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Monotonicity of Avoidance Coupling on K_N

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Answering a question by Angel, Holroyd, Martin, Wilson and Winkler [1], we show that the maximal number of non-colliding coupled simple random walks on the complete graph K_N, which take turns, moving one at a time, is monotone in N. We use this fact to couple $\lceil N/4 \rceil$ such walks on K_N, improving the previous $\Omega(N/\log N)$ lower bound of Angel *et al.* We also introduce a new generalization of simple avoidance coupling which we call partially ordered simple avoidance coupling, and provide a monotonicity result for this extension as well.

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1. Introduction

Let $G = ([N], E)$ be a graph whose vertices are the set of integers $[N] = \{1, \ldots, N\}$. A *simple random walk* on this graph is a Markov chain $(X_t)_{t\in\mathbb{Z}}$ of elements in $[N]$ such that for all $t \in \mathbb{Z}$ the distribution of X_t is uniform on the neighbours of X_{t-1}.

A *simple avoidance coupling* (SAC) of k walks on G is a sequence of random maps $(U_t)_{t\in\mathbb{Z}}$ from $[k]$ to $[N]$ which satisfy two conditions:

$$\forall i \in [k] : (U_t(i))_{t\in\mathbb{Z}} \text{ is a simple random walk on } G, \tag{1.1}$$

$$\forall t \in \mathbb{Z}, 1 \leqslant i < j \leqslant k : \mathbb{P}(U_t(i) = U_t(j)) = \mathbb{P}(U_t(i) = U_{t-1}(j)) = 0. \tag{1.2}$$

Angel, Holroyd, Martin, Wilson and Winkler [1] introduce this notion in order to investigate couplings of k simple random walks which move in turns in discrete time and avoid collision.

One possible application of SACs on the complete graph K_N is semi-synchronous orthogonal frequency hopping. A communication network consists of several transmitters.

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As there are overlaps between the transmission ranges, they wish to use distinct frequencies at every given time. Malicious adversaries, each located in the vicinity of one of these transmitters, are trying to interfere with the communication by noising several frequencies at every given time. Once an adversary hits his target transmitter's frequency he can tell that his interruption succeeded. In order to avoid persistent interference the transmitters wish to change frequencies often. Being unable to perfectly synchronize their clocks, the transmitter must take turns at hopping. In this scenario it is desirable for each transmitter to perform a simple random walk as this would make each of its frequency changes (hops) independent from the past with maximal entropy. Independence is desirable since the adversary has some access to the frequency history of its target transmitter. An ideal hopping scheme in this setting is a SAC.

An important result of [1] is that there exists a SAC of $\Omega(N/\log N)$ walks on K_N. The authors also show in [1, Theorem 6.1] that when $N = 2^\ell + 1$ for some $\ell \in \mathbb{N}$, there exists an avoidance coupling of $2^{\ell-1}$ walks on K_N. Angel *et al.* ask whether the existence of an avoidance coupling of k walks on K_{N-1} implies the existence of an avoidance coupling of k walks on K_N. Our main result is a positive answer to this question.

Theorem 1.1. *If there exists a simple avoidance coupling of k walks on K_{N-1}, then there exists a simple avoidance coupling of k walks on K_N.*

Combining this with [1, Theorem 6.1], we obtain the following improved bound.

Theorem 1.2. *There exists a simple avoidance coupling of $\lceil N/4 \rceil$ walks on K_N.*

An interesting byproduct of the proof of Theorem 1.1 is that in the extended coupling on K_N one can find an additional $(k+1)$th special simple random walk whose movement time changes from round to round. In Section 4 we investigate this observation and discuss possible extensions of avoidance coupling to models where the order by which the walkers move varies from one round to the next, subject to some constraints.

1.1. Markovian couplings

In [1] the authors give special attention to Markovian simple avoidance couplings. These have the property that whenever a walker's turn to move arrives, he only needs to look at the current configuration of walkers to determine the distribution of its next location. In particular, the simple avoidance coupling of $\Omega(N/\log N)$ walkers on K_N constructed in [1] has this property, as does the coupling of $2^{\ell-1}$ walkers on $K_{2^\ell+1}$. Our extension theorem does not preserve this property, so the problem of finding whether, for every $N \in \mathbb{N}$, a Markovian SAC of $\Omega(N)$ walks exists, remains open.

The extension does, however, preserve the following weaker *label Markov* property. Consider a SAC in which each site of the underlying graph K_N is assigned a label. At the end of every round, a random permutation is applied to these labels. Such a SAC is called *label Markovian* if, whenever a walker's turn to move arrives, he only needs to look at the current configuration of the walkers and, in addition, at the current labels of the vertices.

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2. Background

Probabilistic coupling of several stochastic processes sharing the same distribution was introduced into probability theory mainly as a tool to study and prove various properties of that common distribution. Such methods have been successfully used in showing properties such as monotonicity, stochastic dominance and convergence.

Nevertheless, probabilistic coupling can also be a subject of study. In this context, the natural question is 'In what sense is a collection of coupled identically distributed stochastic processes different from a collection of independent processes with the same distribution?' A classical example is that of two random walks on some finite graph G. If two independent random walks move on G, then they collide with high probability after a polynomial number of steps. Collisions occur even if a scheduler is allowed to control the times in which each walk makes his move (see [4, 7]), and can be avoided only if the scheduler has some knowledge of the future of each walk, and only on certain graphs (see [5]). On the other hand, there exist many graphs on which coupled random walks can easily avoid each other. On the cycle graph C_n, for example, two walks which start on non-adjacent vertices can avoid each other by moving in the same direction at every step - either clockwise or anticlockwise. Coupling of walks on K_N, the complete graph on N vertices, appears to be more difficult. In [1], the authors use various techniques inspired by discrete harmonic analysis to create an avoidance coupling of $\Omega(N/\log N)$ walks on K_N and of $N/2-1$ walks for an infinite collection of special Ns. They also investigate avoidance coupling on K_N^*, the complete graph with loops on N vertices, and obtain a lower bound of $N/4$ walks on this graph. The authors further show that no coupling exists for $N-1$ walks on K_N^*, if $N \geqslant 4$.

Research into avoidance couplings is closely related to that of Brownian motions which keep at least a constant distance from each other. This subject and its relation to pursuit-evasion problems is investigated in [2], [3] and [6].

3. Extending an avoidance coupling

This section consists of the proof of Theorem 1.1. Let $\mathcal{U}_k^{N-1} = (U_t(j))_{t\in\mathbb{Z}, j\in[k]}$ be a SAC of k walks on K_{N-1}. Our goal is to define $\mathcal{W}_k^N$, a SAC of k walks on K_N.

3.1. The extended coupling

We begin by introducing an auxiliary sequence of random permutations. Let $P_0 \in S_N$ be a uniformly chosen random permutation in S_N. Let $(a_t)_{t\in\mathbb{Z}}$ be an i.i.d. sequence where a_0 is a uniformly chosen element of $[N-1]$. For $t \in \mathbb{N}$ define $P_t, P_{-t} \in S_N$ inductively as follows:

$$
\begin{aligned}
P_t &:= P_{t-1} \circ (N\, a_t), \\
P_{-t} &:= P_{1-t} \circ (N\, a_{1-t}),
\end{aligned}
$$

where $(a\, b)$ is the transposition of the two elements a and b.

Write $\mathcal{P}^N = (P_t)_{t\in\mathbb{Z}}$. It is straightforward to check that $\mathcal{P}^{\mathcal{N}}$ is a stationary Markov chain on S_N which is independent from $\mathcal{U}_k^{N-1}$.

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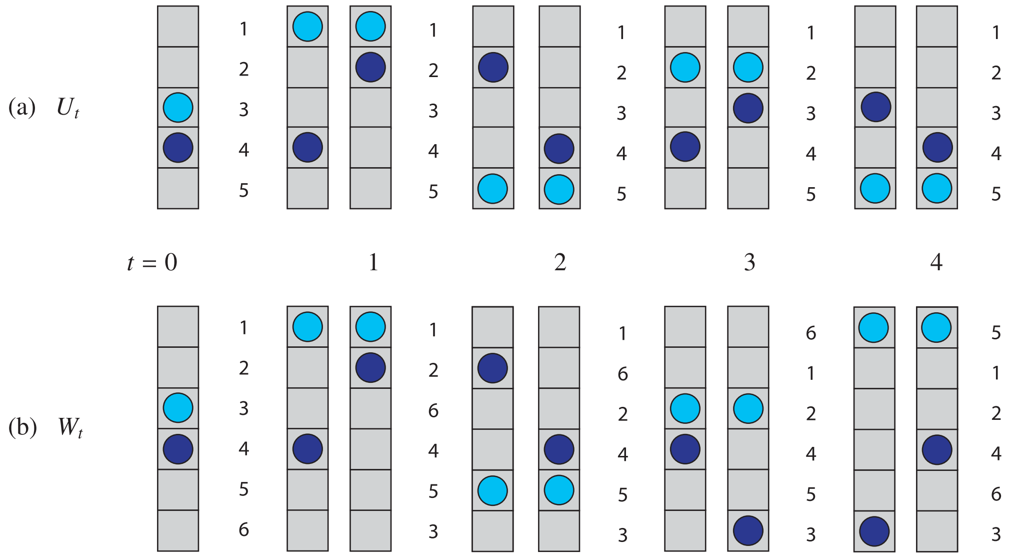

Figure 1. (a) U_t, a SAC of two walks on K_5. (b) W_t, the extended SAC on K_6. The label permutation P_t is given at the end of every time unit. Observe that the light blue walk always moves before the dark one. Also observe how W_t is determined by P_t and U_t.

We define $\mathcal{W}_k^N = (W_t(j))_{t\in\mathbb{Z}, j\in[k]}$ where $W_t : [k] \to [N]$, as follows:

$$W_t(j) = P_t U_t(j), \quad j \in [k],\ t \in \mathbb{Z}.$$

An example of $\mathcal{U}_2^5$, $\mathcal{P}^6$ and $\mathcal{W}_2^6$ is given in Figure 1. Below we prove that $\mathcal{W}_k^N$ is an avoidance coupling of k walks on K_N.

3.2. The extension is a SAC

To show that $\mathcal{W}_k^N$ is a SAC we must show that it satisfies (1.1) and (1.2). We begin by showing (1.2).

Let $t \in \mathbb{Z}$, $1 \leqslant i < j \leqslant k$. We have

$$\mathbb{P}(W_t(i) = W_t(j)) = \mathbb{P}(P_t U_t(i) = P_t U_t(j)) = \mathbb{P}(U_t(i) = U_t(j)) = 0,$$

where the central equality uses the fact that P_t is a permutation and the rightmost equality follows from the fact that $\mathcal{U}_k^{N-1}$ satisfies (1.2).

Recall the definition of the sequence $(a_t)_{t\in\mathbb{Z}}$ and write P'_t for the transposition $(N\,a_t)$. We have

$$\begin{aligned}\mathbb{P}(W_t(i) = W_{t-1}(j)) &= \mathbb{P}(P_t U_t(i) = P_{t-1} U_{t-1}(j)) = \mathbb{P}((P_{t-1} \circ P'_t) U_t(i) = P_{t-1} U_{t-1}(j)) \\ &= \mathbb{P}(P'_t U_t(i) = U_{t-1}(j)) = 0,\end{aligned} \tag{3.1}$$

where the the last equality follows from the fact that $\mathcal{U}_k^{N-1}$ satisfies (1.2), and from the fact that $U_t(i), U_{t-1}(j) \in [N-1]$.

We are left with showing that $\mathcal{W}_k^N$ satisfies (1.1). Fix $j \in [k]$; we must show that $W_t(j)$ is a simple random walk on K_N. Equivalently, for every $\ell \in \mathbb{N}$, every history

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$w_{t-\ell},\dots,w_{t-1}\in[N]$ such that

$$\mathbb{P}(W_{t-1}(j)=w_{t-1},\dots,W_{t-\ell}(j)=w_{t-\ell})>0,$$

and for every $v\neq w_{t-1}$, we have

$$\mathbb{P}(W_t(j)=v\mid W_{t-1}(j)=w_{t-1},\dots,W_{t-\ell}(j)=w_{t-\ell})=\frac{1}{N-1}. \tag{3.2}$$

To obtain this we show a stronger claim. Fix $\ell\in\mathbb{N}$ and let $p=(p_{t-\ell},\dots,p_{t-1})\in(S_N)^\ell$, $u=(u_{t-\ell},\dots,u_{t-1})\in[N]^\ell$. Consider the event

$$A_t^{p,u}=\{U_{t-1}(j)=u_{t-1},\dots,U_{t-\ell}(j)=u_{t-\ell}\text{ and }P_{t-1}(j)=p_{t-1},\dots,P_{t-\ell}(j)=p_{t-\ell}\}.$$

We show that for all p,u such that $\mathbb{P}(A_t^{p,u})\neq 0$ and for all $v\neq p_{t-1}(u_{t-1})$, we have

$$\mathbb{P}(W_t(j)=v\mid A_t^{p,u})=\frac{1}{N-1}. \tag{3.3}$$

Indeed, (3.3) is stronger than (3.2), as the values of $P_{t-1},\dots,P_{t-\ell}$ and $U_{t-1}(j),\dots,U_{t-\ell}(j)$ determine the values of $W_{t-1}(j),\dots,W_{t-\ell}(j)$.

Since $w_{t-1}=p_{t-1}(u_{t-1})\neq p_{t-1}(N)$ and using the fact that by (1.2) we have

$$\sum_{n\in[N]\setminus w_{t-1}}\mathbb{P}(W_t(j)=n\mid A_t^{p,u})=1,$$

it would suffice to show (3.3) in the case $v\neq p_{t-1}(N)$. Thus, let $v\in[N]\setminus\{p_{t-1}(N),w_{t-1}\}$ and use the total probability formula to write

$$\begin{aligned}\mathbb{P}(W_t(j)=v\mid A_t^{p,u})&=\mathbb{P}(W_t(j)=v\mid A_t^{p,u},P_t(N)=v)\,\mathbb{P}(P_t(N)=v)\\&\quad+\mathbb{P}(W_t(j)=v\mid A_t^{p,u},P_t(N)\neq v)\,\mathbb{P}(P_t(N)\neq v)\\&=\mathbb{P}(U_t(j)=N\mid A_t^{p,u},P_t(N)=v)\cdot\frac{1}{N-1}\\&\quad+\mathbb{P}(U_t(j)=P_t^{-1}(v)\mid A_t^{p,u},P_t(N)\neq v)\cdot\frac{N-2}{N-1}\\&=0\cdot\frac{1}{N-1}+\mathbb{P}(U_t(j)=P_t^{-1}(v)\mid A_t^{p,u},P_t(N)\neq v)\cdot\frac{N-2}{N-1}.\end{aligned} \tag{3.4}$$

We now observe that

$$\mathbb{P}(U_t(j)=P_t^{-1}(v)\mid A_t^{p,u},v\neq P_t(N)\})=\mathbb{P}(U_t(j)=p_{t-1}^{-1}(v)\mid A_t^{p,u},v\neq P_t(N)\})=\frac{1}{N-2}. \tag{3.5}$$

where the first equality follows from the fact that, for all $v\notin\{P_t(N),P_{t-1}(N)\}$, we have $P_t^{-1}(v)=P_{t-1}^{-1}(v)$, and the last equality uses our assumption that $v\neq w_{t-1}=P_{t-1}U_{t-1}(j)$.

Plugging (3.5) into (3.4) we deduce (3.3), concluding the proof. □

4. Partially ordered avoidance coupling

Consider the following generalization of an avoidance coupling. Let R be a partial order on $[k]$. An R *partially ordered avoidance coupling* (POSAC) of k walks on G is a sequence

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of random maps

$$U_t : [m] \to [N], t \in \mathbb{Z},$$

such that there exists a sequence of permutations $\sigma_t \in S_m$ which respect R (*i.e.*, $i <_R j \to \sigma(i) < \sigma(j)$) such that $(U_t)_{t\in\mathbb{Z}}$ and σ_t satisfy two conditions:

$$(1)\ \forall i \in [m] : (U_t[i])_{t\in\mathbb{Z}} \text{ is a simple random walk on } G, \tag{4.1}$$

$$(2)\ \forall t \in \mathbb{Z},\ 1 \leqslant \sigma_t(i) < \sigma_t(j) \leqslant m : \mathbb{P}(U_t(i) = U_{t-1}(j)) = \mathbb{P}(U_t(i) = U_t(j)) = 0. \tag{4.2}$$

A POSAC is a generalization of a SAC to a situation where the order in which the walks take turns can change from one round to the next, restricted by some partial order constraint (in the application to orthogonal hopping consider a situation where two transmitters can alter the order of their hops only if they receive each other's signal).

The proof of Theorem 1.1 extends in this case to the following.

Theorem 4.1. *If there exists an R POSAC of k walks on K_{N-1}, then there exists an R POSAC of $k+1$ walks on K_N.*

Observe that in this case, although the extension does not allow adding additional relations it does allow increasing the number of walks.

4.1. Extending a POSAC

This section is dedicated to the proof of Theorem 4.1. Let R be a partial order on $[k]$, let $\mathcal{U}_{k,R}^{N-1}$ be an R POSAC of k walks on K_{N-1}, and let $(s_t)_{t\in\mathbb{Z}}$ be a sequence of permutations which respect R and satisfy (4.2).

Let $\mathcal{P}^N = (P_t)_{t\in\mathbb{Z}}$ and $\mathcal{W}_k^N = (W_t(j))_{t\in\mathbb{Z},j\in[k]}$ be as in Section 3.1, and define

$$\mathcal{W}_{k+1,R}^N = (W_t(j))_{t\in\mathbb{Z},j\in[k+1]}$$

with $W_t(k+1) := P_t(N)$.

Observe that, given $t \in \mathbb{Z}$, for any distinct $i, j \in [k+1]$ we have

$$\mathbb{P}(W_t(i) = W_t(j)) = \mathbb{P}(P_t U_t(i) = P_t U_t(j)) = \mathbb{P}(U_t(i) = U_t(j)) = 0, \tag{4.3}$$

as before. Using this we define $(\sigma_t)_{t\in\mathbb{Z}}$ in the following way. If there exists $b \in [k]$ such that $W_{t-1}(b) = W_t(k+1)$, we set

$$\sigma_t(j) = \begin{cases} s_t(j) & s_t(j) \leqslant s_t(b), \\ s_t(b)+1 & j = m+1, \\ s_t(j)+1 & s_t(j) > s_t(b), \end{cases} \tag{4.4}$$

while otherwise we set

$$\sigma_t(j) = \begin{cases} s_t(j)+1 & j \leqslant m, \\ 1 & j = m+1. \end{cases} \tag{4.5}$$

In other words, the new walk jumps as soon as its target site is unoccupied.

Our purpose is to show that $\mathcal{W}_k^N$ and $(\sigma_t)_{t\in\mathbb{Z}}$ satisfy (4.1) and (4.2). An example of $\mathcal{U}_{2,R}^5$, $\mathcal{P}^6$, $\mathcal{W}_{3,R}^6$ and $(\sigma_t)_{t\in\mathbb{Z}}$ is given in Figure 2.

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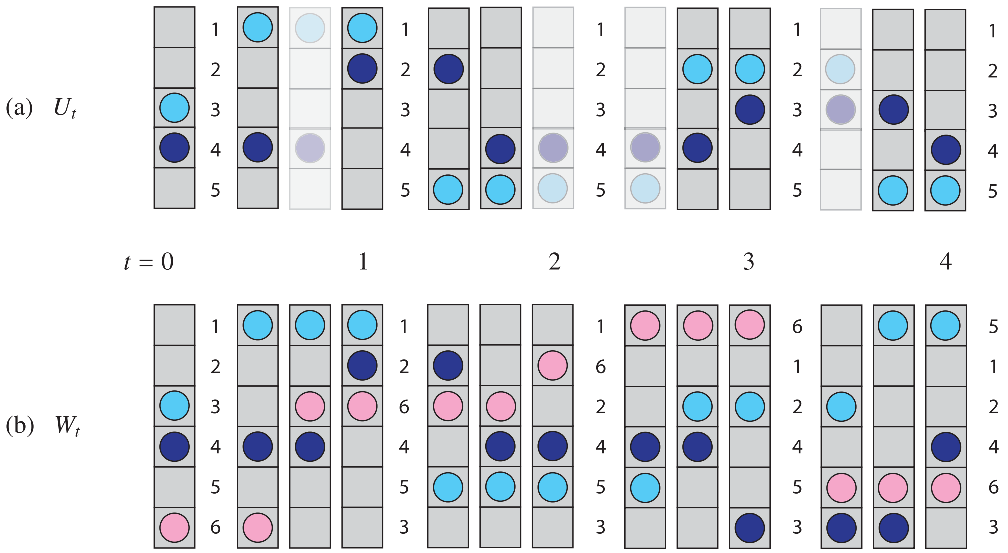

Figure 2. (a) U_t, the same SAC of two walks on K_5 as in Figure 1. Duplicates of previous steps used to synchronize with the extension below are shown faded. (b) W_t, a POSAC of three walks, under the partial order of the light blue walker walking before the dark blue one. The permutation is given at the end of every time unit while the order can be inferred from the diagram. Observe how the order of the blue walk changes with respect to the extended pink walk between different time units. The rule is that the pink walk waits until its new place is clear and then moves. Also notice that the pink walk always ends its motion in place number 6.

We begin by showing (4.1). Since the first k walks of $\mathcal{W}^N_{k+1,R}$ are defined in exactly the same way as those of $\mathcal{W}^N_k$, the proof that each of these walks performs a simple random walk is identical to the proof of this fact for $\mathcal{W}^N_{k+1}$, and we omit it. The fact that $\{W_t(k+1)\}_{t\in\mathbb{Z}}$ is a simple random walk is straightforward from the fact that $W_t(k+1) = P_t(N)$ and from the definition of P_t.

Next let us show that $\mathcal{W}^N_{k+1,R}$ satisfies (4.2). Observe that we have obtained the first part of (4.2) in (4.3). For the second part, consider the event

$$B^{i,j}_t = \{\sigma_t(i) < \sigma_t(j)\},$$

and again write P'_t for the transposition $(N\, a_t)$. For $i, j \in [k+1]$ we have

$$\begin{aligned}\mathbb{P}(W_t(i) = W_{t-1}(j), B^{i,j}_t) &= \mathbb{P}(P_t U_t(i) = P_{t-1} U_{t-1}(j), B^{i,j}_t)\\ &= \mathbb{P}(P_{t-1} \circ P'_t U_t(i) = P_{t-1} U_{t-1}(j), B^{i,j}_t)\\ &= \mathbb{P}(P'_t U_t(i) = U_{t-1}(j), B^{i,j}_t) = 0,\end{aligned}$$

following similar arguments to those used in (3.1).

We are therefore left with the case $k+1 \in \{i, j\}$. However, if $i = k+1$ and $W_t(k+1) = W_{t-1}(j)$, then by the definition of σ_t we would have $\sigma_t(j) = \sigma_t(k+1) = s_t(i) + 1$ and $\sigma_t(i) = s_t(i)$. Thus

$$\mathbb{P}(W_t(k+1) = W_{t-1}(j), B^{i,j}_t) = 0.$$

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Finally consider the case $j = k+1$. If $B_t^{i,k+1}$ holds, then by the definition of σ_t there must exist some $b \in [k]$ which satisfies $B_t^{i,b}$ such that $W_{t-1}(b) = W_t(j) = P_t(N)$. This b satisfies $W_{t-1}(b) = P_{t-1}U_{t-1}(b) = P_t(N)$ and hence, by the definition of P_t, we have $P_t U_{t-1}(b) = P_{t-1}(N)$.

We get that

$$\begin{aligned}\mathbb{P}(W_t(i) = W_{t-1}(k+1), B_t^{i,j}) &= \mathbb{P}(W_t(i) = P_{t-1}(N), B_t^{i,j}) \\ &= \mathbb{P}(\exists b \in [k] : P_t U_t(i) = P_t U_{t-1}(b), B_t^{i,b}) \\ &= \mathbb{P}(\exists b \in [k] : U_t(i) = U_{t-1}(b), B_t^{i,b}) = 0,\end{aligned}$$

where the last equality follows from the fact that $\mathcal{U}_{k,R}^{N-1}$ satisfies (4.2). Theorem 4.1 follows.

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