



Weakly separated Bessel systems of model spaces

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Abstract. We show that any weakly separated Bessel system of model spaces in the Hardy space on the unit disc is a Riesz system and we highlight some applications to interpolating sequences of matrices. This will be done without using the recent solution of the Feichtinger conjecture, whose natural generalization to multidimensional model subspaces of H^2 turns out to be false.

1 Introduction

Let H^2 be the Hardy space on the unit disc \mathbb{D} , that is, the reproducing kernel Hilbert space of those power series centered at the origin with square-summable Taylor coefficients. Its kernel s is the well-studied *Szegő kernel*

$$s_w(z) := \frac{1}{1 - \overline{w}z} \quad w, z \in \mathbb{D},$$

and its multiplier algebra can be identified isometrically with H^∞ , the algebra of bounded holomorphic functions on \mathbb{D} . A key role for the study of the function theory and the hyperbolic geometry of the unit disc is played by *inner functions*, which are those bounded analytic functions on the unit disc with an unimodular radial limit almost everywhere on the unit circle. Given an inner function Θ , one can define the associated *model space*

$$H_\Theta := H^2 \ominus \Theta H^2$$

as the orthogonal complement in the Hardy space of all multiples of Θ in H^2 . A great treatment of the main properties of model spaces, together with their interactions with operator theory on spaces of analytic functions, can be found in [7].

Any function in H^2 that vanishes with multiplicity m at a point λ in \mathbb{D} is divisible in H^2 by a *Blaschke factor*, i.e., an inner function of the form b_λ^m , where

$$b_\lambda(z) := \frac{\lambda - z}{1 - \overline{\lambda}z} \quad z \in \mathbb{D}.$$

Received by the editors February 08, 2021; revised September 24, 2021; accepted September 27, 2021.

Published online on Cambridge Core October 4, 2021.

The author was partially supported by National Science Foundation Grant DMS 1565243.

AMS subject classification: 30J05, 30J10, 47B32.

Keywords: Inner functions of one complex variable, Blaschke products, Linear operators in reproducing-kernel, Hilbert spaces.



Therefore, the model space associated to b_λ^m is m -dimensional and it is spanned by the kernels at λ that represent up to $m - 1$ derivatives of any H^2 function at λ , that is,

$$(1.1) \quad H_{b_\lambda^m} = \text{span} \left\{ s_\lambda, \frac{\partial s_\lambda}{\partial w}, \dots, \frac{\partial^{m-1} s_\lambda}{\partial w^{m-1}} \right\}.$$

Since a model space is a subspace of H^2 generated by an inner function, it comes natural to ask whether function theoretical properties of a sequence of inner functions $(\Theta_n)_{n \in \mathbb{N}}$ translate to Euclidean properties for the sequence $(H_{\Theta_n})_{n \in \mathbb{N}}$. Out of the many results that constitute such a valuable dictionary between operator theory and function theory, one of the most significant for the purposes of this note can be found in [11, Theorem 3.2.14]:

Theorem 1.1 *Let $(\Theta_n)_{n \in \mathbb{N}}$ be a sequence of inner functions such that $\Theta := \prod_{n=1}^\infty \Theta_n$ converges uniformly on any compact subset of \mathbb{D} . The following are equivalent:*

(i) *For any bounded sequence $(\phi_n)_{n \in \mathbb{N}}$ in H^∞ , there exists a function ϕ in H^∞ such that*

$$(1.2) \quad \phi - \phi_n \in \Theta_n H^2, \quad n \in \mathbb{N};$$

(ii) *There exists a $C \geq 1$ such that, for any sequence $(h_n)_{n \in \mathbb{N}}$ of unit vectors in H^2 such that h_n belongs to H_{Θ_n} for any n in \mathbb{N} and for any $(a_n)_{n \in \mathbb{N}}$ in l^2 ,*

$$(1.3) \quad \frac{1}{C^2} \sum_{n \in \mathbb{N}} |a_n|^2 \leq \left\| \sum_{n \in \mathbb{N}} a_n h_n \right\|^2 \leq C^2 \sum_{n \in \mathbb{N}} |a_n|^2;$$

(iii) *There exists a positive δ such that, for any z in \mathbb{D} ,*

$$|\Theta(z)| \geq \delta \inf_{n \in \mathbb{N}} |\Theta_n(z)|.$$

Note that (1.2) is stated in [11, Theorem 3.2.14] as

$$\phi - \phi_n \in \Theta_n H^\infty, \quad n \in \mathbb{N}.$$

This is equivalent to (1.2), since H^∞ is contained in H^2 and since, if for any n there exists a g in H^2 such that

$$\theta_n g = \phi - \phi_n \in H^\infty,$$

then the radial limit of $|g|$ on the unit circle must be bounded by $|\phi - \phi_n|$, since Θ_n is inner, and therefore, g is actually in H^∞ . A sequence of closed subspaces $(H_n)_{n \in \mathbb{N}}$ of a Hilbert space \mathcal{H} that satisfies (ii) is called a **Riesz system**, and the least C for which (1.3) holds is the *Riesz bound* of the sequence. If the least C such that the right-hand side of (1.3) holds is finite then $(H_n)_{n \in \mathbb{N}}$ is a **Bessel systems** with *Bessel bound* C . On the other hand, condition (iii) is equivalent to

$$(1.4) \quad \sup_{n \in \mathbb{N}} \prod_{j \neq n} |\Theta_j(z)| \geq \delta,$$

to hold uniformly in z . Indeed, condition (iii) above states that there exists a positive δ such that, for any z in \mathbb{D} and for any $0 < \varepsilon < \delta$ there exists a n in \mathbb{N} such that

$$|\Theta_n(z)| \leq \frac{|\Theta(z)|}{\delta - \varepsilon},$$

or, equivalently, such that

$$\prod_{j \neq n} |\Theta_j(z)| \geq \delta - \varepsilon,$$

which yields (1.4) (in case $\Theta_n(z) = 0$, take a limit for w going to z and then use continuity). Moreover, condition (iii) is related in [10, Lecture IX] to separation conditions on the subspaces in $(H_{\Theta_n})_{n \in \mathbb{N}}$, being equivalent to asserting that the sine of the angle between any model space and the closure of the linear span of all the others is uniformly bounded below:

$$(1.5) \quad \inf_{n \in \mathbb{N}} \sin \left(H_{\Theta_n}, \overline{\text{span}} \{ H_{\Theta_j} \}_{j \neq n} \right) > 0.$$

Here, the sine between two closed subspaces K_1 and K_2 of a Hilbert space \mathcal{H} is the least sine of the angle between two vectors chosen in K_1 and K_2 or, equivalently,

$$\sin(K_1, K_2) = \|T\|^{-1},$$

where

$$T: \text{span}\{K_1, K_2\} \mapsto \text{span}\{K_1, K_2\}$$

$$T|_{K_1} = Id_{K_1} \quad T|_{K_2} = 0.$$

If (1.5) holds, we say that $(H_{\Theta_n})_{n \in \mathbb{N}}$ is **strongly separated**, whereas **weak separation** will correspond to an uniform bound from below for the angle between any pair of distinct model spaces:

$$(1.6) \quad \inf_{n \neq j} \sin(H_{\Theta_n}, H_{\Theta_j}) > 0.$$

Suppose now that $\Theta_n = b_{\lambda_n}^{m_n}$, for some sequence $(\lambda_n)_{n \in \mathbb{N}}$ in the unit disc and some sequence $(m_n)_{n \in \mathbb{N}}$ of positive integers. Condition (i) of Theorem 1.1 becomes then an *interpolation property*: for any bounded sequence $(\phi_n)_{n \in \mathbb{N}}$ in H^∞ , there exists a bounded analytic function ϕ that, for any n in \mathbb{N} , agrees with ϕ_n at λ_n up to its $m_n - 1^{\text{st}}$ derivative. Theorem 1.1 is therefore a great example of how the deep interconnection between operator theory and function theory greatly helps the studying and the understanding of *interpolating sequences*.

Let \mathcal{H}_k be a reproducing kernel Hilbert space of analytic functions on a domain D of \mathbb{C}^d , and let \mathcal{M}_k be its multiplier algebra. A sequence $Z = (z_n)_{n \in \mathbb{N}}$ in D is **interpolating** for \mathcal{M}_k if for any bounded sequence $(w_n)_{n \in \mathbb{N}}$ in \mathbb{C} there exists a function ϕ in \mathcal{M}_k such that $\phi(z_n) = w_n$ for any n in \mathbb{N} . Intuitively, an interpolating sequence is a separated sequence, as we need to be able to specify the values of ϕ *arbitrarily* at the nodes $(z_n)_{n \in \mathbb{N}}$. It is also not surprising that the separation conditions we will look at depend on the kernel k : Z is a **weakly separated** sequence if there exists a positive M such that, for any $n \neq j$ in \mathbb{N} , there exists a function $\phi_{n,j}$ whose norm in \mathcal{M}_k does not

exceed M and that separates z_n and z_j , that is,

$$\phi_{n,j}(z_n) = 1 \quad \phi_{n,j}(z_j) = 0.$$

A celebrated work of Carleson [4], characterized interpolating sequences for H^∞ :

Theorem 1.2 *A sequence $\Lambda = (\lambda_n)_{n \in \mathbb{N}}$ is interpolating for H^∞ if and only if it is weakly separated and the measure*

$$\mu_\Lambda := \sum_{n \in \mathbb{N}} (1 - |\lambda_n|^2) \delta_{\lambda_n}$$

satisfies the embedding condition

$$(1.7) \quad \|f\|_{L^2(\mathbb{D}, \mu_\Lambda)} \leq C_\Lambda \|f\|_{H^2} \quad f \in H^2.$$

A measure μ on a domain D that embeds continuously a reproducing kernel Hilbert space \mathcal{H}_k on D into $L^2(D, \mu)$ is called a **Carleson measure** for \mathcal{H}_k . One can find in [4] a characterization of Carleson measures for H^2 that involves a one-box condition, and hence the hyperbolic geometry of the unit disc. It turns out that (1.7) holds if and only if the sequence of all lines through the kernels $(s_{\lambda_n})_{n \in \mathbb{N}}$ (i.e., the sequence one-dimensional subspaces spanned by the kernels s_{λ_n}) is a Bessel system [1, Proposition 9.5], highlighting once again a correspondence between the hyperbolic geometry of the unit disc and the Euclidean geometry of the Hardy space. Moreover, such a characterization of Carleson measures in terms of Bessel systems allows to extend Theorem 1.2 to some multiplier algebra other than H^∞ . For example, a class of multiplier algebras for which an analogous of Theorem 1.2 holds is the one associated with *complete Pick kernels*. One of the most important properties that connects interpolating sequence to the study of related Hilbert spaces is the fact that, for any multiplier ϕ in \mathcal{M}_k , any kernel function k_z in \mathcal{H}_k is an eigenfunction of the adjoint of the multiplication operator M_ϕ :

$$(1.8) \quad M_\phi^*(k_z) = \overline{\phi(z)} k_z \quad z \in D,$$

as a straightforward computation using the property of adjoints shows. In particular, if M_ϕ is a contraction and $\phi(z_n) = w_n$ for any n in \mathbb{N} , then the linear map T from $S_Z := \overline{\text{span}}_{n \in \mathbb{N}} \{k_{z_n}\}$ to itself given by

$$(1.9) \quad T(k_{z_n}) := \overline{w_n} k_{z_n} \quad n \in \mathbb{N}$$

is a contraction. A reproducing kernel Hilbert space is said to have the **Pick property** if the existence of such a contraction T is also a sufficient condition for the existence of a function ϕ in the unit ball of \mathcal{M}_k such that $\phi(z_n) = w_n$. In particular, this says that M_ϕ^* is an extension of T to \mathcal{H}_k and $\|M_\phi^*\| = \|T\|$, which implies that any two disjoint sets of points Z_1 and Z_2 can be separated by a function in \mathcal{M}_k of norm at most M if and only if the angle between S_{Z_1} and S_{Z_2} in \mathcal{H}_k is bounded below by $1/M$:

$$(1.10) \quad \sup \{ \|\phi\|_{\mathcal{M}_k} \mid \phi|_{Z_1} = 1, \phi|_{Z_2} = 0 \} = \frac{1}{\sin(S_{Z_1}, S_{Z_2})}.$$

Moreover [1, Theorem 9.19], this implies that Z is interpolating if and only if the sequence of lines through the kernels $(k_{z_n})_{n \in \mathbb{N}}$ is a Riesz system.

Since (1.9) being a contraction is equivalent to the infinite matrix

$$(1 - \overline{w_n} w_j) k_{z_j}(z_n) \quad n, j \in \mathbb{N}$$

being positive semi-definite (i.e., all its finite principal minors are positive semi-definite), one can extend the Pick property to the case of *matrix-valued* functions in \mathcal{H}_k , by defining \mathcal{H}_k to have the $s \times t$ Pick property if whenever z_1, \dots, z_N are points in D and W_1, \dots, W_N are $s \times t$ matrices such that

$$(Id - W_n^* W_j) k_{z_j}(z_n) \geq 0,$$

then there exists a multiplier ϕ in the closed unit ball of

$$\mathcal{M}(\mathcal{H}_k \otimes \mathbb{C}^t, \mathcal{H}_k \otimes \mathbb{C}^s) := \left\{ \phi = (\phi_{l,r}) \mid l = 1, \dots, s, r = 1, \dots, t, \sup_{\mathbf{f} \neq 0} \frac{\|\phi \mathbf{f}\|_{\mathcal{H}_k \otimes \mathbb{C}^s}}{\|\mathbf{f}\|_{\mathcal{H}_k \otimes \mathbb{C}^t}} < \infty \right\},$$

such that $\phi(z_i) = W_i$, for $i = 1, \dots, N$. We say that \mathcal{H}_k has the **complete Pick property** if it has the $s \times t$ Pick property for any positive integers s and t . The Hardy space H^2 has the complete Pick property, as well as some of its natural generalizations, such as the reproducing kernel Hilbert spaces \mathcal{H}_s on \mathbb{D} , $-1 \leq s \leq 0$, defined by the kernels

$$k_w^s(z) := \sum_{n=0}^{\infty} (n+1)^s (\overline{w}z)^n \quad z, w \in \mathbb{D}$$

and the *Drury–Arveson space* H_d^2 on the d -dimensional unit ball \mathbb{B}_d defined by the kernel

$$b_w(z) := \frac{1}{1 - \langle z, w \rangle}, \quad z, w \in \mathbb{B}_d.$$

For instance, see [1, Chapter 7]. In a recent work [2], Aleman et al. extended Theorem 1.2 by showing that any weakly separated sequence Z on a domain D such that the sequence of lines through the kernels $(k_{z_n})_{n \in \mathbb{N}}$ is a Bessel system is an interpolating sequence for \mathcal{M}_k , provided that \mathcal{H}_k has the complete Pick property. This is done using the recent positive answer to the *Feichtinger conjecture*, which states that any Bessel system of one-dimensional subspaces is the disjoint union of finitely many Riesz systems. It has been shown in [5, 13] that the Feichtinger conjecture is equivalent to many other conjectures in operator theory, including the Paving conjecture, who had been proved by the well-known work of Marcus et al. [9].

In [6], the author asked whether the positive answer to the Feichtinger conjecture can be extended to multi-dimensional model spaces of H^2 , that is, if any Bessel system of model spaces is the disjoint union of finitely many Riesz systems. We show in Section 3.2 that this is not the case, though any Bessel systems of model spaces satisfying (1.6) is in fact a Riesz system:

Theorem 1.3 *Any weakly separated Bessel system of model spaces in H^2 is a Riesz system.*

The motivation for Theorem 1.3 is the study of *interpolating sequences of matrices* introduced by the author in [6]. As Section 3 explains in details, we say that a sequence of square matrices $A = (A_n)_{n \in \mathbb{N}}$ with eigenvalues in \mathbb{D} is interpolating if, for any bounded sequence $(w_n)_{n \in \mathbb{N}}$ in \mathbb{C} , there exists a bounded holomorphic function ϕ such that

$$\phi(A_n) = w_n \text{Id}.$$

If P_n is, for any n in \mathbb{N} , the minimal polynomial of A_n , then

$$(1.11) \quad B_n(z) := \frac{P_n(z)}{P_n\left(\frac{1}{\bar{z}}\right)} \quad z \in \mathbb{D}$$

is a Blaschke product with zeros at the eigenvalues of A , and any H^2 function that vanishes at A_n is a multiple of B_n . Let $H = (H_n)_{n \in \mathbb{N}}$ be the sequence of model spaces associated to $(B_n)_{n \in \mathbb{N}}$. The author extended in [6, Theorem 6.6] Theorem 1.2 to sequences of matrices of uniformly bounded dimensions, by showing that A is interpolating if and only if H is a weakly separated Bessel system. Theorem 1.3 can be rephrased to drop the extra assumption on the sizes of the matrices in A :

Theorem 1.4 *A is interpolating if and only if the sequence H is a weakly separated Bessel system.*

Section 2 deals with the proof of Theorem 1.3. Section 3 provides a brief summary of the content of [6] and gives an argument for Theorem 1.4, together with an explicit example of an interpolating sequence of matrices. We also give in Section 3.2 an example of a sequence of matrices whose associated sequence of model spaces $(H_n)_{n \in \mathbb{N}}$ is a Bessel system which can not be written as the disjoint union of finitely many Riesz systems, giving a negative answer to a question posed by the author in [6].

2 The Proof of the Main Result

The first main tool for the proof of Theorem 1.3 can be found in [10, Lecture IX], and relates the sine of the angle between two model spaces H_{Θ_1} and H_{Θ_2} with the constant in Carleson corona Theorem:

Theorem 2.1 *There exists a constant $c \geq 1$ such that, for any Θ_1 and Θ_2 inner functions on \mathbb{D} such that*

$$\inf_{z \in \mathbb{D}} \max\{|\Theta_1(z)|, |\Theta_2(z)|\} = \delta \geq 0$$

then

$$\frac{\delta^3}{c} \leq \sin(H_{\Theta_1}, H_{\Theta_2}) \leq c\delta.$$

We are also going to use the following re-statement of the Bessel system condition:

Proposition 2.2 *A sequence $(H_n)_{n \in \mathbb{N}}$ of closed subspaces of a Hilbert space \mathcal{H} is a Bessel system with Bessel bound M if and only if, for any sequence $(h_n)_{n \in \mathbb{N}}$ of unit vectors*

such that h_n belongs to H_n for any n in \mathbb{N} ,

$$(2.1) \quad \sup_{\|x\|=1} \sum_{n \in \mathbb{N}} |\langle x, h_n \rangle|^2 = M^2.$$

Proof. The idea of the proof comes from [1, Proposition 9.5]. Choose for any n in \mathbb{N} a unit vector h_n in H_n , and suppose first that (2.1) holds. Then, for any finitely supported $(a_n)_{n \in \mathbb{N}}$,

$$\begin{aligned} \left\| \sum_{n \in \mathbb{N}} a_n h_n \right\|^2 &= \sup_{\|x\|=1} \left| \langle x, \sum_{n \in \mathbb{N}} a_n h_n \rangle \right|^2 \\ &= \sup_{\|x\|=1} \left| \sum_{n \in \mathbb{N}} \langle x, h_n \rangle \overline{a_n} \right|^2 \\ &\leq M^2 \sum_{n \in \mathbb{N}} |a_n|^2, \end{aligned}$$

thanks to (2.1) and Cauchy–Schwartz’s inequality. The case of infinitely supported sequences $(a_n)_{n \in \mathbb{N}}$ in l^2 follows by considering the limit case of finitely supported sequences.

Conversely, let M be the Bessel bound for the sequence $(H_n)_{n \in \mathbb{N}}$, and fix a unit vector x in \mathcal{H} . Then set $a_n = \langle x, h_n \rangle$, and observe that

$$\begin{aligned} \sum_{n \in \mathbb{N}} |\langle x, h_n \rangle|^2 &= \left\langle x, \sum_{n \in \mathbb{N}} a_n h_n \right\rangle \\ &\leq \left\| \sum_{n \in \mathbb{N}} a_n h_n \right\| \\ &\leq M \left(\sum_{n \in \mathbb{N}} |a_n|^2 \right)^{\frac{1}{2}} \\ &= M \left(\sum_{n \in \mathbb{N}} |\langle x, h_n \rangle|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

By truncating the sum $\sum_{n \in \mathbb{N}} |\langle x, h_n \rangle|^2$, one sees that this implies that $(a_n)_{n \in \mathbb{N}}$ must be square summable, and this concludes the proof. ■

Remark 2.3 Fixed x in \mathcal{H} , we can choose $(h_n)_{n \in \mathbb{N}}$ so that the sum in (2.1) attains its maximum. We can actually maximize each term of the sum, by setting h_n to be the orthogonal projection onto H_n of x , divided by its norm:

$$f_n(x) := \frac{P_{H_n}(x)}{\|P_{H_n}(x)\|}.$$

Proposition 2.2 then says that $(H_n)_{n \in \mathbb{N}}$ has a finite Bessel bound M if and only if

$$(2.2) \quad \sup_{\|x\|=1} \sum_{n \in \mathbb{N}} (1 - \text{dist}_{H^2}^2(x, H_n)) = \sup_{\|x\|=1} \sum_{n \in \mathbb{N}} |\langle x, f_n(x) \rangle|^2 = M^2.$$

Lastly, we are going to use the one-dimensional case of [6, Theorem 5.1]. For any x in a Hilbert space \mathcal{H} let \hat{x} denote its normalization $x/\|x\|$.

Theorem 2.4 For any inner function Θ on \mathbb{D} ,

$$\text{dist}_{\mathbb{H}^2}(\hat{s}_z, H_\Theta) = |\Theta(z)| \quad z \in \mathbb{D}.$$

Proof. Since the orthogonal projection onto $\Theta\mathbb{H}^2 = \mathbb{H}^2 \ominus H_\Theta$ is $M_\Theta M_\Theta^*$ one gets

$$\text{dist}_{\mathbb{H}^2}(\hat{s}_z, H_\Theta) = \|M_\Theta M_\Theta^*(\hat{s}_z)\| = \|M_\Theta^*(\hat{s}_z)\| = |\Theta(z)|. \quad \blacksquare$$

We are now ready to prove Theorem 1.3:

Proof of Theorem 1.3 Let $(H_{\Theta_n})_{n \in \mathbb{N}}$ be a weakly separated Bessel system. We will show that (1.4) holds, and Theorem 1.1 will conclude the proof. Thanks to (2.2), for any fixed z in \mathbb{D} there exists a positive integer n_z that minimizes the distance between \hat{s}_z and H_n :

$$\text{dist}_{\mathbb{H}^2}(\hat{s}_z, H_{n_z}) = \min_{n \in \mathbb{N}} \text{dist}_{\mathbb{H}^2}(\hat{s}_z, H_n),$$

which thanks to Theorem 2.4 becomes

$$|\Theta_{n_z}(z)| = \min_{n \in \mathbb{N}} |\Theta_n(z)|.$$

Therefore, by Theorem 2.1 and weak separation we have that

$$\inf_{z \in \mathbb{D}} \inf_{n \neq n_z} |\Theta_n(z)| > 0,$$

which implies that

$$\prod_{n \neq n_z} |\Theta_n(z)| = \sup_{n \in \mathbb{N}} \prod_{j \neq n} |\Theta_j(z)|$$

is bounded below uniformly on z if and only if

$$\sum_{n \neq n_z} (1 - |\Theta_n(z)|^2) = \sum_{n \neq n_z} (1 - \text{dist}_{\mathbb{H}^2}^2(\hat{s}_z, H_{\Theta_n}))$$

is uniformly bounded on z , which is true thanks to Remark 2.3. ■

Observe that the proof of Theorem 1.3 used a weaker version of the Bessel system condition, in which the sup in (2.1) is taken only on normalized kernel functions, rather than on all unit vectors in \mathbb{H}^2 . It remains open for us whether such a weaker condition is enough to characterize Bessel systems of model spaces, and a positive answer for the special case in which each Θ_n is a Blaschke product would be of great interest for us, as this is the case that we consider when we apply Theorem 1.3 to interpolating sequences of matrices, as we will see in Section 3:

Question 2.5 Is any sequence of model spaces $(H_{\Theta_n})_{n \in \mathbb{N}}$ in \mathbb{H}^2 satisfying

$$(2.3) \quad \sup_{z \in \mathbb{D}} \sum_{n \in \mathbb{N}} 1 - |\Theta_n(z)|^2 < \infty,$$

a Bessel system? Is it true if Θ_n is, for any positive integer n , a Blaschke product?

Remark 2.6 Question 2.5 has a positive answer whenever $\Theta_n = b_{\lambda_n}$ is, for any positive integer n , a degree-one Blaschke factor at a point λ_n , and therefore, whenever H_{Θ_n} is the line spanned by the Szegő kernel at λ_n [8, Chapter VI, Lemma 3.3].

3 Interpolating matrices

The motivation for Theorem 1.3 is the study of *interpolating sequences of matrices*. The author asked in [6] whether some well-known characterizations for interpolating sequences for H^∞ such as Theorem 1.2 extend to sequences of square matrices $A = (A_n)_{n \in \mathbb{N}}$, without assuming any restriction on the sequence of their dimensions. The fact that a square matrix might have a non trivial algebraic structure invariant under holomorphic functions (its eigenspaces, for example), makes an interpolation problem using matrices a bit trickier than the classic one: given two points λ and w in \mathbb{D} there is no function ϕ in H^∞ that maps

$$A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

to

$$W = \begin{bmatrix} w & 1 \\ 0 & w \end{bmatrix},$$

although both A and W are bounded in the operator norm and the constant function w is a contraction in H^∞ that sends the spectrum of A to the spectrum of W . Since choosing bounded targets in the operator norm makes even a one-point interpolation problem impossible to solve via bounded analytic functions, in order to define interpolating matrices one has to identify a target with a bounded sequence in H^∞ [6, Definition 1.1]:

Definition 3.1 (Interpolating Matrices) A sequence $A = (A_n)_{n \in \mathbb{N}}$ of square matrices with spectra in the open unit disc is interpolating for H^∞ if for any bounded sequence $(\phi_n)_{n \in \mathbb{N}}$ in H^∞ there exists a ϕ in H^∞ such that

$$\phi(A_n) = \phi_n(A_n), \quad n \in \mathbb{N}.$$

Equivalently [6, Theorem 4.1], one can choose diagonal targets, and define A to be interpolating if for any bounded sequence $(w_n)_{n \in \mathbb{N}}$ in \mathbb{C} , there exists a bounded analytic function ϕ such that

$$\phi(A_n) = w_n Id, \quad n \in \mathbb{N}.$$

Here, an analytic function on the unit disc is applied to a square matrix via the Riesz-Dunford functional calculus, hence the assumption on the spectra of the matrices in A . In order to characterize interpolating sequences of matrices, a rather trivial yet important observation is that, for any pair of similar matrices M and N with spectra in \mathbb{D} and for any holomorphic function f on the unit disc then $f(M)$ and $f(N)$ are similar as well, and the matrix that performs both similarities is the same,

$$M = P^{-1}NP \implies f(M) = P^{-1}f(N)P,$$

as an elementary computation using the power series of f shows. As a consequence, we can assume without loss of generality that each matrix of the sequence A is in its *Jordan canonical form*: if, for any positive integer n , $\lambda_{n,1}, \dots, \lambda_{n,k_n}$ are the eigenvalues of A_n , then

$$A_n = \text{diag}(J_{n,1}, \dots, J_{n,k_n}),$$

where $J_{n,j}$ is a Jordan block of size $m_{n,j}$

$$J_{n,j} = \begin{bmatrix} \lambda_{n,j} & 1 & 0 & \dots & 0 \\ 0 & \lambda_{n,j} & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \lambda_{n,j} & 1 \\ 0 & 0 & 0 & 0 & \lambda_{n,j} \end{bmatrix}.$$

Since, for any function f holomorphic in \mathbb{D} ,

$$f(A_n) = \text{diag}(f(J_{n,1}), \dots, f(J_{n,k_n})),$$

where

$$f(J_{n,j}) = \begin{bmatrix} f(\lambda_{n,j}) & f'(\lambda_{n,j}) & \frac{f''(\lambda_{n,j})}{2} & \dots & \frac{f^{(m_{n,j}-1)}(\lambda_{n,j})}{(m_{n,j}-1)!} \\ 0 & f(\lambda_{n,j}) & f'(\lambda_{n,j}) & \dots & \frac{f^{(m_{n,j}-2)}(\lambda_{n,j})}{(m_{n,j}-2)!} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & f(\lambda_{n,j}) & f'(\lambda_{n,j}) \\ 0 & 0 & 0 & 0 & f(\lambda_{n,j}) \end{bmatrix},$$

one realizes that a holomorphic function vanishes at A_n if and only if it vanishes at its eigenvalues with the right multiplicity. Specifically, the multiplicity of $\lambda_{n,j}$ as a zero of f must be the maximal size of a Jordan block of A_n associated to $\lambda_{n,j}$. In particular, any function in H^2 that vanishes at A_n is a multiple of the Blaschke product and the function in (1.11) can be re-written as

$$B_n = \prod_{j=1}^{k_n} \theta_{\lambda_{n,j}}^{m_{n,j}}.$$

Hence, for any n in \mathbb{N} , the subspace

$$(3.1) \quad \begin{aligned} H_n &:= H^2 \ominus \{f \in H^2 \mid f(A_n) = 0\} \\ &= \text{span} \left\{ s_{\lambda_{n,j}}, \frac{\partial s_{\lambda_{n,j}}}{\partial \bar{w}}, \dots, \frac{\partial^{m_{n,j}-1} s_{\lambda_{n,j}}}{\partial \bar{w}^{m_{n,j}-1}} \mid j = 1, \dots, k_n \right\} \end{aligned}$$

containing all the interpolation information of the matrix A_n is in fact a model space:

$$H_n = H_{B_n} \quad n \in \mathbb{N}.$$

Each H_n can be seen also as a *kernel at the matrix A_n* . More precisely, let M be a $m \times m$ square matrix with eigenvalues in the unit disc, and let H_M be the associated model space in H^2 . Let us define, for any u and v in \mathbb{C}^m , the H^2 function

$$K_M(u, v)(z) := \sum_{n \in \mathbb{N}} \langle v, M^n u \rangle_{\mathbb{C}^m} z^n \quad z \in \mathbb{D}.$$

Then, thanks to the definition of the inner product in H^2 ,

$$(3.2) \quad \langle f, K_M(u, v) \rangle_{H^2} = \langle f(M)u, v \rangle_{\mathbb{C}^m} \quad f \in H^2.$$

Equation (3.2) works as a *reproducing property* for the collection

$$X_M := \{K_M(u, v) \mid u, v \in \mathbb{C}^m\}.$$

In particular, since $f(M) = 0$ if and only if the right hand side of (3.2) vanishes for any u and v , we have that X_M coincides with the model space H_M . Moreover, (3.2) implies that the collection of function in X_M is linear in v and conjugate-linear in u , and that for any ϕ in H^∞

$$(3.3) \quad M_\phi^*(K_M(u, v)) = K_M(u, \phi(M)^*v) \quad u, v \in \mathbb{C}^m,$$

extending (1.8) to this matrix setting. Another analogy with the scalar case comes from separation: thanks to (3.3) and the commutant lifting Theorem, [1, case $s = 1$ of Corollary 10.30], (1.10) extends by saying that A is **weakly separated** if the sequence $(H_n)_{n \in \mathbb{N}}$ is weakly separated or, equivalently, if there exists a positive M such that, for any pair of distinct positive integers n and j , there exists a bounded analytic function $\phi_{n,j}$ whose H^∞ norm does not exceed M and that separates A_n and A_j , that is,

$$\phi_{n,j}(A_n) = Id \quad \phi_{n,j}(A_j) = 0.$$

Following the same idea, if $(H_n)_{n \in \mathbb{N}}$ is strongly separated we say that A is **strongly separated**, and the commutant lifting Theorem, together with (3.3), makes it equivalent to asserting the existence of a bounded sequence $(\phi_n)_{n \in \mathbb{N}}$ in H^∞ that separates each A_n with the rest of the sequence, i.e.,

$$\phi_n(A_j) = \delta_{n,j} Id.$$

The scalar case has an even more geometric viewpoint on separation via bounded analytic functions: given two points z and w in the unit disc, there exists a function ϕ whose H^∞ norm does not exceed M that separates z and w (and hence the sine of the angle between s_z and s_w is bounded below by $1/M$) if and only if their *pseudo-hyperbolic distance*

$$\rho(z, w) := |b_w(z)|$$

is bounded below by $1/M$. This extends to the matrix case by looking at the action of the adjoint of the multiplication by a Blaschke product on different model spaces:

Lemma 3.2 *Let A_1 and A_2 be two square matrices corresponding to the Blaschke products B_1 and B_2 , and let H_1 and H_2 be the associated model spaces. Then the sine of the angle between H_1 and H_2 is equal to $\delta > 0$ if and only if the restriction of $M_{B_1}^*$ to H_2 is bounded below by δ , that is,*

$$\inf_{x \in H_2} \|M_{B_1}^*(\hat{x})\| = \sin(H_1, H_2).$$

Proof. Since the orthogonal projection onto $B_1H^2 = H^2 \ominus H_1$ is $M_{B_1}M_{B_1}^*$, one gets

$$\sin(H_1, H_2) = \inf_{x \in H_2 \setminus \{0\}} \|M_{B_1}M_{B_1}^*(\hat{x})\| = \inf_{x \in H_2 \setminus \{0\}} \|M_{B_1}^*(\hat{x})\|,$$

since M_{B_1} is an isometry. ■

The author extended in [6] Carleson's characterizations [3, 4] of interpolating sequences to sequences of square matrices, together with the characterization of interpolating sequences in terms of Riesz systems conditions from [12]. Nevertheless, the analogous of Theorem 1.2 was proven with the additional assumption that the dimensions of the matrices in A are uniformly bounded, and used the solution of the Feichtinger conjecture:

Theorem 3.3 *Let $A = (A_n)_{n \in \mathbb{N}}$ be a sequence of matrices with spectra in the unit disc, and let $H = (H_n)_{n \in \mathbb{N}}$ be the associated sequence of model spaces defined in (3.1). The following are equivalent:*

- (i) A is interpolating for H^∞ ;
- (ii) A is strongly separated; and
- (iii) H is a Riesz system.

Moreover, if the dimensions of the matrices in A are uniformly bounded, (i) (and hence also all the conditions above) is equivalent to

- (iv) A is weakly separated and H is a Bessel system.

Thanks to equivalence between conditions (i)–(iii) in Theorem 3.3, Theorem 1.3 applied to the sequence of model spaces $(H_n)_{n \in \mathbb{N}}$ says, together with Theorem 1.1 that the extra assumption on the dimensions of the matrices in A in condition (iv) of Theorem 3.3 can be dropped. Therefore, Theorem 1.2 extends to sequences of matrices of any sizes.

Theorem 3.4 *A is interpolating if and only if it is weakly separated and H is a Bessel system.*

In particular, Theorem 1.4 follows from Theorem 1.3.

We present below two examples of sequences of matrices having equidistributed eigenvalues. Section 3.1 gives an example of an interpolating sequence of matrices, together with some useful tool to estimate the angle between model spaces arising from Blaschke products. Section 3.2 uses a similar construction in order to exhibit a Bessel system of model spaces which is not the disjoint union of finitely many weakly separated sequences, and hence not the finite union of finitely many Riesz systems. This will give a negative answer to a question posed by the author in [6], where it was asked whether the positive answer to the Feichtinger conjecture can be extended to multi-dimensional model subspaces of the Hardy space.

3.1 Interpolating matrices with equidistributed eigenvalues

Let, for any positive integer n ,

$$\omega_n := e^{\frac{2\pi i}{2^n}}$$

be a primitive 2^n -root of unity, and let

$$W_n := \text{diag}(1, \omega_n, \dots, \omega_n^{2^n-1})$$

be a $2^n \times 2^n$ diagonal matrix having 2^n equidistributed points on the unit circle as its eigenvalues. Let $(r_n)_{n \in \mathbb{N}}$ be a sequence in $(0, 1)$ that rescales the sequence $(W_n)_{n \in \mathbb{N}}$ so that its spectra belong to \mathbb{D} :

$$(3.4) \quad A_n := r_n W_n, \quad n \in \mathbb{N}.$$

We discuss here how fast must $(r_n)_{n \in \mathbb{N}}$ go to 1 in order for $A := (A_n)_{n \in \mathbb{N}}$ to be interpolating. Thanks to Definition 3.1, if $(A_n)_{n \in \mathbb{N}}$ is interpolating, then so is the sequence of radii $(r_n)_{n \in \mathbb{N}}$. Moreover, if A is interpolating it is trivially a *zero sequence*, that is, there exists bounded analytic function on \mathbb{D} that vanishes on A and that does not vanish outside the spectra of the matrices in A . It turns out that those two conditions are enough to characterize interpolating sequences of matrices that look like (3.4):

Theorem 3.5 *The sequence of matrices defined in (3.4) is interpolating if and only if it is a zero sequence and $(r_n)_{n \in \mathbb{N}}$ is interpolating.*

Remark 3.6 Since $(r_n)_{n \in \mathbb{N}}$ approaches the unit circle radially, asking that it is interpolating is actually the same as asking that it is just weakly separated [10, Lecture X, Corollary 5].

The proof of Theorem 3.5 requires that we are able to estimate (from below) the angle between two model spaces arising from Blaschke products. Such a tool is the content of Lemma 3.7. Let $(B_n)_{n \in \mathbb{N}}$ be a sequence of Blaschke products such that $B = \prod_{n \in \mathbb{N}} B_n$ converges uniformly on any compact subset of \mathbb{D} to a nonzero inner function, and let $(H_n)_{n \in \mathbb{N}}$ be the associated sequence of model spaces in H^2 . For any subset σ of \mathbb{N} , we will define, for the sake of brevity,

$$H_\sigma := \overline{\text{span}}_{i \in \sigma} \{H_i\}$$

and

$$B_\sigma = \prod_{i \in \sigma} B_i.$$

Lemma 3.7 *Let σ and τ be two disjoint subsets of \mathbb{N} , and suppose that $(H_i)_{i \in \sigma}$ is a Riesz system with Riesz bound γ . Then*

$$\sin(H_\sigma, H_\tau) \geq \frac{1}{\gamma^2} \inf_{i \in \sigma} \sin(H_i, H_\tau).$$

Proof. For any i in σ let $\delta_i := \sin(H_i, H_\tau)$, and let $\delta := \inf_{i \in \sigma} \delta_i$. It suffices to show that

$$T: H_{\sigma \cup \tau} \rightarrow H_{\sigma \cup \tau}$$

such that

$$T|_{H_\sigma} = \delta \text{Id}|_{H_\sigma} \quad T|_{H_\tau} = 0$$

is bounded by γ^2 . Let T_i be, for any i in σ , the restriction to H_i of $M_{B_\tau}^*$. Thanks to Lemma 3.2, each T_i is bounded below by δ_i . Fix then a vector $x = u + v$ in $H_{\sigma \cup \tau}$, where u is in H_σ and v is in H_τ . There exists a sequence $(h_i)_{i \in \sigma}$ of unit vectors such that h_i is in H_i for any i in σ so that u can be written as a linear combination of $(h_i)_{i \in \sigma}$:

$$u = \sum_{i \in \sigma} \alpha_i h_i.$$

Since each T_i is a contraction and it is bounded below by δ , the sequence $(T_i(h_i))_{i \in \mathbb{N}}$ is bounded above and below. Moreover, $T_i(h_i)$ belongs to H_i , since each H_i is invariant under M_{B_τ} , and therefore,

$$\begin{aligned} \|T(x)\|^2 &= \|T(u)\|^2 = \left\| \sum_{i \in \sigma} \delta \alpha_i h_i \right\|^2 \\ &\leq \gamma^2 \delta^2 \sum_{i \in \sigma} |\alpha_i|^2 \leq \gamma^2 \sum_{i \in \sigma} \delta_i^2 |\alpha_i|^2 \\ &\leq \gamma^2 \sum_{i \in \sigma} |\alpha_i|^2 \|T_i(h_i)\|^2 \leq \gamma^4 \left\| \sum_{i \in \sigma} \alpha_i T_i(h_i) \right\|^2 \\ &= \gamma^4 \|M_{B_\tau}^*(u)\|^2 = \gamma^4 \|M_{B_\tau}^*(x)\|^2. \end{aligned}$$

Since $M_{B_\tau}^*$ is a contraction, this shows that the norm of T does not exceed γ^2 , as we claimed. ■

Remark 3.8 Suppose that also $(H_j)_{j \in \tau}$ is a Riesz system. Then a double application of Lemma 3.7 implies that the distance between H_σ and H_τ is comparable with the minimal distance attained by a model space labeled by an index in σ and one with a label in τ . If the sequence $(H_n)_{n \in \mathbb{N}}$ is weakly separated, this says roughly speaking that the set of *sparse subsequences* of $(H_n)_{n \in \mathbb{N}}$ is a separated set as well.

We are now ready to prove Theorem 3.5. Here, $(B_n)_{n \in \mathbb{N}}$ and $(H_n)_{n \in \mathbb{N}}$ arise from the sequence of matrices A defined in (3.4). We need to show that A is interpolating, provided that it is a zero sequence and that $(r_n)_{n \in \mathbb{N}}$ is weakly separated, thanks to Remark 3.6.

Proof of Theorem 3.5 Let, for any positive integer n ,

$$(3.5) \quad r_n = 1 - \alpha_n 2^{-n}.$$

Since A is a zero sequence, then the spectra of the matrices in A form a zero sequence and therefore

$$\sum_{n \in \mathbb{N}} \alpha_n < \infty.$$

Let γ_n be the Riesz bound of the basis $\{\hat{s}_{r_n}, \dots, \hat{s}_{r_n \omega_n^{2^n-1}}\}$ of H_n . Then $(\gamma_n)_{n \in \mathbb{N}}$ is uniformly bounded if and only if the strong separation constants

$$\prod_{l=1}^{2^n-1} |b_{r_n \omega_n^l}(r_n)|^2$$

are uniformly bounded below. Let, for any j and n in \mathbb{N} ,

$$\begin{aligned} M_j(n) &:= \sum_{l=1}^{2^j} |\langle \hat{s}_{r_j \omega_j^l}, \hat{s}_{r_n} \rangle|^2 \\ (3.6) \quad &= \sum_{l=1}^{2^j} \frac{(1-r_n^2)(1-r_j^2)}{|1-r_n r_j \omega_j^l|^2}. \end{aligned}$$

Since $(\alpha_n)_{n \in \mathbb{N}}$ is bounded, then the weak separation constant of the set $\{r_n, \dots, r_n \omega_n^{2^n-1}\}$ is uniformly bounded below in n . Since also

$$|b_w(z)|^2 = 1 - |\langle \hat{s}_w, \hat{s}_z \rangle|^2 \quad w, z \in \mathbb{D},$$

then $(\gamma_n)_{n \in \mathbb{N}}$ is bounded if and only if $(M_n(n))_{n \in \mathbb{N}}$ is, which is the case, thanks to Lemma 3.9, because $(\alpha_n)_{n \in \mathbb{N}}$ is bounded. We want to show that $(H_n)_{n \in \mathbb{N}}$ is strongly separated:

$$\inf_{n \in \mathbb{N}} \sin \left(H_n, \overline{\text{span}}_{j \neq n} \{H_j\} \right) > 0.$$

Since $(\gamma_n)_{n \in \mathbb{N}}$ is bounded, thanks to Lemma 3.7 the sine between H_n and $\overline{\text{span}}_{j \neq n} \{H_j\}$ is comparable to

$$\sin \left(\hat{s}_{r_n}, \overline{\text{span}}_{j \neq n} \{H_j\} \right) = \prod_{j \neq n} |B_j(r_n)|.$$

In fact, the sine between H_n and $\overline{\text{span}}_{j \neq n} \{H_j\}$ is comparable to

$$\min_{l=0, \dots, 2^n-1} \sin \left(s_{r_n \omega_n^l}, \overline{\text{span}}_{j \neq n} \{H_j\} \right) = \min_{l=0, \dots, 2^n-1} |B_j(r_n \omega_n^l)|.$$

Nevertheless, the minimum is attained at $l = 0$, since

$$B_j(z) = \frac{r_j^{2^j} - z^{2^j}}{1 - r_j^{2^j} z^{2^j}} = b_{r_j^{2^j}}(z^{2^j})$$

and the point on the circle of radius r_n which is closest (with respect the pseudo-hyperbolic distance) to $r_j^{2^j}$ is precisely r_n .

We are then left to show that

$$(3.7) \quad \inf_{n \in \mathbb{N}} \prod_{j \neq n} |B_j(r_n)| > 0.$$

This is true if and only if each term on the product is uniformly bounded below and

$$\sup_{n \in \mathbb{N}} \sum_{j \in \mathbb{N}} M_j(n) < \infty.$$

Such sum converges thanks to Lemma 3.9, since $(\alpha_n)_{n \in \mathbb{N}}$ is summable. Moreover, each term on the product in 3.7 is uniformly bounded below, since $(r_n)_{n \in \mathbb{N}}$ is weakly separated and since, thanks to Lemma 3.7, the pseudo-hyperbolic distance between r_n and r_j is, for any $n \neq j$, comparable to $\sin(\hat{s}_{r_n}, H_j) = |B_j(r_n)|$. ■

A technical tool for the proof of Theorem 3.5 is the following computation, which relates the quantity $M_j(n)$ to the parameters $(\alpha_n)_{n \in \mathbb{N}}$ defined in (3.5).

Lemma 3.9 *Let*

$$r_n := 1 - \alpha_n 2^{-n} \quad n \in \mathbb{N}$$

be a sequence in $(0, 1)$ and let $M_j(n)$ be defined as in (3.6). Then, for any j and n positive integers,

$$M_j(n) \simeq \frac{\alpha_n \alpha_j}{\alpha_n + \alpha_j 2^{n-j} - \alpha_n \alpha_j 2^{-j}}.$$

Proof. Let, for any j in \mathbb{N} and for any $l = -2^{j-1}, \dots, 2^{j-1}$,

$$\theta_l^j := \arg \omega_j^l = \frac{2\pi l}{2^j}.$$

Then

$$1 - \cos(\theta_l^j) \simeq \frac{(\theta_l^j)^2}{2} \simeq \frac{l^2}{2^{2j+1}} \quad l = -2^{j-1}, \dots, 2^{j-1}$$

and therefore,

$$\begin{aligned} M_j(n) &\simeq (1 - r_j)(1 - r_n) \sum_{l=-2^{j-1}}^{2^{j-1}} \frac{1}{|1 - r_n r_j \omega_j^l|^2} \\ &= (1 - r_j)(1 - r_n) \sum_{l=-2^{j-1}}^{2^{j-1}} \frac{1}{(1 - r_n r_j)^2 + 2((1 - \cos(\theta_l^j)))r_n r_j} \\ &\simeq \frac{(1 - r_j)(1 - r_n)}{(1 - r_j r_n)^2} \sum_{l=1}^{2^{j-1}} \frac{1}{1 + \left(\frac{\sqrt{r_n r_j}}{2^j(1 - r_n r_j)} l\right)^2} \\ &\simeq \frac{(1 - r_j)(1 - r_n)}{(1 - r_j r_n)^2} \int_1^{2^{j-1}} \frac{1}{1 + \left(\frac{\sqrt{r_n r_j}}{2^j(1 - r_n r_j)} x\right)^2} dx \end{aligned}$$

$$\begin{aligned}
 &= \frac{2^j(1-r_j)(1-r_n)}{(1-r_jr_n)\sqrt{r_jr_n}} \int_{\frac{\sqrt{r_jr_n}}{2^j(1-r_jr_n)}}^{\frac{\sqrt{r_jr_n}}{(1-r_jr_n)}} \frac{1}{1+x^2} dx \\
 &\approx \frac{2^j(1-r_j)(1-r_n)}{1-r_jr_n} \\
 &= \frac{\alpha_n\alpha_j}{\alpha_n + \alpha_j2^{n-j} - \alpha_n\alpha_j2^{-j}}.
 \end{aligned}$$

■

3.2 Bessel systems of model spaces

Thanks to the positive answer to the Feichtinger conjecture, any Bessel system of lines in a Hilbert space is the disjoint union of finitely many Riesz systems. We show here that this is not the case for multi-dimensional model spaces in H^2 , as we will construct a sequence of matrices A which cannot be written as the disjoint union of finitely many weakly separated sequences and whose associated sequence of model spaces is a Bessel system. This implies that [1, Theorem 9.11] does not extend to multi-dimensional model spaces, and since any Riesz system is weakly separated it will show that the positive answer to the Feichtinger conjecture does not extend to multi-dimensional model spaces whose dimensions are not uniformly bounded.

Pick a divergent sequence $(m_n)_{n \in \mathbb{N}}$, and let $(r_n)_{n \in \mathbb{N}}$ be a sequence of radii that converges to 1, and whose rate of convergence will be determined later. For any such a sequence, there exists a second sequence of radii $(s_n)_{n \in \mathbb{N}}$ such that

$$(3.8) \quad \rho(r_n, s_n) \leq \frac{1}{n+1} \quad n \in \mathbb{N}.$$

For each $n \in \mathbb{N}$, consider m_n diagonal matrices, $A_{n,1}, \dots, A_{n,m_n}$, each one of size $m_n - 1$, and whose eigenvalues are placed as follows: if ω_n is a $\binom{m_n}{2}$ th primitive root of unity, the set of all eigenvalues of $A_{n,1}, \dots, A_{n,m_n}$ is

$$(3.9) \quad \left\{ r_n, r_n\omega_n, \dots, r_n\omega_n^{\binom{m_n}{2}-1} \right\} \cup \left\{ s_n, s_n\omega_n, \dots, s_n\omega_n^{\binom{m_n}{2}-1} \right\}.$$

We now need to assign each of those $m_n(m_n - 1)$ eigenvalues to the matrices $A_{n,1}, \dots, A_{n,m_n}$ so that

$$(3.10) \quad \sin(H_{A_{n,i}}, H_{A_{n,j}}) \leq \frac{1}{n+1} \quad i, j = 1, \dots, m_n,$$

where $H_{n,i}$ is the model space in H^2 associated to the matrix $A_{n,i}$. Observe that this would imply that the sequence

$$A := \{A_{n,j} \mid n \in \mathbb{N}, j = 1, \dots, m_n\}$$

cannot be written as the disjoint union of finitely many weakly separated subsequences, since $(m_n)_{n \in \mathbb{N}}$ is divergent. More importantly, this would hold for any choice of $(r_n)_{n \in \mathbb{N}}$. In order to achieve (3.10), we assign the eigenvalues in (3.9) to the matrices $A_{n,1}, \dots, A_{n,m_n}$ so that the set of nonordered pairs

$$\left\{ (r_n\omega_n^l, s_n\omega_n^l) \mid l = 1, \dots, \binom{m_n}{2} \right\}$$

coincide with the set of non-ordered pairs

$$\{(\lambda, \gamma) \mid \lambda \in \sigma(A_{n,i}), \gamma \in \sigma(A_{n,j}), i \neq j\}.$$

Thus, (3.10) follows from the fact that, for any $i \neq j = 1, \dots, m_n$, there exists an eigenvalue λ of $A_{n,i}$ and an eigenvalue γ of $A_{n,j}$ so that

$$\rho(\lambda, \gamma) \leq \frac{1}{n + 1},$$

thanks to (3.8).

Example 3.10 In order to have an explicit example in mind, let $m_1 = 4$, and let then ω_1 be a primitive sixth root of unity. One can then choose

$$\begin{aligned} r_1 \in \sigma(A_{1,1}) & \quad s_1 \in \sigma(A_{1,2}), \\ r_1 \omega \in \sigma(A_{1,1}) & \quad s_1 \omega \in \sigma(A_{1,3}), \\ r_1 \omega^2 \in \sigma(A_{1,1}) & \quad s_1 \omega^2 \in \sigma(A_{1,4}), \\ r_1 \omega^3 \in \sigma(A_{1,2}) & \quad s_1 \omega^3 \in \sigma(A_{1,3}), \\ r_1 \omega^4 \in \sigma(A_{1,2}) & \quad s_1 \omega^4 \in \sigma(A_{1,4}), \\ r_1 \omega^5 \in \sigma(A_{1,3}) & \quad s_1 \omega^5 \in \sigma(A_{1,4}). \end{aligned}$$

In general, the idea is to assign, for all n , the $m_n(m_n - 1)$ points on the two very close circle of radius r_n and s_n in a way so that the $\binom{m_n}{2}$ pairs of points very close to each other correspond to the set of pairs $\{(A_{n,i}, A_{n,j})\}$.

What is left now to show is that, if $(r_n)_{n \in \mathbb{N}}$ is chosen to be converging to 1 adequately fast, the sequence of model spaces associated with the sequence A is a Bessel system. Let $A' = (z_j)_{j \in \mathbb{N}}$ be the (scalar) sequence of all the eigenvalues of the matrices in A . Observe that we can recursively choose the sequence $(r_n)_{n \in \mathbb{N}}$ to approach 1 fast enough so that

$$(3.11) \quad \sup_{z \in \mathbb{D}} \sum_{j \in \mathbb{N}} 1 - |b_{z_j}(z)|^2 < \infty.$$

Indeed, one can use the same idea of [6, Theorem 2.2] and add the contribution of the eigenvalues in (3.9) once at a time in the sum in (3.11). More precisely, let A'_n be the set in (3.9), and let

$$Q_n(z) := \sum_{j=1}^n \sum_{z_l \in A'_j} 1 - |b_{z_l}(z)|^2.$$

Then, once we choose r_1, \dots, r_n , there exists a radius t_n such that

$$\sup_{t_n \leq |z| < 1} Q_n(z) \leq \frac{1}{2^n},$$

and it suffices then to choose r_{n+1} so that

$$\sum_{z_l \in A'_{n+1}} 1 - |b_{z_l}(z)|^2 \leq \frac{1}{2^n} \quad |z| \leq t_n$$

and so that the Riesz bound of the set of normalized kernels at the points in $A'_{n+1} \cap \{|z| = r_{n+1}\}$ is uniformly bounded (in fact, since m_{n+1} is fixed one can make such Riesz bound arbitrarily close to 1, by choosing r_{n+1} close enough to 1).

Thanks to (3.11) and Remark 2.6 the sequence of lines spanned by the Szegő kernels at the point of A' is a Bessel system. This, together with an extra separation condition on the eigenvalues of the matrices in A , implies that the model spaces associated with the sequence A forms a Bessel system:

Lemma 3.11 *Let $(H_n)_{n \in \mathbb{N}}$ be a sequence of closed subspaces of a Hilbert space \mathcal{H} , and let, for any n in \mathbb{N} , $\{x_n^1, \dots, x_n^{m_n}\}$ be a basis of H_n made of unit vectors such that*

$$(3.12) \quad \sum_{l=1}^{m_n} |c_l|^2 \leq C_n^2 \left\| \sum_{l=1}^{m_n} c_l x_n^l \right\|^2, \quad c_1, \dots, c_{m_n} \in \mathbb{C}.$$

If

$$(3.13) \quad C := \sup_{n \in \mathbb{N}} C_n < \infty$$

and (x_n^l) is a Bessel system with bound M , then $(H_n)_{n \in \mathbb{N}}$ is a Bessel system with bound CM .

Proof. Let $(h_n)_{n \in \mathbb{N}}$ be a sequence of unit vectors in \mathcal{H} such that h_n belongs to H_n for any n in \mathbb{N} , and let $(a_n)_{n \in \mathbb{N}}$ be an l^2 sequence. Write $h_n = \sum_{l=1}^{m_n} b_n^l x_n^l$, and observe that

$$\begin{aligned} \left\| \sum_{n \in \mathbb{N}} a_n h_n \right\|^2 &= \left\| \sum_{n \in \mathbb{N}} \sum_{l=1}^{m_n} a_n b_n^l x_n^l \right\|^2 \\ &\leq M^2 \sum_{n \in \mathbb{N}} |a_n|^2 \sum_{l=1}^{m_n} |b_n^l|^2 \\ &\leq C^2 M^2 \sum_{n \in \mathbb{N}} |a_n|^2, \end{aligned}$$

thanks to (3.12) and (3.13). ■

Since the sequence $(m_n)_{n \in \mathbb{N}}$ is fixed, by eventually increasing the rate of convergence of $(r_n)_{n \in \mathbb{N}}$ to 1, we can assume that the Riesz bound of the kernels basis of each H_{A_n} is uniformly bounded in n , thus in particular condition (3.12) holds. Indeed, although the eigenvalues of a given matrix A_n might belong to two circle very close to each other, our construction ensures that their arguments are separated by at least an angle of $\frac{2\pi}{\binom{m_n}{2}}$. Therefore, thanks to Lemma 3.11, the sequence of model spaces associated to the sequence A is a Bessel system.

Acknowledgments The author would like to thank John Mc Carthy for the valuable suggestions given during all the conversations that led to this work. The author is also indebted to the reviewer for some extremely important remarks and suggestions.

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