q(z)| = a_n|z|^n + o(|z|^n)$ as $z \to \infty$. Thus P can only have an unbounded zero set if $m = n$.

We now look for further conditions on a harmonic polynomial which will ensure that it is a proper map of $\mathbb{R}^2$ to $\mathbb{R}^2$. More generally, given a polynomial map $F = (F_1, \ldots, F_n): \mathbb{R}^n \to \mathbb{R}^n$, let us decompose each component F_k into homogeneous polynomials and let $\mathcal{H}(F_k)$ be the homogeneous term of highest degree in F_k. Write $\mathcal{H}(F)$ for $(\mathcal{H}(F_1), \ldots, \mathcal{H}(F_n))$ so that $\mathcal{H}(F)$ is also a polynomial map of $\mathbb{R}^n$. The following result is from [7, Lemma 10.1.9].

LEMMA 4.1 (L. Andrew Campbell). *If $\mathcal{H}(F)$ does not vanish in $\mathbb{R}^n \setminus \{0\}$, then the map $F: \mathbb{R}^n \to \mathbb{R}^n$ is a proper map, that is, $|F(x)| \to \infty$ as $|x| \to \infty$.*

Lemma 4.1 can be restated in a form adapted to harmonic polynomials in $\mathbb{C}$.

LEMMA 4.2. *Consider a harmonic polynomial $q(z) = \sum_{k=0}^{n}(a_k z^k + b_k \bar{z}^k)$ of degree $n \geq 1$ as a map from $\mathbb{C}$ to $\mathbb{C}$.*

(a) *If $|a_n| \neq |b_n|$, then q is proper.*

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(b) *If* $|a_n| = |b_n|$, *let* $\eta \in \mathbb{T}$ *be such that* $\eta a_n = \overline{\eta b_n}$. *If* $\eta a_k = \overline{\eta b_k}$ *for* $k = 1, \dots, n$, *then* q *is not proper. Otherwise, let* K *be the largest value of* k *such that* $\eta a_k \neq \overline{\eta b_k}$. *If there is no* $z \neq 0$ *such that*

$$\operatorname{Re}(\eta a_n z^n) = 0 = \operatorname{Im}((\eta a_K - \overline{\eta b_K}) z^K),$$

then q *is proper.*

Proof. Part (a) follows from the reverse triangle inequality: that is,

$$|q(z)| \geq ||a_n| - |b_n||\,|z|^n + o(|z|^n) \quad \text{as } n \to \infty.$$

To prove part (b), observe that

$$\operatorname{Im}(\eta q(z)) = \sum_{k=0}^{n} \operatorname{Im}((\eta a_k - \overline{\eta b_k} z^k). \tag{4.2}$$

If $\eta a_k = \overline{\eta b_k}$ for $k = 1, \dots, n$, then $\operatorname{Im}(\eta q)$ is constant, which means that, up to a constant term, ηq is a real-valued harmonic function. By Harnack's inequality, a nonconstant harmonic function $h\colon \mathbb{C} \to \mathbb{R}$ must be unbounded from above and from below, and therefore $q^{-1}(0)$ is an unbounded set. Since q is constant on an unbounded set, it is not a proper map.

Finally, suppose that K, as defined in (b), exists. It follows from (4.2) that

$$\mathcal{H}(\operatorname{Im}(\eta q(z))) = \operatorname{Im}((\eta a_K - \overline{\eta b_K}) z^K).$$

Since also

$$\mathcal{H}(\operatorname{Re}(\eta p(z))) = \operatorname{Re}((\eta a_n + \overline{\eta b_n}) z^n) = 2 \operatorname{Re}(\eta a_n z^n),$$

the last statement in (b) follows by applying Lemma 4.1 to $(\operatorname{Re}(\eta q), \operatorname{Im}(\eta q))$ considered as a map of $\mathbb{R}^2$ to $\mathbb{R}^2$. □

We are now ready to apply Lemma 4.2 to the special case $P(z) = p(z) - p(1/\bar{z})$, where p is a Laurent polynomial. Recall that, in view of Theorem 2.1 and the relation (2.4), the following result describes when the image $p(\mathbb{T})$ has infinite complement in the real algebraic set V containing it.

Theorem 4.3. *Suppose* $p(z) = \sum_{k=-n}^{n} a_k z^n$ *is a Laurent polynomial with* $a_n a_{-n} \neq 0$. *Let* $P(z) = p(z) - p(1/\bar{z})$. *If the zero set of* P *is unbounded, then one of the following holds.*

(a) $p(\mathbb{T})$ *is contained in a line.*

(b) *There exists* $\eta \in \mathbb{T}$ *such that* $\eta a_n + \overline{\eta a_{-n}} = 0$. *Furthermore, there is an integer* $k \in \{1, \dots, n-1\}$ *such that the harmonic polynomial* $\operatorname{Im}((\eta a_k + \overline{\eta a_{-k}}) z^k)$ *is nonconstant and shares a nonzero root with the harmonic polynomial* $\operatorname{Re}(\eta a_n z^n)$.

As a partial converse, if (a) holds, then the zero set of P *is unbounded.*

Although part (b) of Theorem 4.3 is convoluted, it is not difficult to check, in practice, because η is uniquely determined (up to irrelevant sign) and the zero sets of both harmonic polynomials involved are simply unions of equally spaced lines through the origin.

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PROOF. We apply Lemma 4.2 to the polynomial q in (4.1), which means letting $b_k = -a_{-k}$ for $k = 1, \dots, n$. Since q is not proper, part (b) of the lemma provides two possible scenarios, which are considered below.

One possibility is that there exists a unimodular constant η such that $\eta a_k = -\overline{\eta a_{-k}}$ for $k = 1, \dots, n$. Therefore, for $z \in \mathbb{T}$,

$$\mathrm{Re}(\eta p(z)) = \mathrm{Re}(a_0) + \sum_{k=1}^{n} \mathrm{Re}(\eta a_k z^k + \overline{\eta a_{-k}} z^k) = \mathrm{Re}(a_0),$$

which means that $p(\mathbb{T})$ is contained in a line. The converse is true as well. If $p(\mathbb{T})$ is contained in a line, then there exists a unimodular constant η such that $\mathrm{Re}(\eta p)$ is constant on $\mathbb{T}$. Considering the Fourier coefficients of $\mathrm{Re}(\eta p)$, we find $\eta a_k + \overline{\eta a_{-k}} = 0$ for all $1 \le k \le n$.

The other possibility described in Lemma 4.2(b) transforms into part (b) of Theorem 4.3 with the substitution $b_k = -a_{-k}$. □

5. Intersection of polynomial images of the circle

As an application of Theorem 2.1, we establish an upper bound for the number of intersections between two images of the unit circle $\mathbb{T}$ under Laurent polynomials. It is necessary to exclude some pairs of polynomials from consideration because, for example, the images of $\mathbb{T}$ under any two of the Laurent polynomials

$$p_\alpha(z) = z + z^{-1} + \alpha \quad \text{for } -2 < \alpha < 2,$$

have infinite intersection. This is detected by the computation of the polynomial h in Theorem 2.1, according to which $h(x, y) = -4y^2$ regardless of α.

THEOREM 5.1. *Consider two Laurent polynomials*

$$p(z) = \sum_{k=-m}^{n} a_k z^k \quad \text{and} \quad \widetilde{p}(z) = \sum_{k=-r}^{s} b_k z^k,$$

where $m, r \ge 0$, $n, s \ge 1$ and $a_{-m} a_n b_{-r} b_s \ne 0$. Then the intersection $p(\mathbb{T}) \cap \widetilde{p}(\mathbb{T})$ consists of at most $4ns - 2(n - m)(s - r)$ points unless the corresponding polynomials h and $\tilde{h}$ from Theorem 2.1 have a nontrivial common factor.

In the special case of algebraic polynomials, $m = r = 0$, the estimate in Theorem 5.1 simplifies to $2ns$. In this case, the theorem is due to Quine [6, Theorem 3], where the bound $2ns$ is shown to be sharp. A related problem of counting the self-intersections of $p(\mathbb{T})$ was addressed in [5] for algebraic polynomials and in [3] for Laurent polynomials.

PROOF. Let $h_{\mathbb{C}} \in \mathbb{C}[w, \overline{w}]$ be the polynomial associated to p by Theorem 2.1(b). Consider its homogenisation

$$H(w, \overline{w}, \zeta) = \zeta^{2n} h_{\mathbb{C}}(w/\zeta, \overline{w}/\zeta).$$

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Since $h_{\mathbb{C}}$ has degree $m + n$ in the variable w, it follows that H has a zero of order at least $2n - (m + n) = n - m$ at the point $(1, 0, 0)$ of the projective space $\mathbb{CP}^2$. Similarly, it has a zero of order at least $n - m$ at the point $(0, 1, 0)$.

The homogeneous polynomial $\widetilde{H}$ associated with $\widetilde{p}$ has zeros of order at least $s - r$ at the same two points. Therefore the projective curves $H = 0$ and $\widetilde{H} = 0$ intersect with multiplicity at least $(n - m)(s - r)$ at each of the points $(1, 0, 0)$ and $(0, 1, 0)$ [8, Theorem 5.10, page 114].

Bezout's theorem implies that, unless H and $\widetilde{H}$ have a nontrivial common factor, the projective curves $H = 0$ and $\widetilde{H} = 0$ have at most $\deg H \deg \widetilde{H} = 4ns$ intersections in $\mathbb{CP}^2$, counted with multiplicity. Subtracting the intersections at the two aforementioned points, we are left with at most $4ns - 2(n - m)(s - r)$ points of intersection in the affine plane. □

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LEONID V. KOVALEV, 215 Carnegie, Mathematics Department,
Syracuse University, Syracuse, NY 13244, USA
e-mail: lvkovale@syr.edu

XUERUI YANG, 215 Carnegie, Mathematics Department,
Syracuse University, Syracuse, NY 13244, USA
e-mail: xyang20@syr.edu