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A new geometric method for singularity analysis of spherical mechanisms

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SUMMARY
Finding singular configurations (singularities) has an important role during the design, trajectory planning, and control stages of mechanisms because in these configurations, the instantaneous kinematics is locally undetermined. In this paper, a systematic method is presented to obtain singular configurations of spherical mechanisms with input and output links. The method extends the use of instantaneous poles to singularity analysis of spherical mechanisms and offers geometric conditions for any type of singularities occurring in these mechanisms.

KEYWORDS: Spherical mechanisms; Singularity analysis; Instantaneous poles; Angular velocity.

1. Introduction

Study of parallel mechanisms highlighted the importance of identifying singular configurations (singularities). In these configurations, degree of freedom (dof) of the mechanism changes and the mechanism will become scarcely controllable, so these configurations must be found and avoided.

Different approaches have been adopted in dealing with singularity analysis; considering a mechanism as an input–output device, Gosselin and Angeles[1] identified three types of singularities:

Type (I) singularities: singularities where the inverse instantaneous kinematic problem (IIKP) is unsolvable. This type of singularities occurs when at least one out of the input-variable rates can be different from zero even though all the output-variable rates are zero. In one-dof mechanisms, such singularities occur when the output link reaches a dead center, i.e., when an output variable reaches a border of its range.

Type (II) singularities: singularities where the forward instantaneous kinematic problem (FIKP) is unsolvable. This type of singularities occurs when at least one out of the output-variable rates can be different from zero even though all the input-variable rates are zero. These singularities may be present only in mechanisms with closed loops and fall inside the output-variables range (workspace). In one-dof mechanisms, such configurations occur when the input link reaches a dead center.

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Type (III) singularities: singularities where both the IIKP and the FIKP are unsolvable, i.e., when two previous singularities occur simultaneously; in this type of singularities, the input–output instantaneous relationship holds no longer and the mechanism behavior may change. In one-dof mechanisms, these singularities lead to one or more additional uncontrollable dofs.

From different point of view, Hunt[2] identified two classes of singular configurations: (a) stationary configurations where one joint variable is instantaneously inactive and (b) uncertainty configurations where the mechanism locally gains transitory mobility.

One of the geometrical devices to deal with instantaneous kinematics of planar mechanisms is instant centers.[3–11] For instance, they can be used for singularity analysis of planar mechanisms;[7–11] Daniali[9] introduced a fast method to find singular configurations of three-dof planar parallel manipulators, based on the properties of instant centers. Di Gregorio[12] presented an algorithm that computes the instant centers in all single-dof planar mechanisms, and an exhaustive analytical and geometrical study[10] about the singularity analysis of single-dof planar mechanisms. He classified the single-dof planar mechanisms into four groups and derived the explicit expression of the input–output instantaneous relationship for each group through the concept of instant centers. Using the obtained relationships, he found the conditions corresponding to each type of singularities of these mechanisms. He also found[11] singular configurations of *n*-dof planar mechanisms by considering them as the union of *n* one-dof mechanisms.

All of the above mentioned works could be done for spherical mechanisms using instantaneous poles. For example, Deducing from his previous work,[12] Di Gregorio presented an exhaustive algorithm[13] to determine the instantaneous poles' positions of single-dof spherical mechanisms.

There exist some papers that addressed singularity analysis of spherical mechanisms, but the literature is so limited and most of the works are analytical.[14–17] However, Zarkandi *et al.*[18] have recently presented a work in which singularity analysis of single-dof spherical mechanisms was implemented geometrically using the concepts of mechanical advantage and instantaneous poles.

Here, the author extends the methodology presented by Di Gregorio[10,11] to singularity analysis of spherical mechanisms by exploiting the properties of instantaneous poles. This work can be considered as a counterpart of Di

Gregorio's mentioned works for the spherical mechanisms. In contrast to the previous work,[18] the presented method can be used to study the singularities of single- and multi-dof spherical mechanisms having input and output links.

2. Spherical Motion and Instantaneous Poles

In this section, the concepts of spherical motion and instantaneous poles are reviewed based on the work presented by Chiang.[19] Spherical motion is defined as the motion of a rigid body, a unique point of which is permanently fixed in space. Let this point be denoted by O. Because the distance between any point in the body, say A, and the fixed point O is not changeable, the point A can only move within a spherical surface. The point O then becomes the center of the spherical surface. Position of the rigid body will be completely determined if the positions of any two points A and B in the body are determined, provided that O, A, and B are not collinear. In other words, if A and B are on the same spherical surface, the position of the whole body can be fully represented by the positions of A and B. It follows that, as far as kinematics of a rigid body is concerned, it is sufficient to consider the motion of two distinct points of the body, both lying on the same spherical surface. It is also sufficient, in the case of multiply moving bodies with a common sphere center, to consider only the motion of such points of these bodies that lie on a unique spherical surface. Without loss of generality, the radius of this unique sphere may be taken as unity and the sphere is called the reference sphere. Each body, however, may be considered as a spherical shell of negligible thickness, extending over the whole surface of the reference sphere. Corresponding to the straight line on a plane, the shortest line on a curved surface connecting two points is called a *geodetic line*. The geodetic line on a spherical surface between two points is part of a great circle. The great circle is the intersection of the spherical surface with a plane passing through these two points and the sphere center. It is well known that, in plane kinematics, a rigid body can be represented by a straight line passing through two points of the body. Similarly, in spherical kinematics, a rigid body can also be represented by a segment of a great circle passing through two points of the body, on the surface of the reference sphere.

Let us consider the motion of a spherically moving body AB, represented by a piece of a spherical shell as shown in Fig. 1. Assume that the directions of the velocities $\mathbf{v}_A$ and $\mathbf{v}_B$ of the points A and B are known. $\mathbf{v}_A$ must be normal to OA. If we draw a great circle m passing through A and normal to the direction of $\mathbf{v}_A$; then, by the definition of a rigid body, the direction of the velocity of any point on the great circle m must be parallel to $\mathbf{v}_A$. Similarly, the direction of the velocity of any point on the great circle n passing through B and normal to $\mathbf{v}_B$ must be parallel to $\mathbf{v}_B$. Let the intersection point of the two great circles m and n be denoted by P. If there exists a linear velocity $\mathbf{v}_P$ of the point P on the moving body, it must be parallel to both $\mathbf{v}_A$ and $\mathbf{v}_B$. Since this is not possible, $\mathbf{v}_p$ must vanish identically. P is thus called the velocity pole or pole of the moving body and the body is rotating about the pole axis OP. Thus, we see that any

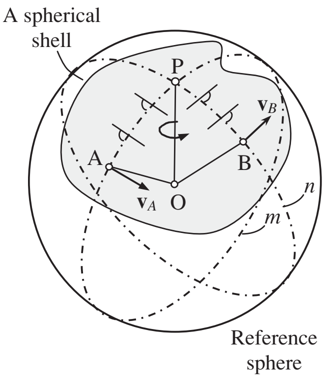

Fig. 1. Pole of a spherical shell moving on surface of the reference sphere and velocity of two point of it.[19]

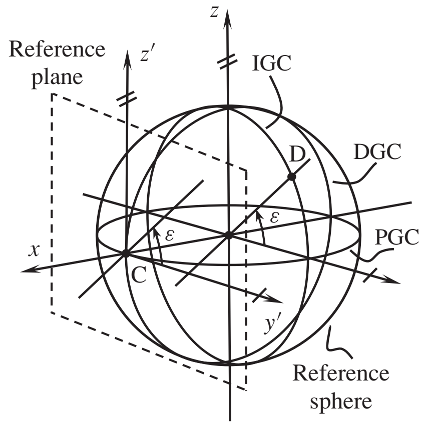

Fig. 2. Reference sphere, reference plane, and Cartesian reference system fixed to the frame: IGC = infinity great circle, PGC = primary great circle, DGC = declination great circle.

spherical motion is a rotation, i.e., an angular motion. OP can be either a fixed or an instantaneous axis of rotation.

The pole of a spherically moving body, as defined above, may be generalized for two cospherically moving shells. There exist between such two shells two instantaneously coincident points, each belonging to the respective shell, the linear velocities of which are identical. This common point is called *instantaneous pole* of the two shells. In fact, for two shells moving about a common sphere center, there are two instantaneous poles located diametrically opposite to each other on the reference sphere, and the two shells are said to rotate instantly relative to each other about an *instantaneous pole Axis* that passes through two instantaneous poles and the sphere center.

3. Notations

Spherical mechanisms can be studied by projecting them through the spherical-motion center onto a sphere (reference sphere) with center at the spherical-motion center, Fig. 2. So doing, instantaneous poles, henceforth referred as the instant poles, become points of the reference sphere and links become spherical shells that move on the reference sphere.

With reference to a Cartesian system (Fig. 2) whose origin is at the spherical motion center, intersections between the reference sphere and xy plane, yz plane, and zx plane will

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be called primary great circle (PGC), infinity great circle (IGC), and declination great circle (DGC), respectively.[13] The IGC cuts the reference sphere into two hemispheres: the one (positive hemisphere) whose points have a positive x-coordinate and the other (negative hemisphere) whose points have a negative x-coordinate. Two great circles with different slopes have only one intersection in the positive (negative) hemisphere, whereas two great circles with the same slope (i.e., that belong to the same pencil of meridians) do not intersect each other in the positive (negative) hemisphere. An instantaneous pole axis that does not lie on the yz plane intersects the reference sphere at two diametrically opposite points: one lying on the positive hemisphere and the other lying on the negative hemisphere. An instantaneous pole axis that lies on the yz plane cuts the IGC into two diametrically opposite points: one either with positive y-coordinate, or with zero y-coordinate and positive z-coordinate, and the other either with negative y-coordinate, or with zero y-coordinate and negative z-coordinate. Therefore, the points of the positive (negative) hemisphere plus the points of the IGC either with positive (negative) y-coordinate or with zero y-coordinate and positive (negative) z-coordinate are sufficient to identify all the possible instant pole. This set of points, which is a subset of the reference sphere, will be called positive (negative) shell.[13]

Since instant poles' positions are sufficient to fully describe first-order kinematics of spherical mechanisms, the first-order kinematics of the spherical mechanisms can be studied by using only one (either positive or negative) shell of the reference sphere. Hereafter, the positive shell will be used.

A bijective mapping can be defined[13] between points of the positive hemisphere and those of the reference plane $x = 1$, which is tangent to reference sphere at point C (1, 0, 0) (see Fig. 2). Moreover, by projecting a straight line with the certain slope passing through point of tangency between the reference plane and reference sphere on the yz plane, a line with the same slope is obtained that can be used to uniquely locate the corresponding IGC point, D, of the positive hemisphere as shown in Fig. 2. Therefore, the first-order kinematics of spherical mechanisms can be fully described by using only points of the reference plane.

As a consequence of the above discussion, any theorem, considering instantaneous kinematics of planar mechanisms, can have a spherical counterpart. For instance, Aronhold–Kennedy theorem in plane kinematics can be adapted for spherical kinematics and stated as follows:

Theorem 1 (Aronhold–Kennedy theorem). Three instant poles of the three cospherically moving links lie on a unique great circle.

Considering a one-dof spherical mechanism as an input–output system, any input–output instantaneous relationship can be written by using only the instant poles among four links: input link (i), output link (o), reference link (f) used to evaluate the rate of input variable, and reference link (k) used to evaluate the rates of all output variables. The input (output) variable is a motion characteristic of the relative motion "if" ("ok") and is a rotation angle about the correspondent instantaneous pole axis; moreover, it is taken positive if counterclockwise with respect to the positive side[1] of the axis.

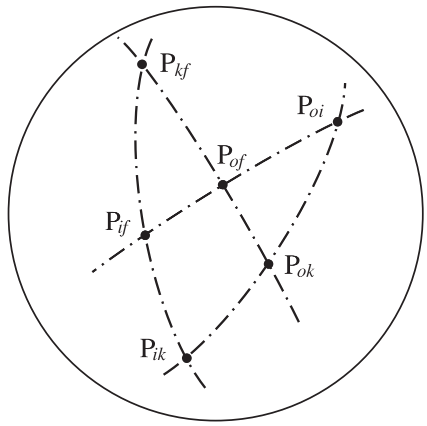

Fig. 3. Great circles that the six instant poles lie on.

For the four bodies of a spherical mechanism, "i," "o," "f," and "k," there exist six different relative motions that will be denoted "if," "ok," "ik," "kf," "oi," and "of" (the second letter indicates the link from which motion of the link, correspond to the first letter, is observed). Instant poles of these six motions will be denoted P_{mn}, where $mn \in \{if, ok, ik, kf, oi, of\}$. $\mathbf{P}_{mn}$ will indicate the position vector that locates the instant pole P_{mn} (it is meant that all the position vectors are defined in a unique reference system fixed to the center of the unit sphere). According to the Aronhold–Kennedy theorem, these instant poles must lie on the great circles shown in Fig. 3.

Velocity of P_{mn}, as a point of link "t" and evaluated in a reference system fixed to link "r," where $r, t \in \{i, o, f, k\}$, will be denoted ${}^{r}\mathbf{v}_{mn|t}$. Angular velocity vector of the relative motion "mn" will be denoted $\boldsymbol{\omega}_{mn}$. With these notations and conventions, the following relationship holds:

$$ {}^{r}\mathbf{v}_{mn|t} = \boldsymbol{\omega}_{tr} \times (\mathbf{P}_{mn} - \mathbf{P}_{tr}). \tag{1} $$

In the following sections, without loosing generality, a single-input–single-output system (SISO system) will be considered. A mechanism[2] with n output variables (i.e., a single-input–multiple-output system (SIMO system)) can be considered as n independent SISO systems that work in parallel.

4. Input–Output Instantaneous Relationship

An input–output instantaneous relationship of a one-dof spherical mechanism with only one output variable is linear and can be generally written as follows:

$$ \mathbf{a} \times (\mathbf{e}_{if}\dot{\theta}_{if}) = \mathbf{b} \times (\mathbf{e}_{ok}\dot{\theta}_{ok}), \tag{2} $$

where $\mathbf{e}_{if}$ ($\mathbf{e}_{ok}$) is a unit vector directed along the positive side of instantaneous pole axis of the input (output) link and

[1] Positive side of an instantaneous pole axis is defined as the side that has intersection with the positive hemisphere.
[2] In this paper, if it is not differently specified, the mechanisms that are considered are spherical mechanisms.

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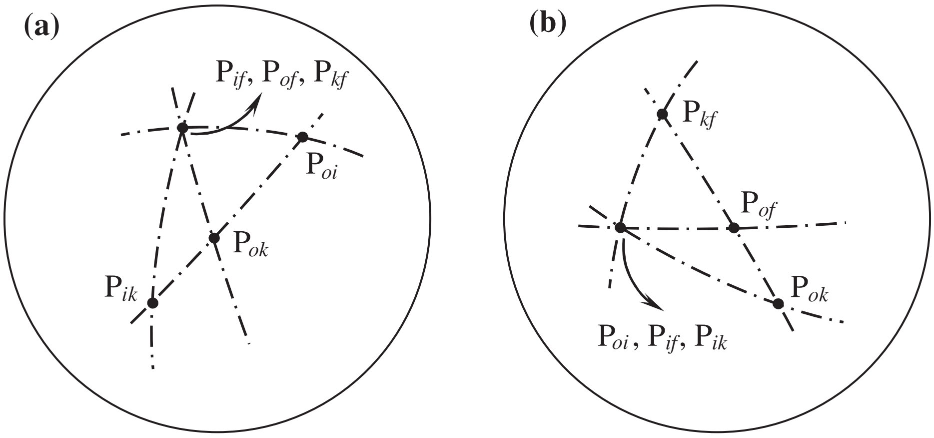

Fig. 4. Type (I) singularities: (a) condition (8a) is matched and (b) condition (8b) is matched.

the reference link f (k); $\dot{\theta}_{if}(\dot{\theta}_{ok})$ is the rate of input (output) variable, i.e., θ_{if} (θ_{ok}); Therefore, $\dot{\theta}_{if}(\dot{\theta}_{ok})$ is equal to angular velocity of relative motion between the input (output) link and the reference link f (k). In other words,

$$\mathbf{e}_{if}\dot{\theta}_{if} = \boldsymbol{\omega}_{if},$$
$$\mathbf{e}_{ok}\dot{\theta}_{ok} = \boldsymbol{\omega}_{ok}.$$

Additionally, **a** and **b** are coefficients that depend only on the mechanism configuration and are obtained as follows. In view of properties of instant poles, the following relationships can be written:

$$^{f}\mathbf{v}_{oi|i} = {^{f}\mathbf{v}_{oi|o}}, \quad (3a)$$

$$^{k}\mathbf{v}_{oi|i} = {^{k}\mathbf{v}_{oi|o}}. \quad (3b)$$

Moreover, the following relationship holds:

$$\boldsymbol{\omega}_{if} + \boldsymbol{\omega}_{ok} = \boldsymbol{\omega}_{of} + \boldsymbol{\omega}_{ik}. \quad (4)$$

Introduction of Eq. (1) into Eq. (3) leads to

$$(\mathbf{P}_{oi} - \mathbf{P}_{if}) \times \boldsymbol{\omega}_{if} = (\mathbf{P}_{oi} - \mathbf{P}_{of}) \times \boldsymbol{\omega}_{of} \quad (5a)$$

$$(\mathbf{P}_{oi} - \mathbf{P}_{ik}) \times \boldsymbol{\omega}_{ik} = (\mathbf{P}_{oi} - \mathbf{P}_{ok}) \times \boldsymbol{\omega}_{ok} \quad (5b)$$

Finally, introduction of Eqs. (5) into Eq. (4) and doing some algebraic manipulation (see Appendix) results in the following input–output instantaneous relationship:

$$(\mathbf{P}_{of} - \mathbf{P}_{if}) \times (\mathbf{P}_{oi} - \mathbf{P}_{ik}) \times \boldsymbol{\omega}_{if} = (\mathbf{P}_{ok} - \mathbf{P}_{ik}) \times (\mathbf{P}_{oi} - \mathbf{P}_{of}) \times \boldsymbol{\omega}_{ok}. \quad (6)$$

Thus, the following expressions are obtained for **a** and **b**:

$$\mathbf{a} = (\mathbf{P}_{of} - \mathbf{P}_{if}) \times (\mathbf{P}_{oi} - \mathbf{P}_{ik}), \quad (7a)$$

$$\mathbf{b} = (\mathbf{P}_{ok} - \mathbf{P}_{ik}) \times (\mathbf{P}_{oi} - \mathbf{P}_{of}). \quad (7b)$$

5. Singularity Analysis of Single-Dof Spherical Mechanisms

Equation (2) is the general form of an input–output instantaneous relationship of a one-dof spherical mechanism with only one output variable (SISO system). With reference to Eq. (2) and considering the physical meanings of the three types of singularities identified by Gosselin and Angeles (see Section 2), type (I) singularities (inverse kinematic singularities) occur when the vector **a** is equal to zero then the IIKP admits an infinite number of solutions for the rate $\dot{\theta}_{if}$ (i.e., the rate $\dot{\theta}_{if}$ is undetermined); type (II) singularities (direct kinematic singularities) occur when the vector **b** is equal to zero then the FIKP results in an infinite number of solutions for the rate $\dot{\theta}_{ok}$ (i.e., the rate $\dot{\theta}_{ok}$ is not determined); finally, type (III) singularities occur when vectors **a** and **b** are both equal to zero. In the following part of this section, the above-deduced explicit relationship will be discussed and conditions that identify the singularities will be given.

Analysis of Eq. (7a) reveals that an inverse problem singularity occurs when at least one out of the following geometric conditions is satisfied (Fig. 4):

$$\mathbf{P}_{of} = \mathbf{P}_{if}, \quad (8a)$$

$$\mathbf{P}_{oi} = \mathbf{P}_{ik}. \quad (8b)$$

On the other side, Eq. (7b) shows that a direct problem singularity occurs when at least one out of the following geometric conditions is matched (Fig. 5):

$$\mathbf{P}_{ok} = \mathbf{P}_{ik}, \quad (9a)$$

$$\mathbf{P}_{oi} = \mathbf{P}_{of}. \quad (9b)$$

Finally, type (III) singularities occur when at least one out of Eqs. (8) together with at least one out of Eqs. (9) are satisfied simultaneously.

In the following subsection, an example will illustrate how the presented procedure can be implemented.

5.1. *Singularity analysis of spherical Shaper mechanism*

In this subsection, the above procedure will be applied to study a six-bar linkage called spherical-shaper mechanism.

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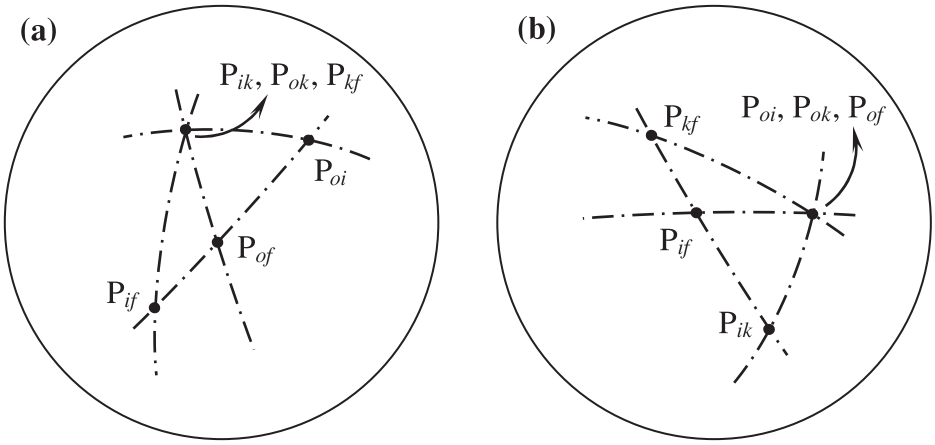

Fig. 5. Type (II) singularities: (a) condition (9a) is matched and (b) condition (9b) is matched.

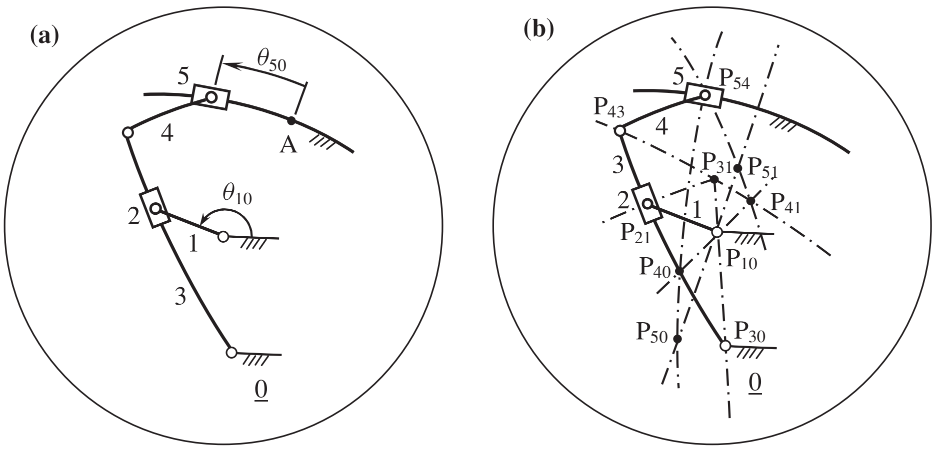

Fig. 6. (a) A spherical-shaper mechanism at a generic configuration and (b) the mechanism with its instant poles.

Figure 6(a) shows a spherical-shaper mechanism together with the notations that will be used. Link 1 is the input link; link 5 is the output link; link 0 is the reference link used to evaluate the rate both of the input variable and the output variable. Therefore, "i" = 1, "o" = 5, "f" = "k" = 0. θ_{10} is the input variable. Arbitrary point A is fixed to link 0. θ_{50} is considered as the output variable and is defined as the convex central angle formed by two radius vectors that pass through the point A and a specific point on output link 5. Instant poles of the mechanism are shown in Fig. 6(b).

Please not that due to the structure of the mechanism the singularity conditions (8a) or (9a) can not be satisfied. Singularity conditions (8b) and (9b) bring to the conclusion that type (I) singularities occur when P_{51} coincides with P_{10}. Considering Fig. 6(b), one can see that coincidence of these two instant poles occurs when the great circle, passing through link 1, is perpendicular to the great circle passing through link 3, which is the geometric condition identifying dead-center positions of link 5 in the mechanism (see Fig. 7).

On the other hand, type (II) singularities occur when P_{51} coincides with P_{50}. Coincidence of P_{51} with P_{50} (i.e., type (II) singularity) occurs when the great circle, passing through link 4, is perpendicular to the curves of correspondent slipping joint of link 5; Fig. 8 shows a configuration of mechanism at this type of singularity.

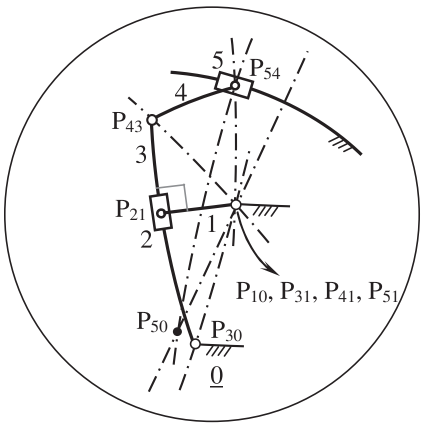

Fig. 7. The spherical-shaper mechanism at a type (I) singularity.

Type (III) singularities are mechanism configurations in which two previous singularities occur simultaneously. Considering Eqs. (8) and (9), one can conclude that at this type of singularity, P_{51} must coincide with P_{10} and P_{50} simultaneously. This condition is satisfied when P_{51} is not determined. Figure 9 shows the mechanism in this type of singularity in which P_{41} and consequently P_{51} are not determined.

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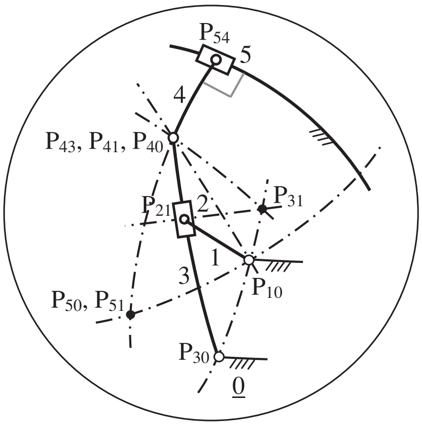

Fig. 8. The spherical-shaper mechanism at a type (II) singularity.

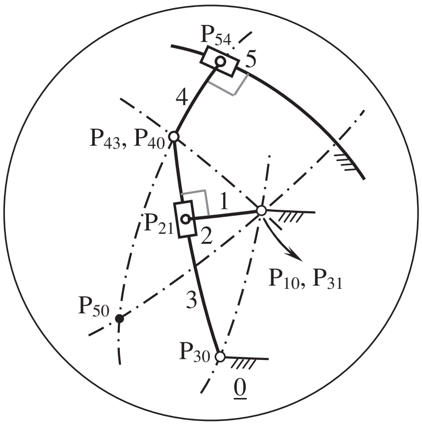

Fig. 9. The spherical-shaper mechanism at a type (III) singularity.

6. Singularity Analysis of Spherical Mechanisms with More Than One Degree of Freedom

As it was said in Section 3, any theorem or result considering instantaneous kinematics of planar mechanisms can have a spherical counterpart. Some results, which are deduced from a work presented by Di Gregorio,[11] are adapted for spherical mechanisms and stated through three statements in the following.

Considering an n-dof spherical mechanism as the union of n single-dof mechanisms where the ith single-dof mechanism is generated from the n-dof mechanism by locking all inputs but the ith one, then for the ith single-dof mechanism, the following relationship holds:

$$\mathbf{a}_i \times \left(\mathbf{e}^i_{if}\dot{\theta}^i_{if}\right) = \mathbf{b}_i \times \left(\mathbf{e}^i_{ok}\dot{\theta}^i_{ok}\right), \tag{10}$$

where $\mathbf{a}_i$, $\mathbf{b}_i$, $\mathbf{e}^i_u$, $\dot{\theta}^i_u$, u ∈ {if, ok} are equivalent to the $\mathbf{a}$, $\mathbf{b}$, $\mathbf{e}_u$, $\dot{\theta}_u$, u ∈ {if, ok} for the ith single-dof mechanism, respectively.

Statement 1. Type (I) singularities of the n-dof spherical mechanisms can be collected into two classes:

(a) Configurations that are singular for at least one single-dof mechanism generated from the n-dof mechanism and occur if the coefficient $\mathbf{a}_i$, $i \in \{1, 2, \ldots, n\}$ is equal to $\mathbf{0}$.

(b) Singularities, occurring in the mechanisms with $n \geq 3$, that are not singularities of any single-dof mechanism generated from that n-dof mechanism and occur, if and only if, three instant poles p_i $i \in \{1, 2, 3\}$ locate on the same great circle and all of the $\mathbf{a}_i$ and $\mathbf{b}_i$ coefficients are different from $\mathbf{0}$

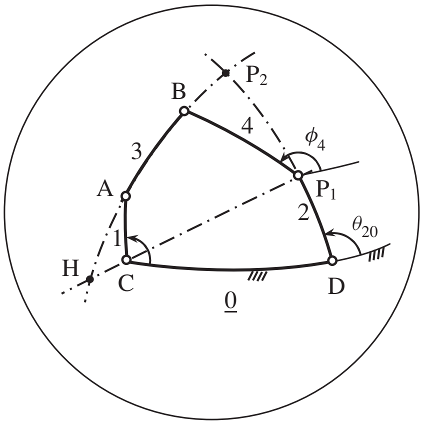

Fig. 10. A 5R spherical mechanism.

Where point P_i is the instant pole of the instantaneous motion that the output link would perform with respect to reference link (o) if it was guided by the ith single-dof mechanism generated from the n-dof mechanism.

Statement 2. Type (II) singularities of the n-dof spherical mechanisms occur if the coefficient $\mathbf{b}_i$, $i \in \{1, 2, \ldots, n\}$ is equal to $\mathbf{0}$.

Statement 3. Coincidence of all the P_i instant poles $i \in \{1, 2, \ldots, n\}$ identifies a particular type (II) singularity of an n-dof spherical mechanism.

In the following subsections, the presented procedure will be applied on two spherical mechanisms: (1) a spherical 5R mechanism and (2) a three-dof 3-RRP spherical fully parallel mechanism in which each of the legs begins with two revolute joint and ends with a slipping joint.

6.1. Singularity analysis of the 5R spherical mechanism

The 5R spherical mechanism is a two-dof mechanism. Figure 10 shows the mechanism at a generic configuration. Links 1 and 2 are the input links; link 4 is chosen as output link and link 0 (reference sphere) is the reference link for both input and output variables. θ_{i0}, $i = 1, 2$ are the input variables. Two single-dof mechanisms generated from the 5R mechanism are depicted in Fig. 11. P_1 is equivalent to the instant pole P_{40} in the first single-dof mechanism that is the one generated by locking θ_{20}, whereas P_2 is the instant pole P_{40} in the second single-dof mechanism that is the one generated by locking θ_{10}. Both the single-dof mechanisms generated from the 5R mechanism are four-bar linkages. In the ith single-dof mechanism, the output link motion is a rotation around the correspondent instantaneous pole axis, which allows the angle ϕ_4 to be chosen as output variable. Additionally, H is the instant pole P_{41} in the first single-dof mechanism (see Fig. 10).

For the ith single-dof mechanism, we have

$$o = 4, i = i, f = k = 0;$$

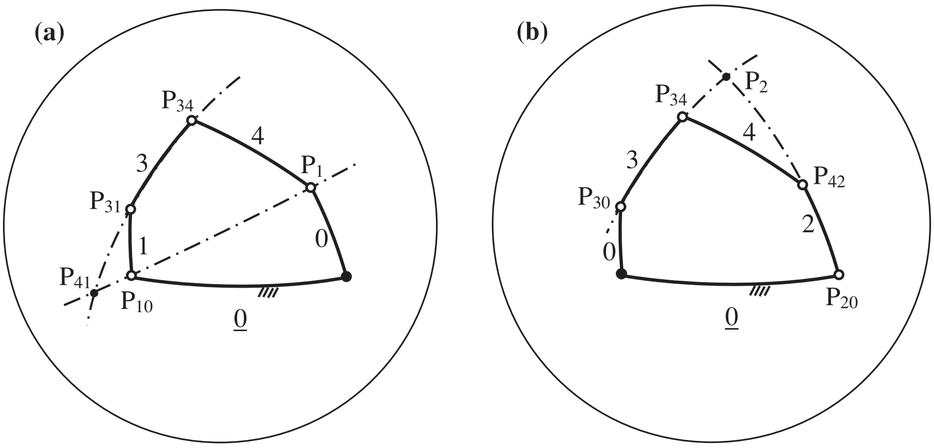

Fig. 11. Two single-dof mechanisms generated from the spherical 5R mechanism: (a) mechanism generated by locking θ_{20} and (b) mechanism generated by locking θ_{10}.

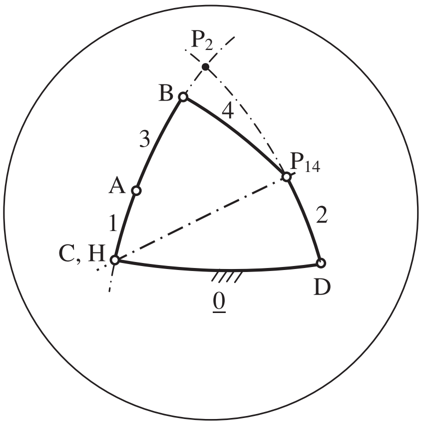

Fig. 12. Example of type (I) singularity in the 5R spherical mechanism, where links 1 and 3 are aligned.

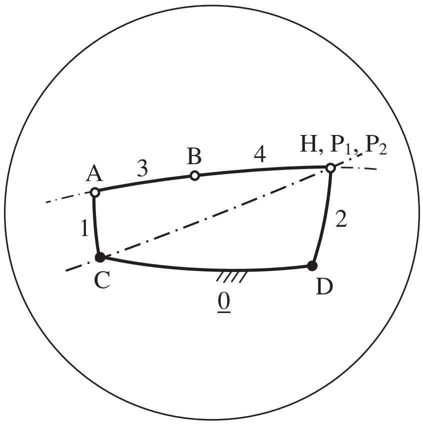

Fig. 13. Example of type (II) singularity in the 5R spherical mechanism, where links 3 and 4 are aligned.

With reference to Eqs. (7a) and (7b), the following expressions are obtained for $\mathbf{a}_i$ and $\mathbf{b}_i$:

$$\mathbf{a}_i = (\mathbf{P}_i - \mathbf{P}_{i0}) \times (\mathbf{P}_{4i} - \mathbf{P}_{i0});$$
$$\mathbf{b}_i = (\mathbf{P}_i - \mathbf{P}_{i0}) \times (\mathbf{P}_{4i} - \mathbf{P}_i), \quad i = 1, 2. \tag{11}$$

Type (I) singularities (i.e., $\mathbf{a}_i = \mathbf{0}$, $i = 1, 2$) occur when at least one of the following conditions is satisfied:

$$\mathbf{P}_i = \mathbf{P}_{i0}, \tag{12a}$$
$$\mathbf{P}_{4i} = \mathbf{P}_{i0}. \tag{12b}$$

Type (II) singularities (i.e., $\mathbf{b}_i = \mathbf{0}$, $i = 1, 2$) occur when at least one out of the following geometric conditions is verified:

$$\mathbf{P}_i = \mathbf{P}_{i0}, \tag{13a}$$
$$\mathbf{P}_{4i} - \mathbf{P}_i. \tag{13b}$$

Conditions (12a) or (13a) do not lead to any singular configuration and will not be considered in singularity analysis of the mechanism. Therefore, type (I) singularities occur when P_{41} coincides with P_{10} or P_{42} coincides with P_{20} for the first and second single-dof mechanism, respectively. Considering Figs. 11(a) and (b), this type of singularity occurs when P_{41} coincides with P_{10} (see Fig. 12). So in this type of singularity, the great circles, passing through links 1 and 3, are coincident and θ_{10} does not produce any motion in the output link, link 4.

Moreover, type (II) singularities occur when P_{41} coincides with P_1 or P_{42} coincides with P_2 for the first and second single-dof mechanisms, respectively. These two conditions lead to a unit configuration as it is shown in Fig. 13.

So in this type of singularity, the great circles, passing through links 3 and 4, are coincident and the output link, link 4, can have finite motion even if all inputs are locked and the mechanism gains one additional dof.

Eventually, type (III) singularities occur when both previous singularities occur simultaneously; so in this case, P_{41} must coincide with P_{10} and P_1 simultaneously; this condition is satisfied when the position of P_{41} is not determined. An example of this type of singularity is depicted in Fig. 14.

6.2. Singularity analysis of the 3-RRP spherical mechanism

The 3-RRP spherical mechanism is a three-dof mechanism. Figure 15 shows the mechanism at a generic configuration.

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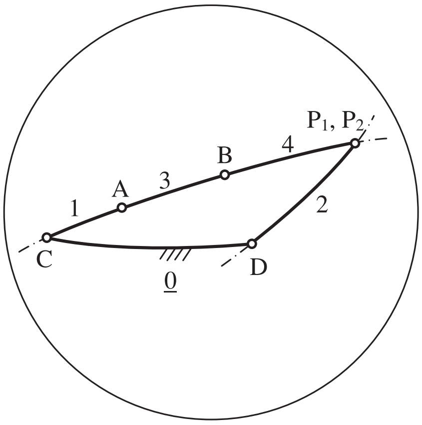

Fig. 14. Example of type (III) singularity in the 5R spherical mechanism, where links 1, 3, and 4 are aligned.

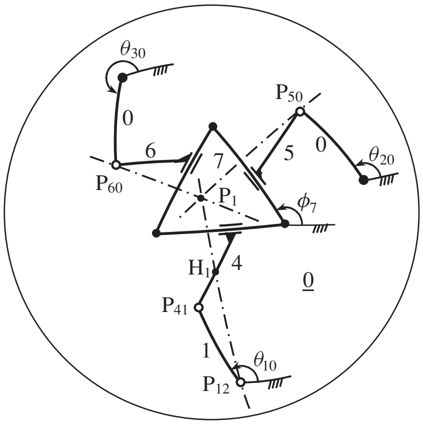

Fig. 16. The first single-dof mechanism generated from the 3-RRP spherical mechanism by locking all inputs but θ_{10}.

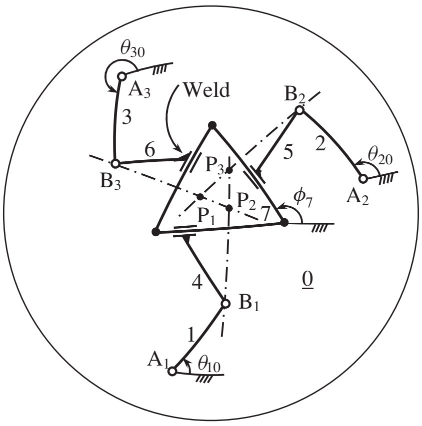

Fig. 15. A spherical 3-RRP mechanism.

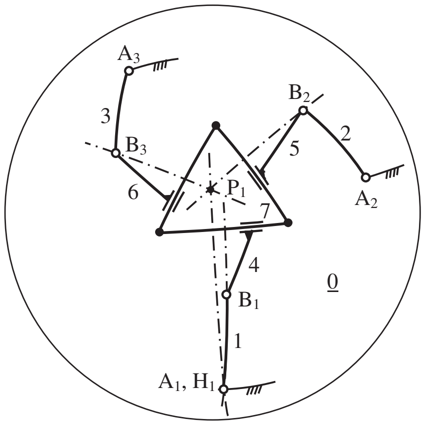

Fig. 17. Example of configuration of the spherical 3-RRP mechanism, where the great circle, passing through link 1, is perpendicular to the curve of slipping joint of the correspondent leg (type (I) singularity).

With reference to Fig. 15, θ_{i0} for $i = 1, 2, 3$ are input variables. Links 1, 2, and 3 are input links. Link 7 is the output link. Link 0 (reference sphere) is chosen as the reference link. In the ith single-dof mechanism, the output link motion is a rotation around the instant pole P_i, so angle ϕ_7 can be chosen as the output variable. As an example, the first single-dof mechanism, i.e., the one generated by locking all the inputs but θ_{10} in the 3-RRP mechanism, is depicted in Fig. 16; P_i for $i = 1, 2, 3$ is the instant pole of the output link in the ith single-dof mechanism; in other words, P_i is equal to P_{70} for the ith single-dof mechanism. Moreover, H_i denotes the instant pole P_{7i} in the ith single-dof mechanism.

For the ith single-dof mechanism, we have

$$o = 7, i = i, f = k = 0.$$

Considering Eqs. (7a) and (7b), following expressions are obtained for $\mathbf{a}_i$ and $\mathbf{b}_i$:

$$\mathbf{a}_i = (\mathbf{P}_i - \mathbf{P}_{i0}) \times (\mathbf{P}_{7i} - \mathbf{P}_{i0});$$
$$\mathbf{b}_i = (\mathbf{P}_i - \mathbf{P}_{i0}) \times (\mathbf{P}_{7i} - \mathbf{P}_i), \quad i = 1, 2, 3. \tag{14}$$

So type (I) singularities of the ith single-dof mechanism (i.e., $\mathbf{a}_i = \mathbf{0}$) occur when the following geometric condition is verified: P_{7i} coincides with P_{i0}. Moreover, the analysis of Fig. 15 reveals that the three instant poles P_i, $i = 1, 2, 3$ locate on the same great circle if and only if they coincide at a common point, which is a geometric condition identifying a particular type (II) singularity (see statement 3). In this case, type (I) singularities of the 3-RRP mechanism that verify conditions of the part (b) of Statement 1 are not present, and the set of the type (I) singularities of the 3-RRP mechanism is the union of three sets of type (I) singularities due to the three single-dof mechanisms. Therefore, first type of singularity in 3-RRP mechanism occurs when the great circle, passing through link i for $i = 1$ or 2 or 3, is perpendicular to the curves of slipping joint of the correspondent leg. Figure 17 shows an example of this type of singularities where P_{71} (or H_1) is coincident with P_{10}.

Regarding Eq. (12), Type-(II) singularities (i.e., $\mathbf{b}_i = \mathbf{0}$) occur when P_{7i} coincides with P_i. Analysis of Fig. 15 reveals that such a geometric condition occurs when three instant poles P_i (and H_i) for $i = 1, 2, 3$ coincide at a common point. A general configuration of mechanism in this type of singularity is illustrated in Fig. 18.

So in the second type of singularity of 3-RRP spherical mechanism, output link can rotate freely about a point even if all inputs are locked and the mechanism gains one additional dof. This point is the intersection of three great circles that

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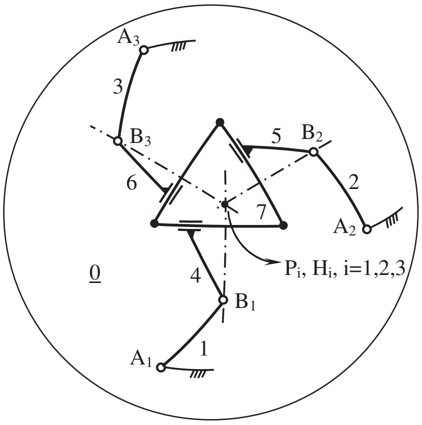

Fig. 18. Example of configuration of the spherical 3-RRP mechanism, where the three instant poles P_i for $i = 1, 2, 3$ coincide at a common point (type (II) singularity).

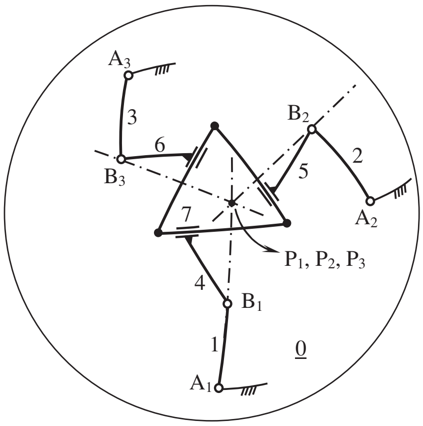

Fig. 19. Example of configuration of the spherical 3-RRP mechanism, where two previous singularities occur simultaneously (type (III) singularity).

are spherically perpendicular to the curves of slipping joint of the legs and pass through the correspondent median joint.

Again, please note that the condition correspondent to coincidence of instant poles P_i with P_{i0}, which is obtained from the common factor $(\mathbf{P}_i - \mathbf{P}_{i0})$ in $\mathbf{a}_i$ and $\mathbf{b}_i$, does not give any singular configuration, so it is not considered in singularity analysis of the mechanism.

Eventually, type (III) singularities occur when both previous singularities occur simultaneously. With reference to Eq. (12), in this type of singularity, P_{7i} must coincides with P_{i0} and P_i simultaneously; this condition is satisfied when P_{7i} (H_i) is not determined. Figure 19 shows a general configuration of this type of singularity.

7. Conclusion

A systematic analysis of singularities occurring in one-dof spherical mechanisms has been presented. In particular, explicit expressions of the input–output instantaneous relationships have been deduced, using the properties of instant poles. Moreover, geometric conditions that identify singularities of this type of mechanisms have been given.

Then, all singular configurations of multi-dof spherical mechanisms have been found by using instant poles of the single-dof mechanisms generated from the multi-dof spherical mechanisms.

In contrast to the analytical methods that can be used only for particular types of the spherical mechanisms, the presented method is a novel and can be used to study the singularities of all types of spherical mechanisms, which have input and output links. However, when we have output point instead of the output link, the method can not be applied. For instance, it can not be used in singularity analysis of the 5R mechanism of Section 6.1 if the position of point B is chosen as the output variable.[14]

Appendix

Multiplying both sides of Eq. (4) in $(\mathbf{P}_{oi} - \mathbf{P}_{of})$ leads to

$$(\mathbf{P}_{oi} - \mathbf{P}_{of}) \times \boldsymbol{\omega}_{if} + (\mathbf{P}_{oi} - \mathbf{P}_{of}) \times \boldsymbol{\omega}_{ok} = (\mathbf{P}_{oi} - \mathbf{P}_{of}) \times \boldsymbol{\omega}_{of} + (\mathbf{P}_{oi} - \mathbf{P}_{of}) \times \boldsymbol{\omega}_{ik}. \quad \text{(A1)}$$

According to Eq. (5a), the term $(\mathbf{P}_{oi} - \mathbf{P}_{of}) \times \boldsymbol{\omega}_{of}$ can be replaced by $(\mathbf{P}_{oi} - \mathbf{P}_{if}) \times \boldsymbol{\omega}_{if}$ in Eq. (A1); In this case, we have

$$(\mathbf{P}_{oi} - \mathbf{P}_{of}) \times \boldsymbol{\omega}_{if} + (\mathbf{P}_{oi} - \mathbf{P}_{of}) \times \boldsymbol{\omega}_{ok} = (\mathbf{P}_{oi} - \mathbf{P}_{if}) \times \boldsymbol{\omega}_{if} + (\mathbf{P}_{oi} - \mathbf{P}_{of}) \times \boldsymbol{\omega}_{ik}. \quad \text{(A2)}$$

Now, we multiply both sides of Eq. (A2) with $(\mathbf{P}_{oi} - \mathbf{P}_{ik})$; doing this along with some manipulation results in

$$(\mathbf{P}_{oi} - \mathbf{P}_{of}) \times (\mathbf{P}_{oi} - \mathbf{P}_{ik}) \times \boldsymbol{\omega}_{if} + (\mathbf{P}_{oi} - \mathbf{P}_{of}) \times (\mathbf{P}_{oi} - \mathbf{P}_{ik}) \times \boldsymbol{\omega}_{ok} = (\mathbf{P}_{oi} - \mathbf{P}_{if}) \times (\mathbf{P}_{oi} - \mathbf{P}_{ik}) \times \boldsymbol{\omega}_{if} + (\mathbf{P}_{oi} - \mathbf{P}_{of}) \times (\mathbf{P}_{oi} - \mathbf{P}_{ik}) \times \boldsymbol{\omega}_{ik}. \quad \text{(A3)}$$

Considering Eq. (5b), we can substitute the terms $(\mathbf{P}_{oi} - \mathbf{P}_{ik}) \times \boldsymbol{\omega}_{ik}$ for $(\mathbf{P}_{oi} - \mathbf{P}_{ok}) \times \boldsymbol{\omega}_{ok}$ in Eq. (A3)

$$(\mathbf{P}_{oi} - \mathbf{P}_{of}) \times (\mathbf{P}_{oi} - \mathbf{P}_{ik}) \times \boldsymbol{\omega}_{if} + (\mathbf{P}_{oi} - \mathbf{P}_{of}) \times (\mathbf{P}_{oi} - \mathbf{P}_{ik}) \times \boldsymbol{\omega}_{ok} = (\mathbf{P}_{oi} - \mathbf{P}_{if}) \times (\mathbf{P}_{oi} - \mathbf{P}_{ik}) \times \boldsymbol{\omega}_{if} + (\mathbf{P}_{oi} - \mathbf{P}_{of}) \times (\mathbf{P}_{oi} - \mathbf{P}_{ok}) \times \boldsymbol{\omega}_{ok}. \quad \text{(A4)}$$

Finally, rearranging the Eq. (A4) leads to the Eq. (6).

References

1. C. M. Gosselin and J. Angles, "Singularity analysis of closed-loop kinematic chains," *IEEE Trans. Robot. Autom.* **6**(3), 281–290 (1990).
2. K. H. Hunt, *Kinematic Geometry of Mechanisms* (Oxford University Press, New York, USA, 1978).
3. A. G. Erdman, G. N. Sandor and S. Kota, *Mechanism Design Volume 1*, 4th ed. (Prentice-Hall, Inc., Upper Saddle River, New Jersey, USA, 2001).
4. H.-S. Yan and L.-I. Wu, "The stationary configurations of planar six-bar kinematic chains," *Mech. Mach. Theory* **23**(4), 287–293 (1988).
5. H.-S. Yan and L.-I. Wu, "On the dead-center positions of planar linkage mechanisms," *ASME J. Mech. Transmissions, Autom. Des.* **111**(1), 40–46 (1989).

https://doi.org/10.1017/S0263574711000385 Published online by Cambridge University Press

6. G. R. Pennock and G. M. Kamthe, "Study of dead-centre positions of single-degree-of-freedom planar linkages using Assur kinematic chains," *Proc. IMechE, Part C, J. Mech. Eng. Sci.* **220**, 1057–1074 (2006).
7. H. P. Devis and T. R. Chase, "Stephenson Chain Branch Analysis: Four Generic Stationary Configurations and One New Linkage Polynomial," *Proceedings of the 1994 ASME Design Engineering Technical Conferences on Mechanism Synthesis and Analysis* (1994) DE-vol. 70, pp. 359–367.
8. D. Sen and T. S. Mruthyunjaya, "A centro-based characterization of singularities in the workspace of planar closed-loop manipulators," *Mech. Mach. Theory* **33**(8) 1091–1104 (1998).
9. H. R. M. Daniali, "Instantaneous center of rotation and singularities of planar parallel manipulators," *Int. J. Mech. Eng. Ed.* **33**(3), 251–259 (2005).
10. R. Di Gregorio, "A novel geometric and analytic technique for the singularity analysis of one-dof planar mechanisms," *Mech. Mach. Theory* **42**(11), 1462–1483 (2007).
11. R. Di Gregorio, "A novel method for the singularity analysis of planar mechanisms with more than one degree of freedom," *Mech. Mach. Theory* **44**(1), 83–102 (2009).
12. R. Di Gregorio, "An algorithm for analytically calculating the positions of the secondary instant centers of indeterminate linkages," *ASME J. Mech. Des.* **130**(4), 1–10 (2008).
13. R. Di Gregorio, "Determination of the Instantaneous Pole Axes in Single-Dof Spherical Mechanisms," *Proceedings of the ASME International Design Engineering Technical Conferences and Computers and Information in Engineering Conference*, Brooklyn, New York, USA (Aug. 3–6, 2008) pp. 1401–1411.
14. J. J. Cervantes-Sánchez, J. C. Hernández-Rodríguez and E. J. González-Galván, "On the 5R spherical, symmetric manipulator: Workspace and singularity characterization," *Mech. Mach. Theory* **39**(4), 409–429 (2004).
15. H. R. M. Danidi, P. Zsombor-Murray and J. Angeles, "Singularity Analysis of Spherical Parallel Manipulators," *Proceedings of the 10th CISM-IFToMM Symposium on the Theory and Practice of Robotic Manipulators*, Gdansk, (1991).
16. J. Sefrioui and C. Gosselin, "Ĕtude et représentation des lieux de singularité des manipulateurs paralléles sphériques á trois degrés de liberté avec actionneurs prismatiques," *Mech. and Mach. Theory* **29**(4), 559–579 (1991).
17. G. Alıcı and B. Shirinzadeh, "Loci of singular configurations of a 3-DOF spherical parallel manipulator", *Robot. Auton. Syst.* **48**(2), 77–91 (2004).
18. S. Zarkandi, M. R. Esmaili and H. R. M. Daniali, "Singularity analysis of single-dof spherical mechanisms using instantaneous poles", *Int. J. Res. Rev. Appl. Sci.* **5**(1), 35–42 (2010).
19. C. H. Chiang, *Kinematics of Spherical Mechanisms* (Cambridge University Press, New York, USA, 1988).