

On-line Ascent Phase Trajectory Optimal Guidance Algorithm based on Pseudo-spectral Method and Sensitivity Updates

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The objective of this paper is to investigate an online method to generate an optimal ascent trajectory for air-breathing hypersonic vehicles. A direct method called the Pseudo-spectral method shows promise for real-time optimal guidance. A significant barrier to this optimisation-based control strategy is computational delay, especially when the solution time of the non-linear programming problem exceeds the sampling time. Therefore, an online guidance algorithm for an air-breathing hypersonic vehicles with process constraints and terminal states constraints is proposed based on the Pseudo-spectral method and sensitivity analysis in this paper, which can reduce online computational costs and improve performance significantly. The proposed ascent optimal guidance method can successively generate online open-loop suboptimal controls without the design procedure of an inner-loop feedback controller. Considering model parameters' uncertainties and external disturbance, a sampling theorem is proposed that indicates the effect of the Lipschitz constant of the dynamics on sampling frequency. The simulation results indicate that the proposed method offers improved performance and has promising ability to generate an optimal ascent trajectory for air-breathing hypersonic vehicles.

KEY WORDS

1. Online optimal guidance.
2. Pseudo-spectral method.
3. Sensitivity analysis.
4. Air-breathing hypersonic vehicles

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1. INTRODUCTION. The air-breathing hypersonic launch vehicle has many inherent features to reduce the cost of payloads for future space transportation (Henry and McLellan, 1971). Optimal steering for an air-breathing hypersonic launch vehicle exhibits completely different behaviour from conventional rocket flight. Both thrust and lift depend on the atmospheric dynamic pressure (Yamamoto

and Kawaguchi, 2007), and an optimal ascent guidance algorithm should be adopted during the flight to achieve a fuel-optimal control trajectory.

Ascent phase trajectory optimisation and guidance of an air-breathing hypersonic vehicle is a difficult non-linear problem with various constraints (Vinh, 1981; Garcia, 2005). Aerodynamics, propulsion, vehicle and attitude are strongly coupled during the ascent phase. The traditional approach for trajectory guidance is online tracking of the optimal reference profile which was designed offline (Li et al., 2014; Saraf et al., 2004), but this method does not have enough capability to deal with external disturbances such as aerodynamic and thrust errors.

When solving optimal guidance trajectories in endo-atmospheric flight, it is difficult to solve the Two Point Boundary Value Problem (TPBVP) in time. Both indirect and direct methods have been mostly used to solve the offline trajectory optimisation problem in previous literature (Conway, 2012). The numerical algorithms of non-linear trajectory optimisation for vehicles were summarised and systematically analysed by Betts (1998) and Huang et al. (2012). Dukeman (2002) and Dukeman and Calise (2003) developed a closed-form ascent guidance algorithm, to cyclically solve the TBBVP during endo-atmospheric flight. The work of Murillo and Lu (2010) adopted an indirect scheme to achieve a proposed closed-loop ascent guidance algorithm by solving TPBVP; they first derived the necessary conditions of the optimal ascent trajectory. However, indirect methods suffer from difficulties in finding an appropriate initial guess and obtaining a convergent solution for the TPBVP.

In recent years, with improvements in on board computer performance, a direct optimal method named Gauss Pseudo-spectral Method (GPM) has been proved to be an effective way to solve the real-time trajectory optimisation problem (Bollino et al., 2006). GPM not only has high precision and fast convergence in solving the optimal control problems, but also proved the equivalence between the Karush-Kuhn-Tucker (KKT) conditions and the Hamiltonian Boundary Value Problem (HBVP) first-order optimality conditions (Benson, 2005; Huntington, 2007). This removes the need for computing analytical gradients of aerodynamic coefficients. This approach to optimal feedback control based on the Carathodory- π trajectory concept is capable of solving industrial-strength problems and it offers the ability to solve the Non-linear Programming (NLP) problem online (Ross et al., 2008). Two algorithms (free sampling frequency and fixed sampling frequency) are proposed in a real-time optimal feedback control method based on a domain transformation technique and a Radau-based Pseudo-spectral method. The techniques are used for a flexible robot arm and a benchmark inverted pendulum problem (Ross et al., 2006) as well. A rapid trajectory optimisation method via GPM was recently studied for air-breathing hypersonic vehicles (Zhang et al., 2012). However, a significant barrier to this optimisation-based control strategy is computational delay, especially when the solution time of the NLP problem exceeds the sampling time.

Recently proposed non-linear programming sensitivity methods have reduced online computational costs and can lead to significantly improved performance. Sensitivity update strategy has been successfully used for Non-linear Model Predictive Control (NMPC) problems (Zavala and Biegler, 2009; Yang and Biegler 2013). NMPC is an approach to feedback design that is based on the solution of an Optimal Control Problem (OCP) at each controller update step (Grüne and Pannek, 2011). The basic sensitivity

strategy for NLP problem solvers is derived through application of the Implicit Function Theorem (IFT) to the KKT conditions of a parametric NLP problem (Fiacco, 1976). However, the air-breathing hypersonic launch vehicle exhibits severe non-linearity, the sample-and-hold technique used in traditional NMPC problems cannot be adopted.

In this paper, an online guidance algorithm for an air-breathing hypersonic vehicles with process constraints and terminal states constraints is proposed based on the Pseudo-spectral method and sensitivity analysis. An approximated closed-loop optimal control problem solution is computed in the following way: in each sampling interval, the nominal non-linear model was used to predict the next sampling states and an optimal control problem is solved based on the predicted states. When the system dynamics are simulated to the next sampling instant, based on the difference between predicted states and measured states, the resulting optimal control sequence which is applied as input for the next sampling interval(s), is corrected via sensitivities. This procedure is then repeated iteratively. The proposed ascent optimal guidance method can successively generate online open-loop suboptimal controls without the design procedure of an inner-loop feedback controller. This new proposed guidance method has the ability of online trajectory reconstruction for the optimal performance index and has high accuracy and robustness via sensitivity updates.

This paper is organised as follows. In Section 2 the nominal differential equations of an air-breathing hypersonic launch vehicle are presented. In Section 3 the online optimal guidance algorithm based on the Pseudo-spectral method and sensitivity updates is proposed. In Section 4, considering model parameters uncertainties and external disturbance, a sampling theorem is proposed which indicates the effect of the Lipschitz constant of the dynamics on sampling frequency. In Section 5, simulation results indicate that the method proposed above offers improved performance and has promising ability for online computation. In Section 6, some conclusions and indications of future research are given.

2. PROBLEM FORMULATION. During the ascent phase for a point-mass vehicle model, the hypersonic vehicle dynamics are modelled using Equations (1) to (4) for a non-rotating Earth assumption (Prasanna et al., 2005).

$$\frac{dh}{dt} = v \sin \gamma \quad (1)$$

$$\frac{dv}{dt} = \frac{T \cos \alpha - D}{m} - \frac{\mu \sin \gamma}{(R_e + h)^2} \quad (2)$$

$$\frac{d\gamma}{dt} = \frac{T \sin \alpha + L}{mv} + \cos \gamma \left[\frac{v}{(R_e + h)} - \frac{\mu}{v(R_e + h)^2} \right] \quad (3)$$

$$\frac{dm}{dt} = -\frac{T}{I_{sp}g} \quad (4)$$

where, h is the altitude, v is the velocity, α is the angle of attack, γ is the flight path angle, m is the mass, T is the aircraft thrust, I_{sp} is the engine specific, D and L are drag and lift, respectively. μ is the gravitational constant, g is the gravitational acceleration and R_e is the radius of the earth.

Lift force L and drag force D are given as follows:

$$L = \frac{1}{2} \rho v^2 S_{ref} C_L(M_a, \alpha) \tag{5}$$

$$D = \frac{1}{2} \rho v^2 S_{ref} C_D(M_a, \alpha) \tag{6}$$

where ρ , S_{ref} are the air density of the current altitude and the reference area respectively. C_L and C_D are the lift and drag coefficients respectively, which are the non-linear functions of the attack angle α and the mach M_a .

Considering the thrust T and aerodynamic forces as a composition of forces, we define the longitudinal, lateral accelerations and dynamic pressure as a_{long} , a_{lat} and Q respectively.

$$a_{long} = \frac{T \cos \alpha - D}{m} - \frac{\mu \sin \gamma}{(R_e + h)^2} \tag{7}$$

$$a_{lat} = \frac{T \sin \alpha + L}{m} + \cos \gamma \left[\frac{v^2}{(R_e + h)} - \frac{\mu}{(R_e + h)^2} \right] \tag{8}$$

$$Q = \frac{1}{2} \rho v^2 \tag{9}$$

The 1976 US Standard Atmosphere is used as the model for the density ρ which is applied to compute lift and drag force. The speed of sound c used to calculate the Mach number M_a is also provided in the model.

Reduction in fuel will make the vehicle climb to a predetermined height; if we put minimum fuel consumption as a objective of ascent flight phase, the cruise segment of flight range will be greatly increased. For these reasons, the optimisation objective is to minimise fuel consumption (or maximise mass at the end) during the ascent phase, that is equivalent to the following formula:

$$\min J = -m_f \tag{10}$$

3. OPTIMAL GUIDANCE ALGORITHM BASED ON PSEUDO-SPECTRAL METHOD AND SENSITIVITY UPDATES.

The idea of a recursive open-loop solution is compelling, because control is available at any point that has been defined as optimal. Once the state is perturbed from the expected optimal path, correcting back to that trajectory is likely not optimal from the disturbed position, and a new optimal path originating from the current state should be introduced. However, a significant barrier to this optimisation-based control strategy is computational delay, especially when the solution time of the optimal control problem exceeds the sampling time. Sensitivity analysis can be adopted to reduce online computational costs.

3.1. *Optimal control problem.* The optimal control problem, which is often turned into a Non-linear Programming (NLP) problem, can be written in the following general form (the 1976 US atmosphere model which as a lookup table was implemented in the optimisation problem (Murillo and Lu, 2010)). The state, $x(t) = [h \ v \ \gamma \ m]$, control, $u(t) = \alpha$, and final time, t_f , that minimise the cost function

$$J = M[x(t_f), t_f] + \int_{t_0}^{t_f} g(x(t), u(t), t) dt \tag{11}$$

subject to the constraints, differential dynamics, terminal constraints and path constraints:

$$\dot{x}(t) = f[x(t), u(t), t] \quad (12)$$

$$\varphi[x(t_0), t_0, x(t_f), t_f] = 0 \quad (13)$$

$$c[x(t), u(t), t] \leq 0 \quad (14)$$

3.2. *The basic principle of the Gauss Pseudo-spectral Method.* The trajectory optimisation problem is formulated over the time interval $[t_0, t_f]$, and the Legendre-Gauss (LG) points lie in the interval $[-1, +1]$. Consider now the following transformation of the independent variable, t , to the variable $\tau \in [-1, +1]$ (Benson, 2005) :

$$t = \frac{t_f - t_0}{2} \tau + \frac{t_f + t_0}{2} \quad (15)$$

Furthermore, we choose the collocation points in the phase to be the set of LG points, (τ_1, \dots, τ_N) , which are the roots of an N th-degree Legendre polynomial, $P_N(\tau)$, given as

$$P_N = \frac{1}{2^N N!} \frac{d^N}{d\tau^N} [(\tau^2 - 1)^N] \quad (16)$$

The state and control variables are approximated by using a basis of $N + 1$ Lagrange interpolating polynomials, $L_i(\tau)$ ($i = 0, 1, \dots, N$)

$$x(\tau(t)) = \sum_{i=0}^N x(\tau_i) L_i(\tau) \quad (17)$$

$$u(\tau(t)) = \sum_{i=1}^N u(\tau_i) L_i(\tau) \quad (18)$$

The derivative of Equation (17) at the LG points τ_k ($k = 1, 2, \dots, N$) can be represented as follows

$$\dot{x}(\tau_k) = \frac{2}{t_f - t_0} \sum_l^N D_{kl} x_l \quad (19)$$

where $D = [D_{kl}]$ is a $(N) \times (N + 1)$ differential approximation matrix,

$$D_{kl} = \dot{L}_l(\tau_k) = \sum_{i=0}^N \frac{\prod_{j=0, j \neq l, i}^N (\tau_k - \tau_j)}{\prod_{j=0, j \neq l}^N (\tau_l - \tau_j)} \quad (20)$$

where $k = 1, \dots, N$ and $l = 0, 1, \dots, N$. So the state dynamic constraint Equation (12) is transcribed into algebraic constraints via the differential approximation matrix as follows

$$\sum_{l=0}^N D_{kl} x_l - \frac{t_f - t_0}{2} f(x_k, u_k) = 0 \quad (21)$$

Also terminal constraints and path constraints are transcribed as follows

$$e(x_0, x_N, \tau_0, \tau_f) = 0 \quad (22)$$

$$c(x_k, u_k) \leq 0 \quad (23)$$

The cost function can be approximated with a Gauss quadrature, resulting in

$$J = E(x_0, x_N, \tau_0, \tau_f) + \frac{t_f - t_0}{2} \sum_{k=1}^N F(x_k, u_k) w_k \quad (24)$$

Here, w_k is the Gauss weights defined as

$$w_k = \frac{2}{1 - \tau_l^2} [\dot{L}_N(\tau_l)]^2 \quad (25)$$

Finally, the optimal control problem is transformed into a NLP problem, it can be summarised as: finding the state parameters $X^N = [x_0, x_1, \dots, x_N]$, control parameters $U^N = [u_1, \dots, u_N]$ to make the cost function of Equation (24) minimised under the condition that all the constraints of Equations (21) to (23) are satisfied.

Although the Pseudo-spectral Method can be used to calculate an open-loop optimal trajectory in a very short time, for the online vehicle ascent phase optimal guidance problem under model parameters uncertainties and external disturbance Equation (26), a significant barrier to this optimisation-based control strategy is computational delay, especially when the solution time of the NLPP exceeds the sampling time.

$$\dot{x} = f(x, u, t; p) + d(t) \quad (26)$$

where p represents the model parameters and $d(t)$ represents the external disturbance.

3.3. NLP Sensitivity Analysis. In recent decades, NLP sensitivity analysis has been applied to numerous applications in process engineering with parameters that represent uncertain data or unknown inputs (Zavala and Biegler, 2009; Yang and Biegler, 2013). In the MPC feedback context, Zavala and Biegler (2009) analysed the impact of sensitivity updates on the stability of the closed loop in the presence of stabilising terminal constraints and Lyapunov-type terminal costs.

After an optimal control problem has been transformed into a NLP problem, we can write the NLP as $P(p)$ which is parametric in the initial state $p: = x_0$ via Equation (27)

$$\begin{aligned} & \min J_N(z, p) \\ & \text{subject to } C_i(z, p) = 0, i = 1, \dots, n_e \\ & C_i(z, p) \geq 0, i = n_e + 1, \dots, n_c \end{aligned} \quad (27)$$

where z denotes the non-linear program variable.

The Lagrangian function can be denoted as Equation (28)

$$L(z, \mu, p) = J_N(z, p) + \mu^T C(z, p) \quad (28)$$

The index set of active constraints is shown as Equation (29)

$$\begin{aligned} A(z, p) = & \{1, \dots, n_e\} \cup \\ & \{i | C_i(z, p) = 0, i = n_e + 1, \dots, n_c\} \end{aligned} \quad (29)$$

we denote $C_{A(z,p)}$ and $\mu_{A(z,p)}$ as the active constraints and the corresponding multipliers respectively.

The NLP Sensitivity Theorem (Fiacco, 1976) states sufficient conditions for the relationship between an optimal solution $z(p)$ and the parameter p . The theorem states:

Lemma 1: Considering Equation (28), suppose that J_N and C are at least twice differentiable in a neighborhood of the nominal solution $z^*(p_0)$. If the Linear Independence Constraint Qualification (LICQ), the Sufficient Second order Optimality Conditions (SSOC) and the Strict Complementarity Condition (SCC) are satisfied in this neighborhood, then we have that:

- $z^*(p_0)$ is an isolated local minimizer and the respective Lagrange multipliers are unique.
- for where p is in a neighborhood of p_0 there exists a unique local minimizer $z^*(p)$ which satisfies LICQ, SSOC and SCC and is differentiable with respect to p .
- there exists a Lipschitz constant L_J such that such that

$$|J_N(p) - J_N(p_0)| \leq L_{Jp} \|p - p_0\| \tag{30}$$

- there exists a Lipschitz constant L_z such that for the solution z

$$|z(p) - z(p_0)| \leq L_{zp} \|p - p_0\| \tag{31}$$

Based on the implicit function theorem (Fiacco, 1976), we can obtain Equation (32)

$$\begin{aligned} & \begin{bmatrix} \nabla_{zz}^2 L(z^*, \mu^*, p_0) & \nabla_z C_{A(z^*, p_0)}(z^*, p_0)^T \\ \nabla_z C_{A(z^*, p_0)}(z^*, p_0) & 0 \end{bmatrix} \bullet \begin{bmatrix} \frac{\partial z}{\partial p}(p_0) \\ \frac{\partial \mu_{A(z^*, p_0)}}{\partial p}(p_0) \end{bmatrix} \\ & = - \begin{bmatrix} \nabla_{zp}^2 L(z^*, \mu^*, p_0) \\ \nabla_p C_{A(z^*, p_0)}(z^*, p_0) \end{bmatrix} \end{aligned} \tag{32}$$

where the matrix $\left[\frac{\partial z}{\partial p}(p_0), \frac{\partial \mu_{A(z^*, p_0)}}{\partial p}(p_0) \right]^T$ is called the sensitivity matrix. It is simple to obtain a first-order approximation of the optimal solution for a perturbed parameter via Equation (33)

$$z(p) = z^*(p_0) + \frac{\partial z}{\partial p}(p_0) \cdot (p - p_0) \tag{33}$$

where $\frac{\partial z}{\partial p}(p_0)$ is the sensitivity. During the computation of the optimal control problem, sensitivity information can be obtained very easily with improvements in fast Newton-based barrier methods. The software Interior Point Optimiser (IPOPT) and Sensitivity IPOPT (sIPOPT) (Pirnay et al., 2012) are used to compute the sensitivity information, where IPOPT solves a non-linear program in the background and

sIPOPT provides an estimate of a neighboring solution. Moreover, in Equation (33), $\frac{\partial z}{\partial p}(p_0)$ is directly available in factored form from the optimal solution by IPOPT, so the sensitivity can be calculated through a simple back solution by sIPOPT.

3.4. *Real-time optimal guidance algorithm.* If the sampling frequency of the system is fixed, an approximated closed-loop solution of an optimal control problem is computed in the following way: in each sampling interval, we use the nominal non-linear model to predict the next sampling states, an optimal control problem is solved based on the predicted states. When the system dynamics are simulated to the next sampling instant, based on the difference between predicted states and measured states, the resulting optimal control sequence is corrected via sensitivities and is applied as input for the next sampling interval(s). This procedure is then repeated iteratively.

We denote t_0 as the initial time, $t_i (i = 1, 2, \dots)$ represents the sampling instant, ΔT is the fixed sampling period; $x(t_0)$ is the initial state, $x(t_i)$ and $\hat{x}(t_i) (i = 1, 2, \dots)$ represent the measured state and the predicted state respectively; $u^*(\hat{x}(t_i), t)$ is the optimal control via GPM and $u(\hat{x}(t_i), t)$ is the corrected control via sensitivity updates; we denote $t_{pi} (i = 1, 2, \dots)$ as the predicted time from t_{i-1} to t_i and $t_{gi} (i = 1, 2, \dots)$ as the computational time of the optimal control problems from t_{i-1} to t_f with the initial state $\hat{x}(t_i)$. The real-time optimal guidance algorithm design is summarised in the following algorithm and is shown in Figure 1.

Step 1: Choose the discrete parameter N , the fixed sampling period ΔT and the first sampling time t_1 ; Collect the initial state $x_0 = x(t_0)$ and set $i = 1$. Use the Pseudo-spectral method to calculate the optimal controller $u_0^* = u^*(x_0, t)$ and apply it to the system until $t_1 = t_0 + \Delta T$ and get the the predicted state $\hat{x}(t_1)$;

Step 2: Collect the real measurement $x(t_1)$, correct the control via sensitivity updates Equation (34)

$$u(x(t_1), t) = u_0^* + \frac{\partial u_0^*}{\partial \hat{x}(t_1)}(x(t_1) - \hat{x}(t_1)) \quad (34)$$

Step 3: Propagate the system dynamics with the control $u(x(t_{i-1}), t)$ with $x(t_{i-1})$ as the initial condition to time t_i and get the the predicted state $\hat{x}(t_i)$;

Step 4: Use the Pseudo-spectral method to calculate the optimal controller $u^*(\hat{x}(t_i), t)$ with $\hat{x}(t_i)$ as the initial condition;

Step 5: If $t_{pi} + t_{gi} \leq \Delta T$, collect the measurement $x(t_i)$, correct the control via sensitivity updates to get the control $u(x(t_i), t)$ as Equation (35) and go to Step 3; If $t_{pi} + t_{gi} > \Delta T$, during time period $[t_i, t_{i+1}]$, apply the control $u(x(t_{i-1}), t)$ and get the predicted state $\hat{x}(t_{i+1})$, go to Step 3;

$$u(x(t_i), t) = u_i^* + \frac{\partial u_i^*}{\partial \hat{x}(t_i)}(x(t_i) - \hat{x}(t_i)) \quad (35)$$

We denote that if the control values violate their corresponding constraints, then the control command $u(t)$ is limited to satisfy the active constraint (Jiang et al., 2012) as Equation (36).

$$u(t) = \{v \in U | c[x(t), u(t), t] = 0\} \quad (36)$$

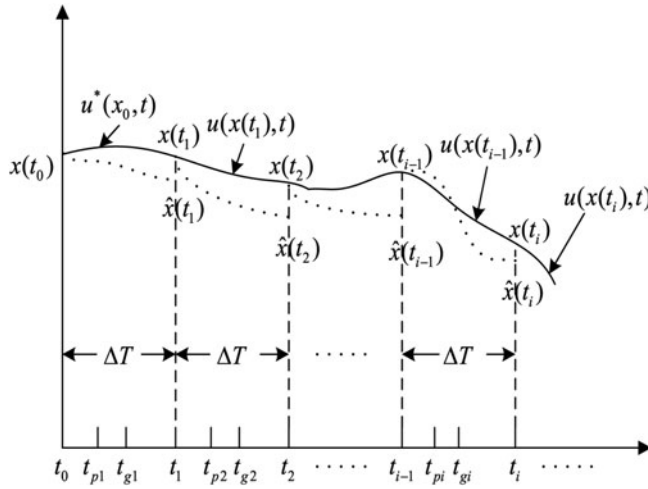


Figure 1. Real-time optimal guidance algorithm.

4. FEEDBACK-BASED REQUIREMENTS ON SAMPLING FREQUENCY.

Consider two trajectories $x(t)$ and $r_i(t)$. $x(t)$ is the system trajectory under the real control $u_i(t) = u(x(t_i), t)$, $r_i(t)$ is the optimal trajectory starting from the same initial condition $x(t_i)$ and driven by the optimal control $u_i^*(t) = u^*(x(t_i), t)$. For each sampling interval $[t_i, t_{i+1}]$, where $x(t)$ satisfies

$$\begin{aligned} \dot{x}(t) &= f(x(t), u_i(t), p) + d(t) \\ x(t_i) &= x(t_i) \end{aligned}$$

and $r_i(t)$ satisfies

$$\begin{aligned} \dot{r}_i(t) &= f(r_i(t), u_i^*(t), p_0) \\ r_i(t_i) &= x(t_i) \end{aligned}$$

Assumption 1: For the non-linear system Equation (26), model parameters uncertainties and external disturbance are bounded, there exist $\epsilon_p > 0$, $\epsilon_d > 0$, such that

$$\|p - p_0\| \leq \epsilon_p, \|d\|_{L_\infty} \leq \epsilon_d$$

Assumption 2: The non-linear vector field $f(x, u, t; p)$ is Lipschitz continuous, i.e., there are constants $L_{fx} > 0$, $L_{fu} > 0$, $L_{fp} > 0$, such that:

$$\|f(x, u, p) - f(y, v, q)\| \leq L_{fx}\|x - y\| + L_{fu}\|u - v\| + L_{fp}\|p - q\|$$

Assumption 3: Given any two initial conditions x_1 and x_2 , we denote $u^*(x_1, t)$ and $u^*(x_2, t)$ as the optimal controls with the corresponding initial conditions. It is assumed that there are constants $L_{ux} > 0$, such that for all $t \in [t_0, \infty)$

$$\|u^*(x_1, t) - u^*(x_2, t)\| \leq L_{ux}\|x_1 - x_2\|$$

This assumption requires that the difference in the optimal controls be linearly bounded by the difference in the initial conditions.

Assumption 4: Errors between the predicted state and the real state caused by model parameters uncertainties and outside disturbance are bounded. There are positive constants δ , such that for all i and $t \in [t_0, \infty)$:

$$\|x(t_i) - \hat{x}(t_i)\| \leq \delta$$

Assumption 5: The relationship between the corrected control via sensitivity updates and the optimal control is as follows, there are positive constants ϵ_u , such that for all i and $t \in [t_i, t_{i+1}]$:

$$\|u(x(t_i), t) - u^*(\hat{x}(t_i), t)\| = \frac{\partial u_i^*}{\partial \hat{x}(t_1)} \|x(t_i) - \hat{x}(t_i)\| \leq \frac{\partial u_i^*}{\partial \hat{x}(t_1)} \delta \leq \epsilon_u$$

where $\frac{\partial u_i^*}{\partial \hat{x}(t_1)}$ is the sensitivity. ϵ_u is used to represent the sensitivity and the computational error in the calculation of the optimal controller.

Definition 1: The multivalued function $W(x)$, $x \in R$, given implicitly by

$$W(z)e^{W(z)} = z \tag{37}$$

is called the Lambert W function (Ross et al., 2008). For $x \geq 0$, $W(x)$ is single-valued.

Lemma 2: Let $[t_0, t_f] \rightarrow y(t) \in R_+$ be an integrable function that satisfies Gronwalls inequality (Ross et al., 2008):

$$y(t) \leq a(t) + \int_{t_0}^t b(s)y(s)ds \tag{38}$$

where $a(t)$, $b(t)$ are continuous, nonnegative, bounded functions, with $t \rightarrow a(t)$ nondecreasing over the interval $[t_0, t_f]$; then

$$y(t) \leq a(t)e^{B(t)} \tag{39}$$

$$B(t) = \int_{t_0}^t b(s)ds \tag{40}$$

The process of the proof is similar to the notes (Ross et al., 2008), but additionally, we consider the effect of model parameters, uncertainties, external disturbance and sensitivities. We note that all of the statements below are under the assumption that $t_{pi} + t_{gi} \leq \Delta T$; if $t_{pi} + t_{gi} > \Delta T$, the ones below should be progressed as indicated in Step 5, Section 3.4.

Theorem 1: Let the controller be designed according to the proposed algorithm. Under Assumption 1–5, for any given $\epsilon > 0$ if

$$\Delta T \leq \Delta T_m \tag{41}$$

where ΔT_m satisfies

$$\Delta T_m = \frac{W(z_m)}{L_{fx}} \tag{42}$$

$$z_m = \frac{\epsilon L_{fx}}{A} \tag{43}$$

$$A = L_{fu}(\epsilon_u + L_{ux}\delta) + L_{fp}\epsilon_p + \epsilon_d \tag{44}$$

then,

$$\|x(t_{i+1}) - r_i(t_{i+1})\| \leq \varepsilon \tag{45}$$

Proof: By the Principle of Optimality, $u^*(\hat{x}(t_i), t)$ is the optimal control with the initial condition $\hat{x}(t_i)$. Therefore, from Assumption 3 and Assumption 5, $\forall t \in [t_i, t_{i+1}]$,

$$\begin{aligned} \|u_i(t) - u_i^*(t)\| &\leq \|u_i(t) - u^*(\hat{x}(t_i), t)\| + \|u^*(\hat{x}(t_i), t) - u_i^*(t)\| \\ &\leq \varepsilon_u + L_{ux}\|\hat{x}(t_i) - x(t_i)\| \leq \varepsilon_u + L_{ux}\delta \end{aligned}$$

By Assumption 1 and Assumption 2, we can obtain

$$\begin{aligned} \|x(t) - r_i(t)\| &\leq L_{fx} \int_{t_i}^t \|x(\tau) - r_i(\tau)\| d\tau + L_{fu}\Delta T \|u_i(t) - u_i^*(t)\| \\ &\quad + L_{fp}\Delta T \|p - p_0\| + \Delta T \|d\|_\infty \\ &\leq L_{fx} \int_{t_i}^t \|x(\tau) - r_i(\tau)\| d\tau + A\Delta T \end{aligned} \tag{46}$$

where $A = L_{fu}(\varepsilon_u + L_{ux}\delta) + L_{fp}\varepsilon_p + \varepsilon_d$. Moreover, by Gronwalls inequality (Ross et al., 2008), from Equation (46) we can get:

$$\|x(t) - r_i(t)\| \leq A\Delta T e^{L_{fx}\Delta T}$$

When $t = t_{i+1}$,

$$\|x(t_{i+1}) - r_i(t_{i+1})\| \leq (A/L_{fx})L_{fx}\Delta T e^{L_{fx}\Delta T} \tag{47}$$

Based on Definition 1 of the Lambert W function, it can be easily verified that for $z_m, c \in R_+$, if $c \leq W(z_m)$, then $ce^c \leq z_m$. Hence letting $c = \Delta TL_{fx}$, from Equations (42) to (44), we have

$$\|x(t_{i+1}) - r(t_{i+1})\| \leq (A/L_{fx})z_m = \varepsilon \tag{48}$$

According to the property of the W function, there exists a maximal ΔT_m satisfying

$$\Delta T_m = \frac{W(\varepsilon L_{fx}/A)}{L_{fx}} \tag{49}$$

Remark 1: From Equations (47) and (48) we can see that errors between the real state and the optimal state of the nominal model are bounded and are affected by the sampling period, the sensitivity, the computational error, model parameters uncertainties and external disturbance.

Remark 2: The sample period is mainly determined by the Lipschitz constants of the system. The W function is an increasing function. The maximal ΔT_m can be obtained by Equation (42).

Remark 3: For the infinite-horizon feedback control problem, optimal control has the advantage of guaranteeing stability through the Bellman value function serving as a natural Lyapunov function (Clarke, 2004). However, the online vehicle guidance problem is a finite-horizon feedback control problem. Thus, if open-loop optimal controls are generated starting at $t_i \in \pi$ for each $x(t_i)$, it is possible to guide a vehicle to its target by generating a Carathodory- π feedback solution (Ross et al., 2008). The proposed method can generate a corrected Carathodory- π feedback solution online.

5. SIMULATION RESULTS AND ANALYSIS

5.1. *Simulation parameters.* In this section, we present the trajectory optimisation problem proposed in the previous sections. Gauss Pseudospectral Optimisation Software (GPOPS) (Rao et al., 2010; Patterson and Rao, 2014) and siPOPT (Pirnay et al., 2012) are used as the simulation software. During the ascent phase of the trajectory, the parameters used in solving the problem are as follows (Prasanna et al., 2005): reference area $S = 1 \text{ m}^2$, the empty mass of the aircraft $m_1 = 2000 \text{ kg}$, the quality of fuel $m_2 = 1600 \text{ kg}$, the initial mass of the aircraft $m_0 = m_1 + m_2 = 3600 \text{ kg}$, the gravitational constant $\mu = 3.9853 \times 10^{14} \text{ m}^3/\text{s}^2$, $g = 9.8 \text{ m/s}^2$ and $R_e = 6378000 \text{ m}$ is the radius of the earth. The number of LG point is 20. The other parameters used in simulation are shown in Tables 1 to 6.

At 32.5 km, let the density be $\rho_{32.5}$, speed of sound be $C_{32.5}$, and the speed corresponding to a given mach number M is $v_{32.5} = MC_{32.5}$. $\dot{m}_{air(32.5)}$ is shown in Table 6.

The hypersonic propulsion system has been modelled using data on air mass flow rate (\dot{m}_{air}) and specific impulse (I_{sp}) as functions of Mach number (M), angle of attack (α) and altitude (h), respectively. From (\dot{m}_{air}) the thrust (T) can be found using Equations (50) and (51), where $\dot{m}_f = \dot{m}_{air}\phi/15$, ϕ is the equivalence ratio and during ascent phase its value is 1. A is the nozzle area.

$$\underbrace{\dot{m}_{air}}_{\text{at given altitude}} = \frac{\dot{m}_{air(32.5)}}{\rho_{32.5} v_{32.5} A} \underbrace{\rho v A}_{\text{at given altitude}} \tag{50}$$

$$T = I_{sp}\dot{m}_f g \tag{51}$$

5.2. *Analysis of the optimised results.* Solving optimal control problems by means of the Pseudo-spectral method implies that the differential equation is approximated by a discrete time counterpart. In order to assess the accuracy of the discretization, we may simulate the system using the optimal control profile as input. With this approach, the state profiles are computed with high accuracy and the result may then be compared with the profiles resulting from optimisation. Notice that this procedure does not verify the optimality of the resulting optimal control profiles, but only the accuracy of the discretization of the dynamics. The comparison of the two states is shown in Figures 2 and 3 and the number of discretization points used in the optimal control solution is 30 in Figures 2 and 3. The magnitude of relative errors between them are $10^{-3} \sim 10^{-4}$, so the optimised results are credible.

5.3. *Simulation of online trajectory reconstruction.* The external disturbance will lead the trajectory to deviate from the standard trajectory during the flight. Based on the current measured states, the trajectory should be re-planned to meet the terminal and path constraints from the proposed method. The sampling period for starting the online trajectory reconstruction algorithm and updating the guidance commands is defined as 5 s. To assess the capability of the proposed method to deal with disturbance, the aerodynamic coefficients variations in the simulations are given by following cases:

- Closed-loop simulation based on Pseudo-spectral method optimal control with sensitivity updates: set drag coefficient error is +10% and lift coefficient error is -10%

Table 1. Boundary Conditions.

	$h(m)$	$v(m/s)$	$\gamma(\text{rad})$	$m(\text{kg})$
$x(t_0)$	16000	1198	0.0698	3600
$x(t_f)$	32000	1840	0	≥ 2000

Table 2. Constraint Conditions.

	$\alpha(\text{deg})$	$\gamma(\text{deg})$	$Q(\text{Pa})$
min	0	-15°	0
max	8°	15°	132000

Table 3. Lift Coefficients.

Mach/ α	0°	2°	4°	6°	8°
3.5	0.178	0.665	1.146	1.628	2.099
4.0	0.189	0.629	1.075	1.533	1.969
5.0	0.186	0.576	0.973	1.379	1.782
6.0	0.180	0.530	0.893	1.259	1.614
6.5	0.165	0.494	0.842	1.203	1.574

Table 4. Drag Coefficients.

Mach/ α	0°	2°	4°	6°	8°
3.5	0.427	0.477	0.571	0.706	0.874
4.0	0.431	0.480	0.561	0.649	0.841
5.0	0.332	0.376	0.452	0.530	0.705
6.0	0.269	0.274	0.342	0.413	0.572
6.5	0.249	0.288	0.352	0.447	0.575

- Closed-loop simulation base on Pseudo-spectral method optimal control without sensitivity updates: set drag coefficient error is +10% and lift coefficient error is -10%
- Open-loop simulation with off-line optimal control command: set drag coefficient error is +10% and lift coefficient error is -10%
- Simulation with off-line optimal control command in the normal case without error

The trajectory without reconstruction is defined as the open-loop trajectory. As a contrast, Table 7 lists the deviation of the re-planning trajectory with sensitivity updates and the re-planning trajectory without sensitivity updates for the terminal constraints and the max trajectory re-planning time during flight. The results are shown from Figure 4 to Figure 8. Figure 4 illustrates the altitude-versus-time profiles in the four cases. Figure 5 illustrates the velocity-versus-time profiles in the four cases. As expected, the flight trajectory with the online trajectory reconstruction algorithm can arrive at the terminal altitude. If there is no trajectory reconstruction, the open-loop

Table 5. Specific impulse (I_{sp} in seconds).

M/h(km)	15	20	25	30	35	40
3.0	1054.4	1044.8	1024.8	1005.6	976.8	943.2
4.0	1054.4	1044.8	1024.8	1005.6	976.8	943.2
5.0	964	952	931.2	909.6	879.2	847.2
6.0	848.8	837.6	816.8	799.2	775.2	749.6
7.0	719.2	712.8	697.6	687.2	668.8	644
8.0	594.4	592.8	580.8	569.6	542.4	499.2

Table 6. Mass flow rate of air (-kg/s) corresponds to altitude 32.5 km.

Mach/ α	0°	2°	4°	6°	8°
3.5	6.9513	7.2387	8.2962	8.9447	9.6047
4.0	8.9355	9.6937	10.6681	11.6734	12.6860
5.0	12.7916	14.7104	16.6331	18.6106	20.3261
6.0	17.7370	21.1998	24.7330	26.8701	29.1919
6.5	21.5142	25.1683	29.6150	33.1770	35.2587

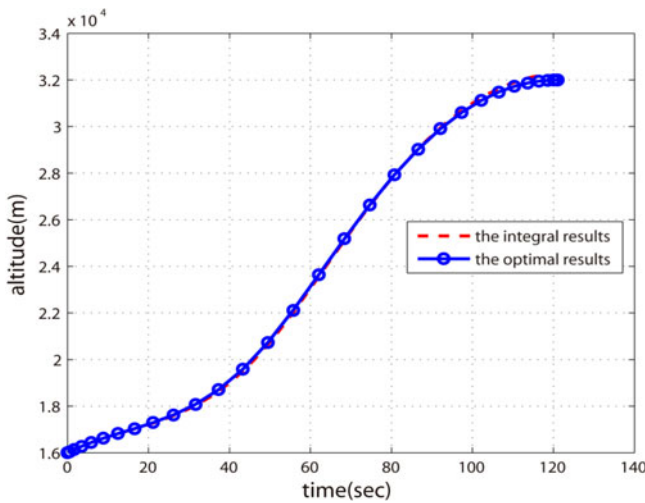


Figure 2. Feasibility analysis of the altitude curve.

flight trajectory cannot meet the terminal altitude constraint. Figure 6 illustrates the control-versus-time profiles in the four cases. The oscillatory behavior of the angle of attack near the end of the trajectory is affected by the sampling period of the online trajectory reconstruction algorithm. In our paper, the sampling period for starting the online trajectory reconstruction algorithm and updating the guidance commands is defined as 5 s. This is because the simulation environment is Matlab; the sampling period should be much shorter in a C or C++ language environment and the oscillatory behavior of the angle of attack near the end of the trajectory should be improved. Figure 7 illustrates the dynamic pressure-versus-time profiles in the four cases. If

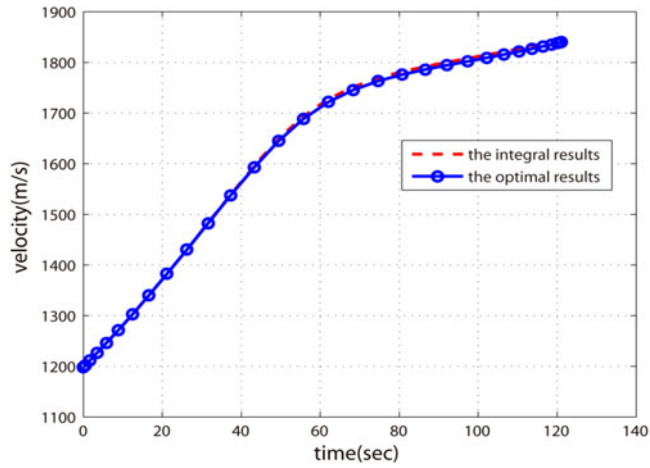


Figure 3. Feasibility analysis of the speed curve.

Table 7. Reconstruction precision and time on disturbance on-line.

	With sensitivity	Without sensitivity
Altitude(m)	31999.14	31998.71
Speed(m/s)	1838.98	1838.32
Flight path angle(deg)	0.0011	0.0013
Mass(kg)	2913.11	2894.20
Max re-planning time	4.13 s	3.94 s

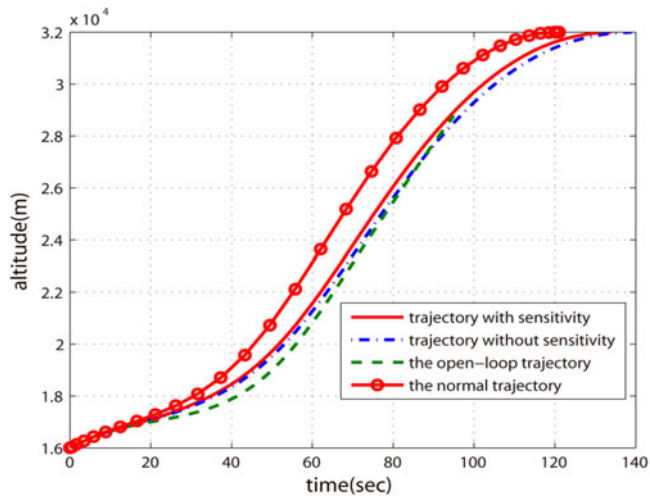


Figure 4. The altitude curve.

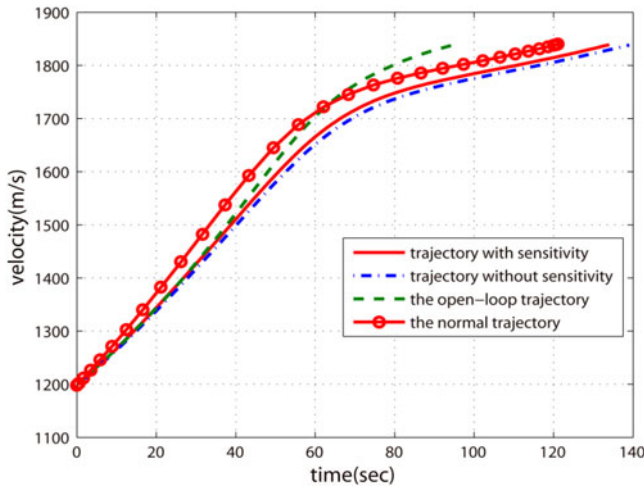


Figure 5. The velocity curve.

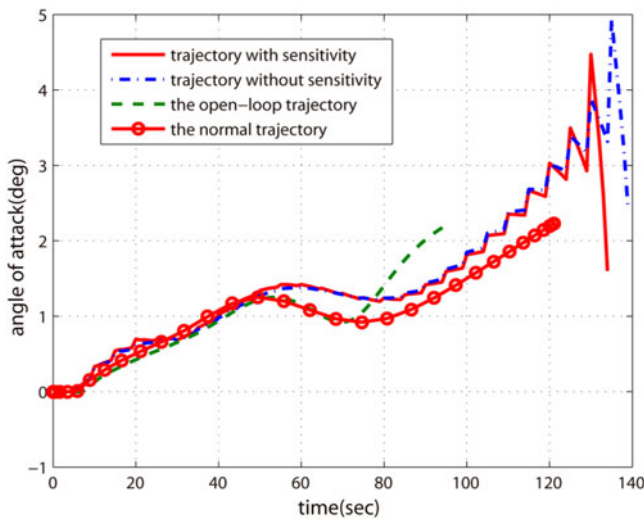


Figure 6. The control curve.

there is no trajectory reconstruction, the open-loop flight trajectory will exceed the path constraints. Figure 8 illustrates the mass-versus-time profiles in the four cases. The online trajectory reconstruction algorithm with sensitivity updates can consume less mass. The time to reconstruct the trajectory is shorter than 5 s, it should be much faster in a C or C++ language environment. The re-planning trajectory with sensitivity updates has indeed improved the performance indexes from the data in Table 7.

6. CONCLUSIONS AND FUTURE WORK. For hypersonic ascent phase optimal guidance problems, an online optimal guidance algorithm based on Pseudo-spectral

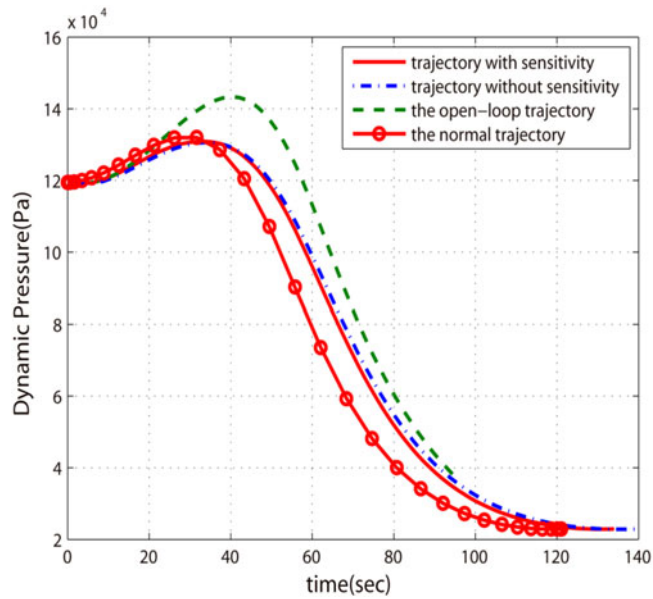


Figure 7. The dynamic pressure curve.

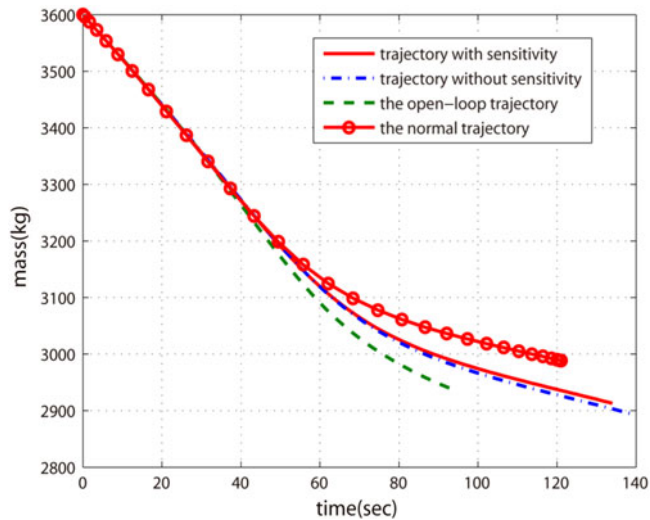


Figure 8. The mass curve.

method and sensitivity analysis is proposed. The proposed closed-loop feedback method can successively generate online open-loop suboptimal controls without the design procedure of an inner-loop feedback controller. Considering model parameter uncertainties and external disturbance, a sampling theorem is proposed which indicates the effect of the Lipschitz constant on the dynamics of sampling

frequency. The simulation results indicate that the method proposed above offers improved performance and has ability for online calculation. Future work will propose a guidance law for tracking the re-planning ascent phase trajectory and estimating aerodynamic parameters and density online to get higher predicted accuracy.

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