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Abelian Surfaces with an Automorphism and Quaternionic Multiplication

Matteo Alfonso Bonfanti and Bert van Geemen

Abstract. We construct one-dimensional families of Abelian surfaces with quaternionic multiplication, which also have an automorphism of order three or four. Using Barth's description of the moduli space of $(2,4)$-polarized Abelian surfaces, we find the Shimura curve parametrizing these Abelian surfaces in a specific case. We explicitly relate these surfaces to the Jacobians of genus two curves studied by Hashimoto and Murabayashi. We also describe a (Humbert) surface in Barth's moduli space that parametrizes Abelian surfaces with real multiplication by $\mathbf{Z}[\sqrt{2}]$.

Introduction

The Abelian surfaces, with a polarization of a fixed type, whose endomorphism ring is an order in a quaternion algebra are parametrized by a curve, called a Shimura curve, in the moduli space of polarized Abelian surfaces. There have been several attempts to find concrete examples of such Shimura curves and of the family of Abelian surfaces over this curve. In [HM], Hashimoto and Murabayashi find two Shimura curves as the intersection, in the moduli space of principally polarized Abelian surfaces, of two Humbert surfaces. Such Humbert surfaces are now known "explicitly" in many other cases (see [BW]), and this might allow one to find explicit models of other Shimura curves. Another approach was taken by Elkies in [E] who characterizes elliptic fibrations on the Kummer surfaces of such Abelian surfaces. See [PS] for yet another approach.

In this paper we consider the rather special case where one of the Abelian surfaces in the family is the selfproduct of an elliptic curve. Moreover, we assume this elliptic curve to have an automorphism (fixing the origin) of order three or four. It is then easy to show that, for a fixed product polarization of type $(1,d)$, the deformations of the selfproduct with the automorphism are parametrized by a Shimura curve. In fact, an Abelian surface with such an automorphism must have a Néron–Severi group of rank at least three, and we show that this implies that the endomorphism algebra of such a surface is in general a quaternion algebra. One can then work out for which d the quaternion algebra is actually a skew field (rather than a matrix algebra). The cases for $d \leq 20$ are listed in Section 1.5.

The remainder of this paper is devoted to the case of an automorphism of order three and a polarization of type $(1,2)$. In that case the general endomorphism ring is a maximal order $\mathcal{O}_6$ of the quaternion algebra of discriminant 6. Barth, in [B], provides

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a description of a moduli space $M_{2,4}$, embedded in $\mathbf{P}^5$, of $(2,4)$-polarized Abelian surfaces with a level structure. Since the polarized Abelian surfaces we consider have an automorphism of order three, the corresponding points in $M_{2,4}$ are fixed by an automorphism of order three of $\mathbf{P}^5$. This allows us to explicitly identify the Shimura curve in $M_{2,4}$ that parametrizes the Abelian surfaces with quaternionic multiplication by the maximal order $\mathcal{O}_6$ in the quaternion algebra with discriminant 6. It is embedded as a line, which we denote by $\mathbf{P}^1_{QM}$, in $M_{2,4} \subset \mathbf{P}^5$. The symmetric group S_4 acts on this line by changing the level structures.

According to Rotger [R], an Abelian surface with endomorphism ring $\mathcal{O}_6$ has a unique principal polarization, which is in general defined by a genus two curve in that surface. We show explicitly how to find such genus two curves, or rather their images in the Kummer surface embedded in $\mathbf{P}^5$ with a $(2,4)$-polarization. These curves were already considered by Hashimoto and Murabayashi in [HM]. We give the explicit relation between the two descriptions in Proposition 4.2. As a byproduct, we find a (Humbert) surface in $M_{2,4}$ that parametrizes Abelian surfaces with $\mathbf{Z}[\sqrt{2}]$ in the endomorphism ring.

In a series of papers (*cf.* [GP1, GP2]), Gross and Popescu studied, both in general and for several small d in particular, explicit maps from moduli spaces of $(1,d)$-polarized Abelian surfaces to projective spaces. The methods we used to find the Shimura curve in $M_{2,4}$ can, in principle, be extended also to these cases.

1 Polarized Abelian Surfaces with Automorphisms

1.1 Abelian Surfaces with a $(1,d)$-polarization

We recall the basic results on moduli spaces of Abelian surfaces with a $(1,d)$-polarization, following [HKW, Chapter 1]. Such an Abelian surface A is isomorphic to $\mathbf{C}^2/\Lambda$, where the lattice Λ can be obtained as the image of $\mathbf{Z}^4$ under the map given by the period matrix Ω, where we consider all vectors as row vectors:

$$A \cong \mathbf{C}^2\Lambda, \qquad \Lambda = \mathbf{Z}^4\Omega, \quad \Omega\colon \mathbf{Z}^4 \longrightarrow \mathbf{C}^2,$$

$$x \longmapsto x\Omega = x\begin{pmatrix} \tau \\ \Delta_d \end{pmatrix} = x\begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \\ 1 & 0 \\ 0 & d \end{pmatrix},$$

where τ is a symmetric complex 2×2 matrix with positive definite imaginary part, so $\tau \in \mathbf{H}_2$, the Siegel space of degree two, and Δ_d is a diagonal matrix with entries $1, d$. The polarization on A is defined by the Chern class of an ample line bundle in $H^2(A,\mathbf{Z}) \cong \wedge^2 H^1(A,\mathbf{Z}) = \wedge^2 \operatorname{Hom}(\Lambda,\mathbf{Z})$, that is, by an alternating map $E_d\colon \Lambda \times \Lambda \to \mathbf{Z}$, which is the one defined by the alternating matrix with the same name (so $E_d(x,y) = xE_d{}^t y$):

$$E_d := \begin{pmatrix} 0 & \Delta_d \\ -\Delta_d & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & d \\ -1 & 0 & 0 & 0 \\ 0 & -d & 0 & 0 \end{pmatrix}.$$

1.2 Products of Elliptic Curves

The selfproduct of an elliptic curve with an automorphism of order three and four respectively provides, for any integer $d > 0$, a $(1,d)$-polarized Abelian surface with an automorphism of the same order whose eigenvalue on $H^{2,0}$ is equal to one.

To see this, let $\zeta_j := e^{2\pi i/j}$ be a primitive j-th root of unity. For $j = 3, 4$, let E_j be the following elliptic curve with an automorphism $f_j \in \mathrm{End}(E_j)$ of order j:

$$E_j := \mathbf{C}/\mathbf{Z} + \mathbf{Z}\zeta_j, \qquad f_j\colon E_j \longrightarrow E_j, \quad z \longmapsto \zeta_j z.$$

Then the Abelian surface $A_j := E_j^2$ has the automorphism

$$\phi_j := f_j \times f_j^{-1} : A_j := E_j \times E_j \longrightarrow A_j.$$

As f_j^* acts as multiplication by ζ_j on $H^{1,0}(E_j) = \mathbf{C}dz$, the eigenvalues of ϕ_j^* on $H^{1,0}(A_j)$ are ζ_j, ζ_j^{-1}. Thus ϕ_j^* acts as the identity on $H^{2,0}(A_j) = \wedge^2 H^{1,0}(A_j)$.

The principal polarization on E_j is fixed by f_j, so the product of this polarization on the first factor with d-times the principal polarization on the second factor is a $(1,d)$-polarization on A_j that is invariant under ϕ_j.

The lattice $\Lambda_j \subset \mathbf{C}^2$ defining A_j is given by the image of the period matrix Ω_j:

$$A_j \cong \mathbf{C}^2/\Lambda_j, \qquad \Lambda_j = \mathbf{Z}^4\Omega_j, \qquad \Omega_j := \begin{pmatrix} \zeta_j & 0 \\ 0 & d\zeta_j \\ 1 & 0 \\ 0 & d \end{pmatrix}.$$

The automorphism ϕ_j determines, and is determined by, the matrices $\rho_r(\phi_j)$ and $\rho_a(\phi_j)$, which give the action of ϕ_j on Λ_j and $\mathbf{C}^2$, respectively. Here we have

$$\rho_r(\phi_j)\Omega_j = \Omega_j\rho_a(\phi_j), \qquad \rho_r(\phi_j) = M_j, \quad \rho_a(\phi_j) = \begin{pmatrix} \zeta_j & 0 \\ 0 & \zeta_j^{-1} \end{pmatrix},$$

where the matrix M_j is given by:

$$M_3 := \begin{pmatrix} -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 \end{pmatrix}, \qquad M_4 := \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

The $(1,d)$-polarization is defined by the alternating matrix E_d from Section 1.1 and is indeed preserved by ϕ_j (so $\phi_j^* E_d = E_d$), since $M_j E_d {}^tM_j = E_d$.

1.3 Deformations of $(A_j, E_{1,d}, \phi_j)$

For a matrix $M \in M_4(\mathbf{R})$ such that $ME_d {}^tM = E_d$ we define

$$M *_d \tau := (A\tau + B\Delta_d)(C\tau + D\Delta_d)^{-1}\Delta_d, \qquad \text{where} \quad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

The fixed point set of M_j for the $*_d$-action on $\mathbf{H}_2$ is denoted by

$$\mathbf{H}_{j,d} := \{\tau \in \mathbf{H}_2 : M_j *_d \tau = \tau\}.$$

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The following proposition shows that the $(1,d)$-polarized Abelian surfaces that are deformations of (A_j,ϕ_j) form a one parameter family that is parametrized by $\mathbf{H}_{j,d}$. We will see in Theorem 1.2 and Table 1.5 that for certain combinations of j and d the general surface in this family is simple and has quaternionic multiplication.

Proposition 1.1 *The $(1,d)$-polarized Abelian surface $(A_{\tau,d} = \mathbf{C}^2/(\mathbf{Z}^4\Omega_\tau), E_d)$, with $\tau \in \mathbf{H}_2$, admits an automorphism ϕ_j induced by M_j if and only if $\tau \in \mathbf{H}_{j,d}$. Moreover, $\mathbf{H}_{j,d}$ is biholomorphic to $\mathbf{H}_1$, the Siegel space of degree one.*

Proof The Abelian surface $A_{\tau,d} = \mathbf{C}^2/(\mathbf{Z}^4\Omega_\tau)$ admits an automorphism induced by M_j if there is a 2×2 complex matrix N_τ such that

$$M_j\Omega_\tau = \Omega_\tau N_\tau, \qquad \Omega_\tau := \begin{pmatrix} \tau \\ \Delta_d \end{pmatrix}.$$

Writing M_j as a block matrix with rows A, B and C, D, the equation $M_j\Omega_\tau = \Omega_\tau N_\tau$ is equivalent to the two equations

$$A\tau + B\Delta_d = \tau N_\Omega, \qquad C\tau + D\Delta_d = \Delta_d N_\tau,$$

hence $N_\tau = \Delta_d^{-1}(C\tau + D\Delta_d)$ and substituting this in the first equation we get:

$$(A\tau + B\Delta_d)(C\tau + D\Delta_d)^{-1}\Delta_d = \tau, \qquad \text{hence} \quad M_j *_d \tau = \tau.$$

Conversely, if $M_j *_d \tau = \tau$, then define $N_\tau := \Delta_d^{-1}(C\tau + D\Delta_d)$, and one finds that $M_j\Omega_\tau = \Omega_\tau N_\tau$.

The fact that this fixed point set is a copy of $\mathbf{H}_1$ in $\mathbf{H}_2$ follows easily from [F, Hilfsatz III, 5.12, p. 196]. ∎

1.4 Polarizations and Automorphisms

Recall that for a complex torus $A = \mathbf{C}^g/\Lambda$ we can identify $\mathbf{C}^g = \Lambda_\mathbf{R} := \Lambda \otimes_\mathbf{Z} \mathbf{R}$. The scalar multiplication by $i = \sqrt{-1}$ on $\mathbf{C}^g$ induces an $\mathbf{R}$-linear map J on $\Lambda_\mathbf{R}$ with $J^2 = -1$. An endomorphism of A corresponds to a $\mathbf{C}$-linear map M on $\mathbf{C}^g$ such that $M\Lambda \subset \Lambda$, equivalently, after choosing a $\mathbf{Z}$-basis for Λ:

$$\mathrm{End}(A) = \{M \in M_{2g}(\mathbf{Z}) : JM = MJ\},$$

where $M_{2g}(\mathbf{Z})$ is the algebra of $2g \times 2g$ matrices with integer coefficients.

The Néron–Severi group of A, a subgroup of

$$H^2(A,\mathbf{Z}) = \wedge^2 H^1(A,\mathbf{Z}) = \wedge^2 \mathrm{Hom}(\Lambda,\mathbf{Z}),$$

can be described similarly:

$$\mathrm{NS}(A) := \{F \in M_{2g}(\mathbf{Z}) : {}^tF = -F, \quad JF^tJ = F\},$$

where the alternating matrix $F \in \mathrm{NS}(A)$ defines the bilinear form $(x,y) \mapsto xF^ty$. Moreover, F is a polarization, *i.e.*, the first Chern class of an ample line bundle, if F^tJ is a positive definite matrix. In particular, F is then invertible (in $M_{2g}(\mathbf{Q})$).

It is now elementary to verify that if $E, F \in \mathrm{NS}(A)$ and E is invertible in $M_{2g}(\mathbf{Q})$, then $FE^{-1} \in \mathrm{End}(A)_\mathbf{Q}$ (*cf.* [BL, Proposition 5.2.1a] for an intrinsic description). This result will be used in the proof of Theorem 1.2.

In Theorem 1.2 we show that if $\tau \in \mathbf{H}_{j,d}$ then the Abelian surface $\mathrm{End}(A_{\tau,d})_{\mathbf{Q}}$ contains a quaternion algebra (and not just the field $\mathbf{Q}(\zeta_j)$!). This is of course well known (see, for example, [BL, Exercise 4, Section 9.4]), but we can also determine this quaternion algebra explicitly. It allows us to find infinitely many families of $(1,d)$-polarized Abelian surfaces whose generic member is simple and whose endomorphism ring is an (explicitly determined) order in a quaternion algebra. To find the endomorphisms, we first study the Néron–Severi group. Notice that in the proof of Theorem 1.2 we do not need to know the period matrices of the deformations explicitly.

Theorem 1.2 *Let $j \in \{3,4\}$ and let $\tau \in \mathbf{H}_{j,d}$, so that the Abelian surface $A_{\tau,d}$ has an automorphism ϕ_j induced by M_j (see Proposition 1.1).*

Then the endomorphism algebra of $A_{\tau,d}$ also contains an element ψ_j with $\psi_j^2 = d$. Moreover, for a general $\tau \in \mathbf{H}_{j,d}$ one has

$$\mathrm{End}(A_{\tau,d}) = \mathbf{Z}[\phi_j,\psi_j], \qquad \mathrm{End}(A_{\tau,d})_{\mathbf{Q}} \cong \tfrac{(-j,d)}{\mathbf{Q}},$$

where $\frac{(a,b)}{\mathbf{Q}} := \mathbf{Q}1 \oplus \mathbf{Q}\mathbf{i} \oplus \mathbf{Q}\mathbf{j} \oplus \mathbf{Q}\mathbf{i}\mathbf{j}$ is the quaternion algebra with $\mathbf{i}^2 = a$, $\mathbf{j}^2 = b$, and $\mathbf{i}\mathbf{j} = -\mathbf{j}\mathbf{i}$.

Proof The Néron–Severi group of an Abelian surface A can also be described as

$$\mathrm{NS}(A) \xrightarrow{\cong} H^2(A,\mathbf{Z}) \cap H^{1,1}(A) \xrightarrow{\cong} \{\omega \in H^2(A,\mathbf{Z}) : (\omega, \omega_A^{2,0}) = 0\},$$

where $(\cdot,\cdot)$ denotes the $\mathbf{C}$-linear extension to $H^2(A,\mathbf{C})$ of the intersection form on $H^2(A,\mathbf{Z})$ and we fixed a holomorphic 2-form on A so that $H^{2,0}(A) = \mathbf{C}\omega_A^{2,0}$.

The intersection form is invariant under automorphisms of A, so $(\phi_j^* x, \phi_j^* y) = (x,y)$ for all $x,y \in H^2(A,\mathbf{Z})$, where $A = A_{\tau,d}$. Moreover, by construction of ϕ_j, we have that $\phi_j^*\omega_A^{2,0} = \omega_A^{2,0}$, so $\omega_A^{2,0} \in H^2(A,\mathbf{C})^{\phi_j^*}$, the subspace of ϕ_j-invariant classes. Therefore any integral class which is orthogonal to the ϕ_j-invariant classes is in particular orthogonal to $\omega_A^{2,0}$ and thus must be in $\mathrm{NS}(A)$:

$$\begin{aligned}\left(H^2(A,\mathbf{Z})^{\phi_j^*}\right)^{\perp} := \big\{\, \omega \in H^2(A,\mathbf{Z}) : (\omega,\theta) = 0,& \\ \text{for all } \theta \in H^2(A,\mathbf{Z}) \text{ with } \phi_j^*\theta = \theta \big\} &\subset \mathrm{NS}(A).\end{aligned}$$

The eigenvalues of ϕ_j^* on $H^1(A,\mathbf{C}) = H^{1,0}(A) \oplus \overline{H^{1,0}(A)}$ are ζ_j and ζ_j^{-1}, both with multiplicity two. Thus the eigenvalues of ϕ^* on $H^2(A,\mathbf{C}) = \wedge^2 H^1(A,\mathbf{C})$ are ζ_j^2, ζ_j^{-2}, with multiplicity one, and 1 with multiplicity 4. In particular, $(H^2(A,\mathbf{Z})^{\phi_j^*})^{\perp}$ is a free $\mathbf{Z}$-module of rank 2, it is the kernel in $H^2(A,\mathbf{Z})$ of $(\phi_3^*)^2 + \phi_3^* + 1$ in case $j = 3$ and of $(\phi_4^*)^2 + 1$ in case $j = 4$. Identifying $H^2(A,\mathbf{Z})$ with the alternating bilinear $\mathbf{Z}$-valued maps on $\Lambda_j \cong \mathbf{Z}^4$, the action of ϕ^* is given by $M_j \cdot F := M_j F^t M_j$, where F is an alternating 4×4 matrix with integral coefficients. It is now easy to find a basis $E_{j,1}, E_{j,2}$ of the $\mathbf{Z}$-module $(H^2(A,\mathbf{Z})^{\phi_j^*})^{\perp}$. Since E_d defines a polarization on A, the matrices $E_d^{-1}E_{j,k}$, $k = 1,2$, are the images under ρ_r of elements in $\mathrm{End}(A)_{\mathbf{Q}}$ (*cf.* [BL, Proposition 5.2.1a]). In this way we found that for any $\tau \in \mathbf{H}_{j,d}$, the Abelian surface $A = A_{\tau,d}$ has an endomorphism ψ_j defined by the matrix $\rho_r(\psi_j)$ below:

$$\rho_r(\psi_3) = \begin{pmatrix} 0 & d & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & d \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad \rho_r(\psi_4) = \begin{pmatrix} 0 & 0 & 0 & -d \\ 0 & 0 & 1 & 0 \\ 0 & d & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

It is easy to check that $\rho_r(\psi_j)^2 = d$ and that $M_4\rho_r(\psi_4) = -\rho_r(\psi_4)M_4$, whereas $(1 + 2M_3)\rho_r(\psi_3) = -\rho_r(\psi_3)(1 + 2M_3)$ (and notice that $(1 + 2M_3)^2 = -3$). Therefore, $(-j, d)/\mathbf{Q} \subset \mathrm{End}(A)_{\mathbf{Q}}$ (in fact, $M_4^2 = -1$, but $(-1, d)/\mathbf{Q} \cong (-4, d)/\mathbf{Q}$). As $(-j, d)/\mathbf{Q}$ is a (totally) indefinite quaternion algebra (so of type II), for general $\tau \in \mathbf{H}_{j,d}$ the Abelian surface $A = A_{\tau,d}$ has $(-j, d)/\mathbf{Q} = \mathrm{End}(A)_{\mathbf{Q}}$ by [BL, Theorem 9.9.1]. Therefore, if $\phi \in \mathrm{End}(A)$, then $\rho_r(\phi)$ is both a matrix with integer coefficients and a linear combination of I, $M_j = \rho_r(\phi_j)$, $\rho_r(\psi_j)$ and $M_j\rho_r(\psi_j)$ with rational coefficients. It is then easy to check that $\mathrm{End}(A)$ is as stated in Theorem 1.2. ∎

1.5 A Table

Using Magma [M], we found that for the following $d \le 20$, the quaternion algebras $(-1, d)/\mathbf{Q}$ and $(-3, d)/\mathbf{Q}$ are skew fields:

d	discriminant $\frac{(-1,d)}{\mathbf{Q}}$
3, 6, 15	6
7, 14	14
11	22
19	38

,

d	discriminant $\frac{(-3,d)}{\mathbf{Q}}$
2, 6, 8, 14, 18	6
5, 15, 20	15
10	10
11	33
17	51

.

Moreover, for $d \le 20$, $\mathrm{End}(A)$ is never a maximal order in $(-1, d)/\mathbf{Q}$, and it is a maximal order in $(-3, d)/\mathbf{Q}$ if and only if $d = 2, 5, 11, 17$.

In particular, for $\tau \in \mathbf{H}_{3,2}$ the Abelian surface $A_{\tau,2}$ has a $(1, 2)$-polarization invariant by an automorphism of order three induced by M_3 and $\mathrm{End}(A_{\tau,2}) = \mathcal{O}_6$, the maximal order in the quaternion algebra with discriminant 6, for general $\tau \in \mathbf{H}_{3,2}$. After a discussion of an equivariant map $\overline{\psi}_D$ of a moduli space of Abelian surfaces to a projective space, we will describe the image of $\mathbf{H}_{3,2}$ in Section 3.

2 The Level Moduli Space

2.1 The Moduli Space of $(1, d)$-polarized Abelian Surfaces

The integral symplectic group with respect to E_d is defined as

$$\widetilde{\Gamma}_d^0 := \{M \in GL(4, \mathbf{Z}) : ME_d{}^tM = E_d\}.$$

This group acts on the Siegel space by [HKW, Equation (1.4)]:

$$\widetilde{\Gamma}_d^0 \times \mathbf{H}_2 \longrightarrow \mathbf{H}_2, \qquad \begin{pmatrix} A & B \\ C & D \end{pmatrix} *_d \tau := (A\tau + B\Delta_d)(C\tau + D\Delta_d)^{-1}\Delta_d.$$

Notice that for $d = 1$ one finds the standard action of the symplectic group on $\mathbf{H}_2$. The quotient space (in general a singular quasi-projective 3-dimensional algebraic

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variety) is the moduli space $\mathcal{A}_d^0$ of pairs (A, H), where A is an Abelian surface and H is a polarization of type $(1, d)$ (see [HKW, Theorem 1.10(i)]).

For the study of this moduli space, and of certain "level" covers of it, we use the standard action of $\mathrm{Sp}(4, \mathbf{R})$ on $\mathbf{H}_2$, which is $*_1$. For this, as in the proof of Proposition 1.1 (*cf.* [HKW, p.11]), we use the 4×4 matrix R_d. Then $\Gamma_{1,d}^0 := R_d^{-1}\widetilde{\Gamma}_d^0 R_d \in \mathrm{Sp}(4, \mathbf{R})$ is a subgroup of the (standard) real symplectic group of the (standard) alternating form E_1, and we have $(R_d^{-1}MR_d) *_1 \tau = M *_d \tau$ for all $M \in \widetilde{\Gamma}_d^0$. Therefore,

$$\mathcal{A}_d^0 := \widetilde{\Gamma}_d^0 \backslash \mathbf{H}_2 \cong \Gamma_{1,d}^0 \backslash \mathbf{H}_2,$$

where the actions are $*_d$ and $*_1$, respectively.

2.2 Congruence Subgroups

We now follow [BL] for the definition of coverings of the moduli space and maps to projective space. Recall that we defined a group $\widetilde{\Gamma}_d^0$ in Section 2.1 of matrices with integral coefficients that preserve the alternating form E_d. We will actually be interested in the form $2E_2$, which is preserved by the same group. With the notation from [BL, 8.1, p. 212] we thus have

$$\widetilde{\Gamma}_2^0 = \Gamma_D = \mathrm{Sp}_4^D(\mathbf{Z}), \qquad D = \mathrm{diag}(2, 4) = 2\Delta_2.$$

It is easy to check that

$$\mathbf{Z}^4 \widetilde{D}^{-1} = \{x \in \mathbf{Q}^4 : x(2E_2)y \in \mathbf{Z}, \forall y \in \mathbf{Z}^4\}, \qquad \widetilde{D} := \begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix}.$$

Let $T(2, 4)$ be the following quotient of $\mathbf{Z}^4$:

$$T(2, 4) = (\mathbf{Z}^4 \widetilde{D}^{-1})/\mathbf{Z}^4 \cong (\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/4\mathbf{Z})^2,$$

The group Γ_D acts on this quotient and we define

$$\Gamma_D(D) := \ker(\Gamma_D \longrightarrow \mathrm{Aut}(T(2, 4))).$$

One verifies easily that

$$\begin{aligned} \Gamma_D(D) &= \{M \in \Gamma_D : \widetilde{D}^{-1}M \equiv \widetilde{D}^{-1} \mod M_4(\mathbf{Z})\} \\ &= \left\{ M = \begin{pmatrix} I + D\alpha & D\beta \\ D\gamma & I + D\delta \end{pmatrix} \in \Gamma_D : \alpha, \beta, \gamma, \delta \in M_2(\mathbf{Z}) \right\}. \end{aligned}$$

This shows that $\Gamma_D(D)$ is the subgroup as defined in [BL, Section 8.3] (see also [BL, Section 8.8]). The alternating form E_2 defines a "symplectic" form $\langle \cdot, \cdot \rangle$ on $T(2, 4)$ with values in the fourth-roots of unity (*cf.* [B, Section 3.1]). For this we write (*cf.* [B, Section 2.1])

$$T(2, 4) = K \times \widehat{K}, \qquad K = \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/4\mathbf{Z}, \quad \widehat{K} = \mathrm{Hom}(K, \mathbf{C}^*) \cong \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/4\mathbf{Z},$$

and the symplectic form is

$$\langle \cdot, \cdot \rangle : T(2, 4) \times T(2, 4) \longrightarrow \mathbf{C}^*, \qquad \big\langle (\sigma, l), (\sigma', l') \big\rangle := l'(\sigma) l(\sigma')^{-1}.$$

We denote by $\mathrm{Sp}(T(2, 4))$ the subgroup of $\mathrm{Aut}(T(2, 4))$ of automorphisms that preserve this form.

Lemma 2.1 *The reduction homomorphism $\Gamma_D \to \mathrm{Sp}(T(2,4))$ is surjective. Hence $\Gamma_D/\Gamma_D(D) \cong \mathrm{Sp}(T(2,4))$, this is a finite group of order $2^9 3^2$.*

Proof As the symplectic form is induced by E_2, we have $\mathrm{im}(\Gamma_D) \subset \mathrm{Sp}(T(2,4))$. In [B, Proposition 3.1] generators ϕ_i, $i = 1,\dots,5$ of $\mathrm{Sp}(T(2,4))$ are given. It is easy to check that the following matrices are in G_D and induce these automorphisms on $T(2,4)$:

$$\begin{pmatrix} 1&0&0&0\\0&0&0&-1\\0&0&1&0\\0&1&0&0 \end{pmatrix}, \quad \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&1&0&1 \end{pmatrix}, \quad \begin{pmatrix} 0&0&-1&0\\0&1&0&0\\1&0&0&0\\0&0&0&1 \end{pmatrix}, \quad \begin{pmatrix} 1&0&0&0\\0&1&0&0\\1&0&1&0\\0&0&0&1 \end{pmatrix}, \quad \begin{pmatrix} 1&0&0&0\\-2&1&0&0\\0&0&1&1\\0&0&0&1 \end{pmatrix}.$$

The order of $\mathrm{Sp}(T(2,4))$ is determined in [B, Proposition 3.1]. ∎

2.3 The Subgroup $\Gamma_D(D)_0$

We define a normal subgroup of $\Gamma_D(D)$ by:

$$\Gamma_D(D)_0 := \ker(\phi\colon \Gamma_D(D) \longrightarrow (\mathbf{Z}/2\mathbf{Z})^4), \qquad \phi(M) = (\beta_0, \gamma_0) := (\beta_{11}, \beta_{22}, \gamma_{11}, \gamma_{22}),$$

where $M \in \Gamma_D(D)$ is as above. Since D has even coefficients, $D = 2\,\mathrm{diag}(1,2)$, it is easy to check that ϕ is a homomorphism. Moreover, ϕ is surjective, since the matrix with $\alpha = \gamma = \delta = 0$ and $\beta = \mathrm{diag}(a,b)$ $(a,b \in \mathbf{Z})$ is in $\Gamma_D(D)$ and maps to $(a,b,0,0)$; similarly, the matrix with $\alpha = \beta = \delta = 0$ and $\gamma = \mathrm{diag}(a,b)$ is also in $\Gamma_D(D)$ and maps to $(0,0,a,b)$. It follows that $\Gamma_D(D)/\Gamma_D(D)_0 \cong (\mathbf{Z}/2\mathbf{Z})^4$.

The groups Γ_D, $\Gamma_D(D)$ and $\Gamma_D(D)_0$ are denoted by $G_{\mathbf{Z}}$, $G_{\mathbf{Z}}(e)$ and $G_{\mathbf{Z}}(e, 2e)$ in [I2, V.2, p. 177]. In [I2, V.2 Lemma 4] one finds that $\Gamma_D(D)_0$ is in fact a normal subgroup of Γ_D. There is an exact sequence of groups:

$$0 \longrightarrow \Gamma_D(D)/\Gamma_D(D)_0 \longrightarrow \Gamma_D/\Gamma_D(D)_0 \longrightarrow \Gamma_D/\Gamma_D(D) \longrightarrow 0.$$

The group Γ_D act on $\mathbf{H}_2$ in a natural way, but to get the standard action $*_1$ one must conjugate these groups by a matrix R_D with diagonal blocks I, D, and one obtains the groups

$$G_D = R_D^{-1}\Gamma_D R_D, \qquad G_D(D) = R_D^{-1}\Gamma_D(D) R_D, \qquad G_D(D)_0 = R_D^{-1}\Gamma_D(D)_0 R_D;$$

see [BL, Sections 8.8, 8.9].

The main result from [BL, section 8.9] is Lemma 8.9.2, which asserts that the holomorphic map given by theta-null values

$$\psi_D\colon \mathbf{H}_2 \longrightarrow \mathbf{P}^7, \qquad \tau \longrightarrow \Big(\dots : \vartheta\big[\begin{smallmatrix} l \\ 0 \end{smallmatrix}\big](0,\tau) : \cdots\Big)_{l\in K},$$

where l runs over $K = D^{-1}\mathbf{Z}^2/\mathbf{Z}^2$ and where the theta functions $\vartheta\big[\begin{smallmatrix} l \\ 0 \end{smallmatrix}\big](v,\tau)$ are defined in [BL, 8.5, Formula (1)], factors over a holomorphic map

$$\overline{\psi}_D\colon \mathcal{A}_D(D)_0 := \mathbf{H}_2/\Gamma_D(D)_0 \cong \mathbf{H}_2/G_D(D)_0 \longrightarrow \mathbf{P}^7.$$

2.4 Group Actions

The finite group $\Gamma_D/\Gamma_D(D)_0$ acts on $\mathcal{A}_D(D)_0$. The Heisenberg group $\mathcal{H}(D)$, a non-Abelian central extension of $T(2,4)$ by $\mathbf{C}^*$, acts on $\mathbf{P}^7$ ([BL, Section 6.6]). This action is induced by an irreducible representation (called the Schrödinger representation) of

$\mathcal{H}(D)$ on the vector space $V(2,4)$ of complex valued functions on the subgroup K of $T(2,4)$ ([BL, Section 6.7])

$$\rho_D: \mathcal{H}(D) \longrightarrow GL(V(2,4)).$$

In [B, Section 2.1]) the action of generators of $\mathcal{H}(D)$ on $\mathbf{P}V(2,4) = \mathbf{P}^7$ are given explicitly. The linear map $\widetilde{\iota} \in GL(V(2,4))$ that sends the delta functions $\delta_l \mapsto \delta_{-l}$ ($l \in K$) is also introduced there (*cf.* Sections 3.1, 3.2).

The normalizer of the Heisenberg group (in the Schrödinger representation) is, by definition, the group

$$N(\mathcal{H}(D)) := \{\gamma \in \mathrm{Aut}(\mathbf{P}V(2,4)) : \gamma\rho_D(\mathcal{H}(D))\gamma^{-1} \subset \rho_D(\mathcal{H}(D))\}.$$

The group $N(\mathcal{H}(D))$ maps onto $\mathrm{Sp}(T(2,4))$ with kernel isomorphic to $T(2,4)$. The elements in this kernel are obtained as interior automorphisms: $\gamma = \rho_D(h)$, for some $h \in \mathcal{H}(D)$. Explicit generators of $N(\mathcal{H}(D))$ are given in [B, Table 8] (but there seem to be some misprints in the action of the generators on $\mathcal{H}(D)$ in the lower left corner of that table). Let $N(\mathcal{H}(D))_2$ be the subgroup of $N(\mathcal{H}(D))$ of elements that commute with $\widetilde{\iota}$. The group $N(\mathcal{H}(D))_2$ is an extension of $\mathrm{Sp}(T(2,4))$ by the 2-torsion subgroup (isomorphic to $(\mathbf{Z}/2\mathbf{Z})^4$) of $T(2,4)$ and $\sharp N(\mathcal{H}(D))_2 = 2^{13}3^2$.

We need the following result.

Proposition 2.2 *There is an isomorphism* $\gamma : G_D/G_D(D)_0 \cong N(\mathcal{H}(D))_2$, $M' \mapsto \gamma_{M'}$ *such that the map* $\overline{\psi}_D$ *is equivariant for the action of these groups. So if we denote by* $\widetilde{\gamma}$ *the composition*

$$\widetilde{\gamma}: \Gamma_D/\Gamma_D(D)_0 \xrightarrow{\cong} G_D/G_D(D)_0 \xrightarrow{\gamma} N(\mathcal{H}(D)),$$

then $\overline{\psi}_D(M * \tau) = \widetilde{\gamma}_M \overline{\psi_D}(\tau)$, *where* $*$ *denotes the action of* $\Gamma(D)$ *on* $\mathbf{H}_2$.

Proof Let $\mathcal{L}_\tau = L(H, \chi_0)$ be the line bundle on $A_{\tau,2} := \mathbf{C}^2/(\mathbf{Z}^4\Omega_\tau)$ that has Hermitian form H with $E_2 = \mathrm{Im}H$ (so it defines a polarization of type $(1,2)$) and the quasi-character χ_0 is as in [BL, 3.1, Formula (3)] for the decomposition $\Lambda = \mathbf{Z}^2\tau \oplus \mathbf{Z}^2\Delta_2$. According to [BL, Remark 8.5.3d], the theta functions $\vartheta[{l \atop 0}](v,\tau)$ are a basis of the vector space of classical theta functions for the line bundle $\mathcal{L}_\tau^{\otimes 2}$. As χ_0 takes values in $\{\pm 1\}$ one has $\mathcal{L}_\tau^{\otimes 2} = L(2H, \chi_0^2 = 1)$, so it is the unique line bundle with first Chern class $2E_2$ and trivial quasi-character. Thus if $M \in G_D$ and $\tau' = M *_1 \tau$, then $\phi_M^*\mathcal{L}_\tau^{\otimes 2} \cong \mathcal{L}_{\tau'}^{\otimes 2}$, where $\phi_M: A_{\tau',2} \to A_{\tau,2}$ is the isomorphism defined by M. Notice that $\mathcal{L}_\tau$ and $\mathcal{L}_\tau^{\otimes 2}$ are symmetric line bundles ([BL, Corollary 2.3.7]).

Let $\mathcal{G}(\mathcal{L}_\tau^{\otimes 2})$ be the theta group ([BL, Section 6.1]); it has an irreducible linear representation $\widetilde{\rho}$ on $H^0(A_{\tau,2}, \mathcal{L}_\tau^{\otimes 2})$ ([BL, Section 6.4]).

A theta structure $b: \mathcal{G}(\mathcal{L}_\tau^{\otimes 2}) \to \mathcal{H}(D)$ is an isomorphism of groups that is the identity on their subgroups $\mathbf{C}^*$. A theta structure b defines an isomorphism β_b, unique up to scalar multiple ([BL, Section 6.7]), which intertwines the actions of $\mathcal{G}(\mathcal{L}^{\otimes 2})$ and $\mathcal{H}(D)$:

$$\beta_b: H^0(A_{\tau,2}, \mathcal{L}_\tau^{\otimes 2}) \longrightarrow V(2,4), \qquad \beta_b\widetilde{\rho}(g) = \rho_D(b(g))\beta_b \quad (\forall g \in \mathcal{G}(\mathcal{L}_\tau^{\otimes 2})).$$

A symmetric theta structure ([BL, Section 6.9]) is a theta structure that is compatible with the action of $(-1) \in \mathrm{End}(A_{\tau,2})$ on the symmetric line bundle $\mathcal{L}_\tau^{\otimes 2}$ and the map $\widetilde{\iota} \in GL(V(2,4))$ defined in [B, Section 2.1].

For $\tau \in \mathbf{H}_2$, define an isomorphism $\beta_\tau: H^0(A_{\tau,2}, \mathcal{L}_\tau^{\otimes 2}) \to V(2,4)$ by sending the basis vectors $\vartheta[{}^l_0](v,\tau)$ to the delta functions δ_l for $l \in K$. From the explicit transformation formulas for the theta functions under translations by points in $A_{\tau,2}$, one finds that for $g \in \mathcal{G}(\mathcal{L}_\tau^{\otimes 2})$ the map $\beta_\tau \widetilde{\rho}(g)\beta_\tau^{-1}$ acts as an element, which we denote by $b_\tau(g)$, of the Heisenberg group $\mathcal{H}(D)$ acting on $V(2,4)$. This map $b = b_\tau: \mathcal{G}(\mathcal{L}_\tau^{\otimes 2}) \to \mathcal{H}(D)$ is a theta structure and $\beta_\tau \widetilde{\rho}(g) = \rho_D(b_\tau(g))\beta_\tau$; moreover, it is symmetric, since $\theta[{}^l_0](-v,\tau) = \theta[{}^{-l}_0](v,\tau)$.

For $M \in G_D$ and $\tau' = M *_1 \tau$ we have an isomorphism $\beta_{\tau'}$ and the composition $\gamma_M := \beta_{\tau'}\phi_M^*\beta_\tau^{-1} \in GL(V(2,4))$ is an element of $N(\mathcal{H})$, since ϕ_M^* induces an isomorphism $\mathcal{G}(\mathcal{L}_\tau^{\otimes 2}) \to \mathcal{G}(\mathcal{L}_{\tau'}^{\otimes 2})$. In fact $\gamma_M \in N(\mathcal{H})_2$, since the theta structures $\beta_\tau, \beta_{\tau'}$ are symmetric and ϕ_M commutes with (-1) on the abelian varieties.

From [BL, Proposition 6.9.4] it follows that the group generated by the γ_M is contained in an extension of $\mathrm{Sp}(T(2,4))$ by $(\mathbf{Z}/2\mathbf{Z})^4$. The map $M \mapsto \gamma_M \in \mathrm{Aut}(\mathbf{P}(V(2,4))$ is thus a (projective) representation of G_D whose image is contained in $N(\mathcal{H})_2$ and which, by construction, is equivariant for $\overline{\psi}_D$. Unwinding the various definitions, we have shown that γ_M maps the point $(\dots : \theta[{}^l_0](v,\tau) : \cdots)$ to the point $(\dots : \theta[{}^l_0]({}^t(C\tau + D)v, M *_1 \tau) : \cdots)$, where M has block form $A, \dots, D$. From the classical theory of transformations of theta functions (as in [BL, Section 8.6]) one now deduces that $M \mapsto \gamma_M$ provides the desired isomorphism of groups. Notice that the element $-I \in G_D$, which acts trivially on $\mathbf{H}_2$, maps to $\widetilde{\iota} \in N(\mathcal{H}(D))_2$, which acts trivially on the subspace $\mathbf{P}^5 \subset \mathbf{P}^7$ of even theta functions. ∎

3 A Projective Model of a Shimura Curve

3.1 Barth's Variety $M_{2,4}$

We choose projective coordinates $x_1, \dots, x_8$ on $\mathbf{P}^7 = \mathbf{P}V(2,4)$ as in [B, §2.1]. The map $\widetilde{\iota} \in \mathrm{Aut}(\mathbf{P}^7)$ is then given by

$$\widetilde{\iota}(x) = (x_1 : x_2 : x_3 : x_4 : x_5 : x_6 : -x_7 : -x_8).$$

It has two eigenspaces that correspond to the even and odd theta functions. The image of $\overline{\psi}_D$ lies in the subspace $\mathbf{P}^5 = \mathbf{P}V(2,4)_+$ of even functions that is defined by $x_7 = x_8 = 0$. We use $x_1, \dots, x_6$ as coordinates on this $\mathbf{P}^5$. Let

$$f_1 := -x_1^2x_2^2 + x_3^2x_4^2 + x_5^2x_6^2, \qquad f_2 := -(x_1^4 + x_2^4) + x_3^4 + x_4^4 + x_5^4 + x_6^4.$$

Then Barth's variety of theta-null values is defined as ([B, (3.9)])

$$M_{2,4} := \{x \in \mathbf{P}^5 : f_1(x) = f_2(x) = 0\}.$$

The image of $\overline{\psi}_D(\mathbf{H}_2)$ is a quasi-projective variety, and the closure of its image is $M_{2,4}$.

3.2 The Heisenberg Group Action

Recall that $T(2,4) = \mathbf{Z}^4\widetilde{D}^{-1}/\mathbf{Z}^4$ and let $\sigma_1, \sigma_2, \tau_1, \tau_2 \in T(2,4)$ be the images of $e_1/2, e_2/4, e_3/2, e_4/4$. We denote certain lifts of the generators $\sigma_1, \dots, \tau_2$ of $T(2,4)$ to $\mathcal{H}(D)$ by $\widetilde{\sigma}_1, \dots, \widetilde{\tau}_2$. These lifts act, in the Schrödinger representation, on $\mathbf{P}^7 =$

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$\mathbf{P}V(2,4)$ as follows (see [B, Table 1]):

$$\begin{aligned}\widetilde{\sigma}_1(x) &= (x_2: \ x_1: \ x_4: \ x_3: x_6: \ x_5: \ x_8: \ x_7),\\ \widetilde{\sigma}_2(x) &= (x_3: \ x_4: \ x_1: \ x_2: x_7: \ x_8: -x_5: -x_6),\\ \widetilde{\tau}_1(x) &= (x_1: -x_2: \ x_3: -x_4: x_5: -x_6: \ x_7: -x_8),\\ \widetilde{\tau}_2(x) &= (x_5: \ x_6: ix_7: \ ix_8: x_1: \ x_2: \ ix_3: \ ix_4),\end{aligned}$$

where $x = (x_1 : \ldots : x_8) \in \mathbf{P}^7$ and $i^2 = -1$. For any $g = (a, b, c, d) \in T(2,4)$ one then finds the action of a lift $\widetilde{g}$ of g by defining $\widetilde{g} := \widetilde{\sigma}_1^a \cdots \widetilde{\tau}_2^d$.

Proposition 3.1 *Let $\widetilde{\mu}_3$ on $\mathbf{P}^7$ be the projective transformation defined as*

$$\widetilde{\mu}_3 : x \longmapsto$$
$$(x_3 - ix_4 : x_3 + ix_4 : \zeta x_5 - \zeta^3 x_6 : \zeta x_5 + \zeta^3 x_6 : x_1 - ix_2 : x_1 + ix_2 : \zeta^3 x_7 + \zeta x_8 : \zeta^3 x_7 - \zeta x_8),$$

where ζ is a primitive 8-th root of unity (so $\zeta^4 = -1$) and $i := \zeta^2$. Then $\widetilde{\mu}_3 \in N(\mathcal{H}(D))_2$ and with M_3 as in Section 1.2 we have $\widetilde{\gamma}_{M_3} = \widetilde{h}\widetilde{\mu}_3\widetilde{h}^{-1}$ for some $\widetilde{h} \in \ker(N(\mathcal{H}(D))_2 \to \mathrm{Sp}(T(2,4))$.

Proof The map $M_3 : \mathbf{Z}^4 \to \mathbf{Z}^4$ from Section 1.2 induces the (symplectic) automorphism $\overline{M}_3$ of $T(2,4)$ given by (recall that we used row vectors, so for example $e_4 M_3 = -e_2 - e_4$ and thus $\tau_2 \mapsto -\sigma_2 - \tau_2$)

$$\sigma_1 \longmapsto -\sigma_1 - \tau_1, \quad \sigma_2 \longmapsto \tau_2, \quad \tau_1 \longmapsto \sigma_1, \quad \tau_2 \longmapsto -\sigma_2 - \tau_2.$$

Now one verifies that, as maps on $\mathbf{C}^8$, one has

$$\widetilde{\mu}_3\widetilde{\sigma}_1\widetilde{\mu}_3^{-1} = i\widetilde{\sigma}_1^{-1}\widetilde{\tau}_1^{-1}, \quad \widetilde{\mu}_3\widetilde{\sigma}_2\widetilde{\mu}_3^{-1} = \widetilde{\tau}_2, \quad \widetilde{\mu}_3\widetilde{\tau}_1\widetilde{\mu}_3^{-1} = \widetilde{\sigma}_1, \quad \widetilde{\mu}_3\widetilde{\tau}_2\widetilde{\mu}_3^{-1} = \zeta\widetilde{\sigma}_2^{-1}\widetilde{\tau}_2^{-1}.$$

Hence, $\widetilde{\mu}_3 \in \mathrm{Aut}(\mathbf{P}^7)$ is in the normalizer $N(\mathcal{H})$ and it is a lift of $\overline{M}_3 \in \mathrm{Sp}(T(2,4))$. One easily verifies that it commutes with the action of $\widetilde{\iota}$ on $\mathbf{P}^7$, so $\widetilde{\mu}_3 \in N(\mathcal{H})_2$. Any other lift of $\overline{M}_3$ to $\mathrm{Aut}(\mathbf{P}^7)$ that commutes with $\widetilde{\iota}$ is of the form $\widetilde{g}\widetilde{\mu}_3$ for some $g \in T(2,4)$ with $2g = 0$. Since $\overline{M}_3^2 + \overline{M}_3 + I = 0$, the map $h \mapsto (\overline{M}_3 + I)h$ is an isomorphism on the two-torsion points in $T(2,4)$. Thus there is an $h \in T(2,4)$, with $2h = 0$, such that $g = (\overline{M}_3 + I)h$. As $\widetilde{\mu}_3\widetilde{h}\widetilde{\mu}_3^{-1} = \widetilde{k}$, where $k = \overline{M}_3 h$ and thus $k = g + h$, it follows that $\widetilde{h}\widetilde{\mu}_3\widetilde{h}^{-1} = \widetilde{g}\widetilde{\mu}_3$. ∎

3.3 Fixed Points and Eigenspaces

The map $\overline{\psi}_D$ is equivariant for the actions of Γ_D and $N(\mathcal{H})_2$. Hence the fixed points of M_3 in $\mathbf{H}_2$, which parametrize abelian surfaces with quaternionic multiplication, map to the fixed points of $\widetilde{\gamma}_{M_3} = \widetilde{h}\widetilde{\mu}_3\widetilde{h}^{-1}$ in $\mathbf{P}^7$. Conjugating M_3 by an element $N \in \Gamma_D$ such that $\widetilde{\gamma}_N = \widetilde{h}$ (as in Proposition 3.1), we obtain an element of order three $M_3' \in \Gamma_D$ whose fixed point locus $\mathbf{H}_2^{M_3'}$ also consists of period matrices of Abelian surfaces with QM by $\mathcal{O}_6$ and the image $\overline{\psi}_D(\mathbf{H}_2^{M_3'})$ consists of fixed points of $\widetilde{\mu}_3$. The following lemma identifies this fixed point set.

Theorem 3.2 *Let $\mathbf{P}^1_{QM} \subset \mathbf{P}^5$ be the projective line parametrized by*

$$\mathbf{P}^1 \xrightarrow{\cong} \mathbf{P}^1_{QM}, \qquad (x:y) \longmapsto p_{(x:y)} := (\sqrt{2}x : \sqrt{2}y : x+y : i(x-y) : x-iy : x+iy).$$

Then $\mathbf{P}^1_{QM} \subset M_{2,4}$ is a Shimura curve that parametrizes Abelian surfaces with QM by $\mathcal{O}_6$, the maximal order in the quaternion algebra of discriminant 6.

The following two elements $\widetilde{\nu}_1, \widetilde{\nu}_2 \in N(\mathcal{H}(D))_2$,

$$\widetilde{\nu}_1(x) = (x_5+x_6, -x_5+x_6, \zeta(x_3-x_4), \zeta(x_3+x_4), x_1+x_2, x_1-x_2, \zeta(-x_7+x_8), \zeta(x_7+x_8))$$

$$\widetilde{\nu}_2(x) = (x_4, -x_3, \zeta^3 x_6, \zeta^3 x_5, ix_1, -ix_2, \zeta^3 x_7, \zeta^3 x_8),$$

restrict to maps in $\mathrm{Aut}(\mathbf{P}^1_{QM})$ which generate a subgroup isomorphic to the symmetric group $S_4 \subset \mathrm{Aut}(\mathbf{P}^1_{QM})$.

Proof The subspace $\mathbf{P}^5$ is mapped into itself by $\widetilde{\mu}_3$. The restriction μ_3 of $\widetilde{\mu}_3$ to $\mathbf{P}^5$ has three eigenspaces on $\mathbf{C}^6$, each 2-dimensional. The eigenspace of μ_3 with eigenvalue $\sqrt{2} := \zeta + \zeta^7$ is the only eigenspace whose projectivization $\mathbf{P}^1_{QM}$ is contained in $M_{2,4}$. Thus $\overline{\psi}_D(\mathbf{H}_2^{M'_3}) \subset \mathbf{P}^1_{QM}$ and we have equality since the locus of Abelian surfaces with QM by $\mathcal{O}_6$ in $\mathcal{A}_D(D)_0$ (in fact in any level moduli space) is known to be a compact Riemann surface.

The maps $\widetilde{\nu}_1$, $\widetilde{\nu}_2$ commute with $\widetilde{\iota}$ and moreover:

$$\begin{aligned}
\widetilde{\nu}_1\widetilde{\sigma}_1\widetilde{\nu}_1^{-1} &= -\widetilde{\sigma}_1\widetilde{\tau}_2^2, & \widetilde{\nu}_1\widetilde{\sigma}_2\widetilde{\nu}_1^{-1} &= i\widetilde{\sigma}_1\widetilde{\sigma}_2^{\,2}\widetilde{\tau}_2,\\
\widetilde{\nu}_2\widetilde{\sigma}_1\widetilde{\nu}_2^{-1} &= -\widetilde{\tau}_1\widetilde{\tau}_2^2, & \widetilde{\nu}_2\widetilde{\sigma}_2\widetilde{\nu}_2^{-1} &= \zeta\widetilde{\sigma}_1\widetilde{\sigma}_2\widetilde{\tau}_2,\\
\widetilde{\nu}_1\widetilde{\tau}_1\widetilde{\nu}_1^{-1} &= -\widetilde{\sigma}_2^2\widetilde{\tau}_1, & \widetilde{\nu}_1\widetilde{\tau}_2\widetilde{\nu}_1^{-1} &= \zeta\widetilde{\sigma}_1\widetilde{\sigma}_2^3\widetilde{\tau}_1\widetilde{\tau}_2,\\
\widetilde{\nu}_2\widetilde{\tau}_1\widetilde{\nu}_2^{-1} &= -\widetilde{\sigma}_1\widetilde{\sigma}_2^2\widetilde{\tau}_2^2, & \widetilde{\nu}_2\widetilde{\tau}_2\widetilde{\nu}_2^{-1} &= \widetilde{\tau}_1\widetilde{\tau}_2^3,
\end{aligned}$$

hence they are in $N(\mathcal{H})_2$. The maps ν_1, ν_2 have order 4 and 3 respectively in $\mathrm{Aut}(\mathbf{P}^7)$ and map $\mathbf{P}^1_{QM}$ into itself. In fact, the induced action on $\mathbf{P}^1_{QM}$ is:

$$\widetilde{\nu}_i p_{(x:y)} = p_{\nu_i(x:y)} \quad \text{with} \quad \nu_1(x:y) := (x:iy), \quad \nu_2(x:y) := (i(x-y):-(x+y)).$$

We verified that $\nu_1, \nu_2 \in \mathrm{Aut}(\mathbf{P}^1)$ generate a subgroup which is isomorphic to the symmetric group S_4 (to obtain this isomorphism, one may use the action of the ν_i on the four irreducible factors in $\mathbf{Q}(\zeta)[x,y]$ of the polynomial g_8 defined in Corollary 3.3). ■

Corollary 3.3 *The images in $\mathbf{P}^1_{QM}$ under the parametrization given in Proposition 3.2 of the zeroes of the polynomials*

$$g_6 := xy(x^4-y^4), \qquad g_8 := x^8+14x^4y^4+y^8, \qquad g_{12} := x^{12}-33x^8y^4-33x^4y^8+y^{12},$$

are the orbits of the points in $\mathbf{P}^1_{QM}$ with a non-trivial stabilizer in S_4. Moreover, the rational function

$$G := g_6^4/g_8^3 \colon \quad \mathbf{P}^1_{QM} \longrightarrow \mathbf{P}^1 \cong \mathbf{P}^1_{QM}/S_4$$

defines the quotient map by S_4.

Proof A nontrivial element σ in $S_4 \subset \mathrm{Aut}(\mathbf{P}^1_{QM})$ has two fixed points, corresponding to the eigenlines of any lift of σ to $GL(2,\mathbf{C})$. The fixed points of σ^k are the same as those of σ whenever σ^k is not the identity on $\mathbf{P}^1_{QM}$. One now easily verifies that the fixed points of cycles of order 3, 4, 2 are the zeroes of g_6, g_8, g_{12}, respectively.

The quotient map $\mathbf{P}^1_{QM} \to \mathbf{P}^1_{QM}/S_4 \cong \mathbf{P}^1$ has degree 24. The rational function $G := g_6^4/g_8^3$ is S_4-invariant and defines a map of degree 24 from $\mathbf{P}^1_{QM}$ to $\mathbf{P}^1$, hence the quotient map is given by G. ∎

4 The Principal Polarization

4.1 Introduction

In the previous section we considered Abelian surfaces whose endomorphism ring contains $\mathcal{O}_6$ endowed with a $(1,2)$-polarization. Rotger proved that an Abelian surface whose endomorphism ring is $\mathcal{O}_6$ admits a unique principal polarization [R, section 7]. As such a surface is simple, it is the Jacobian of a genus two curve. The Abel–Jacobi image of the genus two curve provides the principal polarization. In this section we find the image of such a curve in the Kummer surface. This allows us to relate these genus two curves to the ones described by Hashimoto and Murabayashi [HM] in Section 4.6.

Moreover, we also find an explicit projective model of a surface in the moduli space $M_{2,4}$ that parametrizes $(2,4)$-polarized Abelian surfaces whose endomorphism ring contains $\mathbf{Z}[\sqrt{2}]$; see Section 4.8.

4.2 Polarizations

To explain how we found genus two curves in the $(2,4)$-polarized Kummer surfaces parametrized by $\mathbf{P}^1_{QM}$, it is convenient to first consider the Jacobian $A = \mathrm{Pic}^0(C)$ of one of the genus two curves given in [HM, Theorem 1.3]. In [HM, Section 3.1] one finds an explicit description of the principal polarization E and the maximal order $\mathcal{O}_6$ of

$$\mathrm{End}(A) \cong B_6 \cong \frac{(-6,2)}{\mathbf{Q}}.$$

The element $\eta := (-1+i)/2 + k/4 \in \mathcal{O}_6$ has order three, $\eta^3 = 1$ (with $i^2 = -6, j^2 = 2, k = ij = -ji$). We use the same notation for the endomorphism defined by this element. Then $\eta^* E$ is again a principal polarization, and we obtain a polarization E' that is invariant under η as follows:

$$E' := E + \eta^* E + (\eta^2)^* E, \qquad \text{with} \quad E(\alpha,\beta) := Tr(-i\alpha\beta')$$

(here we identify the lattice in $\mathbf{C}^2$ defining A with $\mathcal{O}_6$ and $\beta \mapsto \beta'$ is the canonical involution on B_6). An explicit computation shows that $E' = 3E''$ and that E'' defines a polarization of type $(1,2)$ on A and $\eta^* E'' = E''$.

Considering E as a class in $H^2(A,\mathbf{Z})$, one has $E^2 = 2$, since E is a principal polarization. As η is an automorphism of A, we also get $(\eta^* E)^2 = ((\eta^2)^* E)^2 = 2$ and $E \cdot (\eta^* E) = (\eta^* E) \cdot ((\eta^2)^*)E = ((\eta^2)^*)E \cdot E$. Then one finds that $(E')^2 = 6 + 2 \cdot 3 \cdot E \cdot (\eta^* E)$ and as E' defines a polarization of type $(3,6)$ we have $(E')^2 = 2 \cdot 3 \cdot 6 = 36$, hence $E \cdot (\eta^* E) = 5$. Moreover, one finds that

$$E \cdot E'' = E \cdot (E + \eta^* E + (\eta^2)^* E)/3 = (2+5+5)/3 = 4.$$

Identify the Jacobian of the genus two curve C with $\mathrm{Pic}^0(C) = A$ and identify C with its image under the Abel–Jacobi map $C \to \mathrm{Pic}^0(C)$, $p \mapsto p - p_0$, where p_0 is a Weierstrass point. If the hyperelliptic involution interchanges the points $q, q' \in C$, then $q + q'$ and $2p_0$ are linearly equivalent and thus $q - p_0 = -(q' - p_0)$. Hence the curve $C \subset \mathrm{Pic}^0(C)$ is symmetric: $(-1)^* C = C$. If $p_1, \dots, p_5$ are the other Weierstrass points of C, then $2p_i$ is linearly equivalent to $2p_0$, hence the five points $p_i - p_0 \in C \subset A$, $i = 1, \dots, 5$ are points of order two in A.

Now let $\mathcal{L}$ be a symmetric line bundle on A defining the $(1,2)$-polarization E'' on A. As $E \cdot E'' = 4$, the restriction of $\mathcal{L}$ to C has degree 4, and thus $\mathcal{L}^{\otimes 2}$ restricts to a degree 8 line bundle on C. The map given by the even sections $H^0(A, \mathcal{L}^{\otimes 2})_+$ defines a 2:1 map from A onto the Kummer surface $A/\pm 1$ of A in $\mathbf{P}^5$. As $(2E'')^2 = 16$, this Kummer surface has degree $16/2 = 8$. In fact, Barth shows that the Kummer surface is the complete intersection of three quadrics; see Section 4.3. The symmetry of C implies that this image is a rational curve and the degree of the image of C is four. But a rational curve of degree four in a projective space spans at most a $\mathbf{P}^4$. Moreover, this $\mathbf{P}^4$ contains at least six of the nodes (the images of the two-torsion points of A) of the Kummer surface that lie on C.

It should be noticed that any $(2,4)$-polarized Kummer surface in $\mathbf{P}^5$ contains subsets of four nodes that span only a $\mathbf{P}^2$ (*cf.* [GS, Lemma 5.3]), these subsets must be avoided to find C.

Conversely, given a rational quartic curve on the Kummer surface which passes through exactly 6 nodes, its inverse image in the Abelian surface will be a genus two curve C. In fact, the general A is simple, hence there are no non-constant maps from a curve of genus at most one to A. The adjunction formula on A shows that $C^2 = 2$, hence C defines a principal polarization on A. Rotger [R, Section 6] proved that an Abelian surface A with $\mathrm{End}(A) = \mathcal{O}_6$ has a unique principal polarization up to isomorphism. Thus C must be a member of the family of genus two curves in given in [HM, Theorem 1.3]. We summarize the results in this section in the following proposition. In Proposition 4.2 we determine the curve from [HM] which is isomorphic to $C = C_x$ on the Abelian surface defined by $x \in \mathbf{P}^1_{QM}$.

Proposition 4.1 *Let A be an Abelian surface with $\mathcal{O}_6 \subset \mathrm{End}(A)$. Then A has a (unique up to isomorphism) principal polarization defined by a genus 2 curve $C \subset A$ that is isomorphic to a curve from the family in* [HM, Theorem 1.3] *(see Section 4.6).*

There is an automorphism of order three $\eta \in \mathrm{Aut}(A)$ such that

$$C + \eta^* C + (\eta^2)^* C = 3E''$$

defines a polarization of type $(3,6)$. Let $\mathcal{L}$ be a symmetric line bundle with $c_1(\mathcal{L}) = E''$. Then the image of C, symmetrically embedded in A, under the map $A \to \mathbf{P}^5$ defined by the subspace $H^0(A, \mathcal{L}^{\otimes 2})_+$, is a rational curve of degree four that passes through exactly six nodes of the Kummer surface of A which lie in a hyperplane in $\mathbf{P}^5$.

Conversely, the inverse image in A of a rational curve that passes through exactly six nodes of the Kummer surface of A is a genus two curve that defines a principal polarization on A.

4.3 A Reducible Hyperplane Section

Now we give a hyperplane $H_x \subset \mathbf{P}^5$ that cuts the Kummer surface K_x for $x \in \mathbf{P}^1_{QM}$ in two rational curves of degree four, the curves intersect in six points that are nodes of K_x.

A general point $x = (x_1 : \ldots : x_6) \in M_{2,4} \subset \mathbf{P}^5$ defines a $(2,4)$-polarized Kummer surface K_x that is the complete intersection of the following three quadrics in $X_1, \ldots, X_6$:

$$\begin{aligned} q_1 &:= (x_1^2 + x_2^2)(X_1^2 + X_2^2) - (x_3^2 + x_4^2)(X_3^2 + X_4^2) - (x_5^2 + x_6^2)(X_5^2 + X_6^2),\\ q_2 &:= (x_1^2 - x_2^2)(X_1^2 - X_2^2) - (x_3^2 - x_4^2)(X_3^2 - X_4)^2 - (x_5^2 - x_6^2)(X_5^2 - X_6^2),\\ q_3 &:= x_1x_2X_1X_2 - x_3x_4X_3X_4 - x_5x_6X_5X_6, \end{aligned}$$

[B, Proposition 4.6]. We used the formulas from [B, p. 68] to replace the λ_i, μ_i by the x_i, but notice that the factors '2' in the formulas for $\lambda_i\mu_i$ should be omitted, so $\lambda_1\mu_1 = x_3^3 + x_4^2$ etc. The 16 nodes of the Kummer surface are the orbit of x under the action of $T(2,4)[2]$; that is, it is the set

$$\mathrm{Nodes}(K_x) = \{p_{a,b,c,d} := (\widetilde{\sigma}_1^a \widetilde{\sigma}_2^{2b} \widetilde{\tau}_1^c \widetilde{\tau}_2^{2d})(x); \quad a,b,c,d \in \{0,1\}\},$$

cf. Section 3.2. We considered the following six nodes:

$$p_{0,0,0,0}, \quad p_{0,0,1,1}, \quad p_{0,1,0,0}, \quad p_{0,1,1,0}, \quad p_{1,1,1,0}, \quad p_{1,1,1,1}.$$

For general $x \in P^1_{QM}$ one finds that these six nodes span only a hyperplane H_x in $\mathbf{P}^5$.

Using Magma we found that over the quadratic extension of the function field $\mathbf{Q}(\zeta)(u)$ of $\mathbf{P}^1_{QM}$ (where $\zeta^4 = -1$ and $u = x/y$) defined by $w^2 = u^8 + 14u^4 + 1$, the intersection of H_x and K_x is reducible and consists of two rational curves of degree four, meeting in the 6 nodes.

We parametrize H_x by $t_1p_{0,0,0,0} + \cdots + t_5p_{1,1,1,0}$. Then Magma shows that the rational function t_4/t_5 restricted to each of the two components is a generator of the function field of each of the two components. Thus t_4/t_5 provides a coordinate on each component and, for each component, we computed the value (in $\mathbf{P}^1 = \mathbf{C} \cup \{\infty\}$) of the coordinate in the 6 nodes. The genus two curve $C = C_x$ is the double cover of $\mathbf{P}^1$ branched in these six points.

4.4 Invariants of Genus Two Curves

A genus two curve over a field of characteristic 0 defines a homogeneous sextic polynomial in two variables, uniquely determined up to the action of $\mathrm{Aut}(\mathbf{P}^1)$. In [I, p. 620], Igusa defines invariants A, B, C, D of a sextic and defines further invariants J_i, $i = 2, 4, 6, 10$, as follows [I, pp. 621–622]:

$$J_2 = 2^{-3}A, \quad J_4 = 2^{-5}3^{-1}(4J_2^2 - B), \; J_6 = 2^{-6}3^{-2}(8J_2^3 - 160J_2J_4 - C), \; J_{10} = 2^{-12}D.$$

In [I, Theorem 6], Igusa showed that the moduli space of genus two curves over $\mathrm{Spec}(\mathbf{Z})$ is a (singular) affine scheme which can be embedded in the affine space $\mathbb{A}^{10}_{\mathbf{Z}}$. Its restriction to $\mathrm{Spec}(\mathbf{Z}[1/2])$ can be embedded into $\mathbb{A}^8_{\mathbf{Z}[1/2]}$ using the functions ([I, p. 642])

$$J_2^5J_{10}^{-1}, \quad J_2^3J_4J_{10}^{-1}, \quad J_2^3J_4^2J_{10}^{-1}, \quad J_2^2J_6J_{10}^{-1}, \quad J_4J_6J_{10}^{-1}, \quad J_2J_6^3J_{10}^{-2}, \quad J_4^5J_{10}^{-2}, \quad J_6^5J_{10}^{-3}.$$

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From this one finds that over Spec$(\mathbf{Q})$ one can embed the moduli space into $\mathbb{A}^8_{\mathbf{Q}}$ using 8 functions $i_1 \dots, i_8$ as above but with $J_2, \dots, J_{10}$ replaced by $A, \dots, D$. In case $A \neq 0$, one can use the three regular functions

$$j_1 := A^5/D, \qquad j_2 := A^3B/D, \qquad j_3 := A^2C/D$$

to express $i_1, \dots, i_8$ as

$$j_1, \quad j_2, \quad j_2^2/j_1, \quad j_3, \quad j_2j_4/j_1, \quad j_4^3/j_1, \quad j_2^5/j_1^3, \quad j_4^5/j_1^2.$$

Thus the open subset of the moduli space over $\mathbf{Q}$ where $A \neq 0$ can be embedded in $\mathbb{A}^3_{\mathbf{Q}}$ using these three functions. In particular, two homogeneous sextic polynomials f, g with complex coefficients and with $A(f), A(g) \neq 0$ define isomorphic genus two curves over $\mathbf{C}$ if and only if $j_i(f) = j_i(g)$ for $i = 1, 2, 3$ (see also [Me, CQ]).

4.5 Invariants of the Curve C_x

With the Magma command "IgusaClebschInvariants" we computed the invariants for each of the two genus curves that are the double covers of the two rational curves in $H_x \cap K_x$. They turn out to be isomorphic as expected from Rotger's uniqueness result. We denote by C_x the corresponding genus two curve. For the general $x \in \mathbf{P}^1_{QM}$ the invariant $A = A(C_x)$ is nonzero and

$$j_1(C_x) = -3^5 2^{-5} \frac{(1 - 64G(x))^5}{G(x)^3}, \quad j_2(C_x) = 3^5 2^{-3} \frac{(1 - 64G(x))^3}{G(x)^2},$$

and

$$j_3(C_x) = 3^4 2^{-3} \frac{(1 - 64G(x))^2(1 - 80G(x))}{G(x)^2}.$$

Notice that the invariants are rational functions in the S_4-invariant function $G = g_6^4/g_8^3$ on $\mathbf{P}^1_{QM}$, as expected. Moreover, the $j_i(C_x)$ actually determine $G(x)$,

$$G(x) = \frac{(j_2(x)/j_3(x)) - 3}{80(j_2(x)/j_3(x)) - 192},$$

hence the classifying map from (an open subset of) $\mathbf{P}^1_{QM}/S_4$ to the moduli space of genus two curves is a birational isomorphism onto its image.

4.6 The Genus Two Curves from Hashimoto–Murabayashi

In [HM, Theorem 1.3], Hashimoto and Murabayashi determine an explicit family of genus two curves $C_{s,t}$ whose Jacobians have quaternionic multiplication by the maximal order $\mathcal{O}_6$. They are parametrized by the elliptic curve

$$E_{HM} : g(t, s) = 4s^2t^2 - s^2 + t^2 + 2 = 0.$$

Using the following rational functions on this curve:

$$P := -2(s + t), \quad R := -2(s - t), \quad Q := \frac{(1 + 2t^2)(11 - 28t^2 + 8t^4)}{3(1 - t^2)(1 - 4t^2)},$$

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the genus two curve $C_{s,t}$ corresponding to the point $(s,t) \in E_{HM}$ is defined by the Weierstrass equation

$$C_{s,t}: \qquad Y^2 = X(X^4 - PX^3 + QX^2 - RX + 1).$$

By the unicity result from [R, section 7] we know that this one parameter family of genus two curves should be the same as the one parametrized by $\mathbf{P}^1_{QM}$. Indeed one has the following proposition.

Proposition 4.2 *The genus two curve C_x defined by $x \in \mathbf{P}^1_{QM}$ is isomorphic to the curve $C_{s,t}$ if and only if $G(x) = H(t)$ (so the isomorphism class of $C_{s,t}$ does not depend on s) where*

$$H(t) := \frac{4(t-1)^2(t+1)^2(t^2+1/2)^4}{27((1-2t)(1+2t))^3}.$$

Proof This follows from a direct Magma computation of the invariants j_i for the $C_{s,t}$. In particular, the classifying map of the Hashimoto–Murabayashi family has degree 12 on the t-line (and degree 6 on the $u := t^2$-line), and this degree six cover is not Galois. ∎

4.7 Special Points

In Section 3.2 we observed that S_4 acts on $\mathbf{P}^1_{QM}$ and has three orbits that have less then 24 elements. They are the zeroes of the polynomials g_d, of degree d, with $d = 6, 8, 12$. In case $d = 12$ one finds that for example $x = \zeta$ is a zero of g_{12}. The invariants $j_i(C_x)$ are the same as the invariants of the curve $C_{s,t}$ from [HM] with $(t,s) = (0, \sqrt{2})$. In [HM, Example 1.5] one finds that the Jacobian of this curve is isogenous to a product of two elliptic curves with complex multiplication by $\mathbf{Z}[\sqrt{-6}]$.

In case $d = 6, 8$ one finds that the invariants $j_i(C_x)$ are infinite, hence these points do not correspond to Jacobians of genus two curves but to products of two elliptic curves (with the product polarization). In case $g_6(x) = 0$ one finds that the intersection of the plane H_x with the Kummer surface K_x consists of four conics, each of which passes through four nodes (and there are now 8 nodes in $H_x \cap K_x$). The inverse image of each conic in the Abelian surface A_x is an elliptic curve that is isomorphic to $E_4 := \mathbf{C}/\mathbf{Z}[i]$, and one finds that $A_x \cong E_4 \times E_4$, but the $(1,2)$ polarization is not the product polarization. The point $(t,s) = (\sqrt{-2}/2, \sqrt{2}/2) \in E_{HM}$ defines the same point in the Shimura curve $\mathbf{P}^1_{QM}/S_4$ as the zeroes of g_6. It corresponds to the degenerate curve $C_{t,s}$ in [HM, Example 1.4], which has a normalization that is isomorphic to E_4.

In case $d = 8$ one has $A_x \cong E_3 \times E_3$ and, with the $(1,2)$-polarization, it is the surface A_3 that we defined in Section 1.2. According to [B, Theorem 4.9] a point $x \in M_{2,4}$ defines an Abelian surface A_x if and only if $r(x) \neq 0$ where $r = r_{12}r_{13}r_{23}$ is defined in [B, Proposition 3.2] (the r_{jk} are polynomials in λ_i^2, μ_i^2 and these again can be represented by polynomials in the x_i, see [B, p. 68]. One can choose these polynomials as follows:

$$r_{12} = -4r_{13} = -4r_{23} = 16(x_1x_6 - x_2x_5)(x_1x_6 + x_2x_5)(x_1x_5 - x_2x_6)(x_1x_5 + x_2x_6),$$

and thus $r = 16r_{12}^3$. Restricting r to $\mathbf{P}^1_{QM}$ and pulling back along the parametrization to $\mathbf{P}^1$, one finds that $r = cg_8^3$, where g_8 is as in Section 3.2 and c is a non-zero constant. More generally, we have the following result.

Proposition 4.3 *The image of the period matrices $\tau \in \mathbf{H}_2$ with $\tau_{12} = \tau_{21} = 0$ in $M_{2,4} \subset \mathbf{P}^5$ is the intersection of $M_{2,4}$ with the Segre threefold, which is the image of the map*

$$S_{1,2}\colon \mathbf{P}^1 \times \mathbf{P}^2 \longrightarrow \mathbf{P}^5, \qquad \big((u_0 : u_1), (w_0 : w_1, w_2)\big) \longmapsto (x_1 : \ldots : x_6),$$

where the coordinate functions are

$$\begin{aligned} x_1 &= u_0 w_0, & x_3 &= u_0 w_1, & x_5 &= u_0 w_2,\\ x_2 &= u_1 w_0, & x_4 &= u_1 w_1, & x_6 &= u_1 w_2. \end{aligned}$$

The image of $S_{1,2}$ intersects $\mathbf{P}^1_{QM}$ in two points that are zeroes of g_8. Moreover, the surface $S_{1,2}(\mathbf{P}^1 \times \mathbf{P}^2) \cap M_{2,4}$ is an irreducible component of $(r = 0) \cap M_{2,4}$.

Proof If $\tau_{12} = \tau_{21} = 0$, then by looking at the Fourier series that define the theta constants, one finds that $\vartheta[\begin{smallmatrix} ab \\ 00 \end{smallmatrix}](\tau) = \vartheta[\begin{smallmatrix} a \\ 0 \end{smallmatrix}](\tau_{11})\vartheta[\begin{smallmatrix} b \\ 0 \end{smallmatrix}](\tau_{22})$. The definition of the x_i's in terms of the standard delta functions in $V(2,4)$, $u_n v_m = \vartheta[\begin{smallmatrix} ab \\ 00 \end{smallmatrix}](\tau)$ with $(a,b) = (n/2, m/4)$ ([B, p.53]), then shows that the map $\mathbf{H}_2 \to \mathbf{P}^5$ restricted to these period matrices is the composition of the map

$$\begin{aligned} &\mathbf{H}_1 \times \mathbf{H}_1 \longrightarrow \mathbf{P}^1 \times \mathbf{P}^2\\ &(\tau_1, \tau_2) \longmapsto \Big((\vartheta[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}](\tau_1) : \vartheta[\begin{smallmatrix} b \\ 0 \end{smallmatrix}](\tau_1)),\\ &\qquad\qquad (\vartheta[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}](\tau_2) + \vartheta[\begin{smallmatrix} b \\ 0 \end{smallmatrix}](\tau_2) : \vartheta[\begin{smallmatrix} a \\ 0 \end{smallmatrix}](\tau_2) + \theta[\begin{smallmatrix} c \\ 0 \end{smallmatrix}](\tau_2) : \theta[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}](\tau_2) - \theta[\begin{smallmatrix} b \\ 0 \end{smallmatrix}](\tau_2))\Big) \end{aligned}$$

with the Segre map as above and $a, b, c = 1/4, 1/2, 3/4$, respectively.

The ideal of the image of $S_{1,2}$ is generated by three quadrics. Restricting these to P^1_{QM} one finds that the intersection of the image with P^1_{QM} is defined by the quadratic polynomial $x^2 + (\zeta^2 - 1)xy + \zeta^2 y^2$, which is a factor of g_8.

The factor $x_1x_6 - x_2x_5$ of r is in the ideal of $S_{1,2}(\mathbf{P}^1 \times \mathbf{P}^2)$, hence this surface is an irreducible component of $(r = 0) \cap M_{2,4}$. ∎

Remark 4.4 The intersection of the image of $S_{1,2}$ with $M_{2,4}$, which is defined by $f_1 = f_2 = 0$ (*cf.* Section 3.1), is the image of the surface

$$\mathbf{P}^1 \times C_F, \quad (\subset \mathbf{P}^1 \times \mathbf{P}^2), \qquad C_F : w_0^4 - w_1^4 - w_2^4 = 0.$$

The curves $\mathbf{P}^1$ and C_F here are both elliptic modular curves (defined by the totally symmetric theta structures associated with the divisors $2O$ and $4O$, where O is the origin of the elliptic curve).

4.8 A Humbert Surface

In Section 4.3 we considered six nodes of the Kummer surface K_x,

$$p_{0,0,0,0}, p_{0,0,1,1}, \ldots, p_{1,1,1,1},$$

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which had the property that for a general $x \in \mathbf{P}^1_{QM}$ these six nodes span only a hyperplane in $\mathbf{P}^5$. For general $x \in M_{2,4}$; however, these nodes do span all of $\mathbf{P}^5$. They span at most a hyperplane if the determinant F of the 6×6 matrix whose rows are the homogeneous coordinates of the nodes, is equal to zero.

$$F = \det \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ -x_2 & x_1 & x_4 & -x_3 & x_6 & -x_5 \\ -x_2 & x_1 & -x_4 & x_3 & x_6 & -x_5 \\ x_1 & -x_2 & x_3 & -x_4 & -x_5 & x_6 \\ x_1 & x_2 & x_3 & x_4 & -x_5 & -x_6 \\ x_1 & -x_2 & -x_3 & x_4 & x_5 & -x_6 \end{pmatrix} = 16\left(x_1^2 x_3^2 x_5 x_6 + \cdots - x_2^2 x_4^2 x_5 x_6\right).$$

Then F is a homogeneous polynomial of degree six in the coordinates of x that has 8 terms. Let D_F be the divisor in $M_{2,4}$ defined by $F = 0$, then $\mathbf{P}^1_{QM}$ is contained in (the support of) D_F. Magma shows that D_F has 12 irreducible components. The only one of these that contains $\mathbf{P}^1_{QM}$ is the surface $S_2 \subset \mathbf{P}^5$ defined by

$$S_2 : \quad x_1^2 - x_2^2 - x_5^2 - x_6^2 = x_1 x_2 - x_4^2 - x_5 x_6 = x_3^2 - x_4^2 - 2x_5 x_6 = 0.$$

Magma verified that S_2 is a smooth surface, hence it is a K3 surface.

Proposition 4.5 *The surface $S_2 \subset M_{2,4}$ parametrizes Abelian surfaces A with* $\mathbf{Z}[\sqrt{2}] \subset \mathrm{End}(A)$.

Proof For a general point x in S_2, the hyperplane spanned by the six nodes intersects K_x in a one-dimensional subscheme that is the complete intersection of three quadrics and that has six nodes. The arithmetic genus of a smooth complete intersection of three quadrics in $H_x = \mathbf{P}^4$ is only five, hence this subscheme must be reducible. In the case $x \in \mathbf{P}^1_{QM}$, this subscheme is the union of two smooth rational curves of degree four intersecting transversally in the six nodes. Thus, for general $x \in S_2$, the intersection must also consist of two such rational curves. Let $C \subset A_x$ be the genus two curve in the Abelian surface A_x defined by x that is the inverse image of one of these components. Then $C^2 = 2$ and $C \cdot \mathcal{L} = 4$, where $\mathcal{L}$ defines the $(1,2)$-polarization. Now we apply [BL, Proposition 5.2.3] to the endomorphism $f = \phi_C^{-1} \phi_{\mathcal{L}}$ of A_x defined by these polarizations. We find that the characteristic polynomial of f is $t^2 - 4t + 2$. As its roots are $2 \pm \sqrt{2}$, we conclude that $\mathbf{Z}[\sqrt{2}] \subset \mathrm{End}(A_x)$. ■

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Dipartimento di Matematica, Università di Milano, 20133 Milano, Italia
e-mail: matteoalfonso.bonfanti@gmail.com lambertus.vangeemen@unimi.it