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A drift of Langmuir waves in a magnetized inhomogeneous plasma

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The concept of informativeness of nonlinear plasma physics scenarios is explained. Natural ideas of developing highly informative models of plasma kinetics are spelled out. They are applied to develop a formula that governs the drift of long Langmuir waves in spatial positions and wave vectors in a magnetized plasma due to the plasma inhomogeneity. Together with previous findings (Erofeev, *Phys. Plasmas*, vol. 22, 2015, 092302), the formula evidences the need for an intelligent generalization of the notion of wave energy density from usual homogeneous plasmas to inhomogeneous ones.

Key words: fusion plasma, plasma nonlinear phenomena, plasma waves

1. Introduction

The most important aspect of a physical theory is the extent to which its predictions of the behaviour of evolving physical systems agree with real pictures of their macrophysical evolutions. More specifically, the longer a theoretical scenario objectively portrays the macrophysical evolution of a system, the better. For a clearer characterization of this aspect of physical-theoretical scenarios, we have suggested the use of the term *informativeness* (Erofeev 2011*a*, 2013, 2014, 2015*a*,*b*, 2016).

As applied to plasma physical studies, we mean that the longer the theoretical scenario adequately depicts the real picture of the plasma macrophysical evolution, the higher the estimate the researcher should suggest for the scenario informativeness. Accordingly, increasing the informativeness of plasma scenarios should be one of the most important motivations for the theory development.

However, the fundamentals of traditional plasma theory prevent researchers from success in pursuing this motivation. Basically, the conventional machineries of the theory provide scenarios of inappropriately low informativeness: most plasma scenarios have an arbitrary correspondence with objective pictures of plasma evolutions in respective physical situations. In other words, usual approaches of the theory allow different and even incompatible versions of a specific plasma phenomenon to be generated in an equally rigorous manner. A rich set of illustrations

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on this point is provided by the nonlinear effects of a weakly turbulent plasma (Erofeev 2011*a*, 2013, 2015*b*).

We have long before clarified two basic reasons of the theory non-informativeness. Here we will consider them again in a somewhat clearer exposition.

To develop scientifically sound ideas of physical phenomena in a high-temperature ionized plasma, it is natural to start with considering the plasma as a mixture of individual classical charged particles. Together with external sources, they generate an electromagnetic field, which exerts Lorentz forces on the particles. The entire variety of the corresponding motions of individual charged particles is described by simultaneous Maxwell (1865) and Klimontovich–Dupree equations (Klimontovich 1958, 1967; Dupree 1963): particle trajectories are characteristics of these simultaneous partial differential equations. It is common to regard the corresponding plasma kinetic model as a full plasma description.[1]

Naturally, the simultaneous integration of the great number of equations of motion for individual particles is technically infeasible. This forces a theorist to employ a model of plasma kinetics that is substantially simpler than the full plasma description. In order for the respective plasma kinetic scenario to provide a fittingly informative description of the plasma evolution, the above simplified plasma kinetic model should be constructed by properly reducing the full plasma description. In view of this, the researcher inevitably loses the ephemeral possibility of adequately modelling the plasma evolution during an infinite time interval, since any of the possible schemes of this reduction implies an essential reduction of the theory informational basis. Further, the reduction inevitably involves the generation of some nonlinear perturbation theory. In any such theory, an increase in the order of consideration entails a factorial increase in the number of terms to be accounted for. Consequently, after a certain order of consideration, the improvement in the accuracy of the scenario (which is the only reason for the increase in the order of consideration!) will inevitably be superseded by a reduction in accuracy: successive iterations begin to diverge due to a growth in the number of summands. It is said that such a perturbation theory exhibits an asymptotic character of convergence. Using respective perturbations, the researcher can equally rigorously develop different scenarios of the plasma evolution from a specified initial state by referring to various versions of leading-order approximation for this initial state. The point is that when one takes different lowest-order approximations of the employed nonlinear perturbation theory, its first sequential orders converge to different conditional limits which correspond to different theoretical scenarios of the plasma macrophysical evolution.

In such a way, the first reason of the non-informativeness of traditional plasma kinetic scenarios is the lack of a proper understanding of the significance of the asymptotic nature of the convergence of successive approximations to a plasma scenario that one generates to reduce the full plasma description to a simpler model of plasma kinetics.

The second reason for the theory non-informativeness is the fallacious tradition of substituting real plasmas by probabilistic ensembles of plasmas. The common practice is to draw, consciously or unintentionally, conclusions on the reciprocal influence of some statistics of the plasma ensemble and to regard them as objective laws of the plasma macrophysical evolution. The laws thus obtained depend substantially on the composition of the ensemble which is defined in a somewhat sensible manner only within the Gibbsian equilibrium statistical thermodynamics (Gibbs 1902).

[1]Conceptually, plasma kinetics is a branch of plasma physics theory that underlies all other branches of this science.

As for studies of the evolution of non-equilibrium physical systems, the ensemble method is flatly useless: the laws derived in the above manner have nothing to do with objective pictures of the macrophysical evolution of particular systems. (This implies that non-equilibrium statistical mechanics is merely a technical discipline that cannot suggest scientifically sound recipes for studying evolving physical systems (Erofeev 2004*b*). Meanwhile, it underlies all the traditional versions of plasma kinetics and respective methods of plasma studies, and plasma theorists continue to rely upon its approaches. Some recent papers where ensemble averaging is implied by the basic equations are those by Yoon (2000, 2005*a*,*b*, 2006), Yoon & Fang (2008), Taguchi (2010), Yoon *et al.* (2012, 2016), Ziebell *et al.* (2012), Plunk (2013), Wang (2013), Hau-Riege & Weisheit (2017), Rozmus *et al.* (2017), Shi, Qin & Fisch (2017), Belyi (2018) and Schoeffler, Loureiro & Silva (2018). Ensemble plasma averaging is also implied by any hydrodynamic modelling of plasma phenomena, and recent examples of such modelling were reported by Araki (2015), Bhattacharjee *et al.* (2015), Stawarz & Pouquet (2015), Squire & Bhattacharjee (2015), Andrés & Sahraoui (2017) and Viciconte, Gréa & Godeferd (2018). Implicitly, plasma ensemble substitutions are present in many numerical plasma simulations (see, e.g. Banks *et al.* 2017; Keenan *et al.* 2017; Hill & Kingham 2018).

The above two reasons of theory non-informativeness are inseparable from each other. (Explanations on this point are given in Erofeev 2009, 2010, 2011*b*,*c*). They dictate the following ideas of developing the most informative of possible plasma scenarios.

First, the researcher should refrain from the plasma ensemble substitution. This necessitates modifying the basic concepts of the theory, the plasma particle distribution functions. The key concept of any constructive model of plasma kinetics is the particle distribution which does not contain data on the positions and momenta of individual particles. (In particular, this is typical of the distribution function in the Vlasov plasma model (Vlasov 1945).) It differs in the latter respect from the Klimontovich's distribution function (Klimontovich 1958, 1967), which objectively characterizes the distribution of charged particles in a classical ionized plasma, $N_\alpha = \sum_n \delta^3(\boldsymbol{r} - \boldsymbol{r}_n(t))\delta^3(\boldsymbol{p} - \boldsymbol{p}_n(t))$. (Here the subscript n numbers the particles of species α, either electrons ($\alpha \mapsto e$) or ions ($\alpha \mapsto i$), and the functions $\boldsymbol{r}_n(t)$ and $\boldsymbol{p}_n(t)$ describe the trajectories of individual particles.) We call the latter distribution a microdistribution. The tradition of plasma kinetic theory implies that the above Vlasovian-type distribution function is obtained from the microdistribution via ensemble averaging. The only possibility of avoiding ensemble averaging is to replace it by a contextually oriented averaging in the phase space of the positions and momenta of plasma particles.

Second, the researcher should directly integrate in time the intermediate evolution equations obtained by reducing the full plasma description. It is this approach that allows one to properly account for available information on the current plasma state and its recent history, and, simultaneously, to diminish the effect of data on the unknown plasma states that are remote in time.

Bearing in mind the above principles, we have developed a technique of highly informative correlation analysis of plasma kinetics for constructing kinetic scenarios of weakly turbulent plasmas (Erofeev 1997, 2009, 2011*a*,*c*, 2013). Initially, it was formulated for plasmas with turbulent fields of potential waves (Erofeev 2011*a*). This helped us to recover the concept of Langmuir wave collisionless dissipation through stochastic plasma electron acceleration (Erofeev 2002*a*,*b*, 2004*b*, 2010), to highlight its implications for the beginning of weak Langmuir turbulence theory (the impossibility

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of formation of the Langmuir condensate in weakly turbulent plasmas (Erofeev 2002*a*), the impossibility of Langmuir wave collapse (Erofeev 2002*b*) and the impossibility of Vedenov–Rudakov's (long-wavelength) plasma modulational instability (Erofeev 2011*a*, 2004*b*)), and to highlight the shortcomings of the traditional understanding of wave scattering by plasma electrons (Erofeev 2000, 2011*b*) and the electron kinetics in the respective scattering process (Erofeev 2003).

An extension of the technique to the case of a macroscopically homogeneous plasma with fields of waves possessing an arbitrary solenoidal component is presented in Erofeev (2014). (See also chap. 2 of Erofeev (2013).) In that paper, we have developed a maximally informative kinetic scenario of the nonlinear conversion of Langmuir waves to electromagnetic ones. (This process is also known as Langmuir wave coalescence (Willes, Robinson & Melrose 1996).) It has been shown that the conventional interpretation of this phenomenon (Terashima & Yajima 1963; Akhiezer, Danelia & Tsintsadze 1964; Al'tshul' & Karpman 1965; Willes *et al.* 1996) substantially distorts the picture of merging of Langmuir waves with long wavelengths. With this, we have indicated the failure of the former 'golden rule'.[2]

Finally, we have adapted our technique to developing highly informative kinetic scenarios for inhomogeneous plasmas with weakly turbulent fields of arbitrary waves (not only potential ones), see Erofeev (2015*b*). Specifically, it has been used to study the inelastic scattering of electromagnetic waves on Langmuir waves. It has been shown that this scattering is a further illustration of the failure of the 'golden rule'. At the same time, it has been shown that the linear drift of electromagnetic waves in space and wave vectors in inhomogeneous plasmas is inconsistent with the traditional understanding of this drift. As a matter of fact, the new rate of wave drift contains an additional term as compared to its traditional counterpart. We have not perceived any sound reason for the appearance of this term. The present study seeks to clarify this issue. We have developed the rate of drift of Langmuir waves in an inhomogeneous magnetized plasma. By Langmuir waves we mean the plasma oscillations pertaining to the wave branch whereat the long-wavelength limit of the natural frequency is the Langmuir frequency. Depending on the wave vector, there may be different ratios of electric and magnetic field strengths in the waves, a property which is of advantage for the study. We shall consider the drift of long waves in a cold, fully ionized plasma using analytical approaches.

The main idea that we have extracted from the results of this research is that there is no rigorous concept in usual plasma theory of the wave energy density in an inhomogeneous plasma. We shall discuss this fact in greater detail at the end of the paper.

The essentials of our technique were described in previous publications (Erofeev 2013, 2014, 2015*b*). In this paper, we clarify its cornerstones with a correction of some previously advanced erroneous propositions regarding the consideration of the plasma inhomogeneity effect. After that, we apply it to the new wave drift problem.

In § 2, we outline the specific features of the physical situation in question and develop a plasma description using simultaneous equations for the evolution of some of the two-time correlation functions involved. In accordance with the adopted concept, it is natural to characterize such a description as some two-time formalism. Its development is the first step in designing a highly informative plasma

[2] A description of this rule can be found, e.g. in Sagdeev & Galeev (1969, p. 23). It provided manual incorporation of a specific symmetric construction from numbers $N_{\boldsymbol{k}}^{\sigma}$ of waves of a given type σ into the wave collision integral. This gave the researcher an opportunity to calculate the 'probability' of wave interaction based on some simple context and then to compose the wave collision integral for the general case.

kinetic scenario. It is followed by reducing the formalism to a description of plasma evolution in terms of functions depending on a single time variable. We shall expose the essentials of the reducing procedure in § 3 via developing the rate of change of the turbulence spectrum due to wave drift in usual space and in wavenumbers. The implementation of this procedure for the case of Langmuir waves in a magnetized plasma will be described in § 4. Final comments on the results obtained will be formulated in § 5.

2. Logic of first step in designing highly informative plasma kinetic scenarios: developing the two-time formalism

We re-emphasize that the concept of a highly informative kinetic plasma scenario strongly depends on the specific features of the physical situation under consideration. The obvious reason of this dependence is the choice of the scheme for averaging the microdistribution in phase space. We shall consider the case of a classical ionized, magnetized, collisionless, weakly turbulent plasma. Let the leading magnetic field be directed along the z axis of a Cartesian coordinate frame. The plasma is weakly inhomogeneous: the typical scale of spatial electron motions in the plasma, the Debye length r_D, is much smaller than the typical length of change in macroscopic plasma density $|\nabla(\ln n_e)|^{-1}$. For simplicity, we assume that the level planes of the plasma density are the yz planes.

A suitable characteristic of the distribution of charged plasma particles of a given species in the plasma, $f_\alpha(\boldsymbol{r}, \boldsymbol{p}, t)$, is their properly averaged microdistribution N_α. To construct it, we do the following. We construct a six-dimensional parallelepiped with centre at $(\boldsymbol{r}, \boldsymbol{p})$ and divide the number of particles in the parallelepiped by the volume of the parallelepiped. Note that the relative variations in the obtained ratio due to the discreetness of plasma particles that take place in wandering in the vicinity of a point can be ignored provided that the volume of the parallelepiped is large enough. (For simplicity, the parallelepipeds can be assumed to be identically shaped for all points of the phase space of the given particles.) That is, the function f_α then becomes a well-defined statistic in the sense of mathematics for the bulk of the phase space occupied by the particles. (This statistic does not possess reliability only in its momentum tails.)

It is noteworthy that in our situation, we do not need the gradations of f_α in the y and z variables. Therefore, the y and z dimensions of the averaging parallelepiped can be chosen rather elongated, which permits obtaining sufficiently small gradations of f_α in x and in the components of the momentum $\boldsymbol{p}$ without loss of the distribution reliability.

A key component of a highly informative plasma kinetic scenario is the evolution law of the distribution function. It should be developed on the basis of the evolution equation of the microdistribution. The latter is just the Klimontovich–Dupree equation which is conceptually the continuity equation for particles of a given species in their phase space. Using the integral consequences of this continuity equation, one finds that the distribution function $f_\alpha(\boldsymbol{r}, \boldsymbol{p}, t)$ is advanced in time by the two-point correlation function $\langle \delta N_\alpha(\boldsymbol{r}, \boldsymbol{p}, t)\delta \boldsymbol{F}(\boldsymbol{r}', t')\rangle$. (Here δN_α is the difference $N_\alpha(\boldsymbol{r}, \boldsymbol{p}, t) - f_\alpha(\boldsymbol{r}, \boldsymbol{p}, t)$ and $\delta \boldsymbol{F}$ is the respective microstructural part of the electromagnetic field tensor (EMF tensor) $\boldsymbol{F}(\boldsymbol{r}, t)$.) The evolution law of the given correlation function can also be developed using the Klimontovich–Dupree equation: the two-point correlation function is advanced in time by the three-point correlation function $\langle \delta N_\alpha(\boldsymbol{r}, \boldsymbol{p}, t)\delta \boldsymbol{F}(\boldsymbol{r}', t') \otimes \delta \boldsymbol{F}(\boldsymbol{r}'', t'')\rangle$. Similarly, the three-point

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correlation function is advanced in time by the four-point correlation function $\langle \delta N_\alpha(\boldsymbol{r},\boldsymbol{p},t)\delta \boldsymbol{F}(\boldsymbol{r}',t')\otimes\delta\boldsymbol{F}(\boldsymbol{r}'',t'')\otimes\delta\boldsymbol{F}(\boldsymbol{r}''',t''')\rangle$, etc.[3]

Thus, the description of the evolution of the distribution function involves integration of the evolution equations for the multipoint correlation functions that constitute an infinite hierarchy. This equation hierarchy resembles the hierarchy of evolution equations of multiparticle distribution functions in the well-known BBGKY theory (Bogoliubov 1962; Born & Green 1949; Kirkwood 1946; Yvon 1935). We emphasize that the BBGKY theory acquires the meaning of a technical tool suitable for studying physical phenomena only within the framework of consideration of rather specific plasma ensembles. Albeit the contents of the corresponding ensembles is never discussed, they are assumed to possess a certain continuity in their distribution over the $6\boldsymbol{N}$-dimensional phase space of an original $\boldsymbol{N}$-body system, and averaging over the ensemble is implied. Meanwhile, being taken out of the ideology of ensemble averaging, Bogolyubov's method of sequential integration of the 'full' N-particle distribution function over the coordinates and momentums of individual charged plasma particles gives no more than the Klimontovich's distribution function $N_\alpha(\boldsymbol{r},\boldsymbol{p},t)$. (The indistinguishability of particles of a given type is implied.) That is, the respective procedure leads no further than to the starting positions in the problem of reducing the full plasma description to informative plasma kinetic scenarios.

The development of somewhat informative scenarios on the basis of our equation hierarchy is possible only when this hierarchy can be truncated at some reasonable order. (In particular, this can be done in the case of a weakly turbulent plasma.) Then the truncated hierarchy can be reduced to a couple of evolution equations: the above evolution equation of the distribution function f_α and the evolution equation of the the two-time correlation function $\boldsymbol{\Phi}(\boldsymbol{r},t,\boldsymbol{r}',t')=\langle\delta\boldsymbol{F}(\boldsymbol{r},t)\otimes\delta\boldsymbol{F}(\boldsymbol{r}',t')\rangle$. (Here the averaging is over the spatial projections of the above six-dimensional parallelepiped and again with a fixed difference between the spatial variables of the two objects during the averaging.) This completes the first step of developing the plasma kinetic model, i.e. the formulation of the two-time formalism. In the next section, we specify aspects important for reducing this formalism to the final kinetic scenario of wave drift in space and wavenumbers.

3. Reduction of the two-time formalism: developing a general picture of wave drift

Conceptually, the two-time correlation function $\Phi^{ijkl}(\boldsymbol{r},t,\boldsymbol{r}',t')$ behaves as the EMF tensor from the viewpoint of both its entry (i.e. in dependence on the indices i and j and the variables $\boldsymbol{r}$ and t) and exit (in dependence on the indices k and l and the variables $\boldsymbol{r}'$ and t'). That is, this tensor should satisfy Maxwell's equations. Accordingly, the properly written Maxwell's equations describe the evolution of the two-time correlation function. For definiteness, we consider the time advances of the function $\boldsymbol{\Phi}(\boldsymbol{r},t,\boldsymbol{r}',t')$ from the viewpoint of its entry. In this sense, the tensor $\Phi^{ijkl}(\boldsymbol{r},t,\boldsymbol{r}',t')=\langle\delta F^{ij}(\boldsymbol{r},t)\delta F^{kl}(\boldsymbol{r}',t')\rangle$ satisfies the vacuum Maxwell equations with the 'external charge current' calculated on the basis of the two-point correlation functions $\langle\delta N_\alpha(\boldsymbol{r},\boldsymbol{p},t)F^{kl}(\boldsymbol{r}',t')\rangle$. Note that in the above two-time $\langle\delta F^{ij}(\boldsymbol{r},t)\delta F^{kl}(\boldsymbol{r}',t')\rangle$

[3] The sign $\otimes$ denotes a direct tensor product; the averaging is over the above parallelepiped with centre at the point $(\boldsymbol{r},\boldsymbol{p})$ in all current and subsequent multipoint correlation functions. The spatial arguments of all objects under the averaging sign vary synchronously with the spatial argument of δN_α: differences between these arguments are fixed in the averaging. Hence, our system of spatial arguments of multipoint correlation functions is somewhat redundant: one of the spatial arguments might be omitted in the notation of each multipoint correlation function when the plasma is homogeneous. We bear in mind this redundancy of the spatial arguments of the functions and still use the respective system of variables because of its internal symmetry.

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and two-point $\langle\delta N_\alpha(\boldsymbol{r},\boldsymbol{p},t)F^{kl}(\boldsymbol{r}',t')\rangle$ correlation functions, the second terms under the averaging signs play the role of a statistical weight. Correspondingly, these correlation functions can be coordinately convoluted at their exits with an arbitrary tensor: the corresponding modification of the 'EMF' and the 'charge distribution' will not violate their relationship given by the above-mentioned Maxwell equations.

Developing a plasma kinetic scenario, one should calculate the two-time correlation function up to a desired accuracy of its representation. We suggest to select some lowest-order approximation of the function and to refine it via iterations on the basis of Maxwell's equations. This supposes that one has at hand the expression for the 'external charge current' in terms of two-time correlation functions. That is, one should express the two-point correlation function in terms of two-time correlation functions. Conceptually, this can be done through integration of the time derivative $\partial\langle\delta N_\alpha(\boldsymbol{r},\,\boldsymbol{p},\,t)F^{kl}(\boldsymbol{r}',\,t')\rangle/\partial t$ (which is extracted from the evolution equation of the function) from $-\infty$ to the current entry time t. Integration of this type was performed in Erofeev (1997). (See also Erofeev 2004*a*, 2011*a*, 2013.) We shall not repeat it here. We only present its result in a linear approximation, which is certainly sufficient for proper modelling of the wave drift. Namely, the two-point correlation function $\langle\delta N_\alpha(\boldsymbol{r},\,\boldsymbol{p},\,t)F^{kl}(\boldsymbol{r}',\,t')\rangle$ in the integrals of the above 'external charge current' can be substituted as follows:

$$\langle\delta N_\alpha(\boldsymbol{r},\boldsymbol{p},t)\delta F^{kl}(\boldsymbol{r}',t')\rangle=\int\mathcal{R}_{\alpha\cdot\gamma}^{\ m\cdot}(\boldsymbol{r},\boldsymbol{p},t,\boldsymbol{r}_1,t_1)\,\mathrm{d}^3\boldsymbol{r}_1\,\mathrm{d}t_1\varPhi_{m\cdots}^{\cdot\gamma kl}(\boldsymbol{r}_1,t_1,\boldsymbol{r}',t'). \tag{3.1}$$

(Latin subscripts and superscripts are used for the co- and contravariant components of 4-vectors and tensors, and Greek letters are used for the spatial components of these 4-vectors and tensors. We also imply the Einstein summation convention. We employ tensor notation just for simplicity of writing and interpreting formulae.)

The tensor $\boldsymbol{\mathcal{R}}_\alpha(\boldsymbol{r},\,\boldsymbol{p},\,t,\,\boldsymbol{r}_1,\,t_1)$ is a response function. Its linear approximation is sufficient for our study:

$$\mathcal{R}_{\alpha\cdot\gamma}^{\ m\cdot}(\boldsymbol{r},\boldsymbol{p},t,\boldsymbol{r}_1,t_1)\approxeq-\frac{e_\alpha}{c}\int\mathrm{d}^3\boldsymbol{p}_1\,{}^0\mathcal{G}_\alpha(\boldsymbol{r},\boldsymbol{p},t,\boldsymbol{r}_1,\boldsymbol{p}_1,t_1)v_1^m\frac{\partial}{\partial p_1^\gamma}f_\alpha(\boldsymbol{r}_1,\boldsymbol{p}_1,t_1). \tag{3.2}$$

The function ${}^0\mathcal{G}_\alpha(\boldsymbol{r},\,\boldsymbol{p},\,t,\,\boldsymbol{r}',\,\boldsymbol{p}',\,t)'$ is the bare Green function of particles of given species α. It satisfies the causality principle: at $t<t'$, the function is identically zero. In the domain $t>t'$, the function evolves according to the equation

$$\left[\frac{\partial}{\partial t}+v^\beta\frac{\partial}{\partial r^\beta}+\frac{e_\alpha}{c}v_i{}^0F^{i\beta}\frac{\partial}{\partial p^\beta}\right]{}^0G_\alpha(\boldsymbol{r},\boldsymbol{p},t,\boldsymbol{r}',\boldsymbol{p}',t')=0, \tag{3.3}$$

with the initial data

$${}^0G_\alpha(\boldsymbol{r},\boldsymbol{p},t'+0,\boldsymbol{r}',\boldsymbol{p}',t')=\delta^3(\boldsymbol{p}-\boldsymbol{p}')\delta^3(\boldsymbol{r}-\boldsymbol{r}'). \tag{3.4}$$

The tensor ${}^0\boldsymbol{F}$ denotes the part of EMF tensor that corresponds to the leading magnetic field.

In view of the above approximation of the two-point correlation function, Maxwell's equations take the form

$$\frac{1}{c}\frac{\partial}{\partial t}\varPhi_{\beta\gamma\cdot\cdot}^{\cdot\cdot kl}(\boldsymbol{r},t,\boldsymbol{r}',t')=-\frac{\partial}{\partial r^\beta}\varPhi_{\gamma0\cdot\cdot}^{\cdot\cdot kl}(\boldsymbol{r},t,\boldsymbol{r}',t')+\frac{\partial}{\partial r^\gamma}\varPhi_{\beta0\cdot\cdot}^{\cdot\cdot kl}(\boldsymbol{r},t,\boldsymbol{r}',t'), \tag{3.5}$$

$$\begin{aligned}\frac{1}{c}\frac{\partial}{\partial t}\varPhi^{\beta0kl}(\boldsymbol{r},t,\boldsymbol{r}',t')\ =\ &-\frac{\partial}{\partial r^\gamma}\varPhi^{\beta\gamma kl}(\boldsymbol{r},t,\boldsymbol{r}',t')\\&-\frac{4\pi}{c}\int\mathrm{d}^3\boldsymbol{r}_1\,\mathrm{d}t_1\sigma_{\cdot\cdot\gamma}^{\beta m\cdot}(\boldsymbol{r},t,\boldsymbol{r}_1,t_1)\varPhi_{m\cdots}^{\cdot\gamma kl}(\boldsymbol{r}_1,t_1,\boldsymbol{r}',t').\end{aligned} \tag{3.6}$$

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Here

$$\sigma^{\beta m\cdot}_{\cdot\cdot\gamma}(\boldsymbol{r},t,\boldsymbol{r}_1,t_1)=\sum_\alpha e_\alpha\int \mathrm{d}^3\boldsymbol{p}v^\beta\mathcal{R}^{\ m\cdot}_{\alpha\cdot\gamma}(\boldsymbol{r},\boldsymbol{p},t,\boldsymbol{r}_1,t_1) \tag{3.7}$$

has the meaning of the conductivity tensor.

Equation (3.5) describes the evolution of the two-time correlation function from the viewpoint of its magnetic entry, and (3.6) describes its evolution from the viewpoint of its electric entry.

In the case of a homogeneous plasma, the spatial Fourier transformation would diagonalize equations (3.5,3.6) over the wave vectors. The spatial inhomogeneity of the plasma leads to some interaction between the spatial harmonics of the two-time function. We should take this into account in the lowest order in the ratio of the typical wavelength in the plasma λ to the spatial scale of the plasma inhomogeneity $l\simeq L(x)=(\mathrm{d}\ln(n_e)/\mathrm{d}x)^{-1}$. For this, it is convenient to introduce respective Fourier transforms as follows:

$$\left.\begin{aligned}\boldsymbol{\Phi}_{\boldsymbol{k}}(\boldsymbol{r},t,t')&=\int\frac{\mathrm{d}^3\boldsymbol{R}}{(2\pi)^3}\exp(-\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{R}))\boldsymbol{\Phi}\left(\boldsymbol{r}+\frac{\boldsymbol{R}}{2},t,\boldsymbol{r}-\frac{\boldsymbol{R}}{2},t'\right),\\ \sigma^{\beta m\cdot}_{\boldsymbol{k}\cdot\cdot\gamma}(\boldsymbol{r},t,t')&=\int \mathrm{d}^3\boldsymbol{R}\exp(-\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{R}))\sigma^{\beta m\cdot}_{\boldsymbol{k}\cdot\cdot\gamma}\left(\boldsymbol{r}+\frac{\boldsymbol{R}}{2},t,\boldsymbol{r}-\frac{\boldsymbol{R}}{2},t'\right).\end{aligned}\right\} \tag{3.8}$$

With definition (3.8), the Fourier transform $\boldsymbol{\Phi}_{\boldsymbol{k}}(\boldsymbol{r},t,t')$ is Hermitian self-adjoint,

$$\boldsymbol{\Phi}_{\boldsymbol{k}}(\boldsymbol{r},t,t')=\boldsymbol{\Phi}^{+}_{\boldsymbol{k}}(\boldsymbol{r},t',t)(\text{i.e. }\Phi^{ijkl}_{\boldsymbol{k}}(\boldsymbol{r},t,t')=[\Phi^{klij}_{\boldsymbol{k}}(\boldsymbol{r},t',t)]^*), \tag{3.9}$$

and it also possesses the property $\Phi^{ijkl}_{\boldsymbol{k}},(\boldsymbol{r},t,t')=[\Phi^{ijkl}_{-\boldsymbol{k}}(\boldsymbol{r},t,t')]^*$ (which stems from the fact that the original is real valued).

In Fourier transforms, equations (3.5), (3.6) in our approximations take the form

$$\frac{1}{c}\frac{\partial}{\partial t}\Phi^{\cdot\cdot kl}_{\boldsymbol{k}\beta\gamma\cdot\cdot}(\boldsymbol{r},t,t')=-\left(\mathrm{i}k_\beta+\frac{1}{2}\frac{\partial}{\partial r^\beta}\right)\Phi^{\cdot\cdot kl}_{\boldsymbol{k}\gamma 0\cdot\cdot}(\boldsymbol{r},t,t')+\left(\mathrm{i}k_\gamma+\frac{1}{2}\frac{\partial}{\partial r^\gamma}\right)\Phi^{\cdot\cdot kl}_{\boldsymbol{k}\beta 0\cdot\cdot}(\boldsymbol{r},t,t'), \tag{3.10}$$

$$\begin{aligned}\frac{1}{c}\frac{\partial}{\partial t}\Phi^{\beta 0kl}_{\boldsymbol{k}}(\boldsymbol{r},t,t')&=-\left(\mathrm{i}k_\gamma+\frac{1}{2}\frac{\partial}{\partial r^\gamma}\right)\Phi^{\beta\gamma kl}_{\boldsymbol{k}}(\boldsymbol{r},t,t')\\ &-\frac{4\pi}{c}\int \mathrm{d}t_1\left\{\sigma^{\beta m\cdot}_{\boldsymbol{k}\cdot\cdot\gamma}(\boldsymbol{r},t,t_1)\Phi^{\cdot\gamma kl}_{\boldsymbol{k}m\cdots}(\boldsymbol{r},t_1,t')-\frac{\mathrm{i}}{2}\frac{\partial}{\partial k_\delta}\sigma^{\beta m\cdot}_{\boldsymbol{k}\cdot\cdot\gamma}(\boldsymbol{r},t,t_1)\frac{\partial}{\partial r^\delta}\Phi^{\cdot\gamma kl}_{\boldsymbol{k}m\cdots}(\boldsymbol{r},t_1,t')\right.\\ &\left.+\frac{\mathrm{i}}{2}\frac{\partial}{\partial r^\delta}\sigma^{\beta m\cdot}_{\boldsymbol{k}\cdot\cdot\gamma}(\boldsymbol{r},t,t_1)\frac{\partial}{\partial k_\delta}\Phi^{\cdot\gamma kl}_{\boldsymbol{k}m\cdots}(\boldsymbol{r},t_1,t')\right\}.\end{aligned} \tag{3.11}$$

The main contributors to the two-time correlation function in a weakly turbulent plasma are natural oscillations. The initial idea of plasma natural oscillation arises from the consideration of homogeneous plasmas where the terms with spatial derivatives fall out of (3.10) and (3.11). Then these equations dictate the following structure of the transform $\boldsymbol{\Phi}_{\boldsymbol{k}}(\boldsymbol{r},t,t')$ at large time delays $t-t'$:

$$\Phi^{ijkl}_{\boldsymbol{k}}(t,t')=\sum_{s,\sigma}F^{\sigma sij}_{\boldsymbol{k}}(t)\exp\left(-\mathrm{i}\int_{t'}^{t}\omega^{\sigma s}_{\boldsymbol{k}}(\tau)\,\mathrm{d}\tau\right)(A^{\sigma skl}_{\boldsymbol{k}}(t'))^*. \tag{3.12}$$

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Here the superscript σ is introduced to characterize the polarization of the oscillation, i.e. the type of wave: Langmuir, fast or slow electromagnetic, ion sound, etc. The superscript $s=\pm$ is used to differentiate between the parts of the transform $\boldsymbol{\Phi}_{\boldsymbol{k}}$ that correspond to waves propagating in opposite directions. The function $\omega_{\boldsymbol{k}}^{\sigma s}(t)\equiv s\omega_{s\boldsymbol{k}}^{\sigma}(t)-\mathrm{i}\gamma_{s\boldsymbol{k}}^{\sigma}(t)$ is the complex natural frequency of the oscillation. The tensor $\boldsymbol{F}_{\boldsymbol{k}}^{\sigma+}(t)\equiv(\boldsymbol{F}_{-\boldsymbol{k}}^{\sigma-}(t))^{*}$ is the wave polarization tensor. Without any loss in generality, it can be normalized by unity,

$$F_{\boldsymbol{k}}^{\sigma s\beta 0}(F_{\boldsymbol{k}\ \beta\cdot}^{\sigma s\cdot 0})^{*}+\tfrac{1}{2}F_{\boldsymbol{k}}^{\sigma s\beta\gamma}(F_{\boldsymbol{k}\ \beta\gamma}^{\sigma s})^{*}=1. \tag{3.13}$$

The natural frequency and the polarization tensor should be chosen to be rather smooth functions of time. We note that Maxwell's equations rigidly prescribe the dependence on time only for the entire structure $\boldsymbol{F}_{\boldsymbol{k}}^{\sigma s}(t)\exp(-\mathrm{i}\int_{t'}^{t}\omega_{\boldsymbol{k}}^{\sigma s}(\tau)\,\mathrm{d}\tau)$. Thus, there remains some arbitrariness in the distribution of this time dependence between the polarization tensor and the natural frequency: the natural frequency is determined up to the time derivative of the phase of $\boldsymbol{F}_{\boldsymbol{k}}^{\sigma s}$. This arbitrariness can be removed by specifying the character of the dependence of the phase of the wave polarization tensor on time. Let us assume that the phase of the polarization tensor is independent of time,

$$(F_{\boldsymbol{k}}^{\sigma s\beta 0})^{*}\frac{\partial}{\partial t}F_{\boldsymbol{k}\ \beta\cdot}^{\sigma s\cdot 0}+\frac{1}{2}(F_{\boldsymbol{k}}^{\sigma s\beta\gamma})^{*}\frac{\partial}{\partial t}F_{\boldsymbol{k}\ \beta\gamma}^{\sigma s}=0. \tag{3.14}$$

This simplifies the calculations.

In spectra of weakly turbulent plasmas, the square root of the dispersion of the real value of the wave natural frequency $\Delta\omega$ is great compared to the characteristic wave damping rate γ^{σ}. (The latter is a typical value of the wave damping rate $\gamma_{\boldsymbol{k}}^{\sigma}(t)$.) We clarify the above statement that the delay $t-t'$ is large: the asymptotics (3.12) of the transform $\boldsymbol{\Phi}_{\boldsymbol{k}}(\boldsymbol{r},t,t')$ holds in the time domain $t-t'\gg(\Delta\omega)^{-1}$. (The latter restriction arises from accounting for the nonlinear terms that we have omitted here, see Erofeev 2014, 2015*b*.)

For further clarification of parameters of natural oscillations, it is useful that the conductivity tensor $\boldsymbol{\sigma}_{\boldsymbol{k}}(\boldsymbol{r},t,t_1)$ decays with increase of $t-t_1$ on a time scale rather small compared to $(\gamma^{\sigma})^{-1}$. (The corresponding rate of tensor decay is of the order $\Delta\omega$.) Correspondingly, using representation (3.12), one can expand the polarization tensor $\boldsymbol{F}_{\boldsymbol{k}}^{\sigma s}(t_1)$ in the second term of the right-hand side of (3.11) in powers of t_1-t and the natural frequency $\omega_{\boldsymbol{k}}^{\sigma s}(\tau)$ in powers of $\tau-t$. Then one can perform a direct integration over the times τ and t_1. This yields the following simultaneous equations for the polarization tensor and the natural frequency:

$$-\frac{\mathrm{i}\omega_{\boldsymbol{k}}^{\sigma s}(t)}{c}F_{\boldsymbol{k}\ \beta\gamma}^{\sigma s}(t)+\mathrm{i}k_{\beta}F_{\boldsymbol{k}\ \gamma 0}^{\sigma s}(t)-\mathrm{i}k_{\gamma}F_{\boldsymbol{k}\ \beta 0}^{\sigma s}(t)=-\frac{1}{c}\frac{\partial}{\partial t}F_{\boldsymbol{k}\ \beta\gamma}^{\sigma s}, \tag{3.15}$$

$$\begin{aligned}&-\frac{\mathrm{i}\omega_{\boldsymbol{k}}^{\sigma s}(t)}{c}F_{\boldsymbol{k}}^{\sigma s\beta 0}(t)+\mathrm{i}k_{\gamma}F_{\boldsymbol{k}}^{\sigma s\beta\gamma}(t)+\frac{4\pi}{c}\left[F_{\boldsymbol{k}\ m\cdot}^{\sigma s\cdot\gamma}(t)+\sum_{n=1}^{\infty}\frac{\partial^{n}F_{\boldsymbol{k}\ m\cdot}^{\sigma s\cdot\gamma}}{\partial t^{n}}\frac{i^{n}}{n!}\frac{\partial^{n}}{\partial\omega^{n}}\right]\\&\quad\times\exp\left(-\mathrm{i}\sum_{n=1}^{\infty}\frac{\partial^{n}\omega_{\boldsymbol{k}}^{\sigma s}}{\partial t^{n}}\frac{i^{n+1}}{(n+1)!}\frac{\partial^{n+1}}{\partial\omega^{n+1}}\right)\sigma_{\boldsymbol{k}\omega\ \cdot\cdot\gamma}^{\ \beta m\cdot}(t)\bigg|_{\omega=\omega_{\boldsymbol{k}}^{\sigma s}(t)}\\&=-\frac{1}{c}\frac{\partial}{\partial t}F_{\boldsymbol{k}}^{\sigma s\beta 0}.\end{aligned} \tag{3.16}$$

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Here we have defined the Laplace transform of the conductivity tensor as follows:

$$\sigma_{\boldsymbol{k}\omega\cdot\cdot\gamma}^{\beta m\cdot}(t)=\int_{-\infty}^{t}\mathrm{d}t_1\sigma_{\boldsymbol{k}\cdot\cdot\gamma}^{\beta m\cdot}(t,t_1)\exp(\mathrm{i}\omega(t-t_1)). \tag{3.17}$$

In conjunction with the requirements (3.13), (3.14), equations (3.15), (3.16) permit obtaining of the leading approximations of the polarization tensor and the natural frequency and their subsequent iterative refinement. In the present study, iterations are not necessary, and also we can ignore the time variation of the polarization tensor.

Finally, we construct the leading order of the two-time correlation function:

$${}^{0}\Phi_{\boldsymbol{k}}^{ijkl}(t,t')=\sum_{s,\sigma}{}^{0}\Phi_{\boldsymbol{k}}^{\sigma sijkl}(t,t'), \tag{3.18}$$

$${}^{0}\Phi_{\boldsymbol{k}}^{\sigma sijkl}(t,t')=F_{\boldsymbol{k}}^{\sigma sij}(F_{\boldsymbol{k}}^{\sigma skl})^{*}\begin{cases}n_{s\boldsymbol{k}}^{\sigma}(t')\exp\left(-\mathrm{i}\int_{t'}^{t}\omega_{\boldsymbol{k}}^{\sigma s}(\tau)\,\mathrm{d}\tau\right) & t>t',\\ n_{s\boldsymbol{k}}^{\sigma}(t)\exp\left(-i\int_{t'}^{t}(\omega_{\boldsymbol{k}}^{\sigma s}(\tau))^{*}\,\mathrm{d}\tau\right) & t<t'.\end{cases} \tag{3.19}$$

(Here we took into account the property (3.9).) The function $n_{\boldsymbol{k}}^{\sigma}$ is real valued and non-negative (Erofeev 2014). We call it the wave spectral density and use it to describe all the effects of plasma natural oscillations.

3.1. *Accounting for the plasma inhomogeneity*

Intending to adapt the leading-order approximation (3.19) to the case of an inhomogeneous plasma, one discovers that this time all its constituents depend on $\boldsymbol{r}$. Accordingly, it becomes necessary to consider the terms of (3.10), (3.11) with derivatives with respect to $\boldsymbol{r}$ and $\boldsymbol{k}$. In view of this, the iterations of the two-time correlation function should be modified. We again use the leading order of the function in the form (3.19). Here the recipe for calculating the natural frequency $\omega_{\boldsymbol{k}}^{\sigma s}$ and the polarization tensor $\boldsymbol{F}_{\boldsymbol{k}}^{\sigma s}$ does not need any modifications; the only thing is that uncertainties appear in the choice of the dependencies of the phase of the polarization tensor on $\boldsymbol{r}$ and $\boldsymbol{k}$. (In homogeneous plasmas, this phase does not influence the final expressions. Further, we shall show that the same is true within the current physical content. However, this is hardly a common case for turbulent wave fields in inhomogeneous plasmas.)

As a matter of fact, the terms with the $\boldsymbol{r}$ and $\boldsymbol{k}$-derivatives of the approximation ${}^{0}\boldsymbol{\Phi}_{\boldsymbol{k}}^{\sigma s}(\boldsymbol{r},t,t')$ within (3.10), (3.11) result in adding some correction $\delta\boldsymbol{\Phi}_{\boldsymbol{k}}^{\sigma s}(\boldsymbol{r},t,t')$ to this approximation. It is natural to regard it as a tensor of some 'forced oscillations'. Their most important part is the one that resembles ${}^{0}\boldsymbol{\Phi}_{\boldsymbol{k}}^{\sigma s}(\boldsymbol{r},t,t')$ in tensorial structure. It is the tensor ${}^{0}\delta\boldsymbol{\Phi}_{\boldsymbol{k}}^{\sigma s}(\boldsymbol{r},t,t')=\boldsymbol{F}_{\boldsymbol{k}}^{\sigma s}\delta\Phi_{\boldsymbol{k}}^{\sigma s}(\boldsymbol{r},t,t')\otimes(\boldsymbol{F}_{\boldsymbol{k}}^{\sigma s})^{\dagger}$, where $\delta\Phi_{\boldsymbol{k}}^{\sigma s}(\boldsymbol{r},t,t')$ is the convolution of the tensor $\delta\boldsymbol{\Phi}_{\boldsymbol{k}}^{\sigma s}(\boldsymbol{r},t,t')$ with the tensor $(\boldsymbol{F}_{\boldsymbol{k}}^{\sigma s})^{\dagger}$ at the entry and with the tensor $\boldsymbol{F}_{\boldsymbol{k}}^{\sigma s}$ at the exit.[4]

From the point of view of carrying out iterations, it is desirable that the tensor ${}^{0}\delta\boldsymbol{\Phi}_{\boldsymbol{k}}^{\sigma s}(\boldsymbol{r},t,t')$ should not compete, in the sense of its norm, with the tensor ${}^{0}\boldsymbol{\Phi}_{\boldsymbol{k}}^{\sigma s}(\boldsymbol{r},t,t')$

[4]By the convolution of the skew-symmetric tensors $\boldsymbol{A}$ and $\boldsymbol{B}^{\dagger}$ we mean the sum of the scalar products of their respective vectors (like $\boldsymbol{E}$ in the EMF tensor) and the pseudovectors (like $\boldsymbol{B}$ in EMF tensor), i.e. the structure $A^{\beta 0}(B_{\beta\cdot}^{\cdot 0})^{*}+A^{\beta\gamma}(B_{\beta\gamma})^{*}/2$. It is a scalar with respect to transformations of the coordinate frame in usual three-dimensional space.

for all values of the input and output time arguments. One can hope to achieve this by properly choosing the time dependence of the wave spectral density $n^{\sigma}_{\boldsymbol{k}}(\boldsymbol{r}, t)$. To analyse this possibility, it is necessary to devise a recipe for calculating $\delta\Phi^{\sigma s}_{\boldsymbol{k}}(\boldsymbol{r}, t, t')$.

Conceptually, the simultaneous equations (3.10), (3.11) constitute a tensorial equation for the evolution of the two-time correlation function $\boldsymbol{\Phi}_{\boldsymbol{k}}(\boldsymbol{r}, t, t')$. We substitute the sum ${}^{0}\boldsymbol{\Phi}^{\sigma s}_{\boldsymbol{k}}(\boldsymbol{r}, t, t') + \boldsymbol{F}^{\sigma s}_{\boldsymbol{k}} \otimes (\boldsymbol{F}^{\sigma s}_{\boldsymbol{k}})^{\dagger}\delta\Phi^{\sigma s}_{\boldsymbol{k}}(\boldsymbol{r}, t, t')$ for the function $\boldsymbol{\Phi}_{\boldsymbol{k}}(\boldsymbol{r}, t, t')$ into this evolution equation and then convolute the equation entry with the tensor $(\boldsymbol{F}^{\sigma s}_{\boldsymbol{k}})^{\dagger}$ and the equation exit with the tensor $\boldsymbol{F}^{\sigma s}_{\boldsymbol{k}}$. This yields a scalar equation that defines the derivative of $\delta\Phi^{\sigma s}_{\boldsymbol{k}}(\boldsymbol{r}, t, t')$ over the entry time t. We use the result of integration of this derivative from $t = t'$ to the current time t for the function $\delta\Phi^{\sigma s}_{\boldsymbol{k}}(\boldsymbol{r}, t, t')$ at $t > t'$. In the integration, the function ${}^{0}\boldsymbol{\Phi}^{\sigma s}_{\boldsymbol{k}}(\boldsymbol{r}, t_1, t')$ can be well approximated by the expression

$$ {}^{0}\boldsymbol{\Phi}^{\sigma s}_{\boldsymbol{k}}(\boldsymbol{r}, t_1, t') = \boldsymbol{F}^{\sigma s}_{\boldsymbol{k}} \otimes (\boldsymbol{F}^{\sigma s}_{\boldsymbol{k}})^{*} n^{\sigma}_{s\boldsymbol{k}}(t') \exp\left(-\mathrm{i}\int_{t'}^{t} \omega^{\sigma s}_{\boldsymbol{k}}(\tau)\,\mathrm{d}\tau\right). \tag{3.20} $$

The point is that the time advances of $\delta\Phi^{\sigma s}_{\boldsymbol{k}}(\boldsymbol{r}, t, t')$ at $t > t'$ depend only on the leading-order approximation ${}^{0}\boldsymbol{\Phi}^{\sigma s}_{\boldsymbol{k}}(\boldsymbol{r}, t_1, t')$ at time delays $t_1 - t' \gtrsim -(\Delta\omega)^{-1}$, due to the rapid decay of the conductivity tensor $\boldsymbol{\sigma}_{\boldsymbol{k}}(t, t_1)$ with increase of $t - t_1$. In the corresponding time domain, the difference between the leading-order approximation and the above approximate expression is insignificant.

Calculation in accordance with the above ideas yields the following formula for function $\delta\Phi^{\sigma s}_{\boldsymbol{k}}(\boldsymbol{r}, t, t')$ at $t > t'$:

$$ \delta\Phi^{\sigma s}_{\boldsymbol{k}}(\boldsymbol{r}, t, t') = \delta\tilde{\Phi}^{\sigma s}_{\boldsymbol{k}}(\boldsymbol{r}, t, t') n^{\sigma}_{s\boldsymbol{k}}(\boldsymbol{r}, t') \exp\left(-\mathrm{i}\int_{t'}^{t} \omega^{\sigma s}_{\boldsymbol{k}}(\boldsymbol{r}, \tau)\,\mathrm{d}\tau\right), \tag{3.21} $$

$$ \begin{aligned} &\delta\tilde{\Phi}^{\sigma s}_{\boldsymbol{k}}(\boldsymbol{r}, t, t') \\ &= \int_{t'}^{t} \frac{\mathrm{d}\tilde{t}}{1 + 4\pi\mathrm{i}(\partial\sigma^{\sigma s}_{\boldsymbol{k}\omega}(\boldsymbol{r}, \tilde{t})/\partial\omega)} \left\{ \frac{c}{2}\left((F^{\sigma s\beta\gamma}_{\boldsymbol{k}})^{*} \frac{\partial}{\partial r^{\beta}} F^{\sigma s}_{\boldsymbol{k}0\gamma} + (F^{\sigma s}_{\boldsymbol{k}0\gamma})^{*} \frac{\partial}{\partial r^{\beta}} F^{\sigma s\beta\gamma}_{\boldsymbol{k}} \right) \right. \\ &\quad + c\mathrm{Re}((F^{\sigma s\beta\gamma}_{\boldsymbol{k}})^{*} F^{\sigma s}_{\boldsymbol{k}0\gamma}) \left[F^{\sigma s\xi 0}_{\boldsymbol{k}} \frac{\partial}{\partial r^{\beta}} (F^{\sigma s\,\cdot\,0}_{\boldsymbol{k}\;\,\xi\,\cdot})^{*} + \frac{1}{2} F^{\sigma s\xi\eta}_{\boldsymbol{k}} \frac{\partial}{\partial r^{\beta}} (F^{\sigma s}_{\boldsymbol{k}\;\,\xi\eta})^{*} \right] \\ &\quad - \mathrm{i}sc\mathrm{Re}((F^{\sigma s\beta\gamma}_{\boldsymbol{k}})^{*} F^{\sigma s}_{\boldsymbol{k}0\gamma}) \int_{t'}^{\tilde{t}} \frac{\partial}{\partial r^{\beta}} \omega^{\sigma}_{s\boldsymbol{k}}(\tau)\,\mathrm{d}\tau + c\mathrm{Re}((F^{\sigma s\beta\gamma}_{\boldsymbol{k}})^{*} F^{\sigma s}_{\boldsymbol{k}0\gamma}) \frac{1}{n^{\sigma}_{s\boldsymbol{k}}(t')} \frac{\partial}{\partial r^{\beta}} n^{\sigma}_{s\boldsymbol{k}}(t') \\ &\quad + 2\pi\mathrm{i}(F^{\sigma s}_{\boldsymbol{k}0\beta})^{*} \frac{\partial}{\partial k_{\delta}} \sigma^{\beta m\gamma}_{\boldsymbol{k}\omega}(\boldsymbol{r}, \tilde{t}) \left(\frac{\partial}{\partial r^{\delta}} F^{\sigma s}_{\boldsymbol{k}m\gamma} + F^{\sigma s}_{\boldsymbol{k}m\gamma} \frac{1}{n^{\sigma}_{s\boldsymbol{k}}(t')} \frac{\partial}{\partial r^{\delta}} n^{\sigma}_{s\boldsymbol{k}}(t') \right) \\ &\quad + 2\pi\mathrm{i}s(F^{\sigma s}_{\boldsymbol{k}0\beta})^{*} \frac{\partial}{\partial\omega} \frac{\partial}{\partial k_{\delta}} \sigma^{\beta m\gamma}_{\boldsymbol{k}\omega}(\boldsymbol{r}, \tilde{t}) F^{\sigma s}_{\boldsymbol{k}m\gamma} \frac{\partial}{\partial r^{\delta}} \omega^{\sigma}_{s\boldsymbol{k}}(\tilde{t}) \\ &\quad + 2\pi s(F^{\sigma s}_{\boldsymbol{k}0\beta})^{*} \frac{\partial}{\partial k_{\delta}} \sigma^{\beta m\gamma}_{\boldsymbol{k}\omega}(\boldsymbol{r}, \tilde{t}) F^{\sigma s}_{\boldsymbol{k}m\gamma} \int_{t'}^{\tilde{t}} \frac{\partial}{\partial r^{\delta}} \omega^{\sigma}_{s\boldsymbol{k}}(\boldsymbol{r}, \tau)\,\mathrm{d}\tau \\ &\quad - 2\pi\mathrm{i}(F^{\sigma s}_{\boldsymbol{k}0\beta})^{*} \frac{\partial}{\partial r^{\delta}} \sigma^{\beta m\gamma}_{\boldsymbol{k}\omega}(\boldsymbol{r}, \tilde{t}) \left(\frac{\partial}{\partial k_{\delta}} F^{\sigma s}_{\boldsymbol{k}m\gamma} + F^{\sigma s}_{\boldsymbol{k}m\gamma} \frac{1}{n^{\sigma}_{s\boldsymbol{k}}(t')} \frac{\partial}{\partial k_{\delta}} n^{\sigma}_{s\boldsymbol{k}}(t') \right) \\ &\quad - 2\pi\mathrm{i}s(F^{\sigma s}_{\boldsymbol{k}0\beta})^{*} \frac{\partial}{\partial\omega} \frac{\partial}{\partial r^{\delta}} \sigma^{\beta m\gamma}_{\boldsymbol{k}}(\boldsymbol{r}, \tilde{t}) F^{\sigma s}_{\boldsymbol{k}m\gamma} \frac{\partial}{\partial k_{\delta}} \omega^{\sigma}_{s\boldsymbol{k}}(\tilde{t}) \\ &\quad - 2\pi s(F^{\sigma s}_{\boldsymbol{k}0\beta})^{*} \frac{\partial}{\partial r^{\delta}} \sigma^{\beta m\gamma}_{\boldsymbol{k}\omega}(\boldsymbol{r}, \tilde{t}) F^{\sigma s}_{\boldsymbol{k}m\gamma} \int_{t'}^{\tilde{t}} \frac{\partial}{\partial k_{\delta}} \omega^{\sigma}_{s\boldsymbol{k}}(\tau)\,\mathrm{d}\tau + 2\pi\mathrm{i}(F^{\sigma s}_{\boldsymbol{k}0\beta})^{*} \frac{\partial}{\partial k_{\delta}} \sigma^{\beta m\gamma}_{\boldsymbol{k}\omega}(\boldsymbol{r}, \tilde{t}) F^{\sigma s}_{\boldsymbol{k}m\gamma} \end{aligned} $$

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$$\times\left[F_{\boldsymbol{k}}^{\sigma s\xi 0}\frac{\partial}{\partial r^{\delta}}(F_{\boldsymbol{k}\ \xi\cdot}^{\sigma s\cdot 0})^{*}+\frac{1}{2}F_{\boldsymbol{k}}^{\sigma s\xi\eta}\frac{\partial}{\partial r^{\delta}}(F_{\boldsymbol{k}\ \xi\eta}^{\sigma s})^{*}\right]-2\pi\mathrm{i}(F_{\boldsymbol{k}0\beta}^{\sigma s})^{*}\frac{\partial}{\partial r^{\delta}}\sigma_{\boldsymbol{k}\omega}^{\beta m\gamma}(\boldsymbol{r},\tilde{t})F_{\boldsymbol{k}m\gamma}^{\sigma s}$$
$$\times\left[F_{\boldsymbol{k}}^{\sigma s\xi 0}\frac{\partial}{\partial k_{\delta}}(F_{\boldsymbol{k}\ \xi\cdot}^{\sigma s\cdot 0})^{*}+\frac{1}{2}F_{\boldsymbol{k}}^{\sigma s\xi\eta}\frac{\partial}{\partial k_{\delta}}(F_{\boldsymbol{k}\ \xi\eta}^{\sigma s})^{*}\right]\Bigg\}_{\omega=\omega_{\boldsymbol{k}}^{\sigma s}(\boldsymbol{r},\tilde{t})}. \tag{3.22}$$

Here

$$\sigma_{\boldsymbol{k}\omega}^{\sigma s}(t)=(F_{\boldsymbol{k}\ \beta\cdot}^{\sigma s\cdot 0})^{*}\sigma_{\boldsymbol{k}\omega\cdot\cdot\gamma}^{\beta m\cdot}(t)F_{\boldsymbol{k}\ m\cdot}^{\sigma s\cdot\gamma}. \tag{3.23}$$

In (3.22), we have replaced the natural frequency $\omega_{\boldsymbol{k}}^{\sigma s}$ under the signs of the $\boldsymbol{k}$ and $\boldsymbol{r}$-derivatives by its real part $s\omega_{s\boldsymbol{k}}^{\sigma}$ since accounting for the corresponding wave damping rates is inessential.

It is worth noting that, according to the logic of its formation, the sum ${}^{0}\boldsymbol{\Phi}_{\boldsymbol{k}}^{\sigma s}(\boldsymbol{r},t,t')+\boldsymbol{F}_{\boldsymbol{k}}^{\sigma s}\otimes(\boldsymbol{F}_{\boldsymbol{k}}^{\sigma s})^{+}\delta\boldsymbol{\Phi}_{\boldsymbol{k}}^{\sigma s}(\boldsymbol{r},t,t')$ surely satisfies (3.10), (3.11) only in the time domain $t-t'\gg(\Delta\omega)^{-1}$. The property (3.9) permits one to construct some approximation of the two-time correlation function with the reverse sequence of the time arguments: $\boldsymbol{\Phi}_{\boldsymbol{k}}^{\sigma s}(\boldsymbol{r},t,t')=(\boldsymbol{\Phi}_{\boldsymbol{k}}^{\sigma s}(\boldsymbol{r},t',t))^{*}$. One may hope that the wave spectral density $n_{\boldsymbol{k}}^{\sigma}$ evolves in such a way that the corresponding approximation is consistent with the bulk of the real two-time correlation function in the corresponding time domain. This provides the key idea for the derivation of the rate of change of $n_{\boldsymbol{k}}^{\sigma}$. Consider (3.10), (3.11) in the asymptotic time domain $t-t'\ll-(\Delta\omega)^{-1}$. Substitute the two-time correlation function on the right-hand sides of the equations by the tensor

$$\boldsymbol{F}_{\boldsymbol{k}}^{\sigma s}\otimes(\boldsymbol{F}_{\boldsymbol{k}}^{\sigma s})^{+}\left[n_{s\boldsymbol{k}}^{\sigma}(t)\exp\left(-\mathrm{i}\int_{t'}^{t}(\omega_{\boldsymbol{k}}^{\sigma s}(\tau))^{*}\,\mathrm{d}\tau\right)+(\delta\boldsymbol{\Phi}_{\boldsymbol{k}}^{\sigma s}(\boldsymbol{r},t',t))^{*}\right]. \tag{3.24}$$

This yields expressions that contain time derivative of the wave spectral density $n_{\boldsymbol{k}}^{\sigma}$. One can try to select a value of this derivative for which the 'motive force' of the extra correction to structure (3.24) is orthogonal to the entry polarization tensor $\boldsymbol{F}_{\boldsymbol{k}}^{\sigma s}$ throughout the domain.[5] However, the kernel of the correction (3.22) contains some integrals that might not permit such a selection. We mean the sum

$$-\mathrm{i}sc\mathrm{Re}((F_{\boldsymbol{k}}^{\sigma s\beta\gamma})^{*}F_{\boldsymbol{k}0\gamma}^{\sigma s})\int_{t'}^{\tilde{t}}\frac{\partial}{\partial r^{\beta}}\omega_{s\boldsymbol{k}}^{\sigma}(\tau)\,\mathrm{d}\tau$$
$$+2\pi s(F_{\boldsymbol{k}0\beta}^{\sigma s})^{*}\frac{\partial}{\partial k_{\delta}}\sigma_{\boldsymbol{k}\omega}^{\beta m\gamma}(\boldsymbol{r},\tilde{t})F_{\boldsymbol{k}m\gamma}^{\sigma s}\int_{t'}^{\tilde{t}}\frac{\partial}{\partial r^{\delta}}\omega_{s\boldsymbol{k}}^{\sigma}(\boldsymbol{r},\tau)\,\mathrm{d}\tau$$
$$-2\pi s(F_{\boldsymbol{k}0\beta}^{\sigma s})^{*}\frac{\partial}{\partial r^{\delta}}\sigma_{\boldsymbol{k}\omega}^{\beta m\gamma}(\boldsymbol{r},\tilde{t})F_{\boldsymbol{k}m\gamma}^{\sigma s}\int_{t'}^{\tilde{t}}\frac{\partial}{\partial k_{\delta}}\omega_{s\boldsymbol{k}}^{\sigma}(\tau)\,\mathrm{d}\tau\Bigg|_{\omega=\omega_{\boldsymbol{k}}^{\sigma s}(\boldsymbol{r},\tilde{t})}. \tag{3.25}$$

If it does not turn to zero, none of the choices of the time dependence of wave spectral density will provide a good approximation of the two-time correlation function in the whole domain $t'-t\gg(\Delta\omega)^{-1}$. Previously, we have tried to formulate a recipe for circumventing the respective difficulty (Erofeev 2015*b*). It consisted in adding an extra phase shift to the argument of the exponent in the leading-order approximation ${}^{0}\boldsymbol{\Phi}_{\boldsymbol{k}}^{\sigma s}(\boldsymbol{r},t,t')$ at $t>t'$. However, this recipe turned out to be inconsistent.

[5]For the case of homogeneous plasma, this provides the uniform smallness of the tensor norm of the correction to the leading-order approximation $\boldsymbol{F}_{\boldsymbol{k}}^{\sigma s}(t')\otimes(\boldsymbol{F}_{\boldsymbol{k}}^{\sigma s}(t))^{+}n_{s\boldsymbol{k}}^{\sigma}(t)\exp(-\mathrm{i}\int_{t'}^{t}(\omega_{\boldsymbol{k}}^{\sigma s}(\tau))^{*}\,\mathrm{d}\tau)$ as compared to the norm of this approximation for all $t-t'\ll-(\Delta\omega)^{-1}$, see Erofeev (2013, 2014).

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It is more likely that the above sum always falls out, at least in its leading order.[6] Note that just this took place in our previous study (Erofeev 2015*b*). The same takes place in the current problem of drift of Langmuir waves in a cold ionized magnetized plasma, which can be checked on the basis of further data on these oscillations. However, currently, we do not have a full-scale proof of the hypothesis that the nullification of the above problematic sum always takes place. We suspect that such a proof can be developed on the basis of the following property of natural oscillations (Erofeev 2014, 2015*b*),

$$\omega_{\boldsymbol{k}}^{\sigma s}(t) - 2k_\gamma c\,\mathrm{Re}[(F_{\boldsymbol{k}}^{\sigma s\beta\gamma})^* F^{\sigma s}_{\boldsymbol{k}\ 0\beta}] = -4\pi\mathrm{i}\sigma^{\sigma s}_{\boldsymbol{k}\omega}(t)|_{\omega=\omega_{\boldsymbol{k}}^{\sigma s}(t)}. \tag{3.26}$$

It arises from the convolution of the entries of the lowest-order approximations of the simultaneous equations (3.15), (3.16) with the tensor $(\boldsymbol{F}_{\boldsymbol{k}}^{\sigma s})^{\dagger}$.

If the above hypothesis is valid, it is easy to calculate the value of $(\partial n_{\boldsymbol{k}}^{\sigma})/(\partial t)$ for which the driving force for the additional correction to the solution $({}^0\boldsymbol{\Phi}_{\boldsymbol{k}}^{\sigma s}(\boldsymbol{r}, t', t) + {}^0\delta\boldsymbol{\Phi}_{\boldsymbol{k}}^{\sigma s}(\boldsymbol{r}, t', t))^{\dagger}$ is orthogonal to the polarization tensor of the natural oscillation. Due to this, the above correction is automatically small compared to $({}^0\boldsymbol{\Phi}_{\boldsymbol{k}}^{\sigma s}(\boldsymbol{r}, t', t) + {}^0\delta\boldsymbol{\Phi}_{\boldsymbol{k}}^{\sigma s}(\boldsymbol{r}, t', t))^{\dagger}$ at all times from the interval $t - t' \ll -(\Delta\omega)^{-1}$. This is the general recipe for developing the wave kinetic equation in inhomogeneous plasmas. It yields the following rate of wave drift in phase space:

$$\begin{aligned}
\frac{\partial n_{s\boldsymbol{k}}^{\sigma}}{\partial t} &= \frac{2\pi}{1+4\pi\mathrm{i}(\partial\sigma^{\sigma s}_{\boldsymbol{k}\omega}/\partial\omega)}\left\{\frac{c}{2\pi}\mathrm{Re}\left[\frac{\partial}{\partial r^\beta}(F^{\sigma s}_{\boldsymbol{k}0\gamma}(F_{\boldsymbol{k}}^{\sigma s\beta\gamma})^*)n^{\sigma}_{s\boldsymbol{k}} + 2(F^{\sigma s}_{\boldsymbol{k}0\gamma}(F_{\boldsymbol{k}}^{\sigma s\beta\gamma})^*)\frac{\partial n^{\sigma}_{s\boldsymbol{k}}}{\partial r^\beta}\right]\right.\\
&\quad - 2\mathrm{Im}\left[(F^{\sigma s}_{\boldsymbol{k}0\beta})^*\frac{\partial}{\partial k_\delta}\sigma_{\boldsymbol{k}\omega}^{\beta m\gamma}\frac{\partial}{\partial r^\delta}F^{\sigma s}_{\boldsymbol{k}m\gamma}n^{\sigma}_{s\boldsymbol{k}} + (F^{\sigma s}_{\boldsymbol{k}0\beta})^*\frac{\partial}{\partial k_\delta}\sigma_{\boldsymbol{k}\omega}^{\beta m\gamma}F^{\sigma s}_{\boldsymbol{k}m\gamma}\frac{\partial n^{\sigma}_{s\boldsymbol{k}}}{\partial r^\delta}\right.\\
&\quad + s(F^{\sigma s}_{\boldsymbol{k}0\beta})^*\frac{\partial}{\partial\omega}\frac{\partial}{\partial k_\delta}\sigma_{\boldsymbol{k}\omega}^{\beta m\gamma}F^{\sigma s}_{\boldsymbol{k}m\gamma}n^{\sigma}_{s\boldsymbol{k}}\frac{\partial\omega^{\sigma}_{s\boldsymbol{k}}}{\partial r^\delta} - (F^{\sigma s}_{\boldsymbol{k}0\beta})^*\frac{\partial}{\partial r^\delta}\sigma_{\boldsymbol{k}\omega}^{\beta m\gamma}\frac{\partial}{\partial k_\delta}F^{\sigma s}_{\boldsymbol{k}m\gamma}n^{\sigma}_{s\boldsymbol{k}}\\
&\quad - (F^{\sigma s}_{\boldsymbol{k}0\beta})^*\frac{\partial}{\partial r^\delta}\sigma_{\boldsymbol{k}\omega}^{\beta m\gamma}F^{\sigma s}_{\boldsymbol{k}m\gamma}\frac{\partial n^{\sigma}_{s\boldsymbol{k}}}{\partial k_\delta} - s(F^{\sigma s}_{\boldsymbol{k}0\beta})^*\frac{\partial}{\partial\omega}\frac{\partial}{\partial r^\delta}\sigma_{\boldsymbol{k}}^{\beta m\gamma}F^{\sigma s}_{\boldsymbol{k}m\gamma}n^{\sigma}_{s\boldsymbol{k}}\frac{\partial\omega^{\sigma}_{s\boldsymbol{k}}}{\partial k_\delta}\\
&\quad + (F^{\sigma s}_{\boldsymbol{k}0\beta})^*\frac{\partial}{\partial k_\delta}\sigma_{\boldsymbol{k}\omega}^{\beta m\gamma}F^{\sigma s}_{\boldsymbol{k}m\gamma}\left[F_{\boldsymbol{k}}^{\sigma s\xi 0}\frac{\partial}{\partial r^\delta}(F^{\sigma s\cdot 0}_{\boldsymbol{k}\ \xi\cdot})^* + \frac{1}{2}F_{\boldsymbol{k}}^{\sigma s\xi\eta}\frac{\partial}{\partial r^\delta}(F^{\sigma s}_{\boldsymbol{k}\ \xi\eta})^*\right]n^{\sigma}_{s\boldsymbol{k}}\\
&\quad - (F^{\sigma s}_{\boldsymbol{k}0\beta})^*\frac{\partial}{\partial r^\delta}\sigma_{\boldsymbol{k}\omega}^{\beta m\gamma}F^{\sigma s}_{\boldsymbol{k}m\gamma}\\
&\quad \left.\left.\times\left[F_{\boldsymbol{k}}^{\sigma s\xi 0}\frac{\partial}{\partial k_\delta}(F^{\sigma s\cdot 0}_{\boldsymbol{k}\ \xi\cdot})^* + \frac{1}{2}F_{\boldsymbol{k}}^{\sigma s\xi\eta}\frac{\partial}{\partial k_\delta}(F^{\sigma s}_{\boldsymbol{k}\ \xi\eta})^*\right]n^{\sigma}_{s\boldsymbol{k}}\right]\right\}\Bigg|_{\omega=s\omega^{\sigma}_{s\boldsymbol{k}}(\boldsymbol{r},t)}.
\end{aligned} \tag{3.27}$$

Equation (3.27) includes a term that contains the convolution of the tensor $\boldsymbol{F}_{\boldsymbol{k}}^{\sigma s}(t')$ with the spatial derivative of the exit polarization tensor $(\boldsymbol{F}_{\boldsymbol{k}}^{\sigma s})^{\dagger}(t')$ of the leading order ${}^0\boldsymbol{\Phi}_{\boldsymbol{k}}^{\sigma s}(\boldsymbol{r}, t, t')$. For our case of Langmuir waves, the dependence of the phase of the polarization tensor on $\boldsymbol{r}$ can be adjusted so that this convolution vanishes:

$$F_{\boldsymbol{k}}^{\sigma s\beta 0}\frac{\partial}{\partial r^\varepsilon}(F^{\sigma s\cdot 0}_{\boldsymbol{k}\ \beta\cdot})^* + \frac{1}{2}F_{\boldsymbol{k}}^{\sigma s\beta\gamma}\frac{\partial}{\partial r^\varepsilon}(F^{\sigma s}_{\boldsymbol{k}\ \beta\gamma})^* = 0. \tag{3.28}$$

Conceptually, the latter restriction implies that the phase of the polarization tensor does not depend on spatial coordinates.

[6]Its higher orders can be nullified by some minor modification of the calculation procedure.

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The situation is different for the dependence of the phase on the wave vector. Unlike in our previous study of electromagnetic waves (Erofeev 2015*b*), no choice of this dependence yields the vanishing of the corresponding convolution:

$$F_{\boldsymbol{k}}^{\sigma s\beta 0}\frac{\partial}{\partial k^{\varepsilon}}(F_{\boldsymbol{k}\ \beta\cdot}^{\sigma s\cdot 0})^{*}+\frac{1}{2}F_{\boldsymbol{k}}^{\sigma s\beta\gamma}\frac{\partial}{\partial k^{\varepsilon}}(F_{\boldsymbol{k}\ \beta\gamma}^{\sigma s})^{*}\neq 0. \tag{3.29}$$

That is, the phase of the polarization tensor cannot be regarded as independent of $\boldsymbol{k}$. Still, terms with the derivative of the wave phase with respect to the wave vector fall out of the wave drift velocity in our physical situation. This fact will be commented in the next section.

We conclude this section by the following remark. We have mentioned above that it is desirable that the tensor of forced oscillations $\delta\boldsymbol{\Phi}_{\boldsymbol{k}}^{\sigma s}(\boldsymbol{r}, t', t)$ be small in comparison with the tensor ${}^{0}\boldsymbol{\Phi}_{\boldsymbol{k}}^{\sigma s}(\boldsymbol{r}, t, t')$. Meanwhile, with increase of time delay $t-t'$ in our calculations, the forced oscillation tensor will ultimately override the tensor of the leading-order approximation ${}^{0}\boldsymbol{\Phi}_{\boldsymbol{k}}^{\sigma s}(\boldsymbol{r}, t, t')$. Still, our leading-order approximation ${}^{0}\boldsymbol{\Phi}_{\boldsymbol{k}}^{\sigma s}(\boldsymbol{r}, t, t')$ represents the bulk of the two-time correlation function in a rather wide piece of the domain $t-t'\gg(\Delta\omega)^{-1}$ provided that the plasma inhomogeneity parameter λ/l is adequately small. Then the obtained picture of wave drift adequately reflects the objective realities of this phenomenon: our calculation is meaningful.

4. Picture of drift in phase space for Langmuir waves

To adapt the above to the problem of drift of Langmuir waves in a magnetized plasma, we first derive the parameters of Langmuir plasma oscillations, i.e. calculate their polarization tensors and natural oscillations. For this, we need the conductivity tensor; its linear approximation is sufficient for our purposes.

4.1. *Conductivity tensor in a magnetized plasma*

Following formulae (3.7) and (3.2),

$$\sigma_{\cdot\cdot\gamma}^{\beta m\cdot}(\boldsymbol{r}, t, \boldsymbol{r}_1, t_1)=-\sum_{\alpha}\frac{e_{\alpha}^{2}}{c}\int \mathrm{d}^3\boldsymbol{p}v^{\beta}\int \mathrm{d}^3\boldsymbol{p}_1{}^{0}G_{\alpha}(\boldsymbol{r}, \boldsymbol{p}, t, \boldsymbol{r}_1, \boldsymbol{p}_1, t_1)v_1^{m}\frac{\partial}{\partial p_1^{\gamma}}f_{\alpha}(\boldsymbol{r}_1, \boldsymbol{p}_1, t_1). \tag{4.1}$$

Bearing in mind that wave drift is due to the plasma inhomogeneity, we analyse the effects of the inhomogeneity on this tensor. Let us assume that the magnetic pressure in the plasma is great compared to the hydrodynamic plasma pressure, i.e.

$$\beta=\frac{nT_e}{B^2/(8\pi)}\ll 1. \tag{4.2}$$

Then spatial changes in the conductivity tensor are predominantly due to inhomogeneities of plasma density or (and) particle momentum distributions rather than due to the inhomogeneity of the confining magnetic field $\boldsymbol{B}(\boldsymbol{r})$: either

$$|\nabla\ln(B)|\ll|\nabla\ln(n_e)| \tag{4.3}$$

or (and)

$$|\nabla\ln(B)|\ll|\nabla\ln(T_e)|. \tag{4.4}$$

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Therefore, we shall neglect the influence of spatial variations of the leading magnetic field on the bare Green function ${}^0G_\alpha(\boldsymbol{r}, \boldsymbol{p}, t, \boldsymbol{r}_1, \boldsymbol{p}_1, t_1)$. Let the z axis of the Cartesian frame be directed along the external magnetic field $\boldsymbol{B}$. Then integration of (3.3) yields the following formula for the Fourier–Laplace transform of the bare Green function:

$$
\begin{aligned}
{}^0G_{\alpha k\omega}(\boldsymbol{p}, \boldsymbol{p}_1) = & -\frac{1}{\omega_{B\alpha}}\delta(p^z - p_1^z)\frac{\delta(p^\perp - p_1^\perp)}{p^\perp}\left[1 - \exp\left(-2\pi\mathrm{i}\frac{\omega - k^z v^z}{\omega_{B\alpha}}\right)\right]^{-1} \\
& \times \exp\left[-\mathrm{i}\frac{\omega - k^z v^z}{\omega_{B\alpha}}(\varphi - \varphi_1) + \mathrm{i}\frac{k^\perp v^\perp}{\omega_{B\alpha}}(\sin(\varphi - \varphi') - \sin(\varphi_1 - \varphi'))\right]. \qquad (4.5)
\end{aligned}
$$

(The transform is defined as ${}^0G_{\alpha k\omega}(\boldsymbol{p}, \boldsymbol{p}') = \int_{t'}^{\infty} \mathrm{d}t \int \mathrm{d}^3\boldsymbol{r}\, {}^0G_\alpha(\boldsymbol{r}, \boldsymbol{p}, t, \boldsymbol{r}', \boldsymbol{p}', t') \exp(\mathrm{i}\omega(t - t') - \mathrm{i}(\boldsymbol{k} \cdot (\boldsymbol{r} - \boldsymbol{r}')))$.) Here we use cylindrical coordinates. The azimuthal angles φ, φ_1 and φ' are the angles between the projections of the vectors $\boldsymbol{p}$, $\boldsymbol{p}_1$ and $\boldsymbol{k}$ onto the plane xy and the axis x, respectively. The sign $\perp$ is used to denote the components of the vectors perpendicular to the z axis. It is assumed that the difference $\varphi - \varphi_1$ is within the interval $[\,0, 2\pi\,[$. The Larmor frequency of the particle is defined as $\omega_{B\alpha} = e_\alpha B/(m_\alpha c)$. (We assume that the particles are non-relativistic in our plasma.)

Generally, taking account of the Larmor rotation of plasma particles in the leading magnetic field, we assume that in our inhomogeneous plasma, the stationary particle distribution f_α has the following structure:

$$
f_\alpha(\boldsymbol{r}, \boldsymbol{p}) = {}^0f_\alpha(x + v_y/\omega_{B\alpha}, p_z, p_\perp). \qquad (4.6)
$$

Here the notation $p_\perp$ is used for the modulus of the component of the particle momentum that is orthogonal to the magnetic field. The function ${}^0f_\alpha(x, p_z, p_\perp)$ characterizes the density of the particles with given $p_\perp$ and p_z that rotate in the external magnetic field around centres located on the current level line of x. This density does not depend on the phase of the rotation.

Taking into account the above structure of the particle distributions, we get

$$
\begin{aligned}
\sigma_{k\omega\cdot\cdot\gamma}^{\beta m\cdot}(t) = & -\sum_\alpha \frac{e_\alpha^2}{c}\int \mathrm{d}^3\boldsymbol{p}\,\mathrm{d}^3\boldsymbol{p}_1 v^\beta v_1^m \\
& \times \left[{}^0G_{\alpha k\omega}(\boldsymbol{p}, \boldsymbol{p}_1) - \frac{\mathrm{i}}{2}\frac{\partial}{\partial k_x}{}^0G_{\alpha k\omega}(\boldsymbol{p}, \boldsymbol{p}_1)\frac{\partial}{\partial x}\right]\frac{\partial}{\partial p_1^\gamma} \\
& \times \left[\left(1 + \frac{p_{1\perp}\sin(\varphi_1)}{m_\alpha\omega_{B\alpha}}\frac{\partial}{\partial x}\right){}^0f_\alpha(x, p_{1z}, p_{1\perp})\right]. \qquad (4.7)
\end{aligned}
$$

The expression with the spatial gradient in the first square brackets accounts for the direct effect of the spatial inhomogeneity of the distribution ${}^0f_\alpha$ (i.e. its dependence on x), and the term in the second square brackets accounts for the anisotropy of the genuine particle distribution f_α in the p_x–p_y plane that stems from the above Larmor rotations of the particles.

The Larmor particle radii are presumably negligible compared to the spatial scale of the plasma inhomogeneity in our cold ionized plasma. Due to this, the difference between the expression in the second square brackets and the distribution ${}^0f_\alpha$ can be ignored. Then we can write

$$
\sigma_{k\omega\cdot\cdot\gamma}^{\beta m\cdot}(\boldsymbol{r}, t) = \left[1 - \frac{\mathrm{i}}{2}\frac{\partial}{\partial k_x}\frac{\partial}{\partial x}\right]{}^0\sigma_{k\omega\cdot\cdot\gamma}^{\beta m\cdot}(\boldsymbol{r}, t), \qquad (4.8)
$$

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$$
\begin{aligned}
&{}^0\sigma_{\boldsymbol{k}\omega\cdot\cdot\gamma}^{\beta m\cdot}(\boldsymbol{r},t)\\
&=-\sum_\alpha \frac{e_\alpha^2}{c}\int \mathrm{d}^3\boldsymbol{p}\,\mathrm{d}^3\boldsymbol{p}_1 v^\beta v_1^m\, {}^0G_{\alpha\boldsymbol{k}\omega}(\boldsymbol{p},\boldsymbol{p}_1)\frac{\partial}{\partial p_1^\gamma}\,{}^0\!f_\alpha(x,p_{1z},p_{1\perp})\\
&=\sum_\alpha \frac{e_\alpha^2}{c\omega_{B\alpha}}\int p_\perp\,\mathrm{d}p_\perp\,\mathrm{d}p_z p_{1\perp}\,\mathrm{d}p_{1\perp}\,\mathrm{d}p_{1z}\delta(p_z-p_{1z})\frac{\delta(p_\perp-p_{1\perp})}{p_\perp}\int_0^{2\pi}\mathrm{d}\varphi\int_{\varphi-2\pi}^{\varphi}\mathrm{d}\varphi_1\\
&\quad\times\frac{\exp[-\mathrm{i}((\omega-k_zv_z)/\omega_{B\alpha})(\varphi-\varphi_1)+\mathrm{i}((k_\perp v_\perp)/\omega_{B\alpha})(\sin(\varphi-\varphi')-\sin(\varphi_1-\varphi'))]}{1-\exp(-2\pi\mathrm{i}(\omega-k_zv_z)/\omega_{B\alpha})}\\
&\quad\times\begin{pmatrix} - \\ v_\perp\cos(\varphi)\\ v_\perp\sin(\varphi)\\ v_z\end{pmatrix}\otimes\begin{pmatrix} c \\ v_{1\perp}\cos(\varphi_1)\\ v_{1\perp}\sin(\varphi_1)\\ v_{1z}\end{pmatrix}\otimes\begin{pmatrix} - \\ (\partial^0\!f_\alpha)/(\partial p_{1\perp})\cos(\varphi_1)\\ (\partial^0\!f_\alpha)/(\partial p_{1\perp})\sin(\varphi_1)\\ (\partial^0\!f_\alpha)/(\partial p_{1z})\end{pmatrix}.
\end{aligned}
\tag{4.9}
$$

The dependence of the tensor on the superscript β is represented in the left column at the bottom of the rightmost side of the given chain of equalities, the dependence on the superscript m is represented in the middle column and the dependence on the subscript γ in the right column.

Within the convolution $\sigma_{\boldsymbol{k}\omega\cdot\cdot\gamma}^{\beta m\cdot}F_{\boldsymbol{k}\ m\cdot}^{\sigma s\cdot\gamma}$, the summands with $m\neq 0$ are small compared to the other summands and these quantities are related as the ratio of the mean square electron velocity in the plasma (akin to the thermal velocity) to the speed of light. In view of this, we can ignore all the summands in this tensor convolution except for those with indices $m=0$. Note also that plasma ions play only the role of macroscopically neutralizing background in Langmuir plasma oscillations: these oscillations depend exclusively on electrons.

Let us assume that the electron distribution ${}^0\!f_e$ is an isotropic Maxwellian distribution with temperature T_e that does not depend on x. Then the matrix of the conductivity tensor takes the form

$$
\begin{aligned}
{}^0\sigma_{\boldsymbol{k}\omega\cdot\cdot\gamma}^{\beta 0\cdot}(x,t)&=-\frac{e^2}{\omega_{Be}T_e}\int_{-\infty}^{\infty}\mathrm{d}p_z\int_0^\infty p_\perp\,\mathrm{d}p_\perp{}^0\!f_e(x,p_z,p_\perp)\\
&\quad\times\frac{-\mathrm{i}\exp(\mathrm{i}\pi a)}{2\sin(\pi a)}\int_0^{2\pi}\mathrm{d}\varphi\int_{\varphi-2\pi}^{\varphi}\mathrm{d}\varphi_1\exp[-\mathrm{i}a(\varphi-\varphi_1)+\mathrm{i}z(\sin(\varphi-\varphi')-\sin(\varphi_1-\varphi'))]\\
&\quad\times\begin{pmatrix} v_\perp^2\cos(\varphi)\cos(\varphi_1) & v_\perp^2\cos(\varphi)\sin(\varphi_1) & v_zv_\perp\cos(\varphi)\\ v_\perp^2\sin(\varphi)\cos(\varphi_1) & v_\perp^2\sin(\varphi)\sin(\varphi_1) & v_zv_\perp\sin(\varphi)\\ v_zv_\perp\cos(\varphi_1) & v_zv_\perp\sin(\varphi_1) & v_z^2\end{pmatrix}.
\end{aligned}
\tag{4.10}
$$

Here we have introduced the notation

$$
a=\frac{\omega-k_zv_z}{\omega_{Be}},\quad z=\frac{k_xv_\perp}{\omega_{Be}}. \tag{4.11a,b}
$$

The integration over azimuthal angles in formula (4.10) yields a somewhat more advanced form of the conductivity tensor. However, it remains rather cumbersome in the general case. We present it for the case $\varphi'=0$ (otherwise $k_y=0$). Then

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$$
{}^{0}\sigma_{\boldsymbol{k}\omega\cdot\cdot\gamma}^{\beta 0\cdot} = \mathrm{i}\frac{\mathrm{e}^2}{T_e\omega_{Be}}\int \mathrm{d}^3\boldsymbol{p} f_e(x, p_z, p_\perp)
\times\begin{pmatrix} v_\perp^2(-a(1+aG)/z^2) & \mathrm{i}v_\perp^2 aG'/(2z) & -v_\perp v_z(1+aG)/z \\ -\mathrm{i}v_\perp^2 aG'/(2z) & v_\perp^2(a(1+aG)/z^2-(zG')'/(2z)-G) & \mathrm{i}v_\perp v_z G'/2 \\ -v_\perp v_z(1+aG)/z & -\mathrm{i}v_\perp v_z G'/2 & -v_z^2 G \end{pmatrix}. \tag{4.12}
$$

Here

$$
\begin{aligned}
G(z) &= -\frac{1}{2\pi}\frac{\exp(\mathrm{i}\pi a)}{2\sin(\pi a)}\int_0^{2\pi}\mathrm{d}\varphi\int_{\varphi-2\pi}^{\varphi}\mathrm{d}\varphi_1\exp[-\mathrm{i}a(\varphi-\varphi_1)+\mathrm{i}z(\sin(\varphi)-\sin(\varphi_1))] \\
&= -\sum_{n=-\infty}^{\infty}\frac{\mathrm{J}_n^2(z)}{a-n} \equiv -\frac{\pi}{\sin(\pi a)}\mathrm{J}_a(z)\mathrm{J}_{-a}(z),
\end{aligned} \tag{4.13}
$$

where $\mathrm{J}_n(z)$ is a first-kind Bessel function of integer order n and $\mathrm{J}_\nu(z)$ is a first-kind Bessel function of irrational (generally, complex) order ν.

Since the electron thermal velocity is small, our waves have lengths much larger than the typical electron Larmor radius $\rho_e \approx v_{Te}/\omega_{Be}$. Then argument z takes small values, and the function G can be expanded in a series in z. For our case of cold plasma, we do not need more than two first orders of the expansion:

$$
\begin{aligned}
G(z) &= -\frac{1}{a}\left[1-\frac{1}{2}z^2\right]-\frac{1}{2}\frac{a}{a^2-1}z^2-\cdots \equiv -\frac{1}{a}\left[1+\frac{1}{4}\left(\frac{1}{a-1}-\frac{1}{a+1}\right)z^2+\cdots\right] \\
&= -\frac{1}{a}-\frac{z^2}{2a(a^2-1)}-\cdots .
\end{aligned} \tag{4.14}
$$

In this case, the tensor ${}^{0}\sigma_{\boldsymbol{k}\omega\cdot\cdot\gamma}^{\beta 0\cdot}$ becomes

$$
\mathrm{i}\frac{\omega_{pe}^2}{4\pi}\begin{pmatrix} \omega/(\omega^2-\omega_{Be}^2) & -\mathrm{i}\omega_{Be}/(\omega^2-\omega_{Be}^2) & 0 \\ \mathrm{i}\omega_{Be}/(\omega^2-\omega_{Be}^2) & \omega/(\omega^2-\omega_{Be}^2) & 0 \\ 0 & 0 & 1/\omega \end{pmatrix}. \tag{4.15}
$$

We see tensor ${}^{0}\sigma_{\boldsymbol{k}\omega\cdot\cdot\gamma}^{\beta 0\cdot}$ does not depend on $\boldsymbol{k}$, so that the tensor $\sigma_{\boldsymbol{k}\omega\cdot\cdot\gamma}^{\beta 0\cdot}$ does not differ from it. It is also invariant with respect to rotation of the coordinate plane k_x–k_y, and this justifies the fact that we have focused our analysis of formula (4.10) on the particular case $k_y = 0$.

The tensor (4.15) is known from the traditional linear hydrodynamic theory of plasma oscillations.

4.2. *Natural frequency and polarization tensor of Langmuir waves*

We first clarify the term long as applied to our Langmuir waves. Here we mean that the wave vector k is small compared to the following three characteristic inverse lengths: ω_p/c, $|\omega_{Be}|/c$ and $\sqrt{|\omega_p^2-\omega_{Be}^2|}/c$. The natural frequency of a Langmuir wave is then

$$
\omega_{\boldsymbol{k}}^{ls} = s\omega_p\left(1+\frac{1}{2}\frac{k_\perp^2c^2}{\omega_p^2}+\frac{1}{2}\frac{k_\perp^2c^2k_z^2c^2}{\omega_p^4}-\frac{1}{8}\frac{k_\perp^4c^4}{\omega_p^4}+O\left(\frac{k^6c^6}{\omega_p^6}\right)\right). \tag{4.16}
$$

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Here $k_\perp$ denotes the wave vector component that is orthogonal to the leading magnetic field.

The 'electric' components of the polarization tensor are

$$\begin{pmatrix} F^{ls}_{\boldsymbol{k}x0} \\ F^{ls}_{\boldsymbol{k}y0} \\ F^{ls}_{\boldsymbol{k}z0} \end{pmatrix} = \begin{pmatrix} -\dfrac{k_x k_z c^2}{\omega_p^2} - \mathrm{i}s\dfrac{k_y k_z c^2}{\omega_p \omega_{Be}} + O\left(\dfrac{k^4 c^4}{\omega_p^4}\right) \\ -\dfrac{k_y k_z c^2}{\omega_p^2} + \mathrm{i}s\dfrac{k_x k_z c^2}{\omega_p \omega_{Be}} + O\left(\dfrac{k^4 c^4}{\omega_p^4}\right) \\ 1 - \dfrac{1}{2}\dfrac{(k_x^2+k_y^2)c^2}{\omega_p^2} + \dfrac{7}{8}\dfrac{(k_x^2+k_y^2)^2 c^4}{\omega_p^4} - \dfrac{1}{2}\dfrac{(k_x^2+k_y^2)k_z^2 c^4}{\omega_p^4}\left(3+\dfrac{\omega_p^2}{\omega_{Be}^2}\right) + O\left(\dfrac{k^6 c^6}{\omega_p^6}\right) \end{pmatrix}, \tag{4.17}$$

and the 'magnetic' components are

$$\begin{pmatrix} F^{ls}_{\boldsymbol{k}yz} \\ F^{ls}_{\boldsymbol{k}zx} \\ F^{ls}_{\boldsymbol{k}xy} \end{pmatrix} = \begin{pmatrix} s\dfrac{k_y c}{\omega_p}\left(1 - \dfrac{(k_x^2+k_y^2-k_z^2)c^2}{\omega_p^2}\right) - \mathrm{i}\dfrac{k_x k_z^2 c^3}{\omega_p^2 \omega_{Be}} \\ -s\frac{k_x c}{\omega_p}\left(1 - \dfrac{(k_x^2+k_y^2-k_z^2)c^2}{\omega_p^2}\right) - \mathrm{i}\dfrac{k_y k_z^2 c^3}{\omega_p^2 \omega_{Be}} \\ \mathrm{i}\dfrac{(k_x^2+k_y^2)k_z c^3}{\omega_p^2 \omega_{Be}} \end{pmatrix} + O\left(\dfrac{k^5 c^5}{\omega_p^5}\right). \tag{4.18}$$

The tensor $\sigma_{\boldsymbol{k}\omega}^{\beta 0\gamma}$ has a simple spatial derivative,

$$\frac{\partial}{\partial r^\delta}\sigma_{\boldsymbol{k}\omega}^{\beta 0\gamma} = \sigma_{\boldsymbol{k}\omega}^{\beta 0\gamma}\frac{\partial}{\partial r^\delta}\ln(n_e). \tag{4.19}$$

Hence, not only can we omit the terms with derivatives of the conductivity tensor with respect to wave vectors in (3.27), but we can also make the substitutions

$$\left.\begin{aligned} (F^{\sigma s}_{\boldsymbol{k}0\beta})^* \frac{\partial}{\partial r^\delta}\sigma_{\boldsymbol{k}\omega}^{\beta 0\gamma} F^{\sigma s}_{\boldsymbol{k}0\gamma} &= \sigma^{\sigma s}_{\boldsymbol{k}\omega}\frac{\partial}{\partial r^\delta}\ln(n_e), \\ (F^{\sigma s}_{\boldsymbol{k}0\beta})^* \frac{\partial}{\partial r^\delta}\frac{\partial}{\partial \omega}\sigma_{\boldsymbol{k}}^{\beta 0\gamma} F^{\sigma s}_{\boldsymbol{k}0\gamma} &= \frac{\partial}{\partial \omega}\sigma^{\sigma s}_{\boldsymbol{k}\omega}\frac{\partial}{\partial r^\delta}\ln(n_e). \end{aligned}\right\} \tag{4.20}$$

Calculations yield

$$\begin{aligned} \sigma^{\sigma s}_{\boldsymbol{k}\omega} = \frac{\mathrm{i}\omega_p^2}{4\pi}\Bigg[&\frac{1}{\omega}\left(1 - \frac{k_\perp^2 c^2}{\omega_p^2} + 2\frac{k_\perp^4 c^4}{\omega_p^4} - \frac{k_\perp^2 k_z^2 c^4}{\omega_p^4}\left(3 + \frac{\omega_p^2}{\omega_{Be}^2}\right)\right) \\ &+ \frac{k_\perp^2 k_z^2 c^4}{\omega_p^4}\frac{\omega - 2s\omega_p + (\omega\omega_p^2/\omega_{Be}^2)}{\omega^2 - \omega_{Be}^2} + O\left(\frac{(kc)^6}{\omega_p^7}\right)\Bigg]. \end{aligned} \tag{4.21}$$

We see that this feature is purely imaginary. Note that the vector

$$F_{\boldsymbol{k}}^{\sigma s\beta 0}\frac{\partial}{\partial k^\varepsilon}(F^{\sigma s\cdot 0}_{\boldsymbol{k}\ \beta\cdot})^* + \frac{1}{2}F_{\boldsymbol{k}}^{\sigma s\beta\gamma}\frac{\partial}{\partial k^\varepsilon}(F^{\sigma s}_{\boldsymbol{k}\ \beta\gamma}) \tag{4.22}$$

is also purely imaginary, so that the dependence of the phase of the wave polarization tensor on $\boldsymbol{k}$ does not affect the kinetics of wave drift in phase space (see (3.27)).

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Finally,

$$\left.\frac{\partial}{\partial\omega}\sigma^{\sigma s}_{k\omega}\right|_{\omega=s\omega^{\sigma}_{sk}(\boldsymbol{r},t)}=-\frac{\mathrm{i}}{4\pi}\left[1-2\frac{k_\perp^2c^2}{\omega_p^2}+4\frac{k_\perp^4c^4}{\omega_p^4}-4\frac{k_\perp^2k_z^2c^4}{\omega_p^4}+O\left(\frac{(kc)^6}{\omega_p^6}\right)\right]. \tag{4.23}$$

For the purpose of the calculation, it is also convenient to use the following consequence of the wave dispersion law (4.16):

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln(n_e)=\frac{2}{\omega_p}\frac{\partial\omega^l_{\boldsymbol{k}}}{\partial x}\left[1+\frac{1}{2}\frac{k_\perp^2c^2}{\omega_p^2}+\frac{3}{2}\frac{k_\perp^2k_z^2c^4}{\omega_p^4}-\frac{1}{8}\frac{k_\perp^4c^4}{\omega_p^4}+O\left(\frac{(kc)^6}{\omega_p^6}\right)\right]. \tag{4.24}$$

Based on the above, we can derive the final equation of Langmuir wave drift.

4.3. *Final equation of Langmuir wave drift*

The rate of change of the spectral density of Langmuir waves due to their drift in phase space is given by

$$\frac{\partial n^l_{\boldsymbol{k}}}{\partial t}=\frac{\mathrm{d}\omega^l_{\boldsymbol{k}}}{\mathrm{d}x}\frac{\partial n^l_{\boldsymbol{k}}}{\partial k_x}-\frac{\partial\omega^l_{\boldsymbol{k}}}{\partial k_\delta}\frac{\partial n^l_{\boldsymbol{k}}}{\partial r^\delta}-n^l_{\boldsymbol{k}}\frac{\mathrm{d}\omega^l_{\boldsymbol{k}}}{\mathrm{d}x}\frac{k_xc^2}{\omega_p^2}\left(\frac{3}{2}-\frac{5}{2}\frac{k_\perp^2c^2}{\omega_p^2}+\frac{5}{2}\frac{k_z^2c^2}{\omega_p^2}\right)+O\left(n^l_{\boldsymbol{k}}\frac{k_xk^4c^6}{l\omega_p^5}\right). \tag{4.25}$$

The reader is reminded that the traditional weak plasma turbulence theory highlights a certain wavenumber density $N^\sigma_{\boldsymbol{k}}$ instead of our wave spectral density $n^\sigma_{\boldsymbol{k}}$. Our predecessors believed that the use of the concept of wavenumber density provides descriptions of the evolution of wave spectra of turbulent plasmas that are maximally adequate to plasma realities.

As a matter of fact, this concept was elaborated for problems of homogeneous plasma wave turbulence. The definition of $N^\sigma_{\boldsymbol{k}}$ stems from the idea that the energy density of a turbulent wave field can be interpreted as the sum of the energies of individual wave quanta (see e.g. Tsytovich 1970). This clarifies the following relationship between the wavenumber density $N^\sigma_{\boldsymbol{k}}$ and the wave spectral density $n^\sigma_{\boldsymbol{k}}$:

$$N^\sigma_{\boldsymbol{k}}=\frac{2\pi^2}{\hbar\omega^\sigma_{\boldsymbol{k}}}\left.\frac{1}{\omega}\frac{\partial}{\partial\omega}(\omega^2(\epsilon^{\beta\gamma}_{\boldsymbol{k}\omega})^H)\right|_{\omega=\omega^\sigma_{\boldsymbol{k}}}[n^\sigma_{\boldsymbol{k}}(F^{\sigma+}_{\boldsymbol{k}\;0\beta})^*F^{\sigma+}_{\boldsymbol{k}\;0\gamma}]. \tag{4.26}$$

Here $(\boldsymbol{\epsilon}^{\beta\gamma}_{\boldsymbol{k}\omega})^H$ is the Hermitian component of the dielectric permittivity tensor, $(\epsilon^{\beta\gamma}_{\boldsymbol{k}\omega})^H\equiv\delta^{\beta\gamma}+4\pi\mathrm{i}(\sigma^{\beta0\gamma}_{\boldsymbol{k}\omega})^A/\omega$.

As we have already noted, addressing the traditional wavenumber densities and understanding of their mutual influence lead away from highly informative nonlinear scenarios of weakly turbulent plasmas (recall failures of the golden rule). The use of wavenumber densities is even less intelligent in descriptions of wave drifts than in problems of nonlinear dynamics of weak wave turbulence of a homogeneous plasma. A major problem here is a generalization of the definition of plasma wave energy to inhomogeneous plasmas. Suppose that we have developed an adequate description of the energy density of the weakly turbulent wave field in an inhomogeneous plasma and that the distribution of the energy density is given by a function $W^\sigma_{\boldsymbol{k}}(\boldsymbol{r},t)$. Then the understandings of geometrical optics yield the following continuity equation of wave energy drift in space and wave vectors:

$$\frac{\partial W^\sigma_{\boldsymbol{k}}}{\partial t}=\frac{\partial\omega^\sigma_{\boldsymbol{k}}}{\partial r^\delta}\frac{\partial W^\sigma_{\boldsymbol{k}}}{\partial k_\delta}-\frac{\partial\omega^\sigma_{\boldsymbol{k}}}{\partial k_\delta}\frac{\partial W^\sigma_{\boldsymbol{k}}}{\partial r^\delta}. \tag{4.27}$$

Here the wave natural frequency is considered as a Hamiltonian that governs the time transformations of the coordinates of the wave package (spatial position) and the conjugated momentum components (leading wave vector of the package). The equation neglects wave dissipation, which is consistent with our physical situation.

It would have been quite natural to use as the wavenumber density $N_{\boldsymbol{k}}^{\sigma}$ the ratio $W_{\boldsymbol{k}}^{\sigma}(\boldsymbol{r},\, t)/(\hbar\omega_{\boldsymbol{k}}^{\sigma}(\boldsymbol{r}))$. Provided that the recipes of geometrical optics are sufficiently correct, we then would have got

$$\frac{\partial N_{\boldsymbol{k}}^{\sigma}}{\partial t} = \frac{\partial \omega_{\boldsymbol{k}}^{\sigma}}{\partial r^{\delta}} \frac{\partial N_{\boldsymbol{k}}^{\sigma}}{\partial k_{\delta}} - \frac{\partial \omega_{\boldsymbol{k}}^{\sigma}}{\partial k_{\delta}} \frac{\partial N_{\boldsymbol{k}}^{\sigma}}{\partial r^{\delta}}. \tag{4.28}$$

However, we have no adequate idea of the wave energy density. We have fairly logically developed the notion of the wave spectral density $n_{k}^{\sigma}(\boldsymbol{r},\, t)$, but how to express the wave energy density $W_{\boldsymbol{k}}^{\sigma}(\boldsymbol{r}, t)$ in its terms?

Suppose that the usual formula for the wave energy in homogeneous plasmas is suitable for inhomogeneous plasmas. Then relation (4.26) should have held. In our situation, it reduces to the equality

$$N_{\boldsymbol{k}}^{l} = \frac{4\pi^2}{\hbar\omega_p} n_{\boldsymbol{k}}^{l} \left(1 - \frac{3}{2}\frac{k_{\perp}^2 c^2}{\omega_p^2} - \frac{5}{2}\frac{k_{\perp}^2 k_z^2 c^4}{\omega_p^4} + \frac{23}{8}\frac{k_z^4 c^4}{\omega_p^4} + O\left(\frac{kc}{\omega_p}\right)^6\right). \tag{4.29}$$

Correspondingly, equation (4.25) yields

$$\frac{\partial N_{\boldsymbol{k}}^{l}}{\partial t} = \frac{\mathrm{d}\omega_{\boldsymbol{k}}^{l}}{\mathrm{d}x}\frac{\partial N_{\boldsymbol{k}}^{l}}{\partial k_x} - \frac{\partial \omega_{\boldsymbol{k}}^{l}}{\partial k_{\delta}}\frac{\partial N_{\boldsymbol{k}}^{l}}{\partial r^{\delta}} + \frac{1}{2}N_{\boldsymbol{k}}^{l}\frac{\mathrm{d}\omega_{\boldsymbol{k}}^{l}}{\mathrm{d}x}\frac{k_x c^2}{\omega_p^2}\left(1 - 3\frac{k_{\perp}^2 c^2}{\omega_p^2} + 3\frac{k_z^2 c^2}{\omega_p^2}\right) + O\left(N_{\boldsymbol{k}}^{l}\frac{k_x k^4 c^6}{l\omega_p^5}\right). \tag{4.30}$$

We have got an equation that differs from (4.28) by an extra term on the right-hand side. It provides insight into the above-mentioned problems with the definition of the wave energy density. Note that the greater the y-component of the magnetic field in the wave, the larger this term. It comprises some piece of the curl of the magnetic field in the wave that generates a piece of electric field in the wave. Undoubtedly, this suggests some contribution to the wave energy density that the relation (4.26) does not take into account.

5. Conclusions

In this paper, we have once again highlighted the logic of developing highly informative plasma kinetic scenarios. It was used to develop an equation that describes the drift of long Langmuir waves in a weakly turbulent inhomogeneous magnetized plasma (4.25). The equation uses the wave spectral density $n_{\boldsymbol{k}}^{l}$ which is suggested as the most natural characteristic of a weakly turbulent wave field. Analysis of this equation has shown that the traditional theory does not contain a concept of energy density suitable for the wave field in an inhomogeneous plasma. We hope that the awareness of this fact will motivate the development of such a concept.

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