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Qualitative behaviour of positive solutions of a predator–prey model: effects of saturation

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We study a predator–prey system with Holling–Tanner interaction terms. We show that if the saturation rate m is large, spatially inhomogeneous steady-state solutions arise, contrasting sharply with the small-m case, where no such solution could exist. Furthermore, for large m, we give sharp estimates on the ranges of other parameters where spatially inhomogeneous solutions can exist. We also determine the asymptotic behaviour of the spatially inhomogeneous solutions as $m \to \infty$, and an interesting relation between this population model and free boundary problems is revealed.

1. Introduction

In this paper, we study positive steady-state solutions of the system

$$\left.\begin{aligned} u_t - d_1\Delta u &= u(a - u - bv/(1+mu)), && x \in D, \quad t > 0, \\ v_t - d_2\Delta v &= v(d - v + cu/(1+mu)), && x \in D, \quad t > 0, \\ \frac{\partial u}{\partial \nu} &= \frac{\partial v}{\partial \nu} = 0, && x \in \partial D, \quad t > 0, \end{aligned}\right\} \tag{1.1}$$

where d_1, d_2, a, b, c, d and m are constants, and they are all positive except d which may be negative, D is a bounded domain in R^N with smooth boundary ∂D, and ν is the unit outer normal vector on ∂D. The system (1.1) is a predator–prey model, where u and v represent the densities of the prey and predator. Thus only non-negative solutions of (1.1) are of interest. The interaction term, $uv/(1+mu)$, is of the Holling–Tanner type, where m measures the prey's ability to evade attack: the more elusive the prey, the larger m becomes (see [31]).

The understanding of (1.1) is quite complete when m is small. For $m = 0$, Leung [26] proved that all positive solutions of (1.1), regardless of the initial data, converge to a constant steady-state solution as time goes to infinity. It follows from [3, 13] that the results of Leung still hold for small positive m.

In this paper, we want to understand how the behaviour of (1.1) changes when m becomes large. For m not small, Hainzl [19, 20] studied the corresponding kinetic

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system of (1.1) in depth. Among other things, his results show that (1.1) can have spatially homogeneous periodic solutions. However, little is known about spatially inhomogeneous steady-state or periodic solutions of (1.1), which are also important in determining the dynamics of (1.1). The main purpose of this paper is to understand all the positive steady-state solutions of (1.1) with large m. Thus we will concentrate on the system

$$\left.\begin{aligned} -\Delta u &= u(a-u-bv/(1+mu)), && x\in D,\\ -\Delta v &= v(d-v+cu/(1+mu)), && x\in D,\\ \frac{\partial u}{\partial \nu} &= \frac{\partial v}{\partial \nu}=0, && x\in \partial D,\end{aligned}\right\} \tag{1.2}$$

which is obtained from (1.1) by using the rescaling

$$u=d_1\tilde{u},\qquad v=d_2\tilde{v},\qquad a=d_1\tilde{a},\qquad b=(d_2/d_1)\tilde{b},$$
$$d=d_2\tilde{d},\qquad c=d_2/d_1\tilde{c},\qquad m=\tilde{m}/d_1.$$

For simplicity, we have dropped the $\tilde{}$ sign in (1.2).

We will use d, the growth rate of the predator, as the main parameter, with other parameters being fixed; in particular, m will be fixed at a large value. Under such assumptions, the constant positive solution set of (1.2) is fully understood as follows: system (1.2) has no constant positive solution if $d\notin(-ca/(1+ma),d^*]$, a unique constant positive solution for $d\in(-ca/(1+ma),a/b]\cup\{d^*\}$, and exactly two constant positive solutions for $d\in(a/b,d^*)$, where $d^*=(a^2/4b)m+o(m)$ is a positive constant depending on a, b, c and m.

It turns out that when m is sufficiently large, non-constant positive solutions of (1.2) can exist only for a more restricted range of d. More precisely, we have the following result.

THEOREM 1.1. *There exists a large $M>0$, depending only on a, b and c, such that if $m\geqslant M$ and $d\notin(a/b,d^*)$, then (1.2) does not have any non-constant positive solutions.*

On the other hand, system (1.2) does have many non-constant positive solutions if m is large and d falls into the range $(a/b,d^*)$. Moreover, as shown in the following result, system (1.2) has at most two types of non-constant positive solutions when m is large.

THEOREM 1.2. *Let $\{(u_n,v_n)\}_{n=1}^{\infty}$ be any non-constant positive solutions of (1.2) with $(d,m)=(d_n,m_n)$ and $m_n\to\infty$. Then $\|v_n-d_n\|_\infty\to 0$ and, subject to choosing a subsequence, one of the following conclusions must hold.*

(i) *$d_n\to d\in(a/b,+\infty)$, $m_nu_n/d_n\to w$ in C^2, where w is a positive smooth solution of*

$$\Delta w+w\left(a-\frac{b}{1/d+w}\right)=0,\qquad \left.\frac{\partial w}{\partial\nu}\right|_{\partial D}=0. \tag{1.3}$$

(ii) *$d_n\to\infty$, $d_n/m_n\to\alpha\in[0,a^2/(4b))$ and $m_nu_n/d_n\to w$ in C^1, where w is a non-negative non-trivial weak solution of the problem*

$$\Delta w+w(a-\alpha w)-b\chi_{\{w>0\}}=0,\qquad \left.\frac{\partial w}{\partial\nu}\right|_{\partial D}=0. \tag{1.4}$$

Theorem 1.2 implies that if (u,v) is a non-constant positive solution of (1.2) with large m, then v is close to d, and mu/d is close to a solution of (1.3) or (1.4); conversely, we can show that non-degenerate positive solutions of (1.3) and (1.4) induce positive solutions of (1.2) for large m (see § 4.3 for details). It is interesting to note that (1.4) is of free boundary type when $\{x \in D : w(x) = 0\}$ is not empty.

This paper is organized as follows. In § 2, we present some general results on (1.1) and (1.2). In § 3, we establish some non-existence results on non-constant positive solutions of (1.2), which includes theorem 1.1 as a special case. A local bifurcation result of Crandall and Rabinowitz [10] will play an important role here. In § 4, we concentrate on the existence of non-constant positive solutions of (1.2): we first establish theorem 1.2 and then determine when (1.3) or (1.4) have non-trivial solutions; finally, we show how non-trivial solutions of (1.2) can be obtained by using solutions of (1.3) and (1.4). Here our arguments rely heavily on critical point theory and topological degree theory.

The system (1.1) with Dirichlet boundary conditions, especially the case $m = 0$, has received extensive studies in the last two decades (see [4,11,12,25,27–29,33] and the references therein); the biologically more interesting case $m > 0$ was studied in [2, 5, 15, 16]. It is known that for small positive m, the system (1.1) behaves similarly as the relatively simple case $m = 0$; for large m, the system (1.1) is much more complicated, e.g. there may exist multiple stable steady-states, and Hopf bifurcation can occur (see [5,15,16]).

The study of predator–prey models has a long history. Apart from the works mentioned above, we refer to Conway [8] and Smoller [32] for references on diffusive predator–prey models; for works on ODE-type predator–prey models, see [9,17,22].

2. Some preliminary results

In this section, we present some general results which can provide us a general look on the problem. First, we study constant positive solutions of (1.2). The function

$$H(u) = b^{-1}(a-u)(1+mu) - cu/(1+mu)$$

turns out to be very useful. In fact, it is easily seen that (u,v) is a constant positive solution of (1.2) if and only if u is a solution of $H(u) = d$ in the interval $(0,a)$. Throughout this paper, we define

$$d^* = \max_{u\in[0,a]} H(u).$$

Note also that

$$\min_{u\in[0,a]} H(u) = H(a) = -ca/(1+ma).$$

By an elementary analysis of the curve $d = H(u)$, which is essentially cubic, we obtain the following result, which gives a rather complete description of the constant solution set of (1.2). Its easy proof is omitted.

THEOREM 2.1. *Let* $m_2 = a^{-1}(1+bc)$ *and*

$$m_1 = \begin{cases} m_2 & \text{if } bc \leqslant 1, \\ a^{-1}[3(bc)^{1/3} - 1] < m_2 & \text{if } bc > 1. \end{cases}$$

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Then the following statements hold.

(i) *For any* $m \geqslant 0$, *then the system (1.2) has no constant positive solution for* $d \notin (-ca/(1+ma), d^*]$, *and at least one constant positive solution for* $d \in (-ca/(1+ma), d^*)$.

(ii) *If* $0 \leqslant m < m_1$, *then* $H(u)$ *is strictly decreasing in* $(0, a)$, $d^* = a/b$ *and (1.2) has a unique constant positive solution if* $d \in (-ca/(1+ma), d^*)$.

(iii) *If* $m > m_2$, *then* $H'(u)$ *changes sign exactly once, from positive to negative, in* $(0, a)$, $d^* > a/b$, *the system (1.2) has no constant positive solution for* $d \notin (-ca/(1+ma), d^*]$, *a unique constant positive solution for* $d \in (-ca/(1+ma), a/b] \cup \{d^*\}$, *and exactly two constant positive solutions for* $d \in (a/b, d^*)$.

Remark 2.2. A complete understanding of the constant positive solutions of (1.2) can also be obtained for the case $m_1 < m < m_2$. For this case, $H'(u)$ changes sign exactly twice in $(0, a)$, and hence there are ranges of d such that (1.2) has three constant positive solutions.

The following result gives a complete understanding of (1.1) for small positive m.

Theorem 2.3. *Suppose that* $m \leqslant a^{-1}$. *Then we have the following.*

(i) *If* $d \in (-ca/(1+ma), a/b)$, *the unique constant positive solution is globally attractive, i.e. any solution of (1.1) with non-trivial non-negative initial data converges to the constant positive solution as time goes to infinity.*

(ii) *If* $d \geqslant a/b$, $(0, d)$ *is globally attractive.*

(iii) *If* $d \leqslant -ca/(1+ma)$, $(a, 0)$ *is globally attractive.*

Proof. We first show that (i) follows from [13]. Following [13], we write (1.2) as

$$-\Delta u = h(u)[f(u) - a(v)], \qquad -\Delta v = k(v)[-g(v) + b(u)],$$

where

$$h(u) = u/(1+mu), \qquad f(u) = (a-u)(1+mu), \qquad a(v) = bv,$$
$$k(v) = v, \qquad g(v) = v - d, \qquad b(u) = cu/(1+mu).$$

Then it is easy to check that the conditions (A1)–(A10) in [13] are satisfied. Then (i) follows from theorem 1 there; for (ii) and (iii), one checks that (A1)–(A10) of [13] are satisfied with $(\hat{u}, \hat{v}) = (0, d)$ and $(a, 0)$, respectively. Again, the conclusions (ii) and (iii) are consequences of theorem 1 there. □

Proposition 2.4. *If* $d \leqslant -ca/(1+ma)$, *then (1.2) has no positive solution.*

Proof. Suppose that (1.2) has a positive solution (u_0, v_0). We show that $d > -ca/(1+ma)$. First, by the equation of u_0, we have $u_0 < a$. This implies that

$$-\Delta v_0 < v_0(d - v_0 + ca/(1+ma)).$$

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By the maximum principle, we have $d + ca/(1 + ma) > v_0 > 0$, i.e.

$$d > -ca/(1 + ma).$$

□

REMARK 2.5. It would be interesting to know whether (1.2) can have a non-constant positive solution for $d > d^*$. By theorem 2.3, the answer is no if $m \leqslant a^{-1}$. We will show in the next section that when m is large, the answer is also no. In fact, from now on, we will concentrate on the large-m case. We remark that $d^*/m \to a^2/(4b)$ as $m \to \infty$, and a proof of this can be found in the next section.

3. Non-constant solutions: non-existence

In this section, we find the ranges of d where (1.2) does not have any non-constant positive solutions. In general, such accurate results are hard to obtain. However, we will show that it can be done when m is large. The main result of this section is as follows.

THEOREM 3.1. *There exist a small $\epsilon_0 > 0$ and a large $M_0 > 0$, both depending only on a, b and c, such that if $m \geqslant M_0$ and $d \notin [a/b + \epsilon_0, m(a^2/(4b) - \epsilon_0)]$, then (1.2) has no non-constant positive solution.*

It is easy to check that theorem 3.1 follows from propositions 3.2 and 3.4. Note that theorem 3.1 includes theorem 1.1 as a special case; moreover, as we shall see in § 4, theorem 3.1 is also crucial in establishing theorem 1.2.

3.1. A lower bound of d

The main result of this subsection is as follows.

PROPOSITION 3.2. *There exists a small $\epsilon_1 > 0$ and a large $M_1 > 0$, depending only on a, b, c, such that if $m \geqslant M_1$ and $d \leqslant a/b + \epsilon_1$, then (1.2) has no non-constant positive solution.*

To prove proposition 3.2, we need the following *a priori* estimates.

LEMMA 3.3. *Let (u_n, v_n) be positive solutions of (1.2) with $(m, d) = (m_n, d_n)$. Suppose that $m_n \to \infty$ and $d_n \to \bar{d} \leqslant a/b$. Then $\|v_n - d_n\|_\infty \to 0$ and either $\|u_n - a\|_\infty \to 0$ or $m_n\|u_n\|_\infty \to 0$. Moreover, if $m_n\|u_n\|_\infty \to 0$, then $\bar{d} = a/b$, and $u_n/\|u_n\|_\infty \to 1$ uniformly.*

Proof. By the maximum principle, we have $d_n \leqslant v_n \leqslant d_n + c/m_n$. Hence we have $\|v_n - d_n\|_\infty \to 0$.

Claim. Passing to a subsequence if necessary, we have that either $m_n\|u_n\|_\infty \to \infty$ or $m_n\|u_n\|_\infty \to 0$.

To prove our assertion, we argue by contradiction. By choosing a subsequence, we may assume that $m_n\|u_n\|_\infty \to \alpha \in (0, \infty)$. Set $\tilde{u}_n = m_n u_n$. Then

$$-\Delta\tilde{u}_n = \tilde{u}_n\left(a - \frac{\tilde{u}_n}{m_n} - \frac{bv_n}{1 + \tilde{u}_n}\right), \qquad \left.\frac{\partial\tilde{u}_n}{\partial\nu}\right|_{\partial D} = 0. \tag{3.1}$$

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By L^p estimates and Sobolev embedding theorems, we may assume that $\tilde{u}_n \to \tilde{u}$ in C^1. Then, passing to the weak limit in (3.1), and by standard elliptic regularity, we can show that $\tilde{u}$ is a classical solution of

$$-\Delta\tilde{u} = \tilde{u}\left(a - \frac{b\bar{d}}{1+\tilde{u}}\right), \qquad \frac{\partial\tilde{u}}{\partial\nu}\bigg|_{\partial D} = 0. \tag{3.2}$$

We show that $\tilde{u} = 0$. If not, since $\bar{d} \leqslant a/b$, the right-hand side of (3.2) is non-negative, and not identically zero. However, this contradicts

$$\int_D \tilde{u}(a - b\bar{d}/(1+\tilde{u}))\,\mathrm{d}x = 0.$$

Hence $\tilde{u} = 0$, i.e. $\tilde{u}_n \to 0$ uniformly. But this is impossible since $\|\tilde{u}_n\|_\infty \to \alpha > 0$. This proves our assertion.

Next we show that $m_n\|u_n\|_\infty \to \infty$ implies that $u_n \to a$ uniformly. Set $\hat{u}_n = u_n/\|u_n\|_\infty$. Then

$$-\Delta\hat{u}_n = \hat{u}_n(a - u_n - bv_n/(1+m_nu_n)), \qquad \frac{\partial\hat{u}_n}{\partial\nu}\bigg|_{\partial D} = 0. \tag{3.3}$$

By standard elliptic regularity, we may assume that $\hat{u}_n \to \hat{u}$ in C^1. Since $bv_n/(1+m_nu_n)$ is bounded, $bv_n/(1+m_nu_n) \to h$ weakly in L^2 and $h \in L^\infty$. Since $\|u_n\|_\infty \leqslant a$, we may assume that $\|u_n\|_\infty \to \alpha$. Passing to the weak limit in (3.3), we have

$$-\Delta\hat{u} = \hat{u}(a - \alpha\hat{u} - h), \qquad \frac{\partial\hat{u}}{\partial\nu}\bigg|_{\partial D} = 0. \tag{3.4}$$

Since $a - \alpha\hat{u} - h$ is in L^∞, by the Harnack inequality, $\hat{u} > 0$ in $\bar{D}$. Therefore, $m_nu_n(x) = m_n\|u_n\|_\infty\hat{u}_n(x) \to \infty$ uniformly. This implies that $h = 0$ almost everywhere. Hence $-\Delta\hat{u} = \hat{u}(a - \alpha\hat{u})$. It then follows that $\hat{u} = a/\alpha = 1$. This implies that $u_n \to a$ in C^1.

Finally, we show that $m_n\|u_n\|_\infty \to 0$ implies that $\bar{d} = a/b$ and $u_n/\|u_n\|_\infty \to 1$. Since $u_n \to 0$ and $m_nu_n \to 0$, we see that $u_n/\|u_n\|_\infty \to \hat{u}$ in C^1 and $\hat{u}$ satisfies $\Delta\hat{u} + \hat{u}(a - b\bar{d}) = 0$. Thus the only possibility is that $\bar{d} = a/b$ and $\hat{u} = 1$. This completes the proof. □

Proof of proposition 3.2. We use an indirect argument. Suppose that there exist $m_n \to \infty$, $d_n \to \bar{d} \leqslant a/b$, such that for $(m,d) = (m_n, d_n)$, system (1.2) has non-constant positive solutions (u_n, v_n). By lemma 3.3, we only need to consider two cases.

Case 1. $u_n \to a$ uniformly. By proposition 2.4 or lemma 3.3, we may assume that $d_n \to \bar{d} \in [0, a/b]$. By theorem 2.1, lemma 3.3 and a simple analysis of the function $H(u)$, it is not hard to see that (1.2) has exactly one constant positive solution, denoted by $(u'_n, v'n)$, with the property that $u'_n \to a$ as $n \to \infty$. Set

$$h'_n = u_n - u'_n, \qquad k'_n = v_n - v'_n,$$

$$h_n = \frac{h'_n}{\|h'_n\|_\infty + \|k'_n\|_\infty}, \qquad k_n = \frac{k'_n}{\|h'_n\|_\infty + \|k'_n\|_\infty}.$$

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Then it is easy to check that

$$-\Delta h_n = \bigg[a - (u_n + u_n') - \frac{bv_n'}{(1+m_n u_n)(1+m_n u_n')}\bigg]h_n \\ - \frac{bu_n}{1+m_n u_n}k_n, \quad \frac{\partial h_n}{\partial \nu}\bigg|_{\partial D} = 0. \quad (3.5)$$

and

$$-\Delta k_n = \bigg[d_n - (v_n + v_n') + \frac{cu_n}{1+m_n u_n}\bigg]k_n \\ + \frac{cv_n'}{(1+m_n u_n)(1+m_n u_n')}h_n, \quad \frac{\partial k_n}{\partial \nu}\bigg|_{\partial D} = 0. \quad (3.6)$$

Since the right-hand sides of (3.5), (3.6) are L^∞ bounded and $\|h_n\|_\infty, \|k_n\|_\infty \leqslant 1$, it follows from the L^p estimates and the Sobolev embedding theorems that $h_n \to h$ and $k_n \to k$ in C^1. Now, by $u_n, u_n' \to a$ and $v_n, v_n' \to \bar{d}$, and passing to the limit in (3.5) and (3.6), we obtain

$$-\Delta h = -ah, \qquad -\Delta k = -\bar{d}k, \qquad \frac{\partial h}{\partial \nu}\bigg|_{\partial D} = 0, \qquad \frac{\partial k}{\partial \nu}\bigg|_{\partial D} = 0.$$

Since $a > 0$, we have $h = 0$. Hence $\bar{d} = 0$ and $k = 1$ or -1. Next we show that $k = 0$. This contradiction would complete the proof of case 1. By Kato's inequality (see [23]),

$$\begin{aligned}-\Delta|k_n| &\leqslant -\frac{k_n}{|k_n|}\Delta k_n \\ &\leqslant \bigg[d_n - (v_n + v_n') + \frac{cu_n}{1+m_n u_n}\bigg]|k_n| + \frac{cv_n'|h_n|}{(1+m_n u_n)(1+m_n u_n')}.\end{aligned}$$

Multiplying this inequality by v_n, integrating it by parts and using $\partial|k_n|/\partial\nu|_{\partial D} = 0$, we obtain

$$\int |k_n|v_n\bigg(d_n - v_n + \frac{cu_n}{1+m_n u_n}\bigg) \\ \leqslant \int\bigg[\bigg(d_n - (v_n + v_n') + \frac{cu_n}{1+m_n u_n}\bigg)|k_n|v_n + \frac{cv_n v_n'|h_n|}{(1+m_n u_n)(1+m_n u_n')}\bigg].$$

Hence

$$\int v_n' v_n |k_n| \leqslant \int \frac{cv_n v_n'}{(1+m_n u_n)(1+m_n u_n')}|h_n|. \quad (3.7)$$

Set $\hat{v}_n = v_n/\|v_n\|_\infty$. Then

$$-\Delta\hat{v}_n = \hat{v}_n\bigg(d_n - v_n + \frac{cu_n}{1+m_n u_n}\bigg), \qquad \frac{\partial \hat{v}_n}{\partial \nu}\bigg|_{\partial D} = 0. \quad (3.8)$$

Since both the right-hand side of (3.8) and $\hat{v}_n$ are L^∞ bounded, it follows from the L^p estimates and the Sobolev embedding theorems that, subject to choosing

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a subsequence, $\hat{v}_n \to \hat{v}$ in C^1. Now passing to the limit in (3.8), since $\bar{d} = 0$, we obtain $\Delta\hat{v} = 0$, $\partial\hat{v}/\partial\nu|_{\partial D} = 0$. This implies that $\hat{v} = 1$, i.e. $\hat{v}_n \to 1$ in C^1. Similarly, $v_n'/\|v_n'\|_\infty \to 1$ in C^1. Now dividing (3.7) by $\|v_n\|_\infty\|v_n'\|_\infty$ and passing to the limit, we obtain

$$\int_D |k| \leqslant 0.$$

Therefore, $k = 0$.

CASE 2. $m_n\|u_n\|_\infty \to 0$. For this case, we first show that $d_n > a/b$. Let

$$w_n = \frac{v_n - d_n}{\|u_n\|_\infty} \quad \text{and} \quad \hat{u}_n = \frac{u_n}{\|u_n\|_\infty}.$$

Then $w_n > 0$ and for any constant d satisfying $0 < d < \bar{d} = a/b$, and all large n,

$$-\Delta w_n + dw_n = (d - v_n)w_n + v_n\frac{c\hat{u}_n}{1 + m_n u_n} \leqslant v_n\frac{c\hat{u}_n}{1 + m_n u_n} \leqslant M,$$

where $M > 0$ is a constant. Hence $\{w_n\}$ is L^∞ bounded. Integrating (3.3) in D, after some rearrangements, we have

$$\int_D \hat{u}_n(a - bd_n) = \int_D \hat{u}_n\left(u_n - \frac{b(v_n - d_n)}{1 + m_n u_n} - bd_n\frac{m_n u_n}{1 + m_n u_n}\right).$$

Dividing this identity by $\|u_n\|_\infty$, and passing it to the limit, noticing that $d_n \to a/b$, $u_n/\|u_n\|_\infty \to 1$, $m_n u_n \to 0$ and w_n is bounded, we obtain

$$\lim_{n\to\infty}\int_D \hat{u}_n\frac{a - bd_n}{\|u_n\|_\infty} = \lim_{n\to\infty}\int_D \hat{u}_n\left(\hat{u}_n - \frac{b}{1 + m_n u_n}w_n - bd_n\frac{\hat{u}_n m_n}{1 + m_n u_n}\right) = -\infty.$$

This shows that $d_n > a/b$ for all large n. Since $d_n > a/b$ and $d_n \to a/b$, it is not difficult to see that one of the two constant positive solutions of (1.2), denoted by (u_n', v_n'), satisfies $m_n u_n' \to 0$. Define h_n and k_n the same as before, and we also arrive at (3.5) and (3.6), by which we see that, by passing to a subsequence, $h_n \to h$, $k_n \to k$ in C^1. By $m_n u_n \to 0$, $m_n u_n' \to 0$, $v_n \to a/b$ and $v_n' \to a/b$, we have

$$-\Delta h = 0, \qquad -\Delta k = (a/b)(-k + ch), \qquad \frac{\partial h}{\partial\nu}\bigg|_{\partial D} = 0, \qquad \frac{\partial k}{\partial\nu}|_{\partial D} = 0.$$

It follows that $h = \text{const.}$ and $k = ch$. Since $\|h\|_\infty + \|k\|_\infty = 1$, we necessarily have $h \neq 0$; on the other hand, using $a - u_n' = bv_n'/(1 + m_n u_n')$ and integrating (3.5) over D, we obtain

$$0 = \int_D\left(-u_n + \frac{bv_n' m_n u_n}{(1 + m_n u_n)(1 + m_n u_n')}\right)h_n - \int_D \frac{bu_n}{1 + m_n u_n}k_n.$$

Thus

$$\int_D \frac{bv_n' u_n/\|u_n\|_\infty}{(1 + m_n u_n)(1 + m_n u_n')}h_n = \frac{1}{m_n}\int_D\left(\frac{u_n}{\|u_n\|_\infty}h_n + \frac{bu_n/\|u_n\|_\infty}{1 + m_n u_n}k_n\right).$$

Passing to the limit, we obtain

$$\int_D h = 0.$$

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Hence $h = 0$, contradicting our earlier conclusion that $h \neq 0$. This completes the proof. □

3.2. An upper bound of d

In this subsection, we find a rather accurate upper bound for d where (1.2) cannot have a non-constant positive solution. The accuracy of this bound will be seen in § 4, where existence of non-constant positive solutions is proved. Since $d^* \to \infty$ as $m \to \infty$, it is much harder to establish this upper bound. Here we will make use of good *a priori* estimates for positive solutions of (1.2), and the bifurcation theory of Crandall and Rabinowitz [10]. Our main result of this subsection is the following.

PROPOSITION 3.4. *There exist a small $\epsilon_2 > 0$ and a large $M_2 > 0$, depending on a, b and c only, such that if $m \geqslant M_2$ and $d \geqslant m[a^2/(4b) - \epsilon_2]$, then (1.2) has no non-constant positive solutions; moreover, $d^* \in (m[a^2/(4b) - \epsilon_2], ma^2/(4b))$ and (1.2) has only two constant positive solutions for $d \in (m(a^2/(4b) - \epsilon_2), d^*)$, one constant positive solution if $d = d^*$ and no positive solution if $d > d^*$.*

In order to prove proposition 3.4, it is convenient to make the following change of variables:

$$d = m\tilde{d}, \qquad u = \tilde{u}, \qquad v = m(\tilde{d} + \tilde{v}).$$

Then (u, v) is a positive solution of (1.2) if and only if $(\tilde{u}, \tilde{v})$ is a positive solution of

$$\left.\begin{aligned} -\Delta u &= u\left(a - u - \frac{bm(\tilde{d} + v)}{1 + mu}\right), && x \in D, \\ -\Delta v &= (\tilde{d} + v)\left(-mv + \frac{cu}{1 + mu}\right), && x \in D, \\ \frac{\partial u}{\partial \nu} &= \frac{\partial v}{\partial \nu} = 0, && x \in \partial D. \end{aligned}\right\} \tag{3.9}$$

We also need several preliminary results.

LEMMA 3.5. *If (3.9) has a positive solution, then $\tilde{d} < (a + 1/m)^2/(4b)$.*

Proof. Let (u, v) be a positive solution of (3.9). Then $0 < u < a$ and $v > 0$. Hence

$$u\left(a - u - \frac{bm(\tilde{d} + v)}{1 + mu}\right) < u\left(a - u - \frac{bm\tilde{d}}{1 + mu}\right). \tag{3.10}$$

Let $f(u) = a - u - bm\tilde{d}/(1 + mu)$. Then one easily finds that f attains its maximum over $(-1/m, \infty)$ at $u_0 = (b\tilde{d})^{1/2} - m^{-1}$ and $f(u_0) = a + 1/m - 2(b\tilde{d})^{1/2}$. Integrating the first equation in (3.9), by (3.10) and the properties of f, we obtain

$$0 = \int u(a - u - \frac{bm(\tilde{d} + v)}{1 + mu}) < \int uf(u) \leqslant \int uf(u_0).$$

Hence $f(u_0) > 0$. That is, $\tilde{d} < (a + 1/m)^2/(4b)$. □

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LEMMA 3.6. *Given any $\delta > 0$, there exist a small $\epsilon_3 > 0$ and a large $M_3 > 0$ such that, if $\tilde{d} > a^2/(4b) - \epsilon_3$, $m > M_3$, then for any positive solution (u, v) of (3.9), $\|u - \frac{1}{2}a\|_\infty < \delta$ and $m\|v\|_\infty < \delta$.*

Proof. By lemma 3.5, it suffices to show that if $d_n \to a^2/(4b)$, $m_n \to \infty$ and (u_n, v_n) is a positive solution of (3.9) with $(\tilde{d}, m) = (d_n, m_n)$, then $u_n \to \frac{1}{2}a$ and $m_n v_n \to 0$ uniformly. By arguments at the beginning of the proof of lemma 3.3, we see that $m_n v_n \to 0$. From

$$-\Delta u_n = u_n\biggl(a - u_n - \frac{bm_n(d_n + v_n)}{1 + m_n u_n}\biggr), \qquad \frac{\partial u_n}{\partial \nu}\bigg|_{\partial D} = 0, \tag{3.11}$$

we see that $\|u_n\|_\infty \geqslant \sigma_0$ for some $\sigma_0 > 0$; otherwise, we may assume $u_n \to 0$, and then the integral of the right-hand side of (3.11) over D is negative for large n, which is impossible. Since the right-hand side of (3.11) is L^∞ bounded, $\|u_n\|_\infty < a$, by L^p estimates and Sobolev embedding theorems, subject to choosing a subsequence, we have that $u_n \to u$ in C^1 and $\|u\|_\infty \geqslant \sigma_0 > 0$. Furthermore, we show that u is a weak solution of

$$-\Delta u = u(a - u) - \tfrac{1}{4}a^2\chi_{\{u>0\}}, \qquad \frac{\partial u}{\partial \nu}\bigg|_{\partial D} = 0, \tag{3.12}$$

where $\chi_{\{u>0\}}$ is the characteristic function of the set

$$D^+ = \{x \in D : u(x) > 0\}.$$

In fact, denoting the right-hand side of (3.11) by h_n, we may assume that $h_n \to h$ weakly in L^2. Then, by passing to the weak limit in (3.11), we find

$$-\Delta u = h, \qquad \frac{\partial u}{\partial \nu}\bigg|_{\partial D} = 0.$$

By elliptic regularity, $u \in W^{2,2}(D)$ and $-\Delta u = h$ a.e. in D (see [18, theorems 8.8 and 8.12]). Since $u = 0$ a.e. on $D \setminus D^+$, $\Delta u = 0$ a.e. on $D \setminus D^+$, we see that $h = 0$ a.e. on $D \setminus D^+$. On the other hand, it is straightforward to see that $h = u(a - u) - \frac{1}{4}a^2$ on D^+. Hence $h = u(a - u) - \frac{1}{4}a^2\chi_{\{u>0\}}$.

We show next that $u = \frac{1}{2}a$. If not, u can not be a constant. Hence

$$\int_D u(a - u)\,\mathrm{d}x < \int_{D^+} (\tfrac{1}{4}a^2)\,\mathrm{d}x.$$

On the other hand, by integrating (3.12) over D, we have

$$0 = \int_D [u(a - u) - (\tfrac{1}{4}a^2)\chi_{D^+}]\,\mathrm{d}x = \int_D u(a - u)\,\mathrm{d}x - \int_{D^+} (\tfrac{1}{4}a^2)\,\mathrm{d}x.$$

This contradiction proves that $u = \frac{1}{2}a$. This implies that the whole sequence $\{u_n\}$ converges to $\frac{1}{2}a$ in C^1. The proof is complete. □

LEMMA 3.7. *There exist a small $\epsilon_4 > 0$ and a large $M_4 > 0$ such that if $d_0 \geqslant a^2/(4b) - \epsilon_4$, $m \geqslant M_4$, and (u_0, v_0) is a degenerate positive solution of (3.9) with $\tilde{d} = d_0$, then all the positive solutions of (3.9) with $\tilde{d}$ near d_0 and (u, v) close*

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to (u_0, v_0) *in* C^1 *norm form a smooth curve* $\{(\tilde{d}(s), u(s), v(s)) : -\delta < s < \delta\}$, $\delta > 0$, *with the properties* $(\tilde{d}(0), u(0), v(0)) = (d_0, u_0, v_0)$, $\tilde{d}'(0) = 0$ *and* $\tilde{d}''(0) < 0$. *In particular, system (3.9) has no positive solution near* (u_0, v_0) *for* $\tilde{d} > d_0$ *but close to* d_0.

Proof. With lemma 3.5 in mind, one sees that it suffices to show that if $d_n \to a^2/(4b)$, $m_n \to \infty$ and (u_n, v_n) is a degenerate positive solution of (3.9) with $(\tilde{d}, m) = (d_n, m_n)$, then, for any fixed large n, all the positive solutions $(\tilde{d}, u, v)$ of (3.9) close to (d_n, u_n, v_n) form a smooth curve

$$\{(d_n(s), u_n(s), v_n(s)) : -\delta_n < s < \delta_n\}, \quad \delta_n > 0,$$

satisfying

$$(d_n(0), u_n(0), v_n(0)) = (d_n, u_n, v_n), \qquad d_n'(0) = 0, \qquad d_n''(0) < 0.$$

Note first that $u_n \to \frac{1}{2}a$ and $m_n v_n \to 0$ in L^∞ by lemma 3.6. Next we use these estimates and theorem 3.2 of [10] to prove the existence of the required curve.

Set

$$X_0 = \{u \in W^{2,p}(D) : \partial u/\partial\nu = 0,\ x \in \partial D\},$$

$X = X_0 \times X_0$ and $Y = L^p(D) \times L^p(D)$, where $p > 1$ is so large that $W^{2,p}$ embeds continuously into C^1. Define $F^n : R^1 \times X \to Y$ by

$$F^n(d, u, v) = \left(\Delta u + u\left(a - u - \frac{bm_n(d+v)}{1+m_n u}\right), \Delta v + (d+v)\left(-m_n v + \frac{cu}{1+m_n u}\right)\right)$$

and define $T_n, \tilde{T}_n, T_0 : X \to Y$ by

$$\begin{aligned}
T_n(h,k) &= (T_n^1(h,k), T_n^2(h,k)),\\
T_n^1(h,k) &= \Delta h + ah - 2u_n h - \frac{bm_n(d_n+v_n)}{(1+m_n u_n)^2}h - \frac{bm_n u_n}{1+m_n u_n}k,\\
T_n^2(h,k) &= m_n^{-1}\Delta k - d_n k - 2v_n k + \frac{c(d_n+v_n)}{m_n(1+m_n u_n)^2}h + \frac{cu_n}{m_n(1+m_n u_n)}k,\\
\tilde{T}_n(h,k) &= (T_n^1(h,k), m_n T_n^2(h,k)),\\
T_0(h,k) &= \left(\Delta h, -\frac{a^2}{4b}k\right).
\end{aligned}$$

Then it is easy to check that $F^n_{(u,v)}(d_n, u_n, v_n) = \tilde{T}_n$. Using $u_n \to \frac{1}{2}a$, $m_n v_n \to 0$ and $d_n \to a^2/(4b)$, one easily checks that $T_n \to T_0$ in the operator norm. It is also easy to see that 0 is an isolated K-simple eigenvalue of T_0 in the sense of [10] with $T_0(1,0) = 0$, where $K : X \to Y$ is the natural injection map. Now it follows from [10, 24] that, for all large n, T_n has a unique K-simple eigenvalue r_n near 0. Moreover, if $T_n(h_n', k_n') = r_n(h_n', k_n')$, then $(h_n', k_n') \to (1,0)$ in X if the norm and sign of (h_n', k_n') are chosen properly.

Since (u_n, v_n) is a degenerate solution, there exists $(h_n, k_n) \neq (0,0)$ such that $\tilde{T}_n(h_n, k_n) = 0$. Using the relationship between T_n and $\tilde{T}_n$, one easily checks that $N(\tilde{T}_n) = N(T_n)$. Thus we must have $r_n = 0$, $(h_n', k_n') = t_n(h_n, k_n)$, $t_n \neq 0$ and $N(\tilde{T}_n) = \text{span}\{(h_n, k_n)\}$ for all large n. We may assume that $(h_n, k_n) \to (1,0)$ in X. Again it follows from the relationship between $\tilde{T}_n$ and T_n, the fact $r_n = 0$ is a K-simple eigenvalue to T_n that $\operatorname{codim} R(\tilde{T}_n) = \operatorname{codim} R(T_n) = 1$ for all large n. Therefore, if we can show that $F_d^n(d_n, u_n, v_n) \notin R(\tilde{T}_n)$ for all large n, then we can use theorem 3.2 of [10] to conclude the following.

If Z is a complement of $\operatorname{span}\{(h_n, k_n)\}$ in X, then, for all large n, the solutions of $F^n(d, u, v) = 0 = F^n(d_n, u_n, v_n)$ near (d_n, u_n, v_n) form a curve

$$(d_n(s), u_n(s), v_n(s)) = (d_n + \tau_n(s), u_n + sh_n + z_n(s), v_n + sk_n + w_n(s)),$$

where $s \to (\tau_n(s), z_n(s), w_n(s)) \in R^1 \times Z$ is a continuously differentiable function near $s = 0$ and $\tau_n(0) = \tau_n'(0) = 0$, $z_n(0) = z_n'(0) = 0$, $w_n(0) = w_n'(0) = 0$. Moreover, if F^n is k-times continuously differentiable, so are $\tau_n(s)$, $z_n(s)$ and $w_n(s)$.

Now it is clear that in order to finish the proof of lemma 3.7, it remains to show two things: (i) $F_d^n(d_n, u_n, v_n) \notin R(\tilde{T}_n)$ for all large n; and (ii) $\tau_n''(0) < 0$ for all large n.

Using perturbation theory for closed linear operators (see [24]), one sees that T_0^*, the adjoint of T_0, has 0 as a K^*-simple eigenvalue. One also checks that $N(T_0^*) = \operatorname{span}\{l_0\}$, where $l_0 \in Y^*$ is given by

$$l_0(u, v) = \int_D u \, \mathrm{d}x.$$

Thus, since $T_n^* \to T_0^*$, $r_n = 0$ is a K^*-simple eigenvalue for T_n^* for all large n. Let $N(T_n^*) = \operatorname{span}\{l_n\}$. Then $l_n \to l_0$ in Y^* if the norm and sign of l_n are chosen suitably.

A simple calculation gives

$$F_d^n(d_n, u_n, v_n) = \left(-\frac{bm_n u_n}{1 + m_n u_n}, -m_n v_n + \frac{cu_n}{1 + m_n u_n}\right) \to (-b, 0).$$

Thus if $F_d^n(d_n, u_n, v_n) \in R(\tilde{T}_n)$, then

$$\left(-\frac{bm_n u_n}{1 + m_n u_n}, -v_n + \frac{cu_n}{m_n(1 + m_n u_n)}\right) \in R(T_n)$$

and

$$0 = l_n\left(-\frac{bm_n u_n}{1 + m_n u_n}, -v_n + \frac{cu_n}{m_n(1 + m_n u_n)}\right) \to l_0(-b, 0) = -\int_D b < 0.$$

This contradiction proves (i).

Next we prove (ii). We differentiate

$$-\Delta u_n(s) = u_n(s)\left(a - u_n(s) - \frac{bm_n(d_n(s) + v_n(s))}{1 + m_n u_n(s)}\right)$$

with respect to s twice at $s = 0$. After some calculations, we obtain

$$\Delta z_n''(0) + az_n''(0) - 2u_n z_n''(0) - \frac{bm_n(d_n + v_n)}{(1 + m_n u_n)^2} z_n''(0) - \frac{bm_n u_n}{1 + m_n u_n} w_n''(0) = \xi_n,$$

$$\xi_n = 2h_n^2 + 2\frac{b_n h_n k_n}{(1 + m_n u_n)^2} - 2\frac{bm_n(d_n + v_n)h_n^2}{(1 + m_n u_n)^3} + \frac{bm_n u_n}{1 + m_n u_n}\tau_n''(0).$$

Similarly, we differentiate

$$-\Delta v_n(s) = (d_n(s) + v_n(s))\biggl(-m_n v_n(s) + \frac{cu_n(s)}{1 + m_n u_n(s)}\biggr)$$

twice with respect to s at $s = 0$ and obtain

$$\begin{aligned}\Delta w_n''(0) - d_n m_n w_n''(0) - 2v_n m_n w_n''(0)&\\ + \frac{c(d_n + v_n)}{(1 + m_n u_n)^2} z_n''(0) + \frac{cu_n}{1 + m_n u_n} w_n''(0) &= \eta_n,\end{aligned}$$

$$\eta_n = 2m_n k_n^2 - \frac{2ch_n k_n}{(1 + m_n u_n)^2} + (d_n + v_n)\frac{2cm_n h_n^2}{(1 + m_n u_n)^3}.$$

Thus $T_n(z_n''(0), w_n''(0)) = (\xi_n, \eta_n/m_n)$ and

$$0 = l_n(\xi_n, \eta_n/m_n) = l_0(\lim \xi_n, \lim \eta_n/m_n) = l_0(2 + b \lim \tau_n''(0), 0).$$

This gives $\lim \tau_n''(0) = -2/b$, i.e. $\tau_n''(0) < 0$ for all large n. The proof is complete. $\square$

The following problem will play an important role in later discussions

$$-\Delta u = u(a - u) - \beta, \qquad \frac{\partial u}{\partial \nu}\bigg|_{\partial D} = 0, \tag{3.13}$$

where $\beta \geqslant 0$ is a constant. It is evident that for $\beta \in [0, \frac{1}{4}a^2)$, problem (3.13) has two constant positive solutions

$$u_1 = \tfrac{1}{2}(a + \sqrt{a^2 - 4\beta}), \qquad u_2 = \tfrac{1}{2}(a - \sqrt{a^2 - 4\beta}),$$

a unique constant positive solution $u = \frac{1}{2}a$ for $\beta = \frac{1}{4}a^2$ and no positive solution otherwise.

Lemma 3.8. *There exists a small $\epsilon_4' > 0$ such that (3.13) has only constant positive solutions for $\beta \geqslant \frac{1}{4}a^2 - \epsilon_4'$.*

Proof. Since (3.13) has no positive solution for $\beta > \frac{1}{4}a^2$, it suffices to show that if u_n is a positive solution of (3.13) with $\beta = \beta_n \to \frac{1}{4}a^2$, $\beta_n \leqslant \frac{1}{4}a^2$, then u_n must be a constant. Since $u_n < a$, by L^p estimates, Sobolev embedding theorems and the equation

$$-\Delta u_n = u_n(a - u_n) - \beta_n, \qquad \frac{\partial u_n}{\partial \nu}\bigg|_{\partial D} = 0, \tag{3.14}$$

we can deduce that $u_n \to u$ in C^1. Passing to the limit in (3.14), we obtain

$$-\Delta u = u(a - u) - \tfrac{1}{4}a^2, \qquad \frac{\partial u}{\partial \nu}\bigg|_{\partial D} = 0.$$

This implies that $u = \frac{1}{2}a$. We show next that u_n is a constant. If not, $h_n = u_n - u_n^1$ is not a constant, where $u_n^1 = \frac{1}{2}(a - \sqrt{a^2 - 4\beta_n})$. One easily checks that

$$-\Delta h_n = [a - (u_n + u_n^1)]h_n, \qquad \frac{\partial h_n}{\partial \nu}\bigg|_{\partial D} = 0. \tag{3.15}$$

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Since $u_n + u_n^1 \to a$ in L^∞, it is easy to see from (3.15) that $|h_n|/\|h_n\|_\infty \to 1$ in C^1. This implies that h_n does not change sign in D for all large n. Similarly, let $u_n^2 = \frac{1}{2}(a + \sqrt{a^2 - 4\beta_n})$, then $k_n = u_n - u_n^2$ does not change sign in D for all large n. Thus for large n we have $u_n < u_n^1$, or $u_n^1 < u_n < u_n^2$, or $u_n > u_n^2$. In any case, $a - (u_n + u_n^1)$ has a fixed sign on D since $a - (u_n^1 + u_n^2) = 0$. This implies that, for large n,

$$\int_D [a - (u_n + u_n^1)]h_n \neq 0,$$

which contradicts (3.15). Thus u_n is a constant for all large n. □

Now we use lemma 3.8 to prove the following result.

LEMMA 3.9. *There exists a small $\epsilon_2 > 0$ such that, for any $\epsilon \in (0, \epsilon_2)$, there exists $M = M(\epsilon) > 0$ large so that, if $m \geqslant M$ and $\tilde{d} \in [a^2/(4b) - \epsilon_2, a^2/(4b) - \epsilon]$, system (3.9) has exactly two positive solutions, and they are both constants.*

Proof. First, we use lemma 3.6 to find $\epsilon_5 > 0$ and $M_5 > 0$ such that $u > \frac{1}{3}a$ for any positive solution (u, v) of (3.9) with $\tilde{d} > a^2/(4b) - \epsilon_5$ and $m > M_5$. Then choose $\epsilon_2 = \min\{\epsilon_5, \epsilon_4, \epsilon_4'/b\}$, with ϵ_4 and ϵ_4' from lemmas 3.7 and 3.8. We show that the conclusion of lemma 3.9 holds for the above chosen ϵ_2. Otherwise, we can find some $\epsilon \in (0, \epsilon_2)$, $m_n \to \infty$ and $d_n \in [a^2/(4b) - \epsilon_2, a^2/(4b) - \epsilon]$ so that (3.9) with $(\tilde{d}, m) = (d_n, m_n)$ has a non-constant positive solution (u_n, v_n). This is because, for large m, $d^* > m(a^2/(4b) - \epsilon)$. Hence, by theorem 2.1, there are two constant positive solutions of (3.9) for $\tilde{d} \in [a^2/(4b) - \epsilon_2, a^2/(4b) - \epsilon]$.

We may assume that $m_n > M_5$. Then $\frac{1}{3}a \leqslant u_n < a$. As before, $m_n\|v_n\|_\infty \to 0$. We may also assume that $d_n \to d \in [a^2/(4b) - \epsilon_2, a^2/(4b) - \epsilon]$. Now u_n satisfies (3.11). By L^p estimates and Sobolev embedding theorems, we deduce that, subject to choosing a subsequence, $u_n \to u$ in C^1, and u is a positive solution of

$$-\Delta u = u(a - u) - bd, \qquad \frac{\partial u}{\partial \nu}\bigg|_{\partial D} = 0.$$

Here we have used $\frac{1}{3}a \leqslant u_n < a$ and $m_n\|v_n\|_\infty \to 0$. By our choice of ϵ_2, we have $bd \geqslant \frac{1}{4}a^2 - \epsilon_4'$. Thus it follows from lemma 3.8 that

$$u = u^1 = \tfrac{1}{2}(a + \sqrt{a^2 - 4bd}) \quad \text{or} \quad u = u^2 = \tfrac{1}{2}(a - \sqrt{a^2 - 4bd}).$$

Therefore, by choosing a subsequence, either (i) $u_n \to u^1$, or (ii) $u_n \to u^2$.

Let (u_n^1, v_n^1), (u_n^2, v_n^2) with $u_n^1 > u_n^2$ be the two constant positive solutions of (3.9) with $(\tilde{d}, m) = (d_n, m_n)$. Then $u_n^1 \to u^1$, $u_n^2 \to u^2$ and $m_n v_n^1$, $m_n v_n^2 \to 0$.

When case (i) occurs, we let

$$h_n' = u_n - u_n^1, \qquad k_n' = v_n - v_n^1,$$
$$h_n = \frac{h_n'}{\|h_n'\|_\infty + \|k_n'\|_\infty}, \qquad k_n = \frac{k_n'}{\|h_n'\|_\infty + \|k_n'\|_\infty}.$$

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Then

$$- \Delta h_n = \left[a - (u_n + u_n^1) - \frac{b m_n (d_n + v_n^1)}{(1 + m_n u_n)(1 + m_n u_n^1)} \right] h_n - \frac{b m_n u_n}{1 + m_n u_n} k_n, \quad \frac{\partial h_n}{\partial \nu}\bigg|_{\partial D} = 0 \quad (3.16)$$

and

$$- \Delta k_n = \left[-m_n d_n - m_n (v_n + v_n^1) + \frac{c m_n u_n}{1 + m_n u_n} \right] k_n + \frac{c m_n (d_n + v_n^1)}{(1 + m_n u_n)(1 + m_n u_n^1)} h_n, \quad \frac{\partial k_n}{\partial \nu}\bigg|_{\partial D} = 0. \quad (3.17)$$

Applying Kato's inequality on (3.17),

$$\begin{aligned} -\Delta |k_n| &\leqslant \left[-m_n d_n - m_n (v_n + v_n^1) + \frac{c m_n u_n}{1 + m_n u_n} \right] |k_n| + \frac{c m_n (d_n + v_n^1)}{(1 + m_n u_n)(1 + m_n u_n^1)} |h_n| \\ &\leqslant -|k_n| + \frac{c m_n (d_n + v_n^1)}{(1 + m_n u_n)(1 + m_n u_n^1)} |h_n|, \end{aligned}$$

for all large n. Hence

$$|k_n| \leqslant (-\Delta + 1)^{-1} \frac{c m_n (d_n + v_n^1)}{(1 + m_n u_n)(1 + m_n u_n^1)} |h_n| \to 0.$$

This implies that $\|k_n\|_\infty \to 0$. Since the right-hand side of (3.16) and h_n are both uniformly bounded, by L^p estimates and Sobolev embedding theorems, up to a subsequence, we have $h_n \to h$ in C^1. Passing to the limit in (3.16), we obtain

$$-\Delta h = (a - 2u^1) h, \qquad \frac{\partial h}{\partial \nu}\bigg|_{\partial D} = 0.$$

By shrinking ϵ_2 if necessary, we can assume $0 < a - 2u^1 < \lambda_2$, where we use $0 = \lambda_1 < \lambda_2 < \cdots$ to denote the distinct eigenvalues of the operator $-\Delta$ under zero Neumann boundary conditions. Hence $h = 0$. Thus $\|h_n\|_\infty + \|k_n\|_\infty \to 0$, contradicting $\|h_n\|_\infty + \|k_n\|_\infty = 1$. This completes the proof for case (i). If case (ii) occurs, we can derive the contradiction similarly. The proof is now complete. □

Now we are ready to prove proposition 3.4.

Proof of proposition 3.4. Suppose that M_4, ϵ_2 and $M(\epsilon)$ are from lemmas 3.7 and 3.9. We define $M_2 = \max\{M_4, M(\frac{1}{2}\epsilon_2)\}$ and show that proposition 3.4 is true for such chosen ϵ_2 and M_2. By lemma 3.9, $d^* \in (m(a^2/(4b) - \epsilon_2), ma^2/(4b))$ if $m \geqslant M_2$. Now we fix $m \geqslant M_2$. Let

$$\hat{d} = \sup\{\tilde{d} : \text{system (3.9) has a positive solution}\}.$$

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Note that $\hat{d} < \infty$. Clearly, $\hat{d} \geqslant d^*/m$. It follows from an easy compactness argument that (3.9) has a positive solution $(\hat{u}, \hat{v})$ at $\tilde{d} = \hat{d}$. Thus we use lemma 3.7 and see that all the positive solutions $(\tilde{d}, u, v)$ of (3.9) near $(\hat{d}, \hat{u}, \hat{v})$ form a smooth curve, and the curve bends at this point to the left of $\hat{d}$. Along this curve, there can not be any other degenerate point as long as $\tilde{d} > a^2/(4b) - \epsilon_2$. This is because, by lemma 3.7, near any such point, system (3.9) has no positive solution for nearby and larger $\tilde{d}$. Thus we can use the implicit function theorem to continue the initial curve near $(\hat{d}, \hat{u}, \hat{v})$ leftwards until $\tilde{d}$ reaches $a^2/(4b) - \epsilon_2$. For $\tilde{d}$ in the range $(a^2/(4b) - \epsilon_2, a^2/(4b) - \frac{1}{2}\epsilon_2)$, due to lemma 3.9, the positive solutions (u, v) on the extended curve can be nothing but the two constant positive solutions. Now we claim that our extended curve must be the constant positive solution curve. If our assertion fails, then the two curves must intersect at a point which is a degenerate solution of (3.9) with $\tilde{d} \geqslant a^2/(4b) - \frac{1}{2}\epsilon_2$. By lemma 3.7, we see that (3.9) cannot have a positive solution near this intersection point with $\tilde{d}$ nearby and larger, which is impossible. The proof is now complete. □

4. Non-constant solutions: existence and profiles

In this section, we study the existence and profiles of non-constant positive solutions of (1.2) with large m. By theorem 3.1, we see that this is possible only if $d \in [a/b + \epsilon_0, m(a^2/(4b) - \epsilon_0)]$.

4.1. Profiles of solutions of (1.2)

In this subsection, we study the limiting behaviour of all possible non-constant positive solutions of (1.2) as $m \to \infty$.

THEOREM 4.1. *Suppose that $\{(u_n, v_n)\}_{n=1}^{\infty}$ are non-constant positive solutions of (1.2) with $(d, m) = (d_n, m_n)$ and $m_n \to \infty$. Then $\|v_n - d_n\|_\infty \to 0$ and one of the following conclusions must hold.*

(i) *$d_n \to d \in (a/b, +\infty)$, $m_n u_n/d_n \to w$ in C^2, where w is a positive smooth solution of*

$$-\Delta w = w\left(a - \frac{b}{1/d + w}\right), \qquad \frac{\partial w}{\partial \nu}\bigg|_{\partial D} = 0. \tag{4.1}$$

(ii) *$d_n \to \infty$, $d_n/m_n \to \alpha \in [0, a^2/(4b))$ and $m_n u_n/d_n \to w$ in C^1, where w is a non-negative non-trivial weak solution of the free boundary problem*

$$-\Delta w = w(a - \alpha w) - b\chi_{\{w>0\}}, \qquad \frac{\partial w}{\partial \nu}\bigg|_{\partial D} = 0. \tag{4.2}$$

Proof. It is easy to show that $d_n \leqslant v_n \leqslant d_n + c/m_n$. Hence $v_n - d_n \to 0$ uniformly. By theorem 3.1, $a/b + \epsilon_0 \leqslant d_n \leqslant m_n(a^2/(4b) - \epsilon_0)$.

STEP 1. We claim that the sequence $\{m_n\|u_n\|_\infty/d_n\}_{n=1}^{\infty}$ is bounded.

To prove the assertion, we argue by contradiction: passing to a subsequence if necessary, we may assume that $m_n\|u_n\|_\infty/d_n \to \infty$. We show that $u_n \to a$ in C^1. Since $\|u_n\|_\infty \leqslant a$ and $m_n\|u_n\|_\infty/d_n \to \infty$, thus $m_n/d_n \to \infty$. Hence, by the

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equation of u_n and standard elliptic regularity, we can show that $u_n \to u$ uniformly, where u is a non-negative smooth solution of

$$\Delta u + u(a-u) = 0, \qquad \frac{\partial u}{\partial \nu}\bigg|_{\partial D} = 0.$$

Therefore, $u = 0$ or $u = a$. We first exclude the case $u = 0$. Set $w_n = u_n/\|u_n\|_\infty$. Then

$$\Delta w_n + w_n(a-u_n) - \frac{bv_n}{d_n}\frac{w_n}{1/d_n + (m_n\|u_n\|_\infty/d_n)w_n} = 0.$$

If $u_n \to 0$ uniformly, then, by standard elliptic regularity, $w_n \to w$ in $C^1(\bar{\Omega})$, where w is a non-negative solution of the equation $\Delta w + aw = 0$, with $\partial w/\partial\nu|_{\partial D} = 0$. Hence $w \equiv 0$, which contradicts $\|w\|_\infty = 1$. Therefore, $u = a$, i.e. $u_n \to a$ uniformly. Set $\delta_n = 1/d_n$ and

$$f_n(u,z) = u\left(a - u - \frac{bz}{\delta_n + (m_n/d_n)u}\right), \qquad g_n(u,z) = z\left(1 - z - \frac{c\delta_n u}{1+m_n u}\right).$$

Let $z_n = v_n/d_n$. Then u_n and z_n satisfy

$$\Delta u_n + f_n(u_n,z_n) = 0, \qquad \delta_n\Delta z_n + g_n(u_n,z_n) = 0, \qquad \frac{\partial u_n}{\partial \nu}\bigg|_{\partial D} = \frac{\partial z_n}{\partial \nu}\bigg|_{\partial D} = 0.$$

Denote

$$\bar{u}_n = \frac{1}{|D|}\int_D u_n\,\mathrm{d}x, \qquad \bar{z}_n = \frac{1}{|D|}\int_D z_n\,\mathrm{d}x.$$

Multiplying the equation of u_n by $u_n - \bar{u}_n$, and integrating in D, we have

$$\begin{aligned}\int_D |\nabla u_n|^2 &= \int_D [f_n(u_n,z_n) - f_n(\bar{u}_n,\bar{v}_n)](u_n - \bar{u}_n)\\ &= \int_D [f_n(u_n,z_n) - f_n(\bar{u}_n,z_n)](u_n - \bar{u}_n)\\ &\qquad + \int_D [f_n(\bar{u}_n,z_n) - f_n(\bar{u}_n,\bar{z}_n)](u_n - \bar{u}_n)\\ &= \int_D \frac{\partial f_n}{\partial u}(\eta_n(x),z_n)(u_n-\bar{u}_n)^2\\ &\qquad + \int_D \frac{\partial f_n}{\partial z}(\bar{u}_n,\zeta_n(x))(u_n-\bar{u}_n)(z_n-\bar{z}_n).\end{aligned}$$

where $\eta_n(x)$ lies between $u_n(x)$ and $\bar{u}_n$, and $\zeta_n(x)$ lies between $\bar{z}_n$ and $z_n(x)$. Since $u_n \to a$, $z_n \to 1$, $m_n \to \infty$, $m_n/d_n \to \infty$, it is easy to check that

$$\frac{\partial f_n}{\partial u}(\eta_n(x),z_n) \to -a \quad \text{and} \quad \frac{\partial f_n}{\partial z}(\bar{u}_n,\zeta_n(x)) \to 0$$

uniformly as $n \to \infty$. Therefore, for any small $\epsilon > 0$, there exists a large $N_1(\epsilon)$ such that, if $n \geqslant N_1$, we have

$$\int_D |\nabla u_n|^2 \leqslant -\tfrac{1}{2}a\int_D (u_n-\bar{u}_n)^2 + \epsilon\int_D (z_n-\bar{z}_n)^2.$$

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By a similar argument on z_n, we can show that

$$\delta_n \int_D |\nabla z_n|^2 \leqslant -\frac{1}{2}\int_D (z_n - \bar{z}_n)^2 + \epsilon \int_D (u_n - \bar{u}_n)^2$$

for all large n. Add the above two inequalities together and set $\epsilon = \min\{\frac{1}{4}a, \frac{1}{4}\}$. Then, for all large n, u_n and v_n are constants. However, this contradicts the assumption that (u_n, v_n) are non-constant solutions. Therefore, the sequence $\{m_n u_n/d_n\}_{n=1}^{\infty}$ is bounded.

STEP 2. We claim that $m_n\|u_n\|_\infty/d_n \geqslant \delta$ for some small $\delta > 0$ and all large n.

We again argue by contradiction. Suppose that $m_n\|u_n\|_\infty/d_n \to 0$. Since $d_n \geqslant a/b + \epsilon_0$, $v_n \geqslant d_n$, we have

$$\begin{aligned} a - u_n - bv_n/(1 + m_n u_n) &\leqslant a - b(v_n/d_n)/(m_n u_n/d_n + 1/d_n) \\ &\leqslant a - b/[m_n\|u_n\|_\infty/d_n + 1/(a/b + \epsilon_0)] \to -b\epsilon_0 < 0. \end{aligned}$$

Thus $a - u_n - bv_n/(1 + m_n u_n)$ is negative for all large n. However, this contradicts

$$\int_D u_n(a - u_n - bv_n/(1 + m_n u_n)) = 0.$$

STEP 3. In the following we consider two cases.

CASE 1. $\{d_n\}_{n=1}^{\infty}$ is bounded. Since $d_n \geqslant a/b + \epsilon_0$, we may assume that, subject to choosing a subsequence, $d_n \to d \in (a/b, \infty)$. Set $w_n = m_n u_n/d_n$. By step 1, we see that w_n is uniformly bounded, and $u_n \to 0$ uniformly. Therefore, by standard elliptic regularity, w_n has a subsequence converging to some w in C^1 and w is a non-negative smooth solution of the equation

$$\Delta w + w(a - b/(1/d + w)) = 0, \qquad \left.\frac{\partial w}{\partial \nu}\right|_{\partial D} = 0.$$

By step 2, $\|w_n\|_\infty \geqslant \delta > 0$. Hence $\|w\|_\infty > 0$. By the strong maximum principle, $w > 0$.

CASE 2. $d_n \to \infty$. Again, set $w_n = m_n u_n/d_n$. Then

$$\Delta w_n + w_n\left[a - (d_n/m_n)w_n - \frac{b(v_n/d_n)}{1/d_n + w_n}\right] = 0, \qquad \left.\frac{\partial w_n}{\partial \nu}\right|_{\partial D} = 0.$$

Since $d_n/m_n \leqslant a^2/(4b) - \epsilon_0$, we may assume that $d_n/m_n \to \alpha \in [0, a^2/(4b))$. By standard elliptic regularity and Sobolev embedding theorems, we see that, subject to a subsequence, $w_n \to w$ in C^1, and a similar consideration to that leading to (3.12) shows that w is a weak solution of the free boundary problem

$$\Delta w + w(a - \alpha w) - b\chi_{\{w>0\}} = 0, \qquad \left.\frac{\partial w}{\partial \nu}\right|_{\partial D} = 0.$$

Again, by step 2, $\|w_n\|_\infty \geqslant \delta > 0$. Thus w is a non-trivial solution. This completes the proof of theorem 4.1. □

4.2. The profile equations (1.3) and (1.4)

In this subsection, we study positive solutions of (1.3) and (1.4). By theorem 4.1, we see that this is crucial in understanding the non-constant positive solutions of (1.2).

Let us begin with some simple observations. By integrating the equation over D, using

$$\int_D \Delta w = 0,$$

it is not hard to show that (1.3) has a positive solution if and only if $d > a/b$. By the convexity of the nonlinearity, one sees further that any positive solution of (1.3) is unstable. Moreover, if $a \leqslant \lambda_2$ and $d > a/b$, then the only positive solution of (1.3) is given by $w_0 \equiv -1/d + b/a$. Indeed, if there is a positive solution w different from w_0, then by integrating (1.3) we find that $w - w_0$ must change sign. We also have

$$-\Delta(w - w_0) + \frac{bd}{(1 + dw)(1 + dw_0)}(w - w_0) = a(w - w_0), \qquad \left.\frac{\partial(w - w_0)}{\partial\nu}\right|_{\partial D} = 0.$$

Thus a is an eigenvalue of the operator $-\Delta + bd/((1 + dw)(1 + dw_0))$ under Neumann boundary conditions corresponding to an eigenfunction which changes sign. It follows that a is at least the second eigenvalue, which is greater than λ_2 since

$$\frac{bd}{(1 + dw)(1 + dw_0)} > 0.$$

Our first existence result is as follows.

THEOREM 4.2. *If $a > \lambda_2$ and $d > a^2[(a - \lambda_2)b]^{-1}$, then (1.3) has at least one non-constant positive solution; if further we assume that $d \neq a^2[(a - \lambda_k)b]^{-1}$ for any $k > 2$, then (1.3) has at least two non-constant positive solutions.*

Proof. Let $f(u) = u[a - b/(1/d + u_+)]$ and consider

$$-\Delta u = f(u), \qquad \left.\frac{\partial u}{\partial\nu}\right|_{\partial D} = 0. \tag{4.3}$$

It is easy to see that f is C^1, and that any solution of (4.3) is non-negative (we use $d > a/b$ here). Therefore, any solution of (4.3) is a non-negative solution of (1.3). Define

$$F(u) = \int_0^u f(s)\,\mathrm{d}s.$$

Then any critical point of $I : H^1(D) \to R$,

$$I(u) = \int [\tfrac{1}{2}|\nabla u|^2 - F(u)]$$

is a non-negative solution of (1.3).

We show that I satisfies the Palais–Smale (PS) condition. Given a PS sequence $\{u_n\} \subset H^1(D)$,

$$I(u_n) \to c, \qquad \int [\nabla u_n \nabla\phi - f(u_n)\phi] = o(\|\phi\|) \quad \forall\, \phi \in H^1, \tag{4.4}$$

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we want to show there is a convergent subsequence. By a standard argument, it suffices to show that $\{u_n\}$ is bounded. Setting $\phi = u_n$ in (4.4), we have

$$\|u_n\|_{H^1}^2 \leqslant C(1 + \|u_n\|_2^2) + o(\|u_n\|). \tag{4.5}$$

Hence we need only prove that $\{\|u_n\|_2\}$ is bounded. If not, we may assume that $\|u_n\|_2 \to \infty$. Choose $\phi = u_n^-$ in (4.4). Then

$$\int |\nabla u_n^-|^2 - (a - bd)(u_n^-)^2 = o(\|u_n^-\|).$$

Hence $\|u_n^-\| \to 0$. Let $v_n = u_n/\|u_n\|_2$. Then from (4.4) it is easy to see that $\{\|v_n\|\}$ is bounded. Hence we may assume that v_n converges weakly in H^1 and strongly in L^2 to some $v \in H^1$ with $\|v\|_2 = 1$. Since $u_n^- \to 0$, we have $v \geqslant 0$. Now dividing (4.5) by $\|u_n\|_2$ and passing to the limit, we obtain

$$\int \nabla v \nabla \phi - av\phi = 0 \quad \forall\, \phi \in H^1.$$

Hence $v \geqslant 0$ is a non-trivial solution of $-\Delta v = av$, $\partial v/\partial \nu|_{\partial D} = 0$. Since $a > \lambda_2$, this is impossible. This proves that I satisfies the PS condition.

By linearizing (1.3), it is easy to check that $u = 0$ is a local minimum of I and its critical groups are given by

$$C_q(I, 0) = \delta_{q,0} G,$$

where $\delta_{q,p} = 0$ when $q \neq p$, $\delta_{q,p} = 1$ when $q = p$ and G is the coefficient group. One can also easily check that I is unbounded from below on $H^1(D) : I(t) \to -\infty$ as $t \to +\infty$. Hence it follows from the mountain pass theorem that I has a critical point of mountain pass type. We may assume that I has finitely many critical points. Then, by a well-known result of Hofer (see [7]), we can choose a critical point u_1 from the mountain pass theorem such that

$$C_q(I, u_1) = \delta_{q,1} G.$$

By our assumption above, $u_0 = w_0 = -1/d + b/a$ is also an isolated critical point of I. Now consider the critical groups of u_0. Note first that the linearization of (1.3) at u_0 is given by

$$-\Delta h = \left[a - \frac{a^2}{bd}\right] h, \qquad \frac{\partial h}{\partial \nu}\bigg|_{\partial D} = 0.$$

Since $a - a^2/(bd) > \lambda_2$, it follows from the shifting theorem that

$$C_q(I, u_0) = 0 \quad \text{for } q \leqslant 1 \leqslant \gamma_2 - 1,$$

where we define

$$\gamma_k = m_1 + \cdots + m_k,$$

where $m_1, m_2, \dots, m_k, \dots$ denote the algebraic multiplicities of the eigenvalues $0 = \lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots$, respectively. Hence 0, u_0 and u_1 are different critical points of I.

Next we show that there is one more critical point if $d \neq a^2/[(a - \lambda_k)b]$, i.e. $a - a^2/(bd) \neq \lambda_k$. Though this can also be proved by using Morse theory, it is

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easier to use a degree argument. Let P denote the natural positive cone in $C(\bar{D})$. Choose $K > 0$ such that $f(u) + Ku$ is increasing in $[0, \infty)$ and define

$$Au = (-\Delta + K)^{-1}[f(u) + Ku].$$

Then it follows from standard arguments that A maps P, $C(\bar{D})$ and $H^1(D)$ to themselves and is completely continuous. Moreover, non-negative solutions of (1.1) correspond to fixed points of A. Since

$$\lim_{u \to 0} f(u)/u = a - bd < 0 = \lambda_1, \qquad \lim_{u \to +\infty} f(u)/u = a > 0 = \lambda_1,$$

it follows from well-known results in [1] that, for all small $r > 0$ and large $R > 0$,

$$\deg_P(I - A, B_r \cap P, 0) = 1, \qquad \deg_P(I - A, B_R \cap P, 0) = 0. \tag{4.6}$$

Here, $B_\rho = \{u \in C(\bar{D}) : \|u\|_C < \rho\}$. From the linearization of u_0 and the Leray–Schauder formula, we have, for $r > 0$ small,

$$\deg_C(I - A, u_0 + B_r, 0) = (-1)^{\gamma_m},$$

where m satisfies $\lambda_m < a - a^2/(bd) < \lambda_{m+1}$. But $u_0 + B_r \subset P$ for r small. Hence

$$\deg_P(I - A, u_0 + B_r, 0) = (-1)^{\gamma_m}. \tag{4.7}$$

As before, we suppose that the mountain pass solution u_1 is isolated in H^1 (and hence it is isolated in C by regularity). Then we may assume that its critical groups are given by $C_q(I, u_1) = \delta_{q,1} G$. It follows from a well-known result in critical point theory (see [7]) that, for any small neighbourhood $U \subset H^1$ of u_1,

$$\deg_{H^1}(I - A, U, 0) = -1.$$

Since A maps H^1 to C compactly, by the commutativity property of the degree, for any small neighbourhood $V \subset C$ of u_1,

$$\deg_C(I - A, V, 0) = \deg_{H^1}(I - A, U, 0) = -1.$$

But $V \subset P$ as $u_1 > 0$ on $\bar{D}$ by the maximum principle. Hence

$$\deg_P(I - A, V, 0) = \deg_C(I - A, V, 0) = -1. \tag{4.8}$$

Now, if there is no other fixed point of A in P, then, by the additivity of the degree and (4.6)–(4.8),

$$\begin{aligned} 0 &= \deg_P(I - A, B_R \cap P, 0) \\ &= \deg_P(I - A, B_r \cap P, 0) + \deg_P(I - A, u_0 + B_r, 0) + \deg_P(I - A, V, 0) \\ &= 1 + (-1)^{\gamma_m} - 1, \end{aligned}$$

which is impossible. The proof is complete. □

By integrating the equation over D, using

$$\int_D \Delta w = 0,$$

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we can easily show that (1.4) has a positive solution if and only if $a > 2\sqrt{\alpha b}$; moreover, if $a \in (0, \lambda_2)$, then (1.4) has only positive constant solutions. Regarding the existence of non-constant solutions of (1.4), we have the following result.

THEOREM 4.3. *If $a > \sqrt{4\alpha b + \lambda_2^2}$, then (1.4) has at least one non-constant positive solution.*

REMARK 4.4. Our proof below shows that if $\alpha = 0$ and $a = \lambda_k$ for some $k \geqslant 2$, then (1.4) has infinitely many non-constant positive solutions. If $\alpha = 0$, $a > \lambda_2$ but $a \neq \lambda_k$ for any $k \geqslant 2$, then any non-constant positive solution w of (1.4) must have non-trivial zero set, i.e. the set $\{w = 0\}$ has positive measure, for otherwise, we easily deduce $w = b/a$.

Proof of theorem 4.3. We first treat the case $\alpha = 0$. Let ϕ_k be an eigenfunction corresponding to λ_k and $a = \lambda_k$, $k \geqslant 2$. Then clearly $w = b/a + t\phi_k$ are non-constant positive solutions of (1.4) for all small non-zero t. Hence we only need to consider the case $a > \lambda_2$ and $a \neq \lambda_k$ $\forall\, k \geqslant 2$. For small $\epsilon > 0$, define

$$f_\epsilon(u) = \begin{cases} au - b, & u \in [\epsilon, \infty), \\ (a - b/\epsilon)u, & u \in (-\infty, \epsilon]. \end{cases}$$

Clearly, $u \to f_\epsilon(u)$ is locally Lipschitz continuous. Define

$$I_\epsilon(u) = \int |\nabla u|^2 - \int F_\epsilon(u),$$

where

$$F_\epsilon(u) = \int_0^u f_\epsilon(s)\,\mathrm{d}s.$$

Then it is well known that the critical points of I_ϵ in $W^{1,2}(D)$ are weak solutions to

$$-\Delta u = f_\epsilon(u), \qquad \frac{\partial u}{\partial \nu}\bigg|_{\partial D} = 0. \tag{4.9}$$

Clearly, system (4.9) has two constant solutions, $u_0 = 0$ and $u_1 = b/a$. It is easy to check that u_0 is a local minimum of I_ϵ and u_1 has Morse index at least 2. Thus u_1 can not be a critical point of I_ϵ obtained by the mountain pass theorem. If u is a solution of (4.9), then multiplying (4.9) by u^- and integrating it over D we obtain

$$\int_D |\nabla u^-|^2 = (a - b/\epsilon) \int_D (u^-)^2.$$

Since $a - b/\epsilon < 0$, we have $u^- = 0$, i.e. $u \geqslant 0$. Now it is clear that if we can find a mountain pass solution of (4.9), then it must be a non-constant positive solution. Since

$$\lim_{u \to +\infty} f_\epsilon(u)/u \to a \neq \lambda_k, \qquad \lim_{u \to -\infty} f_\epsilon(u)/u = a - b/\epsilon < 0,$$

one can check that I_ϵ satisfies the PS condition. It is also easy to see that $I_\epsilon(t) \to -\infty$ as $t \to \infty$. Hence the conditions of the mountain pass theorem are satisfied and we obtain a non-constant positive solution u_ϵ of (4.9) by the mountain pass theorem. Let $\epsilon_n \to 0$ and $u_n = u_{\epsilon_n}$. We show that u_n has a subsequence converging to a non-constant positive solution of (1.4). First, we claim that u_n is bounded in L^∞. Otherwise, we may assume that $\|u_n\|_\infty \to \infty$. Set $w_n = u_n/\|u_n\|_\infty$. Then $-\Delta w_n = f_{\epsilon_n}(u_n)/\|u_n\|_\infty$. By standard elliptic regularity, we may assume that $w_n \to w$ in C^1. Then $f_{\epsilon_n}(u_n)/\|u_n\|_\infty \to aw$ uniformly. Hence

$$-\Delta w = aw, \qquad \frac{\partial w}{\partial \nu}\bigg|_{\partial D} = 0, \qquad \|w\|_\infty = 1.$$

As $a \neq \lambda_k$, this is impossible. Thus u_n is uniformly bounded. Again, by elliptic regularity applied to (4.9) with $(\epsilon, u) = (\epsilon_n, u_n)$, noticing that $f_{\epsilon_n}(u_n)$ is bounded uniformly for n, we have $u_n \to u \geqslant 0$ in C^1. We may assume that $f_{\epsilon_n}(u_n) \to h$ weakly in L^2. Then, passing to the limit in (4.9) with $(\epsilon, u) = (\epsilon_n, u_n)$, we obtain

$$-\Delta u = h, \qquad \frac{\partial u}{\partial \nu}\bigg|_{\partial D} = 0.$$

From this equation, we see that $h = 0$ a.e. on the set $\{u = 0\}$; on the set $\{u > 0\}$, it is easily seen that $h = au - b$. Hence $h = au - b\chi_{\{u>0\}}$, i.e. u is a solution of (1.4). By the equation of u_n, we see that $f_{\epsilon_n}(u_n(x))$ changes sign in D. This implies that $\max u_n > b/a > \min u_n$. Hence $\|u\|_\infty \geqslant b/a$. If $u = b/a$, then for all large n, $u_n > \frac{1}{2}b/a$ and thus

$$-\Delta u_n = au_n - b, \qquad \frac{\partial u_n}{\partial \nu}\bigg|_{\partial D} = 0.$$

This implies $u_n = b/a$ as $a \neq \lambda_k$, contradicting our observation that b/a cannot be a mountain pass type critical point of I_ϵ. This finishes the proof when $\alpha = 0$.

For the case $\alpha > 0$, set

$$f_\epsilon(u) = \begin{cases} -au - b, & u \in [2a/\alpha, \infty), \\ u(a - \alpha u) - b, & u \in [\epsilon, 2a/\alpha], \\ (a - b/\epsilon - \alpha\epsilon)u, & u \in (-\infty, \epsilon]. \end{cases}$$

Again, $f_\epsilon(u)$ is locally Lipschitz continuous. It is also easy to check that I_ϵ is coercive and has $u_0 = 0$, $u_1 = (a - \sqrt{a^2 - 4\alpha b})/(2\alpha)$, $u_2 = (a + \sqrt{a^2 - 4\alpha b})/(2\alpha)$ as critical points. Moreover, u_0 is a local minimizer of I_ϵ and u_2 is a global minimizer of I_ϵ. Thus I_ϵ satisfies the PS condition, and it has a mountain pass critical point u_ϵ with $I_\epsilon(u_\epsilon) > I_\epsilon(u_0)$. By the definition of f_ϵ and the maximum principle, we find that $u_\epsilon \geqslant 0$. Since any positive constant $M > u_2$ is an upper solution of (4.9), we have $u_\epsilon \leqslant u_2$. Thus $f_\epsilon(u_\epsilon)$ is bounded uniformly for all small ϵ. It then follows from standard elliptic regularity and Sobolev embedding theorems that, for some $\epsilon_n \to 0$, $u_{\epsilon_n} \to \tilde{u}$ in C^1. Now, passing to the limit in (4.9) with $(\epsilon, u) = (\epsilon_n, u_{\epsilon_n})$, we find that $\tilde{u}$ solves (1.4). Moreover, $\tilde{u}$ is not a constant since $\min u_\epsilon < u_1 < \max u_\epsilon$, and $\tilde{u} = u_1$ implies that u_{ϵ_n} can never be a mountain pass solution. This last statement follows from the fact that u_{ϵ_n} has Morse index at least 2, due to the assumption that $a^2 - 4\alpha b > (\lambda_2)^2$ and the closeness of u_{ϵ_n} to u_1. This finishes the proof. □

4.3. Non-constant solutions of (1.2)

In this subsection, we show how to obtain non-constant positive solutions of (1.2) by using the limiting problems (1.3) and (1.4).

Let $E = C(\bar{D})$ and define $A_1 : E \to E$ by

$$A_1 u = (-\Delta + 1)^{-1}[u(a + 1 - bd/(1 + du))].$$

Then u is a non-negative solution of (1.3) if and only if it is a non-negative fixed point of A_1. Let U be a bounded open set in E such that $A_1 u \neq u$ for $u \in \partial U$. Then, as A_1 is completely continuous, it is well known that the Leray–Schauder degree $\deg_E(I - A_1, U, 0)$ is well defined. Moreover, system (1.3) has at least one solution in U if $\deg_E(I - A_1, U, 0) \neq 0$; conversely, if u_0 is a non-degenerate solution of (1.3), then $\deg_E(I - A_1, U, 0) \neq 0$ for every small neighbourhood U of u_0.

THEOREM 4.5. *Let U be a bounded open set of E contained in*

$$E^+ = \{u \in E : u(x) > 0 \ \forall\, x \in \bar{D}\},$$

and $\deg_E(I - A_1, U, 0) \neq 0$. Then there exists $M > 0$ large such that for any $m > M$, system (1.2) has at least one positive solution (u, v) satisfying $(m/d)u \in U$.

Proof. We use a degree argument and the homotopy

$$\left.\begin{aligned} -\Delta u &= u(a - tdu/m - b(tv + (1-t)d)/(1 + du)),\\ -\Delta v &= v(d - v + tcdu/[m(1 + du)]),\\ \frac{\partial u}{\partial \nu}\bigg|_{\partial D} &= \frac{\partial v}{\partial \nu}\bigg|_{\partial D} = 0. \end{aligned}\right\} \tag{4.10}$$

It suffices to show that (4.10) has a positive solution (u, v) with u in U for all large m and $t = 1$.

Since $\deg_E(I - A_1, U, 0)$ is defined, $A_1 u \neq u$ for $u \in \partial U$. This means that (1.3) has no solution on ∂U. We show in the following that (4.10) has no solution on $\partial(U \times B_\delta(d))$ for any $t \in [0, 1]$ and m large, where $B_\delta(d) = \{x \in E : \|x - d\|_E < \delta\}$. Otherwise, we can find $t_n \in [0, 1]$, $m_n \to \infty$ and $(u_n, v_n) \in \partial(U \times B_\delta(d))$ such that (u_n, v_n) solves (4.10) with $(t, m) = (t_n, m_n)$. By standard elliptic regularity, $v_n \to d$, $u_n \to u$ in E for some u. By passing to the limit in (4.10), one sees that u is a solution of (1.3). Since $(u_n, v_n) \in \partial(U \times B_\delta(d))$ and v_n is in the interior of $B_\delta(d)$, we must have $u_n \in \partial U$ for all large n. It follows that (1.3) has a solution u on ∂U; a contradiction.

Define $F_t : E \times E \to E \times E$ by

$$F_t(u, v) = \biggl((-\Delta + 1)^{-1} u\biggl(a + 1 - \frac{tdu}{m} - \frac{b(tv + (1-t)d)}{(1 + du)}\biggr),\\ (-\Delta + 1)^{-1} v\biggl(d + 1 - v - \frac{tcdu}{m(1 + du)}\biggr)\biggr).$$

Our above discussion shows that for all large m,

$$F_t(u, v) \neq (u, v) \quad \forall\, t \in [0, 1] \quad \forall\, (u, v) \in \partial(U \times B_\delta(d)).$$

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Since F_t is completely continuous, $\deg_{E^2}(I - F_t, U \times B_\delta(d), 0)$ is well defined and is independent of $t \in [0,1]$. In particular,

$$\deg_{E^2}(I - F_1, U \times B_\delta(d), 0) = \deg_{E^2}(I - F_0, U \times B_\delta(d), 0).$$

But $F_0(u,v) = (A_1 u, (-\Delta + 1)^{-1} v(d + 1 - v))$. Hence, by the product formula of the degree,

$$\deg_{E^2}(I - F_0, U \times B_\delta(d), 0) = \deg_E(I - A_1, U, 0) \cdot \deg_E(I - A_2, B_\delta(d), 0),$$

where $A_2 v = (-\Delta + 1)^{-1} v(d + 1 - v)$.

It is well known that $v = d$ is the unique positive solution of $A_2 v = v$. By the Leray–Schauder formula, we have

$$\deg_E(I - A_2, B_\delta(d), 0) = 1.$$

Thus

$$\deg_{E^2}(I - F_1, U \times B_\delta(d), 0) = \deg_E(I - A_1, U, 0) \neq 0.$$

This implies that (1.2) has a solution (u, v) with $(mu/d, v) \in U \times B_\delta(d)$. The proof is complete. □

COROLLARY 4.6. *If (1.3) has a non-degenerate non-constant positive solution u_0, then there exists a large $M > 0$ such that, for any $m > M$, system (1.2) has a non-constant positive solution (u, v), with $(m/d)u$ close to u_0 in the C^1 norm.*

Proof. By the maximum principle, it is easy to see that $u_0 > 0$ on $\bar{D}$. Since u_0 is non-degenerate, we can find a small neighbourhood U of u_0 in E^+ such that u_0 is the only solution of (1.3) in U. Moreover, by the Leray–Schauder formula, $\deg_E(I - A_1, U, 0) = (-1)^\sigma \neq 0$. Hence, by theorem 4.5, for all large m, we can find a positive solution (u, v) of (1.2) with $mu/d \in U$. Since u_0 is not a constant and U is small, $mu/d \in U$ implies that u is not constant. This finishes the proof. □

REMARK 4.7. By a result of Henry [21], for generic domains D, any solution of (1.3) is non-degenerate. Hence corollary 4.6 implies that there are many non-constant positive solutions of (1.2) for generic domains.

The following result gives non-constant positive solutions of (1.2) for any fixed domain.

COROLLARY 4.8. *Let γ_k be the sum of algebraic multiplicities of the eigenvalues $\lambda_1 < \lambda_2 < \cdots < \lambda_k$. If γ_k is even and $\lambda_k < a - a^2/(bd) < \lambda_{k+1}$, then there exists a large $M > 0$ such that, for any $m > M$, system (1.2) has a non-constant positive solution.*

Proof. Following the degree argument in the proof of theorem 4.2, we find that (4.6) still holds and $\deg_P(I - A_1, u_0 + B_r, 0) = 1$ as γ_k is even. Hence, if we denote $\Omega = (B_R \setminus B_r) \cap P \setminus (u_0 + B_r)$,

$$\deg_P(I - A_1, \Omega, 0) = 0 - 1 - 1 \neq 0.$$

This means that the set of solutions of (1.3) in Ω is non-empty. Let S denote this set. Then it is compact in P by the elliptic regularity and boundedness of Ω. By

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the maximum principle, any solution in S is positive on $\bar{D}$. Hence there exists a small neighbourhood U of S such that U lies in the interior of P. Moreover, we can choose U such that $u \in U$ implies u is not constant and

$$\deg_P(I - A_1, \Omega, 0) = \deg_P(I - A_1, U, 0) = \deg_{C(\bar{D})}(I - A_1, U, 0).$$

Hence we can use theorem 4.5 to conclude. □

THEOREM 4.9. *Let E and E^+ be defined as in theorem 4.5 and*

$$A_2 u = (-\Delta + 1)^{-1}[u(a + 1 - \alpha u) - b],$$

where $\alpha \in (0, a^2/(4b))$. Let U be a bounded open set of E whose closure is contained in E^+ such that $\deg_E(I - A_2, U, 0) \neq 0$. Then there exist a large $M > 0$ and a small $\delta > 0$ such that, for any $m > M$ and $d \in (m(\alpha - \delta), m(\alpha + \delta))$, system (1.2) has at least one positive solution (u, v) with $\alpha^{-1} u \in U$.

Proof. We use the following homotopy:

$$\left.\begin{aligned} -\Delta u &= u(a - \alpha u - tbv/(1 + \alpha m u)) - (1 - t)b, && x \in D,\\ -\Delta v &= v(d - v + tc\alpha u/(1 + \alpha m u)), && x \in D,\\ \frac{\partial u}{\partial \nu} &= \frac{\partial v}{\partial \nu} = 0, && x \in \partial D. \end{aligned}\right\} \tag{4.11}$$

Define $G_t : E^2 \to E^2$ by

$$G_t(u, v) = \biggl((-\Delta + 1)^{-1}\biggl[u\biggl(a + 1 - \alpha u - \frac{tbv}{1 + \alpha m u}\biggr) - (1 - t)b\biggr], \\ (-\Delta + 1)^{-1}\biggl[v\biggl(d + 1 - v + \frac{tc\alpha u}{1 + \alpha m u}\biggr)\biggr]\biggr).$$

We show that, for m large and d/m close to α,

$$F_t(u, v) \neq (u, v) \quad \forall\, t \in [0, 1] \quad \forall\, (u, v) \in \partial(U \times I_d), \tag{4.12}$$

where $I_d = \{v \in E : d - 1 < v < d + 1\}$. Otherwise, we can find $m_n \to \infty$, $d_n/m_n \to \alpha$, $t_n \in [0, 1]$ and $(u_n, v_n) \in \partial(U \times I_{d_n})$ such that $F_{t_n}(u_n, v_n) = (u_n, v_n)$, i.e. (u_n, v_n) solves (4.11) with $(m, d, t) = (m_n, d_n, t_n)$. By the equation of v_n, we easily obtain $0 \leqslant v_n - d_n \leqslant c/m_n$. Hence $u_n \in \partial U$ and $v_n/m_n \to \alpha$ in E. This implies that the right-hand side of the equation of u_n in (4.11) is uniformly bounded in E. By elliptic regularity, we may assume that $u_n \to u \in \partial U$ in C^1. Then, passing to the limit in the equation of u_n in (4.11), we have

$$-\Delta u = u(a - \alpha u) - b, \qquad \frac{\partial u}{\partial \nu}\bigg|_{\partial D} = 0.$$

This shows that $A_2 u = u$ has a solution $u \in \partial U$, which is impossible by our assumption. Hence (4.12) holds. By the homotopy invariance of the Leray–Schauder topological degree,

$$\deg_{E^2}(I - F_1, U \times I_d, 0) = \deg_{E^2}(I - F_0, U \times I_d, 0) \tag{4.13}$$

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when m is large and d/m is close to α. But $F_0(u,v)=(A_2u,Bv)$, where

$$Bv=(-\Delta+1)^{-1}[v(d+1-v)].$$

Hence, by the product formula for the Leray–Schauder degree,

$$\deg_{E^2}(I-F_0,U\times I_d,0)=\deg_E(I-A_2,U,0)\cdot\deg_E(I-B,I_d,0). \tag{4.14}$$

It is easy to check that $v=d$ is the only solution of $Bv=v$ in I_d and

$$\deg_E(I-B,I_d,0)=1.$$

Hence it follows from (4.13) and (4.14) that

$$\deg_{E^2}(I-F_1,U\times I_d,0)=\deg_E(I-A_2,U,0)\neq 0.$$

This implies that $F_1(u,v)=(u,v)$ has a solution in $U\times I_d$, i.e. system (1.2) has a positive solution with $u/\alpha\in U$. The proof is complete. □

The following result follows from an argument similar to that of corollary 4.8.

Corollary 4.10. *If u_0 is a positive non-degenerate solution of system (1.4), then there exist a large $M>0$ and a small $\delta>0$ such that, for any $m>M$ and $d\in(m(\alpha-\delta),m(\alpha+\delta))$, system (1.2) has at least one non-constant positive solution.*

Remark 4.11. Non-degenerate positive solutions of (1.4) like u_0 in corollary 4.10 can often be obtained by looking at solutions of (1.4) which bifurcate from the constant solution $u_c=(a-\sqrt{a^2-4\alpha b})/(2\alpha)$. For example, if $\sqrt{a^2-4\alpha_0 b}$ is an eigenvalue of $-\Delta u=\lambda u$ under homogeneous Neumann boundary conditions, then by a well-known bifurcation result for operators with variational structure, system (1.4) has solutions (u,α) bifurcating from (u_c,α_0) in a small neighbourhood of (u_c,α_0). These are necessarily non-constant positive solutions of (1.4). By [21], for generic domains D, these are non-degenerate solutions satisfying the conditions in corollary 4.10.

Remark 4.12. If the domain D is an interval, then the results in this section can be significantly sharpened. For example, one can use phase plane argument to obtain much sharper existence results for (1.3) and (1.4). Moreover, one can use local and global bifurcation theory to look for non-constant positive solutions of (1.3) and (1.4) which bifurcate from constant solutions. By following the global bifurcation branches, one can find many sets U such that the degree conditions in theorems 4.5 and 4.9 are satisfied, and hence obtain non-constant positive solutions of (1.2).

Remark 4.13. It would be interesting to see how to obtain non-constant positive solutions of (1.2) from non-negative solutions of (1.4) with non-trivial zero sets. The degree method above does not seem to work for this case.

Remark 4.14. For large m, it is easy to show that for $d\in(-ca/(1+ma),a/b]$, the unique constant positive solution of (1.2) is linearly stable; and for $d\in(a/b,d^*)$, one of the constant positive solutions is linearly stable while the other is linearly unstable. It would be interesting to know whether there can be *stable non-constant* positive solutions of (1.2) when m is large.

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