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Towards an algorithmic construction of cut-elimination procedures†

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We investigate cut elimination in propositional substructural logics. The problem is to decide whether a given calculus admits (reductive) cut elimination. We show that for commutative single-conclusion sequent calculi containing generalised knotted structural rules and arbitrary logical rules the problem can be decided by resolution-based methods. A general cut-elimination proof for these calculi is also provided.

1. Introduction

Gentzen sequent calculi have been the central tool in many proof-theoretical investigations and applications of logic in algebra and computer science. A key property of these calculi is cut elimination (*Gentzen's Hauptsatz*), which was first established by Gentzen (Gentzen 1935) for the sequent calculi **LK** and **LJ** for classical and intuitionistic first-order logic. The removal of cuts corresponds to the elimination of intermediate statements (lemmas) from proofs resulting in calculi in which proofs are *analytic* in the sense that all statements in the proofs are subformulae of the result. Analytic proof calculi for logics are not only an important theoretical tool that is useful for understanding relationships between logics and proving metalogical properties like consistency, decidability, admissibility of rules and interpolation, but are also the key to developing automated reasoning methods. These calculi also provide an alternative representation of varieties of algebras (see, for example, Galatos and Ono (2006)), which can then be used to give syntactic proofs of algebraic properties, for example, amalgamation, for which (in particular cases) semantic methods are not known. Cut elimination is also a powerful tool for proving the completeness of a given analytic sequent calculus with respect to a logic formalised using Hilbert style systems, as the *cut rule* simulates *modus ponens.*

Cut-elimination proofs have been provided for very many sequent calculi, though mainly on a case-by-case basis (even when the arguments for a given calculus are similar to that of another) and using heavy syntactic arguments that are usually written without filling in the details. This makes proof checking difficult and the whole process of eliminating cuts rather opaque.

In this paper we perform a resolution-based analysis of cut elimination in *knotted commutative calculi*. These are propositional single-conclusion sequent calculi containing

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arbitrary logical rules (satisfying suitable conditions), the permutation rule and (possibly) unary structural rules generalising both the weakening and contraction rules in Gentzen's **LJ**. The structural rules considered are a generalisation of the *knotted structural rules* in Hori *et al.* (1994), whose (n, k) type is of the form: from $\Gamma, A, \ldots, A$ (n times) $\rightarrow C$ infer $\Gamma, A, \ldots, A$ (k times)$\rightarrow C$, for all $n \geqslant 0$ and $k \geqslant 1$. The (n, k) rule is a restricted form of weakening when $n < k$, and of contraction when $k < n$. Extensions of intuitionistic linear logic without the exponential connectives **ILL** and its implicational fragment **BCI** with (n, k) were investigated in Hori *et al.* (1994) from both the syntactic and semantic point of view. It was shown that **BCI** extended with the (n, k) rule admits cut elimination if and only if $k = 1$. Moreover, **BCI** extended with both weakening and the $(n + 1, n)$ rule admits cut elimination if and only if $n = 1$, while **BCI** extended with contraction and the $(n, n + 1)$ rule admits cut elimination if and only if $n = 0$. A cut-elimination proof working for these cases was also presented. The analysis of Hori *et al.* (1994) applies only to calculi consisting of *one* knotted structural rule and a *fixed* set of connectives: those of **ILL**.

A larger class of single-conclusion calculi, containing arbitrary structural rules and logical rules satisfying some restrictions was considered in Ciabattoni and Terui (2006), where necessary and sufficient conditions for *reductive cut elimination* were provided. Reductive cut elimination is a naturally strengthened version of cut elimination in the presence of axioms (see, for example, Buss (1998)), which encompasses the 'standard' cut-elimination methods working by:

1 shifting up cuts; and
2 replacing them with smaller cuts, when the cut formula is introduced by logical rules in both premisses.

The syntactic conditions defined in Ciabattoni and Terui (2006) (*reductivity* and *weak substitutivity*) formalise Steps 1 and 2 above. No decision procedure to check whether a calculus admits reductive cut elimination was defined in Ciabattoni and Terui (2006).

However, a decision procedure for cut elimination is contained in Avron and Lev (2005) for multiple-conclusion calculi with *all* structural rules (weakening, exchange and contraction). Each calculus belonging to this class admits cut elimination if and only if its logical rules are *coherent*, that is, for each set of rules introducing a connective, the formulae in their premisse(s) from which the principal formula derives form an inconsistent set of clauses. For example, the set of clauses $\{\vdash \alpha_1; \vdash \alpha_2; \alpha_1, \alpha_2 \vdash\}$, corresponding to the rules for conjunction in **LK** is inconsistent. The analysis in Avron and Lev (2005), which is based on semantic techniques (non-deterministic matrices), relies strongly on the presence of all structural rules. The same holds for Basin and Ganzinger (2001), which uses ordered resolution to prove cut elimination and decide rule dependency in **LK**.

Miller and Pimentel (2002; 2005) extended Avron and Lev's analysis to first-order sequent calculi possibly without the weakening rules and/or the contraction rules. In particular, they introduced a *sufficient* condition for any such calculus to admit cut elimination together with an algorithm (based on the encoding of the calculi considered into a linear logic based framework) to check them. Moreover, they provided a decision procedure for derivability of rules in these calculi. However, their analysis does not apply to calculi with additional structural rules other than standard weakening and contraction, and, in particular, fails for knotted structural rules (even of the form $(n, 1)$ for some $n > 2$).

In this paper we provide tools for deciding whether a knotted commutative calculus admits reductive cut elimination and for automating cut-elimination proofs in these calculi. We define algorithms to check whether rules of knotted commutative calculi satisfy reductivity and weak substitutivity – the necessary and sufficient conditions in Ciabattoni and Terui (2006). To decide reductivity, we develop a substructural resolution calculus and make use of normalisation of clauses and of subsumption, while for weak substitutivity we use combinatorial arguments; the latter also serve to decide the dependency (derivability) of structural rules, thus obtaining a method that transforms knotted commutative calculi that (by their form) do not admit reductive cut elimination into others that do. Finally, we provide a constructive proof of reductive cut elimination for knotted commutative calculi satisfying reductivity and weak substitutivity.

The long-term aim is to develop a uniform method for proving (or disproving) cut elimination for a wide class of substructural logics. The advantage of such a method would be twofold:

1 it becomes easier to prove (or disprove) cut-elimination theorems for new sequent type logic calculi; and
2 the construction of the cut-elimination methods can be automatized – provided the general method is computational.

2. Basic notions

We will use $\star_1, \star_2, \star_3, \ldots$ to indicate logical connectives of suitable arity. A *formula* A is either a propositional variable or a *compound formula* of the form $\star(A_1, \ldots, A_m)$ where $A_1, \ldots, A_m$ are formulae. Let $\Gamma, \Delta, \Pi, \Sigma, \ldots$ stand for (possibly empty) multisets of formulae and S, T for arbitrary sequents. To specify inference rules as rule schemata, we will use *meta-variables* (or *formula-variables*) $\alpha, \beta, \ldots$, standing for arbitrary formulae, and (possibly empty) multisets $\Theta, \Xi, \Phi, \Psi, \Upsilon, X, Y, x, y, \ldots$ of meta-variables. ϵ will always denote the empty multiset (of formulae or meta-variables).

When $n \geqslant 0$, we use Γ^n and x^n to denote $\Gamma, \ldots, \Gamma$ and $x, \ldots, x$ (n times), respectively.

Given a (meta)sequent $\Gamma \Rightarrow \Delta$ ($\Theta \Rightarrow \Xi$)

— Γ (Θ) is called the *antecedent*, and Δ (Ξ) the *consequent*.
— The (meta)sequent is said to be *single-conclusion* if its consequent contains at most one formula (meta-variable).
— The (meta)sequent is called a *clause* if it does not contain logical connectives. A single-conclusion clause is called a *Horn clause*. A clause with at most two atoms is called a *Krom clause*.

A sequent calculus is *single-conclusion* if all its sequents are.

Definition 2.1. A *basic calculus* is a single-conclusion sequent calculus that consists of

— axiom schema of *identity* $\alpha \vdash \alpha$;

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— the (multiplicative version of the) *cut rule* (CUT) and the *permutation* (left) rule

$$\frac{\Theta \vdash \alpha \quad \alpha\,\Theta_1 \vdash \Xi}{\Theta_1\,\Theta \vdash \Xi}\,(CUT) \qquad \frac{\Theta\,\beta\,\alpha\,\Theta' \vdash \Xi}{\Theta\,\alpha\,\beta\,\Theta' \vdash \Xi}\,(e,l)$$

where $\Theta, \Theta_1, \Theta'$ and Ξ are arbitrary (thus (CUT) and (e,l) actually consist of countable sets of inference rules);

— possibly, weakening (w,l) and/or *generalised knotted structural rules*

$$\frac{\Theta' \vdash \Xi'}{\Theta'\,\Theta'' \vdash \Xi'}\,(w,l) \qquad \frac{\Theta\,\alpha_1^{n_1}\ldots\alpha_j^{n_j} \vdash \Xi}{\Theta\,\alpha_1^{k_1}\ldots\alpha_j^{k_j} \vdash \Xi}\,((n_1,k_1),\ldots(n_j,k_j))$$

for any $k_1,\ldots k_j, n_1,\ldots,n_j \geqslant 1$, $\Theta,\Theta',\Xi,\Xi' \neq \epsilon$, $l \geqslant 1$ and $\Xi \notin \Theta$;

— for each logical connective $\star$, *left logical rules* $\{(\star,l)_j\}_{j\in\Lambda_1}$ and *right logical rules* $\{(\star,r)_k\}_{k\in\Lambda_2}$ (Λ_1,Λ_2 can be empty):

$$\frac{\Upsilon_1 \Rightarrow \Psi_1 \quad \cdots \quad \Upsilon_n \Rightarrow \Psi_n}{\Theta\ \star(\vec{\alpha}) \Rightarrow \Xi}\,(\star,l)_j \qquad \frac{\Upsilon_1 \Rightarrow \Psi_1 \quad \cdots \quad \Upsilon_{n^\circ} \Rightarrow \Psi_{n^\circ}}{\Theta \Rightarrow \star(\vec{\alpha})}\,(\star,r)_k$$

(where $\vec{\alpha} \equiv \alpha_1,\ldots,\alpha_l$ and $n, n^\circ \geqslant 0$) satisfying the conditions

(**log0**) Any meta-variable in $\Upsilon_1,\ldots,\Upsilon_{n^{(\circ)}}$ (respectively, $\Psi_1,\ldots,\Psi_{n^{(\circ)}}$) is either an α_i, with $i = 1,\ldots l$, or it does occur in Θ (respectively, Ξ).

(**log1**) Each meta-variable occurs *at most once* in Θ.

Instances of the identity axiom schema and rules are obtained by substituting arbitrary formulae for meta-variables.

In logical rules, the meta-variables (formulae) of the form α_i are called *active meta-variables* (*active formulae*) and the introduced $\star(\vec{\alpha})$ (or formula of the form $\star(A_1,\ldots,A_l)$) is called the *principal formula*, the remaining meta-variables (formulae) are called *contexts*. In the generalised knotted structural rules the meta-variable (or their instances) in Θ are called *contexts*. Moreover, the two occurrences of the formula instantiating the meta-variable α in (CUT) are called left and right *cut formulae*.

Remark 2.1. From now on, we will identify (meta)sequents differing only in the order of (meta) formulas in their antecedents and we will therefore not consider explicitly anymore the permutation rule (e,l).

To formalise the class of calculi we will deal with, we define the closure under cuts of two (meta)sequents. Let S, T be sequents of the form $S = \Gamma \vdash A$ and $T = A^m, \Pi \vdash \Lambda$. We define

$$\begin{aligned} Cut_l^0(S,T) &:= \{T\}, \\ Cut_l^{i+1}(S,T) &:= Cut_l^i(S,T) \cup \{A^{m-i-1}, \Gamma^{i+1}, \Pi \vdash \Lambda\}, \\ Cut_l^*(S,T) &:= Cut_l^m(S,T). \end{aligned}$$

We may cut also from the other side. In this case we define

$$\begin{aligned} Cut_r^i(S,T) &:= Cut_l^i(T,S),\ i = 0,1 \\ Cut_r^*(S,T) &:= Cut_l^1(T,S). \end{aligned}$$

The definition above also applies to meta-sequents.

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Definition 2.2. Let $\mathscr{R}$ be a set of unary structural rules and $\rho \in \mathscr{R}$. We define $S \rightarrow_\rho S'$ if S' can be obtained from S by one application of ρ. We define $S \rightarrow_{\mathscr{R}} S'$ if there exists a $\rho \in \mathscr{R}$ such that $S \rightarrow_\rho S'$. We define $\rightarrow_\rho^*$ to be the reflexive transitive closure of $\rightarrow_\rho$, and $\rightarrow_{\mathscr{R}}^*$ to be that of $\rightarrow_{\mathscr{R}}$.

Definition 2.3. A *knotted commutative* calculus K is a basic calculus in which each instance of a logical rule ρ with premisses $S_1, \ldots, S_n$, conclusion S and principal formula A, satisfies the additional conditions:

(log2) For each single conclusion sequent T and each $S' \in Cut_l^*(T,S)$ such that the principal formula $A \in S'$, there are $S_1' \in Cut_l^*(T,S_1), \ldots, S_n' \in Cut_l^*(T,S_n)$ such that $S_1', \ldots, S_n' \rightarrow_\rho S'$.

(log3) for each single conclusion sequent T. and each $S' \in Cut_r^*(T,S)$ such that the principal formula $A \in S'$, there are $S_1' \in Cut_r^*(T,S_1), \ldots, S_n' \in Cut_r^*(T,S_n)$ such that $S_1', \ldots, S_n' \rightarrow_\rho S'$.

Remark 2.2. Condition **(log0)** ensures that logical rules satisfy the subformula property and do not allow meta-variables (that are not active meta-variables) to move from antecedent to consequent of sequents, and *vice versa*. Conditions **(log2)** and **(log3)** ensure that logical rules allow any (CUT) on a context formula to be replaced by (CUT) on its premisse(s) (and one application of the rule).

Definition 2.4. Let $\mathscr{R}$ be the set of structural rules of a knotted commutative calculus. Each $\rho \in \mathscr{R}$ is called *regular* and $\mathscr{R}$ is called the *regular set*. If $\mathscr{R}$ contains (w,l), it is called *w-regular*, otherwise it is *wf-regular* (weakening free regular).

Notice that each generalised knotted structural rule $((n_1,k_1), \ldots (n_j,k_j))$ can be simulated by j knotted structural rules (n_i,k_i), for $i = 1, \ldots, j$

Example 2.1 (Knotted commutative calculi). Many well-known sequent calculi fit into our framework. Among them are propositional **LJ** (Gentzen 1935) and the calculi investigated in Hori *et al.* (1994), which are intuitionistic linear logic without the exponentials **ILL**, and its implicational fragment extended with the knotted structural rules of the form (n,k) and both (n,k) and (k,n). Note that $(2,1)$ is the contraction rule left in **LJ**, $(1,2)$ is expansion (see van Benthem (1991)) and $(n+1,n)$ is the so-called n-contraction rule. The latter, which was investigated in Prijatelj (1996), is sound for the logic of Łukasiewicz with n truth-values.

Some further examples of knotted commutative calculi are:

— the calculus **LBC-** of Baaz *et al.* (2004), whose axioms and rules are exactly those of **ILL** apart from the right rule of the $\wedge$ connective, which, in the case of **LBC-**, is split into the rules

$$\frac{\Theta \vdash \alpha_1 \quad \Theta' \alpha_1 \vdash \alpha_2}{\Theta\, \Theta' \vdash \alpha_1 \wedge \alpha_2}\ (\wedge, r)_1 \qquad \frac{\Theta \vdash \alpha_2 \quad \Theta' \alpha_2 \vdash \alpha_1}{\Theta\, \Theta' \vdash \alpha_1 \wedge \alpha_2}\ (\wedge, r)_2$$

— the calculus K_1 consisting of the rules

$$\frac{\Theta \vdash \alpha_1 \quad \Theta \vdash \alpha_2}{\Theta \vdash \alpha_1 \bar{\wedge} \alpha_2}\ (\bar{\wedge}, r) \qquad \frac{\Theta\, \alpha_1\, \alpha_2 \vdash \Xi}{\Theta\, \alpha_1 \bar{\wedge} \alpha_2 \vdash \Xi}\ (\bar{\wedge}, l).$$

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Definition 2.5. *Canonic $\star$ cut-derivation schema φ:*

$$\dfrac{\dfrac{\Theta_1 \vdash \Xi_1 \quad \ldots \quad \Theta_n \vdash \Xi_n}{\gamma \vdash \star(\bar{\alpha})}\,(\star, r)_j \qquad \dfrac{\Phi_1 \vdash \Xi'_1 \quad \ldots \quad \Phi_m \vdash \Xi'_m}{\star(\bar{\alpha})\,\gamma' \vdash \delta}\,(\star, l)_i}{\gamma\,\gamma' \vdash \delta}\,(CUT)$$

The set $\{\Theta_i \vdash \Xi_i, \Phi_j \vdash \Xi'_j\}$ for $i = 1, \ldots, n$ and $j = 1, \ldots, m$ is called the *reduction set* of φ. A *canonic $\star$ cut-derivation* φ' corresponding to a $\star$ cut-derivation schema φ is an instance of the schema where the (instances of) $\Theta_i \vdash \Xi_i, \Phi_j \vdash \Xi'_j$ are replaced by derivations.

2.1. *Conditions for (reductive) cut elimination*

Ciabattoni and Terui (2006) provides a characterisation of the cut-elimination methods for a large class of propositional single-conclusion sequent calculi that proceed following the ‘standard’ steps:

1 locate canonic cut-derivations and replace them by derivations with ‘smaller’ cuts; and
2 shift inferences to achieve canonic cut-derivations.

(CERES (Baaz and Leitsch 2000) provides an example of an alternative cut-elimination method.)

Necessary and sufficient conditions have been defined for these calculi to admit *reductive cut elimination*, which is a naturally strengthened version of cut elimination in the presence of axioms (see, for example, Buss (1998)), which, in addition, aims to shift non-eliminable cuts upward *as much as possible*. The defined conditions (*reductivity* and *weak substitutivity*) are recalled below and applied to knotted commutative calculi. Intuitively, logical rules are reductive if they allow the replacement of cuts by ‘smaller’ cuts (this formalises Step 1 above), and a structural rule is weakly substitutive when any cut can be permuted upward (*cf.* Step 2). Note that logical rules of knotted commutative calculi are weakly substitutive by definition.

Let K be a knotted commutative calculus and $\mathcal{S}$ be a set of sequents (considered as non-logical axioms). A *derivation* in K of a sequent S_0 from $\mathcal{S}$ is a labelled tree whose root is labelled by S_0, the leaves are labelled by an instance of an identity axiom, by an instance of a logical K-rule without premisses or by a sequent in $\mathcal{S}$, and the inner nodes are labelled in accordance with the instances of the K-rules. A derivation in K of a meta-sequent σ from a set of meta-sequents is defined similarly. When there exists such a derivation, we say that S_0 (or σ) is *derivable* from $\mathcal{S}$ in K.

Definition 2.6. An occurrence of (CUT) in a derivation is said to be *reducible* if one of the following holds:

(i) Both cut formulae are the principal formulae of logical rules.
(ii) At least one of the two cut formulae is a context formula of a rule other than (CUT) or an identity axiom.

We say that a knotted commutative sequent calculus K admits *reductive cut elimination* if whenever a sequent S_0 is derivable in K from a set $\mathcal{S}$ of non-logical axioms, S_0 has a derivation in K from $\mathcal{S}$ without any reducible cuts.

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Notice that in a derivation without non-logical axioms, uppermost cuts are always reducible. Hence reductive cut elimination implies the usual cut elimination.

Definition 2.7. Let K be a knotted commutative sequent calculus, and $(\star, l)_i$ and $(\star, r)_j$ be rules of K introducing a connective $\star$ on the left and right, respectively. These rules are *pairwise reductive* in K if for each canonic $\star$ cut-derivation schema Φ there exists a derivation μ in K of its conclusion $\gamma\gamma' \vdash \delta$ (see Definition 2.5) from the reduction set of Φ using no logical rules and all cut-formulae appearing in μ are the active meta-variables of $\star(\bar{\alpha})$. The rules for $\star$ are *reductive* in K if each left and right rule for $\star$ is pairwise reductive.

Remark 2.3. Reductivity corresponds to the principal formula condition given in Restall(1999) and to the coherence criterion of Miller and Pimentel (2002; 2005).

Proposition 2.1. If rules for $\star$ are reductive in K, the end sequent of any canonic $\star$ cut-derivation Φ' can be derived in K from the instances of the reduction set of Φ using no logical rules and all cut-formulae appearing in the derivation are among the active formulas instantiating $\star(\bar{\alpha})$.

Proof. The required derivation is an instance of the derivation μ of Definition 2.7. □

Definition 2.8. Let K be any knotted commutative calculus with structural rules $\mathscr{R}$. $\rho \in \mathscr{R}$ is said to be *weakly substitutive* in K if for all sequents $S, S_1, S_2 \in K$ such that $S_1 \to_\rho S_2$, we have

$(*)$ for all $S' \in Cut_c^*(S, S_2)$, $c \in \{l, r\}$, there exists an $S'' \in Cut_c^*(S, S_1)$ such that $S'' \to_{\mathscr{R}}^* S'$.

$\mathscr{R}$ is said to be *weakly substitutive* if all $\rho \in \mathscr{R}$ are.

Proposition 2.2. Let K be a knotted commutative calculus with structural rules $\mathscr{R}$. $\rho \in \mathscr{R}$ is weakly substitutive if and only if for all $S, S_1, S_2 \in K$ such that $S_1 \to_\rho S_2$, we have

$(*)'$ for all $S' \in Cut_l^*(S, S_2)$ there exists an $S'' \in Cut_l^*(S, S_1)$ such that $S'' \to_{\mathscr{R}}^* S'$.

Proof. The statement follows trivially from the presence of the 'passive' contexts Θ or Θ' in the structural rules considered (Definition 2.1). □

Remark 2.4. An equivalent definition of weak substitutivity was given in Ciabattoni and Terui (2006) using rule *schemas* instead of rule *instances*.

Theorem 2.1. A knotted commutative calculus admits reductive cut elimination if and only if its logical rules are reductive and its structural rules are weakly substitutive.

Proof. The statement follows from Ciabattoni and Terui (2006). □

Given any sequent calculus K'. It is easy to see whether K' belongs to our framework. This can be checked by eye for structural rules and the conditions **(log0)** and **(log1)** for logical rules. Moreover, conditions **(log2)** and **(log3)** for logical rules of K' can be checked in finite time since $Cut_l^*(T, S)$ and $Cut_l^*(T, S_i)$, $i = 1, \ldots, n$, are finite sets.

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3. On regular sets

Let K be a single conclusion calculus and $\mathscr{R} = \{\rho_1, \ldots, \rho_n\}$ be a set of unary (that is, one premiss) structural rules of K. Consider the following problem: is a unary structural rule ρ dependent on (or, equivalently, derivable from) $\mathscr{R}$? More formally, the unary rule ρ *depends on* $\mathscr{R}$ if for any S_1 and S_2 such that $S_1 \rightarrow_\rho S_2$ we have $S_1 \rightarrow^*_{\mathscr{R}} S_2$.

Since rules in $\mathscr{R}$ can be formulated as Horn/Krom clauses, this problem corresponds to the problem of whether a set of Krom clauses S implies another Krom clause. The latter was shown to be undecidable in Schmidt-Schauss (1988) when S contains at least two elements.

Using combinatorial arguments, we prove below that when K is a knotted commutative calculus with regular set $\mathscr{R}$ and ρ is any regular rule, the problem above is decidable. Using this result, we define a procedure to decide whether a regular set is weakly substitutive. A characterisation of weakly substitutive regular sets is also provided.

Definition 3.1. Two regular sets $\mathscr{R}$ and $\mathscr{R}'$ are equivalent (in symbols $\mathscr{R} \sim \mathscr{R}'$) if each $\rho \in \mathscr{R}$ depends on $\mathscr{R}'$ and each $\rho' \in \mathscr{R}'$ depends on $\mathscr{R}$.

Our analysis proceeds by cases according to whether the weakening rule does or does not belong to the regular set $\mathscr{R}$.

3.1. *W-regular systems*

We assume that $\mathscr{R} = \{(w, l), \rho_i\}$, for $i = 1, \ldots, n$, where ρ_i is

$$\frac{x^{k_i} y \vdash z}{x^{l_i} y \vdash z}\ \rho_i .$$

Lemma 3.1. Without loss of generality, let $l_1 = \min\{l_1, \ldots, l_n\}$. Then a knotted structural rule (k, l), $k \neq l$, is derivable from $\mathscr{R}$ if and only if $l \geqslant l_1$.

Proof. Consider $\mathscr{R}$. Let $p_1 = k_1 - l_1$. We may assume that $k_i > l_i$ for all i, otherwise ρ_i is the identity or an instance of weakening.

$l \geqslant l_1$: Hence there exists a number r such that $l + r * p_1 > k$. We define the derivation

$$\cfrac{\cfrac{x^k y \vdash z}{x^{l+r*p_1} y \vdash z}\ w{:}l}{x^l y \vdash z}\ \rho_1^*$$

As a consequence, all rules $\rho_2, \ldots, \rho_n$ are themselves dependent on ρ_1. If $l_1 = 1$, ordinary contraction can be simulated and hence all unary structural rules!

$l < l_1$: By the presence of weakening, we can derive $x^{k_i} y \vdash z$ from $x^{l_i} y \vdash z$. So the rules can be represented by an equational theory

$$\begin{aligned} \mathscr{E} = \{ & x^{k_1} = x^{l_1}, \ldots, x^{k_n} = x^{l_n}, \\ & xy = yx,\ x(yz) = (xy)z,\ x\epsilon = x,\ \epsilon x = x\} . \end{aligned}$$

There exists a model of $\mathscr{E}$ with the domain $D = \{\epsilon, d_1, \ldots, d_{l_1}\}$ such that $d_i = d^i$ for $i < l_1$ and $d_i = d_{l_1}$ for $i \geqslant l_1$. In this interpretation the equation $d^l = d^k$

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does not hold. In particular, $x^l y = x^k y$ is falsified in this model of $\mathscr{E}$. Clearly, the derivability of $x^l y$ from $x^k y$ implies the equation $x^l y = x^k y$; hence the rule is not derivable. □

Proposition 3.1. Let ρ be a unary rule of the form

$$\frac{x_1^{p_1} \dots x_r^{p_r} y^* \vdash z'}{x_1^{m_1} \dots x_r^{m_r} y^* z^* \vdash z'}$$

where y^* denotes either y or ϵ (and z^* similarly). Hence, determining whether ρ depends on $\mathscr{R}$ is decidable.

Proof. It is clear that ρ depends on $\mathscr{R}$ if and only if

$$(+) \quad \min\{m_1, \dots, m_r\} \geqslant l_1 \, .$$

Indeed, if (+) holds, we simulate the rule r-times as in Lemma 3.1; otherwise, for $m_i < l_i$ we create a rule instance where all x_j are set to ϵ for $j \neq i$. Then the result follows from Lemma 3.1. □

Remark 3.1. The proposition's claim does not ask for the existence of an $i = \{1, \dots, r\}$ such that $p_i = m_i$, so ρ might not be a general knotted structural rule.

3.2. *Wf-regular systems*

We assume that $\mathscr{R} = \{\rho_i\}$, for $i = 1, \dots, n$, where each ρ_i is (k_i, l_i). We can assume without loss of generality that $k_i \neq l_i$ for all $i \in \{1, \dots, n\}$, otherwise the corresponding rule is redundant.

We distinguish 3 cases:

(1) $l_i < k_i$ for all $i = 1, \dots, n$ (that is all the rules are *contractive*);
(2) $l_i > k_i$ for all $i = 1, \dots, n$;
(3) there exist $i, j < n$ and $i \neq j$ such that $l_i < k_i$ and $l_j > k_j$.

The decidability of rule dependency in cases (1) and (2) is easy: in case (1) only finitely many derivations are possible on any sequent; in case (2) we observe the following feature.

Let ρ be the rule

$$\frac{x_1^{q_1} \dots x_m^{q_m} \vdash y}{x_1^{p_1} \dots x_m^{p_m} \vdash y} \, .$$

First note that any rule with different sets of meta-variables on the left-hand side in the premiss and conclusion is not derivable: clearly no meta-variable may vanish, and there is no weakening producing additional ones. Thus we may indeed restrict our analysis to rules ρ of the form above. According to the structure of rules in case (2), the sum of powers of the x_i in sequents are strictly increasing with every rule application. So let us assume we have derived

$$s : x_1^{r_1} \dots x_m^{r_m} \vdash y$$

from $x_1^{p_1} \dots x_m^{p_m} \vdash y$ such that

$$\sum_{i=1}^{n} r_i > \sum_{i=1}^{n} q_i \, .$$

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Then s is a dead end, as there is no way to reach the rule consequent $x_1^{q_1} \dots x_m^{q_m} \vdash y$ from s as the sum of the powers increases strictly. On the other hand, there are only finitely many derivations ending in sequents $x_1^{r_1} \dots x_m^{r_m} \vdash y$ with

$$\sum_{i=1}^{n} r_i \leqslant \sum_{i=1}^{n} q_i .$$

So rule dependency is also decidable in case (2).

We still need to investigate case (3).

Definition 3.2. Let $Q\colon \{q_1, \dots, q_n\}$ be a set of integers such that $q_i \neq 0$ for all i. We say that a number r is *representable* by Q if there exist non-negative integers $k_1, \dots, k_n$ such that

$$r = k_1 * q_1 + \dots + k_n * q_n .$$

Proposition 3.2. Let $\mathscr{R}$ be any regular system of rules fulfilling restriction (3) above. Then a rule of the form

$$\frac{x_1^{s_1} \dots x_m^{s_m} \vdash y}{x_1^{r_1} \dots x_m^{r_m} \vdash y}$$

is derivable from $\mathscr{R}$ if and only if $r_i - s_i = 0 \bmod q$ for all $i = 1, \dots, m$ and a number q depending on $\mathscr{R}$.

Proof. Let $\mathscr{R} = \{\rho_1, \dots, \rho_n\}$ where each ρ_i is a knotted structural rule of the form (k_i, l_i). Assume, without loss of generality, that $l_1 < k_1$ and $l_2 > k_2$, and let q be the greatest common divisor of the set $Q\colon \{q_1, \dots, q_n\}$ for $q_i = l_i - k_i$ $(i = 1, \dots, n)$. Then, by elementary number theory, a number r is representable by $\{q_1, \dots, q_n\}$ if and only if $r = 0 \bmod q$. Note that if all q_i were positive or all were negative, then the representability would hold only above a certain bound. But, due to the presence of different signs, every r with the appropriate modularity is representable.

Now let σ be a generalised knotted structural rule $((s_1, r_1), \dots (s_m, r_m))$. Then σ is a derivable rule if and only if $r_i - s_i = 0 \bmod q$. Because of the existence of contexts in the rules of $\mathscr{R}$, we can restrict the problem of derivability to simpler rules of the form

$$\frac{x^s \vdash y}{x^r \vdash y} .$$

Obviously, the conclusion is derivable from the premiss using the rules in $\mathscr{R}$ if and only if $r = s + k_1 * q_1 + \dots + k_n * q_n$ for non-negative numbers k_i. But this is the case if and only if $r - s$ is representable by Q. However, we have already seen that $r - s$ is representable by Q if and only if $r - s = 0 \bmod q$. □

Example 3.1. Let $\mathscr{R} = \{(3,1), (1,5)\}$. Then $Q = \{-2, 4\}$ and $\gcd(\{-2, 4\}) = 2$. So the rule $(1,3)$ is derivable from $\mathscr{R}$ (note that $2 = 0 \bmod 2$) by

$$\cfrac{\cfrac{xy \vdash z}{x^5 y \vdash z}\ (1,5)}{x^3 y \vdash z}\ (3,1)$$

On the other hand, the rule $(6,1)$ is not derivable from $\mathscr{R}$ as $5 \neq 0 \bmod 2$.

For $\mathscr{R} = \{(1,3),(6,1)\}$ we obtain $\gcd(\{2,-5\}) = 1$. Therefore, all rules (n,k) with $n \geqslant k$ can be simulated. We show the simulation of ordinary left-contraction:

$$\cfrac{\cfrac{\cfrac{x^2 y \vdash z}{x^4 y \vdash z}\,(1,3)}{x^6 y \vdash z}\,(1,3)}{xy \vdash z}\,(6,1)$$

Theorem 3.1. Let $\mathscr{R}$ be a set of structural rules of a knotted commutative calculus. Then rule dependency from $\mathscr{R}$ is decidable.

Proof. The statement follows from Sections 3.1 and 3.2. □

As a consequence of this result, we have a decision procedure for shifting up (possibly multiple) cuts over regular rules (*cf.* condition $(*)'$ in Proposition 2.2). Indeed, we have the following theorem.

Theorem 3.2. Let K be a knotted commutative calculus whose set of structural rules is $\mathscr{R}$ and let $\rho \in \mathscr{R}$. Let S, S_1, S_2 be sequents in K and $S_1 \to_\rho S_2$. For each $S' \in Cut_l^*(S, S_2)$ one can decide whether there exists $S'' \in Cut_l^*(S, S_1)$ such that $S'' \to_{\mathscr{R}}^* S'$.

Proof. First note that $Cut_l^*(S, S_1)$ is a finite set. The claim then follows by Theorem 3.1. □

3.3. *Deciding weak substitutivity*

Theorem 3.2 ensures that for each regular rule ρ and sequent S condition $(*)'$ in Proposition 2.2 can be checked. However, to conclude that ρ is weakly substitutive, such checking should be done *for all* instances of the rule and *all* sequents $S \in K$. This 'brute force' approach results in a semi-decision procedure that eventually finds out if ρ is not weakly substitutive and does not terminate otherwise. To avoid checking possibly infinite instances of regular rules, we introduce below the notion of the *most general instance* of a generalised knotted structural rule ρ consisting of a rule schema $\sigma_1 \to_\rho \sigma_2$ such that,

($\star$) for each instance $S_1 \to_\rho S_2$ and sequent S, for each $S' \in Cut_l^*(S, S_2)$ obtained via cut(s) with (at least a) cut-formula not in the context, there exists $\sigma' \in Cut_l^*(\sigma, \sigma_2)$ where σ is a meta-sequent and S_1, S_2, S and S' are obtained by suitably replacing any meta-variables in $\sigma, \sigma_1, \sigma_2, \sigma'$ with formulae, where common meta-variables in $\sigma, \sigma_1, \sigma_2, \sigma'$ are substituted consistently.

Let ρ be the rule (m,l), and σ a meta-sequent $w \vdash x$. For example, the rule schema

$$\frac{x^m y \vdash z}{x^l y \vdash z}\,(m,l)$$

does not satisfy condition ($\star$). To see this, let $S_1 = A^{m+1}\,\Gamma \vdash \Delta$, $S_2 = A^{l+1}\,\Gamma \vdash \Delta$, $S = \Sigma \vdash A$ and $S' = \Sigma^{l+1}\,\Gamma \vdash \Delta$.

Definition 3.3. The *most general instance* of $\rho = (m,l)$ is obtained by setting $\sigma_1 = x^{m+\mathscr{K}}\, y \vdash z$ and $\sigma_2 = x^{l+\mathscr{K}}\, y \vdash z$ where σ_1, σ_2 represent a sequence of meta-sequents for $\mathscr{K}$ ranging

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over $\mathbb{N}$. Corresponding to $\sigma = w \vdash x, \sigma_1, \sigma_2$, we define

$$\begin{aligned} \tau_1 : \text{cuts}(\sigma, \sigma_1) &= w^I x^{m+\mathscr{K}-I} y \vdash z,\ I \leqslant m + \mathscr{K} \\ \tau_2 : \text{cuts}(\sigma, \sigma_2) &= w^J x^{l+\mathscr{K}-J} y \vdash z,\ J \leqslant l + \mathscr{K} . \end{aligned}$$

cuts(σ, σ_1) and cuts(σ, σ_2) are schemata representing sequences in $CUT_l^*(\sigma, \sigma_1)$ and $CUT_l^*(\sigma, \sigma_2)$, where m, l are fixed and I, J and $\mathscr{K}$ range over $\mathbb{N}$ with the indicated constraints. We call cuts(σ, σ_1) the *first* and cuts(σ, σ_2) the *second cut-schema with respect to* ρ. An instance of the schema is a (meta-) sequent obtained by instantiating I, J and $\mathscr{K}$.

Definition 3.4. Let $\mathscr{R}$ be a regular set and τ_1, τ_2 be two meta-sequents. We define $\tau_1 \leqslant_{\mathscr{R}} \tau_2$ if for every instance τ_2' of τ_2 there exists an instance τ_1' of τ_1 such that $\tau_1' \rightarrow_{\mathscr{R}}^* \tau_2'$ (where the common meta-variables of τ_1, τ_2 have to be substituted consistently). Otherwise, we call τ_2' a *counterexample schema* and write $\tau_1 \nleqslant_{\mathscr{R}} \tau_2$.

Proposition 3.3. Let ρ be a rule in $\mathscr{R}$. Then ρ is weakly substitutive if and only if cuts$(\sigma, \sigma_1) \leqslant_{\mathscr{R}}$ cuts(σ, σ_2).

Proof. By Proposition 2.2 and the presence of Θ (see Definition 2.1), ρ is weakly substitutive if and only if for all $S, S_1, S_2 \in K$ such that $S_1 \rightarrow_\rho S_2$, for each $S' \in Cut_l^*(S, S_2)$ obtained via cut(s) with (at least one) cut-formula not in the context, there exists an $S'' \in Cut_l^*(S, S_1)$ such that $S'' \rightarrow_{\mathscr{R}}^* S'$. Each instance of ρ has the form $A^{m+p}\,\Gamma \vdash \Delta \rightarrow_\rho A^{l+p}\,\Gamma$, for some $p \in \mathbb{N}$. Hence, let S be $\Sigma \vdash A$, and each such S' and S'' have the form $\Sigma^i\, A^{l+p-i}\,\Gamma \vdash \Delta$ and $\Sigma^j\, A^{l+p-j}\,\Gamma \vdash \Delta$ for some $i, j \in \mathbb{N}$. The claim then follows by replacing $\mathscr{K}$ with p in cuts(σ, σ_1) and cuts(σ, σ_2). □

To find out which regular rules are 'good' for reductive cut elimination (that is, weakly substitutive) and which are 'bad', we distinguish two cases according to whether the regular system $\mathscr{R}$ does or does not contain weakening. We start by considering the latter case, which requires an analysis of the subcases (1)–(3) identified in Section 3.2.

Every regular set of rules can be transformed to an equivalent set of rules in 'minimal form'. These minimal representations will be needed in the proofs of the propositions characterising weak substitutivity.

Definition 3.5. We define an ordering on rules of type (n, m): let $\rho_1 = (n_1, m_1)$ and $\rho_2 = (n_2, m_2)$. We say that ρ_1 is smaller than ρ_2 (notation $\rho_1 \sqsubset \rho_2$) if the following two conditions hold:

(a) $n_1 \leqslant n_2$ and $m_1 \leqslant m_2$; and
(b) either $n_1 < m_1$ or $n_2 < m_2$.

Let $\mathscr{R}'$ be a finite set of rules. We say that $\mathscr{R}' \sqsubset \rho$ if $\rho' \sqsubset \rho$ for all $\rho' \in \mathscr{R}'$.

Definition 3.6. Let $\mathscr{R}$ be a regular system. Then:

— $\mathscr{R}$ is called *minimal* if for all finite sets of rules $\mathscr{R}'$ that are derivable from $\mathscr{R}$, there exists no $\rho \in \mathscr{R}$ such that $\mathscr{R}' \sqsubset \rho$ and ρ is derivable from $\mathscr{R}'$.
— $\mathscr{R}$ is called *normal contractive* if all structural rules (apart from (e, l)) are of the form $(m_1, 1), \ldots (m_n, 1)$.

Note that any regular system $\mathscr{R}$ can be algorithmically transformed into an equivalent minimal system $\mathscr{R}_0$. Indeed, for all $\rho \in \mathscr{R}$, the set of rules $\mathscr{R}'$ with $\mathscr{R}' \sqsubset \rho$ is finite, and the derivability of rules is decidable by Theorem 3.1.

Example 3.2. Let $\mathscr{R} = \{(w,l),(4,1)\}$. Then $\mathscr{R}$ is not minimal as $(2,1)$ is derivable from $\mathscr{R}$, $(2,1) \sqsubset (4,1)$ and $(4,1)$ is derivable from $(2,1)$. The corresponding minimal system $\mathscr{R}_0$ is $\{(w,l),(2,1)\}$.

From now on, we will only consider minimal regular sets.

Proposition 3.4 (type-1-bad). Let $\mathscr{R}$ be a system of type (1) with the following properties:

(a) There exists a rule $(m,l) \in \mathscr{R}$, with $l > 1$.
(b) $\mathscr{R}$ is minimal.
(c) No rule (r,s) with $r < m$ and $s > 1$ is derivable in $\mathscr{R}$

Let $\sigma_1 = x^{m+\mathscr{K}}y \vdash z$, $\sigma_2 = x^{l+\mathscr{K}}y \vdash z$, $\sigma = w \vdash x$, and

$$\begin{aligned} \tau_1 &= \text{cuts}(\sigma,\sigma_1) = w^I x^{m+\mathscr{K}-I} y \vdash z,\ I \leqslant m+\mathscr{K} \\ \tau_2 &= \text{cuts}(\sigma,\sigma_2) = w^J x^{l+\mathscr{K}-J} y \vdash z,\ J \leqslant l+\mathscr{K} \end{aligned}$$

(see Definition 3.3). Then $\tau_1 \not\leqslant_{\mathscr{R}} \tau_2$.

Proof. Let σ,σ_1,σ_2 be as above. We instantiate τ_2 to τ_2' by setting $\mathscr{K} = 0$ and $J = l-1$. Then

$$\tau_2' = w^{l-1}xy \vdash z\,.$$

We have to consider the instance

$$\tau' = \tau_1\{\mathscr{K} \to 0\} = w^I x^{m-I} y \vdash z\,.$$

We prove that $\tau' \not\leqslant_{\mathscr{R}} \tau_2'$ and hence that τ_2' is a counterexample schema.

Let $(m,l) \in \mathscr{R}$ with $l > 1$ (such a rule exists by (a)). We distinguish two cases:

$l = 2$: Hence $\tau_2' = wxy \vdash z$. As $\mathscr{R}$ contains no weakening, $I > 0$ and $I < m$ are necessary constraints for the substitution of I.
If $I = 1$, then, necessarily, $x^{m-1} \to^*_{\mathscr{R}} x$, so $(m-1,1)$ would be a derivable rule. But $(m-1,1) \sqsubset (m,2)$ and $(m,2)$ is derivable from $(m-1,1)$ contradicting (b).
So let $I = i$ for $i > 1$. Then, necessarily, $w^i \to^*_{\mathscr{R}} w$ and $x^{m-i} \to^*_{\mathscr{R}} x$. But this is only possible if the rules $(i,1)$ and $(m-i,1)$ are derivable. But for $\mathscr{R}' = \{(i,1),(m-i,1)\}$, we have $\mathscr{R}' \sqsubset (m,2)$, and $(m,2)$ is derivable from $\mathscr{R}'$, which again contradicts (b).

$l > 2$: We check whether there is an i such that

$$S_i\colon\ w^i x^{m-i} y \vdash z \to^*_{\mathscr{R}} w^{l-1}xy \vdash z\,.$$

First, $i \geqslant l-1$ as $\mathscr{R}$ is of type (1).
So let $i = l-1$. Then $S_{l-1} = w^{l-1}x^{m-l+1}y \vdash z$. $S_{l-1} \to^*_{\mathscr{R}} w^{l-1}xy \vdash z$ requires $x^{m-l+1} \to^*_{\mathscr{R}} x$. But this implies that $\rho'\colon (m-l+1,1)$ is derivable in $\mathscr{R}$. But $\rho' \sqsubset (m,l)$ and (m,l) is derivable from ρ', contradicting (b).
Assume $i > l-1$. Then $S_i \to^*_{\mathscr{R}} w^{l-1}xy \vdash z$ requires $w^i \to^*_{\mathscr{R}} w^{l-1}$. But then the rule $(i,l-1)$ is a derivable rule with $l-1 > 1$, and as $i < m$, this contradicts (c). □

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Proposition 3.5 (Type-1-good). Let $\mathscr{R}$ be a normal contractive system of type (1). Let $\rho \in \mathscr{R}$ and τ_1, τ_2 be the cut-schemata corresponding to ρ. Then $\tau_1 \leqslant_{\mathscr{R}} \tau_2$.

Proof. ρ must be of the form $(m, 1)$. Let $\sigma, \sigma_1, \sigma_2, \tau_1, \tau_2$ be as in Definition 3.3 (with $l = 1$). Then for every instance $\{J \to i, \mathscr{K} \to k\}$ we get

$$w^i x^{m+k-i} y \vdash z \to^*_{\mathscr{R}} w^i x^{1+k-i} y \vdash z$$

for $i \leqslant k$ (setting I to i), and for $i = k+1$ we substitute I by $m+k$ to get $w^{m+k} y \vdash z$, but

$$w^{m+k} y \vdash z \to^*_{\mathscr{R}} w^{k+1} y \vdash z. \qquad \square$$

Proposition 3.6 (Type-2-bad). Let $\mathscr{R}$ be a minimal system of type (2), $\rho \in \mathscr{R}$ where $\rho = (m, l)$ such that $m = \min\{k \mid (k, k') \in \mathscr{R}\}$, $m < l$ and assume no rule (r, s) with $r < m$ is derivable in $\mathscr{R}$. Let τ_1, τ_2 be the cut-schemata corresponding to ρ. Then $\tau_1 \nleqslant_{\mathscr{R}} \tau_2$.

Proof. Let $\sigma, \sigma_1, \sigma_1, \tau_1, \tau_2$ be as in Definition 3.3.

$m = 1$: We instantiate τ_2 by $\{\mathscr{K} \to 0, J \to 1\}$, and the corresponding instance is S': $wx^{l-1} y \vdash z$ (note that $l > 1$). The only possible instances of τ_1 under $\mathscr{K} = 0$ are

$$xy \vdash z, \ wy \vdash z$$

Let S be one of these two sequents. Then it is clear that $S \not\to^*_{\mathscr{R}} S'$ ($(w, l) \notin \mathscr{R}$).

$m > 1$: In this case, since $S' = wx^{l-1} y \vdash z$, we must instantiate I to 1 (there is no contractive rule in $\mathscr{R}$). But then

$$S'' \colon \tau_1\{\mathscr{K} \to 0, J \to 1\} = wx^{m-1} y \vdash z\,.$$

If $S'' \to^*_{\mathscr{R}} S'$, then $(m-1, l-1)$ must be derivable in $\mathscr{R}$; but $(m-1, l-1) \sqsubset (m, l)$ and (m, l) is derivable from $(m-1, l-1)$, contradicting the minimality of $\mathscr{R}$. $\square$

Proposition 3.7 (Type-3-bad). Let $\mathscr{R}$ be a minimal system of type (3). Let $\rho \in \mathscr{R}$ such that $\rho = (1, k)$ for $k > 1$. Let τ_1, τ_2 be the cut-schemata corresponding to ρ. Then $\tau_1 \nleqslant_{\mathscr{R}} \tau_2$.

Proof. As in Proposition 3.6 (type-2-bad), we select the instance

$$S' \colon wx^{k-1} y \vdash z$$

from the schema τ_2. Again, the only possible instances of τ_1 are

$$S_1 \colon xy \vdash z, \ S_2 \colon wy \vdash z\,.$$

Clearly, $S_i \not\to^*_{\mathscr{R}} S'$ for $i = 1, 2$. $\square$

Corollary 3.1. A wf-regular set $\mathscr{R}$ is weakly substitutive if and only if $\mathscr{R}$ is normal contractive.

Proof. Note that every regular system $\mathscr{R}$ of type (1) can be transformed to $\mathscr{R}'$ such that $\mathscr{R} \sim \mathscr{R}'$ and either $\mathscr{R}'$ is normal contractive or $\mathscr{R}'$ satisfies the properties (a)–(c) of Proposition 3.4. Moreover, if $(m, l) \in \mathscr{R}$ for some $m < l$, then the rule $(1, l-m+1)$ is derivable in $\mathscr{R}$ by the existence of a characteristic number q (Proposition 3.2). Hence we can always assume that a system of type (3) contains a rule $(1,k)$ for $k > 1$. The claim then follows by Propositions 3.4–3.7. □

We now consider the case $(w,l) \in \mathscr{R}$.

Proposition 3.8. A w-regular set $\mathscr{R}$ is weakly substitutive if and only if:

(a) $\mathscr{R} = \{(w,l)\}$; or
(b) $\mathscr{R}$ contains at least a rule $(n,1)$, with $n > 1$.

Proof.

$\Longrightarrow$ Note that in case (b), by Lemma 3.1, ordinary contraction can be derived.
$\Longleftarrow$ Assume that $\mathscr{R}$ does not contain any $(n,1)$ rule, with $n > 1$. By Lemma 3.1, the rules in $\mathscr{R}$ are interderivable with those in $\mathscr{R}' = \{(w,l),(l+1,l)$ for some $l > 1$. In $\mathscr{R}'$, no rule $(k+1,k)$ for $k < l$ is derivable. It is not hard to see that $(l+1,l)$ is not weakly substitutive. Indeed, let

$$\tau_1 : \mathrm{cuts}(\sigma,\sigma_1) \;=\; w^I x^{l+1+\mathscr{K}-I} y \vdash z,\; I \leqslant (l+1)+\mathscr{K}$$
$$\tau_2 : \mathrm{cuts}(\sigma,\sigma_2) \;=\; w^J x^{l+\mathscr{K}-J} y \vdash z,\; J \leqslant l+\mathscr{K}\,.$$

Select the instance $S' = wx^{l-1}y \vdash z$ by instantiating $\{\mathscr{K} \to 0, J \to 1\}$ in τ_2. Then from $\mathscr{K} \to 0$ in τ_1, we obtain $S = w^I x^{(l+1)-I} y \vdash z$ from τ_1. There is no instance $S''ofS$ such that $S'' \to^*_{\mathscr{R}} S'$. Indeed, $I \to 1$ is necessary as $w^{i+1} \not\to^*_{\mathscr{R}} w$ in $\mathscr{R}'$. But with $I = 1$ we obtain $wx^l y \vdash z$ and $x^l \not\to^*_{\mathscr{R}} x^{l-1}$, and hence $S'' \not\to^*_{\mathscr{R}} S'$.

Remark 3.2. Since weakly substitutivity is a necessary condition for reductive cut elimination in knotted commutative calculi (Proposition 2.1), if a regular set cannot be transformed into an equivalent one that is either normal contractive or of the form (a) or (b) (see Theorem 3.8) the corresponding calculus does not admit reductive cut elimination, no matter what its logical rules are. However, this only says that in such a calculus cuts cannot be removed following Steps 1 and 2 described in Section 2.1 and *not* that applications of (CUT) cannot be removed at all. For instance, consider a calculus whose only structural rule (beside, of course, (e,l)) is $(3,2)$, and whose logical rules are

$$\frac{\vdash}{\Theta\, \alpha_1 \star \alpha_2 \vdash \Xi}\,(\star,l) \qquad \frac{\vdash}{\Theta \vdash \alpha_1 \star \alpha_2}\,(\star,r)\,.$$

The only sequents provable in this calculus are instances of the identity axiom schema. Hence (CUT) in this calculus is trivially admissible, even though its structural rules are not weakly substitutive.

Note that the counterexample schemas in the propositions above can be turned into counterexamples to cut admissibility along the line of Hori *et al.* (1994), if the calculus contains, for example, the implication connective of **ILL**.

4. Deciding reductivity

A knotted commutative calculus K admits reductive cut elimination if and only if:

(a) its structural rules are weakly substitutive; and
(b) its logical rules are reductive.

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Given K, a decision procedure for establishing whether (a) holds is contained in the previous section. Here we investigate knotted commutative calculi whose structural rules are weakly substitutive and provide algorithms to decide whether (b) holds, thus deciding the admissibility of reductive cut elimination for knotted commutative calculi. Our approach is based on substructural (propositional) resolution.

Given a regular set $\mathscr{R}$, we define a structural resolution calculus based on an operator $R_{\mathscr{R}}$.

Definition 4.1. Let $S_1\colon\ X \vdash \alpha$ and $S_2 = \alpha, Y \vdash z$ be Horn clauses, where α is a formula variable and X, Y, z are multisets of formula variables (z contains at most one element). Then the clause

$$XY \vdash z$$

is called the *resolvent* of S_1, S_2 and is denoted by $\mathscr{R}es(S_1, S_2)$.

Definition 4.2. Let $\mathscr{R}$ be a system of unary structural rules and $\mathscr{S}$ be a set of clauses. Then we define

$$\begin{aligned} X_{\mathscr{R}}(\mathscr{S}) &= \{S' \mid \text{ there exists an } S \in \mathscr{S} \text{ such that } S \rightarrow_{\mathscr{R}} S'\} \\ \mathscr{R}es(\mathscr{S}) &= \bigcup\{\mathscr{R}es(S_1, S_2) \mid S_1, S_2 \in \mathscr{S}\} \\ R_{\mathscr{R}}(\mathscr{S}) &= \mathscr{S} \cup X_{\mathscr{R}}(\mathscr{S}) \cup \mathscr{R}es(\mathscr{S}). \end{aligned}$$

The deductive closure under $R_{\mathscr{R}}$ is defined by

$$\begin{aligned} R^0_{\mathscr{R}}(\mathscr{S}) &= \mathscr{S} \\ R^{i+1}_{\mathscr{R}}(\mathscr{S}) &= R_{\mathscr{R}}(R^i_{\mathscr{R}}(\mathscr{S})) \\ R^*_{\mathscr{R}}(\mathscr{S}) &= \bigcup_{i \in \mathbb{N}} R^i_{\mathscr{R}}(\mathscr{S}). \end{aligned}$$

Remark 4.1. If $\mathscr{S}$ is a set of clauses, $R^*_{\mathscr{R}}(\mathscr{S})$ is the set of all clauses derivable by cut (on formula variables) and the structural rules of $\mathscr{R}$.

We distinguish two cases according to whether the weakening rule is or is not in $\mathscr{R}$.

4.1. $(w, l) \notin \mathscr{R}$

Let $\mathscr{R}$ be a normal contractive system (otherwise, by Corollary 3.1, $\mathscr{R}$ is not weakly substitutive). As the permutation rules are always available, we define two clauses to be equal if they are permutation variants of each other.

Definition 4.3. Let C_1, C_2 be Horn clauses where $C_1 = U \vdash \alpha$ and $C_2 = \alpha, V \vdash \gamma$. Then we write the resolvent

$$C\colon\ U, V \vdash \gamma$$

of C_1, C_2 as C_1C_2. We say that C is the product of C_1 and C_2; we call C_1 the *active clause* of the product and C_2 the *passive clause*. If C_1, C_2 have no resolvent with C_1 as active clause, we say that C_1C_2 is undefined.

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Remark 4.2. The multiplication defined above is neither associative nor commutative. For example, if $C_1 = \beta \vdash \alpha$ and $C_2 = \alpha, \alpha \vdash \gamma$, then $C_1(C_1C_2)$ is defined and is $\beta, \beta \vdash \gamma$, but $(C_1C_1)C_2$ is undefined, and, clearly, C_1C_2 is defined, but C_2C_1 is not.

On the other hand, C_1C_2 is unique if it exists, which justifies the notation as a binary function.

Although the product is not associative, it is *semi-associative* in the following sense.

Lemma 4.1. Let C_1, C_2, C_3 be Horn clauses such that $(C_1C_2)C_3$ is defined. Then

$$(C_1C_2)C_3 = C_1(C_2C_3).$$

Proof. The product $(C_1C_2)C_3$ is only defined if the clauses are of the following form

$$\begin{aligned} C_1 &= U \vdash \alpha \\ C_2 &= \alpha, V \vdash \beta \\ C_3 &= \beta, W \vdash \gamma. \end{aligned}$$

But then $(C_1C_2)C_3 = C_1(C_2C_3) = UVW \vdash \gamma$. □

Note that the product of Horn clauses represents resolution without structural rules (except permutation, which is built in).

Definition 4.4. Let $\mathscr{S}$ be a set of Horn clauses. Then we define

$$R(\mathscr{S}) = \{C_1C_2 \mid C_1, C_2 \in \mathscr{S} \text{ and } C_1C_2 \text{ is defined}\}.$$

Furthermore, we define the deductive closure:

$$\begin{aligned} R^0(\mathscr{S}) &= \mathscr{S} \\ R^{i+1}(\mathscr{S}) &= R(R^i(\mathscr{S})) \cup R^i(\mathscr{S}) \\ R^*(\mathscr{S}) &= \bigcup_{i \in \mathbb{N}} R^i(\mathscr{S}). \end{aligned}$$

The following lemma shows that every derivable Horn clause is a product of another derivable clause and an input clause. This is a standard result in automated deduction implying that there is always an input refutation of a set of Horn clauses, see, for example, Leitsch (1997).

Definition 4.5. A product of clauses $C_1, \ldots, C_n$ is said to be in *right-parenthesis form* if

$$C = C_1(C_2 \ldots (C_{n-1}C_n)).$$

Lemma 4.2. Every product of Horn clauses can be transformed into right-parenthesis form.

Proof. We use induction on the number n of clauses occurring in the product. The case $n = 1$ is trivial.

As the induction hypothesis, assume the lemma holds for n.

Let C be a product of $n + 1$ Horn clauses that is defined. Then $C = DE$, where D, E are products of $\leqslant n$ Horn clauses $D_1, \ldots, D_k$ and $E_1, \ldots, E_m$ with $k + m = n + 1$.

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By the induction hypothesis, $D = D_1D'$ for some D' (which is a product of $k-1$ Horn clauses), so $C = (D_1D')E$. By semi-associativity, we get $C = D_1(D'E)$. But $D'E$ is a product of n Horn clauses, so we apply the induction hypothesis again. By iteration of the argument, we eventually get

$$C = D_1(D_2 \ldots (E_1(E_2 \ldots (E_{m-1}E_m) \ldots).$$

□

Corollary 4.1. Let $\mathscr{S}$ be a set of Horn clauses and $C \in R^*(\mathscr{S})$. Then C can be represented in right-parenthesis form over clauses in $\mathscr{S}$.

Proof. By Lemma 4.2, every $C \in R^*(\mathscr{S})$ can be represented in right-parenthesis form. Clearly, all the clauses appearing in the product occur in $\mathscr{S}$. □

Definition 4.6. Let $\mathscr{R}$ be a normal contractive system. A clause C is in $\mathscr{R}$ *normal form* if no rule in $\mathscr{R}$ is applicable to C.

Let $\mathscr{S}$ be a set of Horn clauses. Then $v_{\mathscr{R}}(\mathscr{S})$ is the set of clauses in normal form that can be obtained by reduction via $\mathscr{R}$.

Note that for normal contractive systems, $v_{\mathscr{R}}(\mathscr{S})$ is always finite for finite $\mathscr{S}$. But there is even more, as given by the following proposition.

Proposition 4.1. Let $\mathscr{R}$ a normal contractive system and $\mathscr{S}$ be a (possibly infinite) set of Horn clauses over a finite set of variables (formula- and/or multisets of formula-variables). Then $v_{\mathscr{R}}(\mathscr{S})$ is finite.

Proof. Let $V : \{x_1, \ldots, x_n\}$ be the set of all variables in $\mathscr{S}$. Then every clause over V is of the form

$$C:\ x_1^{k_1} \ldots x_n^{k_n} \vdash x_j^p$$

for $k_i \in \mathbb{N}$ and $p \in \{0, 1\}$ (if $k_i = 0$ we omit the element $x_i^{k_i}$ from the sequent). Now let k be the maximal number such that $(k, 1) \in \mathscr{R}$. Then $v(C)$ consists only of clauses

$$D:\ x_1^{r_1} \ldots x_n^{r_n} \vdash x_j^p$$

for $r_i \leqslant k$ for $i = 1, \ldots, n$. Indeed, any larger power of an x_i can be reduced via $\mathscr{R}$. But the number of such clauses is finite and $\leqslant k^{n+1}$. □

Lemma 4.3. Let $\mathscr{R}$ be a normal contractive system and $\mathscr{S}$ be a set of Horn clauses. Then

$$v_{\mathscr{R}}(R^*_{\mathscr{R}}(\mathscr{S})) = v_{\mathscr{R}}(R^*(\mathscr{S})).$$

Proof. It is enough to show that for all $C \in R^*_{\mathscr{R}}(\mathscr{S})$ there exists a $D \in R^*(\mathscr{S})$ with $D \to^*_{\mathscr{R}} C$ (then, clearly, $v_{\mathscr{R}}(C) \subseteq v_{\mathscr{R}}(D)$). We prove this property for $C \in R^i_{\mathscr{R}}(\mathscr{S})$ by induction on i. The case $i = 0$ is trivial, as $\mathscr{S} \subseteq R^*(\mathscr{S})$.

As induction hypothesis, assume that for all $C \in R^i_{\mathscr{R}}(\mathscr{S})$ there exists a $D \in R^*(\mathscr{S})$ with $D \to^*_{\mathscr{R}} C$.

Now let $C \in R^{i+1}_{\mathscr{R}}(\mathscr{S}) - R^i_{\mathscr{R}}(\mathscr{S})$.

(a) If $C' \to_{\mathscr{R}} C$ for $C' \in R^i_{\mathscr{R}}(\mathscr{S})$ then, by the induction hypothesis, there exists a $D' \in R^*(\mathscr{S})$ such that $D' \to^*_{\mathscr{R}} C'$. But then clearly $D' \to^*_{\mathscr{R}} C$.

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(b) Let $C = C_1 C_2$ for $C_1, C_2 \in R^i_{\mathscr{R}}(\mathscr{S})$ and

$$C_1 = U \vdash \alpha,\ C_2 = \alpha^m Y \vdash z \text{ for some } \alpha \in FV$$

and α not in Y. By the induction hypothesis, there exist clauses $D_1 : U_0 \vdash \alpha$ and $D_2 : \alpha^M Y_0 \vdash z$ with $D_1, D_2 \in R^*(\mathscr{S})$ and $D_1 \to^*_{\mathscr{R}} C_1$, $D_2 \to^*_{\mathscr{R}} C_2$. In particular, we have

$$U_0 \to^*_{\mathscr{R}} U,\ \alpha^M \to^*_{\mathscr{R}} \alpha^m \text{ and } Y_0 \to^*_{\mathscr{R}} Y\ .$$

(b1) $m > 1$:
Hence $C = \alpha^{m-1} U Y \vdash z$. Clearly $\alpha^{M-1} \to^*_{\mathscr{R}} \alpha^{m-1}$, so the product D of D_1, D_2 fulfills

$$D = \alpha^{M-1} U_0 Y_0 \vdash z \to^*_{\mathscr{R}} C.$$

So $D \in R^*(\mathscr{S})$ and $D \to^*_{\mathscr{R}} C$.

(b2) $m = 1$:
Hence $C = UY \vdash z$. Note that $\alpha^M \to^*_{\mathscr{R}} \alpha$ and, more generally, $X^M \to^*_{\mathscr{R}} X$ for all sequences X. Resolving D_1 with D_2 M-times, that is, constructing the product

$$D_1(D_1 \ldots (D_1 D_2) \ldots) \text{ with } M \text{ occurrences of } D_1,$$

results in the clause $D = U_0^M Y_0 \vdash z$, which is in $R^*(\mathscr{S})$. But

$$U_0^M Y_0 \vdash z \to^*_{\mathscr{R}} U_0 Y_0 \vdash z \to^*_{\mathscr{R}} UY \vdash z\,,$$

so $D \to^*_{\mathscr{R}} C$. □

Lemma 4.4. Let $\mathscr{S}$ be a set of Horn clauses and $\mathscr{R}$ be a normal contractive system. Then there exists an algorithm constructing $v_{\mathscr{R}}(R^*(\mathscr{S}))$.

Proof. We know that all clauses in $R^*(\mathscr{S})$ can be written in right-parenthesis form (Lemma 4.2). Let $\mathscr{S} = \{C_1, \ldots, C_n\}$. We construct a search tree as follows:

— Let T_0 be the root.

— T_1 is defined by n edges $E_1, \ldots E_n$ spreading from the root and labelled with the clauses $C_1, \ldots, C_n$. For every end-node N_i in T_1 corresponding to the edge E_i, we define $\gamma(N_i) = C_i$, stop$(N_i) =$ false.

— Let T_n be already constructed. We define T_{n+1} as follows. To every end-node M of T_n for which stop$(M) =$ false, attach n edges labelled by the clauses $C_1, \ldots, C_n$. For the corresponding end-nodes $N(M, C_i)$, we define

$$\gamma(N(M, C_i)) = C_i \gamma(N(M, C_i))\,,$$

provided the product is defined. If the product is undefined, we delete $N(M, C_i)$.
For every end-node N that is not deleted, we check whether there exists a predecessor N' on the path from the root to N with $v_{\mathscr{R}}(\gamma(N)) = v_{\mathscr{R}}(\gamma(N'))$; if the last equation holds, we define stop$(N) =$ true.

As $v_{\mathscr{R}}(R^*(\mathscr{S}))$ is finite, the production of the tree will stop after finitely many steps. Indeed, an infinite path $(N_i)_{i \in \mathbb{N}}$ in the tree can only be constructed if $v_{\mathscr{R}}(N_i) \neq v_{\mathscr{R}}(N_j)$ for all i, j with $i \neq j$. This is impossible as the set of all subsets of $v_{\mathscr{R}}(R^*(\mathscr{S}))$ is finite. If $\gamma(N_i) = \gamma(N_j)$ for $i < j$, we may stop the production of new edges as no new normal forms

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of clauses will be produced. The tree T^* produces all clauses in $R^*(\mathscr{S})$ as it produces all products in right-parenthesis form, which is sufficient. □

Corollary 4.2. Let $\mathscr{S}$ be a set of Horn clauses and $\mathscr{R}$ be a normal contractive system. Then there exists an algorithm constructing $v_{\mathscr{R}}(R^*_{\mathscr{R}}(\mathscr{S}))$.

Proof. The proof is immediate from Lemmas 4.4 and 4.3. □

4.2. $(w, l) \in \mathscr{R}$

By Proposition 3.8, $\mathscr{R}$ is weakly substitutive if and only if $\mathscr{R} = \{(w, l)\}$ or at least a rule $(n_i, 1) \in \mathscr{R}$, with $n_i > 1$. In both cases, reductivity could be checked using the results in Miller and Pimentel (2002; 2005). We give below an alternative proof using resolution.

Theorem 4.1. Let $\mathscr{R} = \{(w, l), (n_i, 1)\}$ for $n_i > 1$. Then $R^*_{\mathscr{R}}(\mathscr{S})$ is decidable.

Proof. We have shown in Section 3 that in this case ordinary contraction can be simulated. Thus the resolution calculus is that of ordinary classical resolution, which is decidable. Indeed, using the contraction normal form of clauses (no repetition of occurring atoms), only finitely many clauses can be derived, or, more formally, $RN^*(\mathscr{S})$ is finite for the corresponding normal resolution operator RN. Then a clause C is in $R^*_{\mathscr{R}}(\mathscr{S})$ if either the normal form C^* of C occurs in $RN(\mathscr{S})$ or C^* can be obtained from $RN(\mathscr{S})$ through weakening (that is, subsumption). It is obvious that this test can be done algorithmically. □

For the proof of the theorem, we need the subsumption principle from automated deduction adapted to our purposes.

Definition 4.7. Let $\mathscr{S}_1, \mathscr{S}_2$ be sets of Horn clauses. Then $\mathscr{S}_1 \leqslant_{ss} \mathscr{S}_2$ if for every $D \in \mathscr{S}_2$ there exists a $C \in \mathscr{S}_1$ such that D can be obtained from C by (possibly multiple) applications of (w, l).

Theorem 4.2. Let $\mathscr{R} = \{(w, l)\}$. Then $R^*_{\mathscr{R}}(\mathscr{S})$ is decidable.

Proof.

(a) There are no positive unit clauses in $\mathscr{S}$.
We check whether a clause C is in $R_{\mathscr{R}} * (\mathscr{S})$. To this end, we produce $R^*_{\mathscr{R}}(\mathscr{S})$ but stop the production on clauses D such that $l(D) > l(C) + 1$ (where $l(C)$ is the length of C, that is, the number of variables occurring in C). Indeed, if we obtain a clause D with $l(D) > l(C) + 1$, it cannot contribute to a derivation of C. Note that the length can only be decreased by resolution with *negative* unit clauses; but these are only the last elements of resolution products, reducing the length by at most one. So let $\mathscr{S}'$ be the set of clauses produced as indicated above. Then $\mathscr{S}'$ is finite and can be produced in finitely many steps. Finally, $C \in \mathscr{S}'$ if and only if $C \in R^*_{\mathscr{R}}(\mathscr{S})$.

(b) There are positive unit clauses in $\mathscr{S}$.
Produce $R^*_{+}(\mathscr{S})$, the set of all positive clauses in $R^*(\mathscr{S})$. $R^*_{+}(\mathscr{S})$ is finite and can be constructed by hyperresolution (see Leitsch (1997, Chapter 3.4)).

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Let $\mathscr{S}_1 = \mathscr{S}_0 \cup R^*_+(\mathscr{S})$ such that $\mathscr{S}_0$ consists of the non-positive clauses in $\mathscr{S}$. Now perform resolution only between clauses in $R^*_+(\mathscr{S})$ and $\mathscr{S}_0$ until all formula variables in $R^*_+(\mathscr{S})$ are cut out from (the antecedents of) the clauses in $\mathscr{S}_0$. The result is a finite set of clauses

$$\mathscr{S}_2 = \mathscr{S}' \cup R^*_+(\mathscr{S})$$

such that no resolvents are definable between $R^*_+(\mathscr{S})$ and $\mathscr{S}'$, and $\mathscr{S}'$ consists only of non-positive clauses. By the definition of $\mathscr{S}_2$, we have

$$\mathscr{S}' \leqslant_{ss} \mathscr{S}_0 .$$

But then $\mathscr{S}_2 \leqslant_{ss} \mathscr{S}_1$ and, by the subsumption principle in resolution (see Leitsch (1997, Chapter 4.2)),

$$R^*(\mathscr{S}_2) \leqslant_{ss} R^*(\mathscr{S}_1) = R^*(\mathscr{S}) \leqslant_{ss} R^*_{\mathscr{R}}(\mathscr{S}) .$$

By the transitivity of subsumption, we get

$$R^*(\mathscr{S}_2) \leqslant_{ss} R^*_{\mathscr{R}}(\mathscr{S}) .$$

Moreover, by the definition of $\mathscr{S}_2$, we have

$$R^*(\mathscr{S}_2) = R^*(\mathscr{S}') \cup R^*_+(\mathscr{S}) .$$

Now we check whether a clause C is in $R^*_{\mathscr{R}}(\mathscr{S})$. Clearly, $R^*(\mathscr{S}_2)$ subsumes a clause C if either

(i) $R^*_+(\mathscr{S}) \leqslant_{ss} C$; or

(ii) $R^*(\mathscr{S}') \leqslant_{ss} C$.

(i) can be checked directly as $R^*_+(\mathscr{S})$ is finite. For (ii), we apply the same method as in (a) (in fact there are no positive unit clauses in $\mathscr{S}'$). □

Theorem 4.3. Reductivity is decidable for knotted commutative calculi with weakly substitutive regular sets.

Proof. Reductivity for normal contractive systems is decidable. Indeed, let $C(\varphi)$ be the reduction set of a $\star$-cut-derivation schema (see Definition 2.5) φ, and $S(\varphi)$ be the end-sequent of φ. Then $S(\varphi)$ is in $\mathscr{R}$-normal form. Therefore $S(\varphi) \in v_{\mathscr{R}}(R^*_{\mathscr{R}}(C(\varphi)))$ if and only if $S(\varphi) \in R^*_{\mathscr{R}}(C(\varphi))$.

By Lemma 4.3,

$$v_{\mathscr{R}}(R^*_{\mathscr{R}}(C(\varphi))) = v_{\mathscr{R}}(R^*(C(\varphi))) .$$

By Lemma 4.4, the finite set $v_{\mathscr{R}}(R^*(C(\varphi)))$ can be constructed algorithmically. This gives a decision procedure for reductivity. The claim then follows by Section 3.3, Theorem 4.1 and Theorem 4.2. □

Remark 4.3. If rules introducing a connective $\star$ are not reductive, the corresponding knotted commutative calculus does not admit reductive cut elimination. As in the case of weak substitutivity, this is not enough to conclude that the calculus does not admit cut elimination at all. For example, the rules for $\bar{\wedge}$ in the calculus K_1 of Example 2.1 are not reductive, see, for example, Ciabattoni and Terui (2006). However, K_1 trivially admits cut elimination since the rule $(\bar{\wedge}, l)$ cannot appear in any derivation and the only sequents provable in K_1 are instances of the identity axiom schema.(Note that applications of (CUT) on these sequents can be easily eliminated).

5. A general cut-elimination procedure

Here we provide a constructive proof of cut elimination for knotted commutative calculi whose structural rules are weakly substitutive and logical rules reductive. From now on, K will denote any such calculus.

Definition 5.1. The *length* $|d|$ of a derivation d is the maximal number of inference rules + 1 occurring on any branch of d. The *complexity* $|A|$ of a formula A is defined to be the number of occurrences of its connectives. The *cut rank* $\rho(d)$ of d is (the maximal complexity of the cut-formulae in d) + 1 ($\rho(d) = 0$ if d has no cuts).

Our cut-elimination procedure for K proceeds by removing cuts that are topmost among all cuts with cut rank equal to the rank of the whole deduction. For example, let

$$\frac{\begin{array}{cc}\vdots\, d_r & \vdots\, d_l \\ \Gamma \vdash A & \Sigma\, A \vdash \Pi\end{array}}{\Gamma, \Sigma \vdash \Pi}\,(\mathrm{CUT})$$

be a subderivation ending in such a cut. Roughly speaking, using the fact that rules are weakly substitutive, our strategy is as follows:

(1) We shift this cut up over d_r *as much as possible* until we meet an axiom (or a logical rule with no premiss) or a logical rule introducing the cut formula A (Lemma 5.2). In the former case the cut is easily eliminated. In the latter case we shift this cut upward over d_l, and go to (2).
(2) We replace this cut by cuts with smaller complexity, when we meet a rule introducing the cut formula A (Lemma 5.1). By Proposition 2.1 this can be done since the logical rules are reductive.

Remark 5.1. Using Theorems 3.2 and 4.3 to automate Steps (1) and (2), the above strategy (in fact, the formal proof below) can lead to a mechanical construction of a cut-free proof from any proof in a given K.

From now on we will write $d \vdash_K S$ if d is a derivation in K of S.

Lemma 5.1 (Logical connectives). Let K be any knotted commutative sequent calculus whose structural rules are weakly substitutive. Let $d_l \vdash_K \Sigma\, A \vdash B$, and $d_r \vdash_K \Gamma \vdash A$, with $\rho(d_l)$, $\rho(d_r) \leqslant |A|$. If $A = \star(A_1, \ldots, A_n)$ is the principal formula of the last rule in the derivation d_r and the rules for the connective $\star$ are reductive in K, then for all $T' \in Cut^*_l(\Gamma \vdash A, \Sigma\, A \vdash B)$, we can find a derivation $d \vdash_K T'$ with $\rho(d) \leqslant |A|$.

Of course, one could derive T' by applying (CUT), but the resulting derivation would then have cut rank $|A| + 1$.

Proof. We proceed by induction on $|d_l|$.

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Base case: $|d_l| = 1$.

Then $\Sigma A \vdash B$ is an instance of either an axiom or of a logical rule with no premisses. In the former case $Cut^*_l(\Gamma \vdash A, \Sigma A \vdash B) = \{\Sigma A \vdash B,\ \Gamma \vdash A\}$, and the required derivation is either d_l or d_r while in the latter case we distinguish two (sub)cases: the rule does not introduces A in the antecedent or it does. In the former (sub)case by condition **(log2)** the required derivation is T' while in the latter (sub)case the claim follows by Proposition 2.1, as rules for $\star$ are reductive.

Induction hypothesis: $|d_l| > 1$.

Let (r) be the last inference rule applied in d_l. Without loss of generality, assume that (r) has the form

$$\frac{S_1 \ \dots \ S_m}{S}\ .$$

— If (r) is neither (CUT) nor a logical rule introducing A in the antecedent, by applying the induction hypothesis m times for all $T' \in \bigcup_{i=1}^m Cut^*_l(\Gamma \vdash A, S_i)$, we can find a derivation $d' \vdash_K T'$ with $\rho(d') \leqslant |A|$. Now by weak substitutivity (and **(log2)**, if (r) is a logical rule), every $T' \in Cut^*_l(\Gamma \vdash A, \Sigma A \vdash B)$ is cut-free derivable from $\bigcup_{i=1}^m Cut^*_l(\Gamma \vdash A, S_i)$. Hence T' has a derivation d in K with $\rho(d) \leqslant |A|$.

— If $(r) = (CUT)$, the claim follows by the induction hypothesis and an application of (CUT) (note that by hypothesis the cut-formula of this cut is of smaller complexity than A).

— Suppose that (r) is a rule introducing A in the antecedent. In this case, by condition **(log2)**, the rule

$$\frac{S'_1 \ \dots \ S'_m}{S'}\ ,$$

where S' is obtained by λ consecutive applications of (CUT) between S and $\Gamma \vdash A$ (where $\lambda + 1$ is the number of occurrences of A on the antecedent in S) and $S'_i \in Cut^*_l(\Gamma \vdash A, S_i)$, for each $1 \leqslant i \leqslant m$, is an instance of (r). Hence the claim follows by Proposition 2.1, being the rules for $\star$ reductive.

Lemma 5.2 (Shifting Lemma). Let K be any knotted commutative calculus in which:

(a) logical rules are reductive; and
(b) structural rules are weakly substitutive.

Let $d_r \vdash_K \Gamma \vdash A$ and $d_l \vdash_K \Sigma A \vdash B$ with $\rho(d_r), \rho(d_l) \leqslant |A|$. For all $T' \in Cut^*_r(\Sigma A \vdash B, \Gamma \vdash A)$, we can find a derivation $d \vdash_K T'$ with $\rho(d) \leqslant |A|$.

Proof. We proceed by induction on $|d_r|$, as in the previous proof.

The main difference is that condition **(log3)** is used and, when the last inference rule (r) applied is a rule (with premisses) introducing A on the consequent, the claim follows by Lemma 5.1. □

Theorem 5.1 (Cut elimination). Any knotted commutative calculus K in which:

(a) logical rules are reductive; and
(b) structural rules are weakly substitutive, admits cut elimination.

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Proof. Let d be a derivation in K with $\rho(d) > 0$. The proof proceeds by a double induction on $(\rho(d), n\rho(d))$, where $n\rho(d)$ is the number of cuts in d with cut rank $\rho(d)$. Indeed, let us take in d an uppermost cut with cut rank $\rho(d)$. By applying Lemma 5.2 to its premisses $\Gamma\, A \vdash \Delta$ and $\Sigma \vdash A$, either $\rho(d)$ or $n\rho(d)$ decreases. □

Remarkably, our cut-elimination procedure can be applied to any single-conclusion sequent calculus whose logical rules, satisfying conditions **(log2)** and **(log3)**, are reductive, and structural rules are weakly substitutive. In particular, it does work for the *simple sequent calculi* considered in Ciabattoni and Terui (2006). The same does not hold for the well-known cut-elimination procedures *à la* Gentzen and *à la* Schütte-Tait (Schütte 1960; Tait 1968). Indeed, Gentzen's method can be applied only when suitable *ad hoc* derivable generalisations of the cut rule (mix-style) are found. These generalisations, which are needed to cope with rules duplicating formulas, are not always easy to define. As an example, consider the calculus obtained by extending **ILL** with weak contraction, that is, the rule

$$\frac{\Theta\, \alpha^2 \vdash \epsilon}{\Theta\, \alpha \vdash \epsilon}\,.$$

On the other hand, the applicability of the Schütte-Tait cut-elimination method relies on the inversion of (at least) one of the premises of each canonic cut-derivation. This cannot always be done in calculi that admit reductive cut elimination. For example, neither of the premisses of a canonic $\wedge$ cut-derivation can be inverted in the usual way in the calculus **LBC-** of Baaz *et al.* (2004) (see Example 2.1), so the Schütte-Tait procedure does not apply to **LBC-** (although its logical rules satisfy conditions **(log2)** and **(log3)** and are reductive, while its structural rules are weakly substitutive).

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