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EQUIVALENCE BETWEEN LOGARITHMIC SOBOLEV INEQUALITY AND HYPERCONTRACTIVITY IN A PROBABILITY GAGE SPACE

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Abstract In this paper, we prove the equivalence between logarithmic Sobolev inequality and hypercontractivity of a class of quantum Markov semigroup and its associated Dirichlet form based on a probability gage space.

Keywords: Quantum Markov semigroup; Dirichlet form; hypercontractivity; logarithmic Sobolev inequality

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1. Introduction

The study of hypercontractivity in bosonic system was pioneered by Nelson [26]. Gross [16] establishes an equivalence relationship between hypercontractivity and the logarithmic Sobolev inequality. Then Gross [15] generalized it to the non-commutative case, i.e., fermonic system, the corresponding relation was obtained under certain condition. In this case, the optimal time was finally obtained by Carlen and Lieb in [10]. Biane [4] extended Carlen's and Lieb's work to q-Ornstein-Uhlenbeck semigroup $(-1 < q < 1)$ (see § 2), but he did not give the equivalent relationship between hypercontractivity and corresponding logarithmic Sobolev inequality. The motivation of this paper is to establish the hypercontractivity of a class of quantum Markov semigroups and corresponding logarithmic Sobolev inequality in the framework of a probability gage space, that is, a finite von Neumann algebra with a faithful normal trace on it, including the above q-Ornstein-Uhlenbeck semigroup $(-1 < q < 1)$ as a special case.

Hypercontractivity and non-commutative functional inequalities of quantum Markov semigroups based on various specific non-commutative probability spaces are in the ascendant, refer to [3, 5, 9, 13, 17, 19–25, 28–30], etc., and they have been applied to the study of quantum information and quantum computation (refer to [2, 27]). Therefore, this study has theoretical significance and application value.

This paper is organized as follows. Section 1 gives a brief introduction to the relevant background and research significance; Section 2 is devoted to prove the equivalence between logarithmic Sobolev inequality and hypercontractivity of a class of quantum Markov semigroup and associated Dirichlet form in a probability gage space.

2. Equivalence between Logarithmic Sobolev inequality and hypercontractivity in a probability gage space

2.1. Markov semigroup and associated Dirichlet form

In this subsection, we recall briefly the main concepts and conclusions about non-commutative L^p-spaces, Markov semigroup and associated Dirichlet form in this non-commutative setting, more details refer to [1, 12]:

Let (A, τ) be a probability gage space, thus A is a finite von Neumann algebra and τ is a faithful, normal trace on it. For $1 \leq p < \infty$, $L^p(A, \tau)$ is the completion of A with respect to the norm $\|x\|_p = (\tau(|x|^p))^{1/p}, x \in A$, and $L^\infty(A, \tau) = A$ equipped with the operator norm. These spaces share all the functional analytic features of the classical L^p-spaces, such as the uniform convexity for $p \in [1, \infty)$, duality between $L^p(A, \tau)$ and $L^{p'}(A, \tau)$ with $p^{-1} + p'^{-1} = 1$, and Riesz–Thorin interpolation, Hölder's and Clarkson's inequalities.

Furthermore, the Markov semigroup and its associated Dirichlet form are based on the standard form $(A, L^2(A, \tau), L^2_+(A, \tau), J)$ of the von Neumann algebra A, where $L^2_+(A, \tau)$ is a closed convex cone in $L^2(A, \tau)$, inducing an anti-linear isometry J (the modular conjugation) on $L^2(A, \tau)$ which is the extension of the involution $a \to a^*$ of A. The subspace of J-invariant elements (called real) will be denoted by $L^2_h(A, \tau)$.

When a is real, the symbol $a \wedge 1$ will denote the projection onto the closed and convex subset $\{a \in L^2_+(A, \tau) : a \leq 1\}$, where 1 is the unit of A.

Definition 2.1.1. A weak $*$- continuous semigroup $\{T_t\}_{t \geq 0}$ of bounded linear operators defined on $L^\infty(A, \tau)$ is said to be quantum Markov semigroup, if it satisfies the following conditions:

(1) (symmetric property): $\tau(T_t(x)y) = \tau(xT_t(y))$;

(2) (completely Markovian property): if $\{T_t \otimes I_n\}$ is Markovian on $L^\infty(A, \tau) \otimes M_n(\mathbb{C})$, that is, if $0 \leq x \leq 1 \otimes I_n$ implies that $0 \leq T_t \otimes I_n(x) \leq 1 \otimes I_n$, for all $n \in \mathbb{N}$, where 1 is the unit of A and I_n is the identity map on matrix algebra $M_n(\mathbb{C})$, respectively.

Remark 2.1.2. (1) Quantum Markov semigroup $\{T_t\}_{t \geq 0}$ is said to be conservative, if $T_t(1) = 1, \forall t \geq 0$;

(2) By using [12, Proposition 3.1] quantum Markov semigroup on $L^\infty(A, \tau)$ can be extended to completely Markov semigroup
on $L^p(A, \tau)$ for all $p \geq 1$.

Definition 2.1.3. A closed, densely defined and non-negative quadratic form $(\epsilon, D(\epsilon))$ on $L^2(A, \tau)$ is said to be

(1) real, if for $a \in D(\epsilon)$ then $J(a) \in D(\epsilon)$ and $\epsilon[J(a)] = \epsilon[a]$;

(2) a Dirichlet form, if it is positive on $L^2_h(A, \tau)$, real and $\epsilon[a \wedge 1] \leq \epsilon[a]$ for $a \in D(\epsilon) \cap L^2_h(A, \tau)$. Furthermore, it is conservative in case $1 \in D(\epsilon)$;

(3) a regular Dirichlet form if in addition $A \cap D(\epsilon)$ is norm dense in A and is also dense in $D(\epsilon)$ with respect to the graph norm: $|||x|||_1^2 = \epsilon[x] + \tau(|x|^2)$;

(4) a completely Dirichlet form if the canonical extension $(\epsilon^n, D(\epsilon^n))$ to $L^2(A,\tau) \otimes M_n(\mathbb{C}), \tau^n)$: $\epsilon^n[[a_{ij}]_{i,j=1}^n] := \sum_{i,j=1}^n \epsilon[a_{ij}]$, is a Dirichlet form for all $n \geq 1$, where $[a_{ij}]_{i,j=1}^n \in D(\epsilon^n) := D(\epsilon) \otimes M_n(\mathbb{C})$, and $\tau^n = \tau \otimes tr_n$ is the faithful, normal trace on the von Neumann algebra $A \otimes M_n(\mathbb{C})$, here tr_n is a normalized trace on $M_n(\mathbb{C})$.

Proposition 2.1.4 (see [1, Lemma 2.3; 12, Proposition 2.12; and 11 Proposition 4.5 and Proposition 4.10]). *Let $(\epsilon, D(\epsilon))$ be a closed, densely defined, non-negative real quadratic form. Then the following statements are equivalent:*

(1) $(\epsilon, D(\epsilon))$ is a Dirichlet form;

(2) For every real-valued Lipschitz function $\varphi : \mathbb{R} \to \mathbb{R}$, which satisfies $|\varphi(t) - \varphi(s)| \leq c_\varphi |t - s|, \forall t, s \in \mathbb{R}$ and $\varphi(0) = 0$, where c_φ is a positive constant, one has $\epsilon[\varphi(x)] \leq c_\varphi^2 \epsilon[x]$ whenever $x \in D(\epsilon) \cap L_h^2(A,\tau)$.
Furthermore, if $(\epsilon, D(\epsilon))$ is conservative, then the above items (1) and (2) are equivalent to the following item:

(3) $\epsilon(1,x) \geq 0$ for all $x \in D(\epsilon) \cap L_+^2(A,\tau)$, and $\epsilon[|x|] \leq \epsilon[x]$ for all $x \in D(\epsilon) \cap L_h^2(A,\tau)$.

In the end of this subsection, the well-known Beurling–Deny type criterion in the non-commutative context is given:

Theorem 2.1.5 (Beurling-Deny). *(see [1, Theorems 2.7, 2.8; and 12, Theorem 3.3, 11, Theorem 4.11) Given a strongly continuous symmetric semigroup $T_t = e^{-tL}$ with infinitesimal generator L, and the associated quadratic form $\epsilon[x] = <\sqrt{L}x, \sqrt{L}x>$, for $x \in D(\epsilon) = D(\sqrt{L})$. Then the following are equivalent:*

(1) The form ϵ is a (completely) Dirichlet form;

(2) The semigroup $T_t = e^{-tL}$

is (completely) Markovian.

Remark 2.1.6. (1) From Theorem 2.1.5, we see that $\{T_t\}$ is conservative quantum Markovian if and only if the associated quadratic form is conservative completely Dirichlet form.

(2) For a given Dirichlet form $\epsilon[,]$, by [12, Proposition 1.2], the following inequality holds true:

$$\epsilon[|x|] \leq 2\epsilon[x] \quad \text{for all} \quad x \in D(\epsilon).$$

When $x \in D(\epsilon) \cap L_h^2(A,\tau)$, from Proposition 2.1.4 item (2) it is easy to check that the coefficient 2 on the right-hand side of the above inequality can be replaced by 1, that is, $\epsilon[|x|] \leq \epsilon[x]$.

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2.2. Hypercontractivity and logarithmic Sobolev inequality

Definition 2.2.1. We are given a quantum Markov semigroup $\{T_t\}_{t\geq 0}$ in the interpolating family $L^p(A,\tau)$ for all $p>1$, and constants $a>0$ and $b\geq 0$. Let $p(t)=1+(p-1)e^{2t/a}$, if

$$\|T_t x\|_{p(t)} \leq e^{b(1/p-1/p(t))}\|x\|_p,$$

for all $x\in L^p(A,\tau), p>1,\ t\geq 0$, then it is said to have hypercontractivity.

Definition 2.2.2. A quantum Markov semigroup $\{T_t\}$ is said to be regular, if its associated Dirichlet form is regular.

From now on, we consider the regular quantum Markov semigroup $\{T_t\}_{t\geq 0}=\{e^{-Lt}\}_{t\geq 0}$ on the above probability gage space (A,τ), then its associated Dirichlet form $\epsilon[x] = <\sqrt{L}x, \sqrt{L}x>$. Let $\mathrm{Ent}(x)=\tau(x\log x)-\|x\|_{L^2}\log\|x\|_{L^2}$ denote the relative entropy of a positive element x. We obtain the following main result:

Theorem 2.2.3. *For each $p>1$, let $p(t)=1+(p-1)e^{2t/a}, b(t)=b(1/p)-1/p(t))$, where $a(>0)$ and $b(\geq 0)$ are constants. Then the following statements are equivalent:*

(1) $\|T_t x\|_{p(t)}\leq e^{b(t)}\|x\|_p$ *for all* $x\in A,\ \forall p>1,\ \forall t\geq 0$;

(2) $Ent(|x|^2)\leq 4\ a\ \epsilon[x]+b\ \|x\|_2^2$ *for all* $x\in D(\epsilon)$*. When* $x\in D_h:=D(\epsilon)\cap L_h^2(A,\tau)$*, the coefficient* $4a$ *in the inequality can be replaced by* $2a$*.*

We need the following lemma which plays a crucial role for proving Theorem 2.2.3. The special cases in the Clifford algebra and mixed spin systems setting was proved by Gross [15] (see [15, Lemma 1.1]) and Biane [4] (see [4, Lemma 3]), respectively.

Lemma 2.2.4. *For all invertible positive $x\in A\cap D(\epsilon)$, and $1<p<\infty$, one has*

$$\epsilon[x^{p/2}]\leq \frac{p^2}{4(p-1)}\epsilon(x,x^{p-1}).$$

That is

$$\leq \frac{p^2}{4(p-1)}.$$

Proof. First, notice that x is invertible and positive, then there exists a constant $c>0$ such that Spec $(x)\subseteq[c,\|x\|]$. Hence, by Proposition 2.1.4 item (2) combining the function calculus of x, it is easy to check that $x^{p/2}$ and x^{p-1} are in $A\cap D(\epsilon)$. By the definition and spectrum decomposition of L,

$$<x^{p/2}, Lx^{p/2}> = \lim_{t\to 0}\frac{1}{t}\tau[(x^{p/2}-T_t x^{p/2})x^{p/2}];$$

$$<x, Lx^{p-1}> == \lim_{t\to 0}\frac{1}{t}\tau[(x-T_t x)x^{p-1}].$$

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So, it suffices to prove

$$\tau[(x^{p/2}-T_tx^{p/2})x^{p/2}]\le \frac{p^2}{4(p-1)}\tau[(x-T_tx)x^{p-1}]. \tag{2.1}$$

For any fixed $t>0$. Since T_t is symmetric and Markovian, then for φ and ψ positive and continuous functions on $\mathbb{R}$, $\tau[\varphi(x)T_t(\psi(x))]$ is positive and linear in φ and ψ. Moreover, for φ and ψ such that $\varphi(\alpha)\le c|\alpha|$ and $\psi(\alpha)\le c'|\alpha|$, where c,c' are constants. As a general property for normal traces on von Neumann algebras (see [1, 12]), there exists a positive measure μ_x on $\mathbb{R}\backslash\{0\}\times\mathbb{R}\backslash\{0\}$ with support contained in Spec$(x)\times$Spec(x) such that $\mu_x(\alpha,\beta)=\mu_x(\beta,\alpha)$ and

$$\tau[\varphi(x)T_t(\psi(x))]=\int\int\varphi(\alpha)\psi(\beta)d\mu_x(\alpha,\beta).$$

Since $1-T_t(1)$ is positive, we have also that there is a positive measure ν_x on $\mathbb{R}\backslash\{0\}$ with support contained in Spec(x) such that

$$\tau[\varphi(x)(1-T_t(1))]=\int\varphi(\alpha)d\nu_x(\alpha).$$

Consider now the quadratic form $\tau[x(1-T_t)x]$, we then have

$$\tau[\varphi(x)(1-T_t)\varphi(x)]=\tau[\varphi(x)^2(1-T_t(1))]+\tau[\varphi(x)^2\ T_t(1)-\varphi(x)T_t(\varphi(x))].$$

Therefore, we have

$$\tau[\varphi(x)(1-T_t)\varphi(x)]=\int\varphi(\alpha)^2\mathrm{d}\nu_x(\alpha)+\frac{1}{2}\int\int[\varphi(\alpha)-\varphi(\beta)]^2\mathrm{d}\mu_x(\alpha,\beta). \tag{2.2}$$

In the following, let $\varphi(\alpha)=\alpha^{p/2},\alpha\in$ Spec (x). Take $\varphi(x)=x^{p/2}$ in Equation (2.2), we obtain

$$\tau[(x^{p/2}-T_tx^{p/2})x^{p/2}]=\int\alpha^p\mathrm{d}\nu_x(\alpha)+\frac{1}{2}\int\int(\alpha^{p/2}-\beta^{p/2})^2\mathrm{d}\mu_x(\alpha,\beta).$$

Similarly,

$$\tau[x^{p-1}(x-T_tx)]=\int\alpha^p\mathrm{d}\nu_x(\alpha)+\frac{1}{2}\int\int(\alpha-\beta)(\alpha^{p-1}-\beta^{p-1}\,\mathrm{d}\mu_x(\alpha,\beta)).$$

Then (2.1) holds true from the above two formulas combining the following fact

$$(a^{p/2}-b^{p/2})^2\le\frac{p^2}{4(p-1)}(a-b)(a^{p-1}-b^{p-1}),\quad a,b\ge 0,\ \ p>1.$$

□

The Proof of Theorem 2.2.3. First, since each T_t is completely positive, from [29, Remark 9] we see that the norm of T_t from $L^p_h(A,\tau)$ to $L^q_h(A,\tau)$ $(p,q>1)$ is achieved on the positive cone $L^p_+(A,\tau)$. Hence, it is sufficient to consider hypercontractivity on

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positive cones. Given an invertible positive $x \in A \cap D(\epsilon)$, put $\varphi(t) = e^{-b(t)}\|T_t x\|_{p(t)}$. By [14, Lemma 2] and [18, Lemma 3.1], a straightforward calculus shows that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\log\varphi(t) &= \frac{\mathrm{d}}{\mathrm{d}t}(-b(t) + \log\|T_t x\|_{p(t)})\\ &= -b'(t) + \frac{1}{\|T_t x\|_{p(t)}}\frac{\mathrm{d}\|T_t x\|_{p(t)}}{\mathrm{d}t}\\ &= -b'(t) + \frac{1}{\|T_t x\|_{p(t)}}\frac{\mathrm{d}}{\mathrm{d}t}[\tau(T_t x)^{p(t)}]^{1/p(t)}. \end{aligned} \tag{I}$$

Since

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}[\tau(T_t x)^{p(t)}]^{1/p(t)} &= \|T_t x\|_{p(t)}\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{p(t)}\log\tau_q(T_t x)^{p(t)}\right]\\ &= \|T_t x\|_{p(t)}\left[-\frac{p'(t)}{p^2(t)}\log\|T_t x\|_{p(t)}^{p(t)} + \frac{1}{p(t)}\frac{1}{\|T_t x\|_{p(t)}^{p(t)}}\frac{\mathrm{d}\|T_t x\|_{p(t)}^{p(t)}}{\mathrm{d}t}\right],\end{aligned}$$

and since

$$\begin{aligned}\frac{\mathrm{d}\|T_t x\|_{p(t)}^{p(t)}}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\tau[(T_t x)^{p(t)}]\\ &= \tau\left[(T_t x)^{p(t)}\left(p'(t)\log T_t x + p(t)(T_t x)^{-1}\frac{\mathrm{d}T_t x}{\mathrm{d}t}\right)\right],\end{aligned}$$

then take the above equation to formula (I), we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\log\varphi(t) &= -b'(t) - \frac{p'(t)}{p^2(t)}\log\|T_t x\|_{p(t)}^{p(t)}\\ &\quad + \frac{1}{p(t)\|T_t x\|_{p(t)}^{p(t)}}\tau\left[(T_t x)^{p(t)}\left(p'(t)\log T_t x + p(t)(T_t x)^{-1}\frac{\mathrm{d}T_t x}{\mathrm{d}t}\right)\right]\\ &= -b'(t) - \frac{p'(t)}{p^2(t)}\log\|T_t x\|_{p(t)}^{p(t)} + \frac{1}{p(t)\|T_t x\|_{p(t)}^{p(t)}}\tau[p'(t)(T_t x)^{p(t)}\log T_t x]\\ &\quad + \tau\left[p(t)(T_t x)^{p(t)-1}\frac{\mathrm{d}T_t x}{\mathrm{d}t}\right].\end{aligned} \tag{II}$$

Notice that $\mathrm{d}T_t x/\mathrm{d}t = -L(T_t x)$, so that

$$\tau\left[(T_t x)^{p(t)-1}\frac{\mathrm{d}T_t x}{\mathrm{d}t}\right] = -\epsilon((T_t x)^{p(t)-1}, T_t x).$$

On the other hand,

$$\begin{aligned}\mathrm{Ent}((T_t x)^{p(t)}) &= \tau[(T_t x)^{p(t)}\log(T_t x)^{p(t)}] - \tau[(T_t x)^{p(t)}\log\tau[(T_t x)^{p(t)}]]\\ &= \tau[(T_t x)^{p(t)}\log(T_t x)^{p(t)}] - \|T_t x\|_{p(t)}^{p(t)}\log\|T_t x\|_{p(t)}^{p(t)}.\end{aligned}$$

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Combing with formula (II), one can obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\log\varphi(t) = -b'(t) + \frac{p'(t)}{p(t)^2}\frac{1}{\|T_tx\|_{p(t)}^{p(t)}}\mathrm{Ent}((T_tx)^{p(t)}) - \frac{1}{\|T_tx\|_{p(t)}^{p(t)}}\epsilon((T_tx)^{p(t)-1}, T_tx). \tag{III}$$

Assume (1). Since $b(0) = 0$ and $p(0) = p$, it follows that $\varphi(0) = \|x\|_p$, then by the hypercontractivity of T_t implies that $\varphi'(0) \leq 0$, which gives, via formula (III)

$$\mathrm{Ent}((T_tx)^{p(t)}) \leq \frac{p^2(t)\|T_tx\|_{p(t)}^{p(t)}}{p'(t)}[b'(t) + \frac{1}{\|T_tx\|_{p(t)}^{p(t)}}\epsilon((T_tx)^{p(t)-1}, T_tx)].$$

Let $t = 0$ and $p = 2$, from which follows that $p(0) = 2$, $p'(0) = 2/a$, $b'(0) = 1/2a$. Therefore, from the above inequality, it implies that

$$\mathrm{Ent}(x^2) \leq 2a\epsilon[x] + b\|x\|_2^2.$$

Now, for any given positive element $x \in D(\epsilon)$. Since $\epsilon[,]$ is regular, then there exists a sequence (x_n) consisting of positive invertible elements in $A \cap D(\epsilon)$ such that $|||x_n - x|||_1 \to 0$ as $n \to \infty$, where $|||x|||_1 = \epsilon[x] + \tau(|x|^2)$ is the graph norm. It follows that $\epsilon[x_n] \to \epsilon[x]$ and $\|x_n\|_2 \to \|x\|_2$, as $n \to \infty$. For any fixed $n \in \mathbb{N}$, by the above proof, we have

$$\mathrm{Ent}(x_n^2) \leq 2a\epsilon[x_n] + b\|x_n\|_2^2.$$

Letting $n \to \infty$, and combining the continuity of norm, we obtain

$$\mathrm{Ent}(x^2) \leq 2a\epsilon[x] + b\|x\|_2^2.$$

Hence, for any $y \in D(\epsilon) \cap L_h^2(A, \tau)$, from the above inequality, we have

$$\mathrm{Ent}(|y|^2) \leq 2a\epsilon[|y|] + b\|y\|_2^2.$$

Since $\epsilon[|y|] \leq \epsilon[y]$ (see Remark 2.1.6 (2)), then combining the above inequality implies that

$$\mathrm{Ent}(|y|^2) \leq 2a\epsilon[y] + b\|y\|_2^2.$$

Finally, notice that $\epsilon[|z|] \leq 2\ \epsilon[z]$ for all $z \in D(\epsilon)$ (see Remark 2.1.6 (2) again), similar to the above proof, we have

$$\mathrm{Ent}(|z|^2) \leq 4\ a\epsilon[z] + b\|z\|_2^2.$$

Conversely, assume (2). For a given invertible positive $x \in A \cap D(\epsilon)$, we have

$$\mathrm{Ent}(x^2) \leq 2a\epsilon[x] + b\|x\|_2^2.$$

Replace x with $x^{p/2}$, one can get

$$\mathrm{Ent}(x^p) \leq 2a\epsilon[x^{p/2}] + b\|x^{p/2}\|_2^2.$$

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By Lemma 2.2.4, it implies that

$$\mathrm{Ent}(x^p) \leq \frac{ap^2}{2(p-1)}\epsilon(x^{p-1}, x) + b\|x\|_p^p.$$

Set x be $T_t x$ and p be $p(t)$ in the above formula, we have

$$\mathrm{Ent}((T_t x)^{p(t)}) \leq \frac{ap(t)^2}{2(p(t)-1)}\epsilon((T_t x))^{p-1}, T_t x + b\|T_t x\|_{p(t)}^{p(t)}. \tag{IV}$$

Notice that formula (III),

$$\frac{\mathrm{d}}{\mathrm{d}t}\log\varphi(t) = \frac{p'(t)}{p^2(t)\|T_t x\|_{p(t)}^{p(t)}}\Big[\mathrm{Ent}((T_t x)^{p(t)}) - \frac{p(t)^2}{p'(t)}\epsilon((T_t x)^{p(t)-1}, T_t x) - \frac{b'(t)p(t)^2}{p'(t)}\|T_t x\|_{p(t)}^{p(t)}\Big]. \tag{V}$$

Since $b(t) = b(1/p - 1/p(t))$, $p(t) = 1 + (p-1)e^{2t/a}$, then $b'(t) = p'(t)/p(t)^2$, $p'(t) = (2/a)(p-1)e^{2t/a}$, hence, $p(t)^2/p'(t) = ap(t)^2/2(p(t)-1)$, and $b'(t)p(t)^2/p'(t) = b$. Compare (IV) and (V), then we have $(\mathrm{d}/\mathrm{d}t)\log\varphi(t) \leq 0$, it follows that $(\mathrm{d}/\mathrm{d}t)\varphi(t) \leq 0$. Therefore, $\varphi(t) \leq \varphi(0) = \|x\|_p$. Since $\epsilon[,]$ is regular, it is easy to prove that the subset of invertible positive elements in $A \cap D(\epsilon)$ is dense in all $L^p(A,\tau)$ with respect to the L^p-norms for all $p > 1$, and since $\varphi(t)$ is continuous in the operator norm, which in turn yields the hypercontractivity of $\{T_t\}$.

It is well known that in the abelian case, when 4a is replaced by 2a in the item (2) of Theorem 2.2.3 for all $x \in D(\epsilon)$, the conclusion still holds. On further analysis, we see that the key point is $\epsilon[|J(x)|] = \epsilon[|x|]$ for all $x \in D(\epsilon)$. Inspired by this, with this restriction, it can be extended to non-commutative case:

Corollary 2.2.5. *The notation is as in Theorem 2.2.3. For a given quantum Markov semigroup $\{T_t\}_{t\geq 0}$ and its associated regular Dirichlet form $\epsilon[,]$ on $L^2(A,\tau)$. If $\epsilon[|J(x)|] = \epsilon[|x|]$ for all $x \in D(\epsilon)$, then the following statements are equivalent*:

(1) $\|T_t x\|_{p(t)} \leq e^{b(t)}\|x\|_p$ for all $x \in A$, $\forall p > 1$, $\forall t \geq 0$;

(2) Ent$(|x|^2) \leq 2\ a\ \epsilon[x] + b\|x\|_2^2$ for all $x \in D(\epsilon)$.

Proof. Using Theorem 2.2.3 and the regularity of $\epsilon[,]$, one only needs to prove $\epsilon[|x|] \leq \epsilon[x]$ for all $x \in A \cap D(\epsilon)$. Indeed, since $\epsilon[x] =$ is completely Dirichlet form, so that $(\epsilon^2, D(\epsilon^2))$ is Dirichlet form (see the above Definition 2.1.3). Put $\tilde{x} = \left[\begin{smallmatrix} 0 & x^* \\ x & 0 \end{smallmatrix}\right]$, it is easy to see that $\tilde{x}$ is a self-adjoint in $D(\epsilon^2) := D(\epsilon) \otimes M_2(\mathbb{C})$. By Remark 2.1.6 (2) we have $\epsilon^2[|\tilde{x}|] \leq \epsilon^2[\tilde{x}]$. By routine calculation, $|\tilde{x}| = (\tilde{x}^*\tilde{x})^{1/2} = \left[\begin{smallmatrix} |x| & 0 \\ 0 & |x^*| \end{smallmatrix}\right]$. Hence,

$$\epsilon[|x|] + \epsilon[|x^*|] = \epsilon^2\left[\left[\begin{matrix} |x| & 0 \\ 0 & |x^*| \end{matrix}\right]\right] = \epsilon^2[|\tilde{x}|] \leq \epsilon^2[\tilde{x}] = \epsilon^2\left[\left[\begin{matrix} 0 & x^* \\ x & 0 \end{matrix}\right]\right] = \epsilon[x^*] + \epsilon[x].$$

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Furthermore, by the condition of this claim, $\epsilon[|x^*|] = \epsilon[|x|]$ (Remark: $J(x) = x^*$ when $x \in A$), and notice that $\epsilon[x^*] = \overline{\epsilon[x]} = \epsilon[x]$ (see Definition 2.1.3). Therefore, from the above inequality, we can obtain $2\ \epsilon[|x|] \leq 2\ \epsilon[x]$, that is, $\epsilon[|x|] \leq \epsilon[x]$. □

3. Application to the q-Ornstein–Uhlenbeck semigroup $(-1 < q < 1)$

In the last section, we use Corollary 2.2.5 to characterize the equivalence of strictly hypercontractivity and the logarithmic Sobolev inequality for $q(-1 < q < 1)$-Ornstein–Uhlenbeck semigroup introduced by Bozèjko and Speicher [7].

For this purpose, we first recall briefly the construction of q-Gaussian von Neumann algebra and q-Ornstein–Uhlenbeck semigroup on it. Since the cases $q = \pm 1$ are well known, in the following, we shall only consider the cases for $-1 < q < 1$. More details can refer to [4, 6–8].

Let $\mathcal{H}$ be an infinite dimensional real separable Hilbert space with complexification $\mathcal{H}_{\mathcal{C}}$. Let Ω be a unit vector in a 1-dimensional complex Hilbert space (disjoint from $\mathcal{H}_{\mathcal{C}}$). We refer to Ω as the vacuum, and by convention define ${\mathcal{H}_{\mathcal{C}}}^{\otimes 0} \equiv \mathbb{C}\Omega$. The algebraic Fock space $\mathcal{F}(\mathcal{H})$ is defined as

$$\mathcal{F}(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}_{\mathbb{C}}^{\otimes n},$$

where the direct sum and tensor product are algebraic. We then define a Hermitian form $< \cdot, \cdot >_q$ in $\mathcal{F}(\mathcal{H})$ as below:

For $\xi, \eta \in \mathcal{F}(\mathcal{H})$

$$< \xi, \eta >_q = < \xi, P_q \eta >,$$

where $<, >$ is the usual scalar product and $P_q = \bigoplus_{n=0}^{\infty} P_q^{(n)}$, in which $P_q^{(n)}(f_1 \otimes f_2 \otimes \cdots \otimes f_n) = \sum_{\pi \in S_n} q^{i(\pi)} f_{\pi(1)} \otimes f_{\pi(2)} \otimes \cdots \otimes f_{\pi(n)}$ for all $f_k \in \mathcal{F}(\mathcal{H}), k = 1, 2, \ldots, n$, $\forall\ n \in \mathbb{N}$.

Here S_n is the symmetric group on n symbols, and $i(\pi)$ counts the number of inversions in π, that is

$$i(\pi) = \sharp\{(i,j) : 1 \leq i\pi(j)\}.$$

Indeed, the above Hermitian form is the conjugate-linear extension of

$$< \Omega, \Omega >_q = 1;$$
$$_q = \delta_{mn} \Sigma_{\pi \in S_n} q^{i(\pi)}$$
$$\cdots,$$

for $f_i, g_j \in \mathcal{H}$, $i = 1, 2, \ldots, m; j = 1, 2, \ldots, n, \forall\ m, n \in \mathbb{N}$.

It is remarkable that, for $-1 < q < 1$, the form $< \cdot, \cdot >_q$ is always non-degenerate on $\mathcal{F}(\mathcal{H})$. The q-Fock space $\mathcal{F}_q(\mathcal{H})$ is defined as the completion of $\mathcal{F}(\mathcal{H})$ with respect to the inner product $< \cdot, \cdot >_q$.

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For any vector $f \in \mathcal{H} \subset \mathcal{H}_{\mathbb{C}}$, define the creation operator $c_q(f)$ on $\mathcal{F}_q(\mathcal{H})$ to extend

$$c_q(f)\Omega = f$$
$$c_q(f) f_1 \otimes \cdots \otimes f_k = f \otimes f_1 \otimes \cdots \otimes f_k.$$

The annihilation operator $c_q^*(f)$ is its adjoint, which satisfies

$$c_q^*(f)\Omega = 0$$
$$c_q^*(f) f_1 \otimes \cdots \otimes f_k = \Sigma_{j=1}^{k} q^{j-1} f_1 \otimes \cdots \otimes f_{j-1} \otimes f_{j+1} \otimes \cdots \otimes f_k.$$

The operators $c_q(f)$ and $c_q^*(f)$ are bounded on $\mathcal{F}_q(\mathcal{H})$ with

$$\|c_q(f)\| = \|c_q^*(f)\| = \begin{cases} \|f\|(1-q)^{-1/2}, & \text{if } 0 \leq q < 1, \\ \|f\|, & \text{if } -1 < q < 0. \end{cases}$$

On the above bases, define the q-Gaussian von Neumann algebra $\Gamma_q(\mathcal{H})$ which is generated by the self-adjoint q-Gaussian operators $\omega(f) = c_q(f) + c_q^*(f), f \in \mathcal{H}$ on $\mathcal{F}_q(\mathcal{H})$. It was proved in [8, Theorem 2.10] that $\Gamma_q(\mathcal{H})$ is a II_1-factor and $\tau_q(a) = < \Omega, a\Omega >_q$ for $a \in \Gamma_q(\mathcal{H})$, is the unique faithful normal trace on $\Gamma_q(\mathcal{H})$.

Next, we will introduce the concept of q-Ornstein–Uhlenbeck semigroup on the above q-Gaussian von Neumann algebra $\Gamma_q(\mathcal{H})$ for $-1 < q < 1$:

Let $T : \mathcal{H} \to \mathcal{H}$ be a contraction on the real Hilbert spaces $\mathcal{H}$ with complexification $T_{\mathbb{C}}$, then the linear map defined on elementary tensors by

$$F_q(T)(f_1 \otimes \cdots \otimes f_n) = T_{\mathbb{C}} f_1 \otimes \cdots \otimes T_{\mathbb{C}} f_n$$

extends to a contraction

$$F_q(T) : \mathcal{F}_q(\mathcal{H}) \to \mathcal{F}_q(\mathcal{H}).$$

Now we define q-Gaussian functor Γ_q as a map

$$\Gamma_q(T) : \Gamma_q(\mathcal{H}) \to \Gamma_q(\mathcal{H})$$

as follows

(1) $\Gamma_q(T)\omega(f) = \omega(Tf)$, for $f \in \mathcal{H}$;

(2) $(\Gamma_q(T)(X))\Omega = F_q(T)(X\Omega)$.

By [8, Theorem 2.11] $\Gamma_q(T)$ is a unique bounded, normal, unital, completely positive, and the trace preserving map, and is a covariant functor, that is, if T_1 and T_2 are contractions on $\mathcal{H}$, then

$$\Gamma_q(T_1 T_2) = \Gamma_q(T_1)\Gamma_q(T_2).$$

Furthermore, let $T_t = e^{-t} I_{\mathcal{H}}, t \geq 0$, where $I_{\mathcal{H}}$ is the identity on H. Then the q-Ornstein–Uhlenbeck semigroup is defined to be $\{U_t^{(q)}\}_{t\geq 0} = \{\Gamma_q(T_t)\}_{t\geq 0}$ on $\Gamma_q(\mathcal{H})$ (see Ref. [8]).

From the above construction, it is easy to see that $\{U_t^{(q)}\}_{t\geq 0}$ is a conservative, completely Markov semigroup on $\Gamma_q(\mathcal{H})$. Remark 2.1.2 tells us that $\{U_t^{(q)}\}_{t\geq 0}$ can be extended to completely Markov semigroup on all non-commutative $L^p(\Gamma_q(\mathcal{H}), \tau_q)$-spaces. Its generator N^q on $L^2(\Gamma_q(\mathcal{H}), \tau_q)$ is the number operator given by:

$$N^q\Omega = 0;$$

and

$$N^q(f_1 \otimes \cdots \otimes f_n) = n(f_1 \otimes \cdots \otimes f_n), f_j \in \mathcal{H}_{\mathbb{C}}(j = 1, 2, \ldots, n).$$

In formally, we write $U_t^{(q)} = e^{-tN^q}$. From Theorem 2.1.5, the corresponding quadratic form $\epsilon[x] =< \sqrt{N^q}x, \sqrt{N^q}x >_q$ for all $x \in D(\sqrt{N^q})$, is a conservative, completely Dirichlet form.

Biane [4] described the strictly hypercontractivity of the q-Ornstein–Uhlenbeck semigroup $\{U_t^{(q)}\}_{t\geq 0}$ (see [4, Theorem 2]) and derived a logarithmic Sobolev inequality from strictly hypercontractivity (see [4, Corollary 1]). The following statements show that strictly hypercontractivity and the logarithmic Sobolev inequality in [4] are equivalent:

Theorem 2.2.6. *The notation is as in Theorem 2.2.3. Given the q-Ornstein–Uhlenbeck semigroup $\{U_t^{(q)}\}_{t\geq 0} = \{e^{-tN^q}\}_{t\geq 0}$ and its associated Dirichlet form $\epsilon[x] =$ based on $\Gamma_q(\mathcal{H})$, then the following statements are equivalent*:

(1) $\|U_t^{(q)}x\|_{p(t)} \leq \|x\|_p$ *for all* $x \in \Gamma_q(\mathcal{H}),\ \forall p > 1,\ \forall t \geq 0;$

(2) $Ent(|x|^2) \leq 2\ \epsilon[x]$ *for all* $x \in D(\epsilon)$.

Proof. In order to prove the above claim, using Corollary 2.2.5 it suffices to prove the Dirichlet form $\epsilon[x] =_q, x \in D(\epsilon)$ is regular and satisfies the condition: $\epsilon[|J(x)|] = \epsilon[|x|]$ for all $x \in D(\epsilon)$, where the operator J ia an anti-linear isometry on $L^2(A, \tau)$ (see § 2.1 for the details).

Indeed, given a fixed orthonormal basis $\{e_i\}_{i\in\mathbb{N}}$ in $\mathcal{H}$. For any subset $I = \{i_1, i_2, \ldots, i_n\} \subseteq \mathbb{N}$, by [8, Proposition 2.7] one can construct the q-Wick product $\psi_I = \psi(e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_n}) \in \Gamma_q(\mathcal{H})$ by induction as below:

$$\psi(e_{i_k}) = \omega(e_{i_k}) = c_q(e_{i_k}) + c_q^*(e_{i_k});$$

$$\psi(e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_n}) = \omega(e_{i_1})\psi(e_{i_2} \otimes \cdots \otimes e_{i_n}) - \sum_{k=1}^{n} q^{k-1}\psi(e_{i_1} \otimes \cdots \otimes \check{e_{i_k}} \otimes \cdots \otimes e_{i_n}),$$

where the symbol $\check{e_{i_k}}$ means that e_{i_k} has to be deleted in the product.

Define

$$\Phi : \Gamma_q(\mathcal{H}) \to \mathcal{F}_q(\mathcal{H})$$

through $\Phi(a) = a(\Omega)$. Since τ_q is a faithful trace, then this map is a continuous imbedding of $\Gamma_q(\mathcal{H})$ into $\mathcal{F}_q(\mathcal{H})$ which extends to an unitary isomorphism of $L^2(\Gamma_q(\mathcal{H}), \tau_q)$ with $\mathcal{F}_q(\mathcal{H})$. By routine calculation we have

$$\Phi(\psi_I) = \psi_I\Omega = e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_n}.$$

Hence, from the construction of the q-Ornstein–Uhlenbeck semigroup $U_t^{(q)} = e^{-tN^q}$, we have $U_t^{(q)}\psi_I = e^{-tn}\psi_I$, it follows that $N^q\psi_I = n\psi_I$, which shows that $\psi_I \in D(\epsilon)$. Denote by $\widetilde{\Gamma_q(\mathcal{H})}$ the set of all finite linear combinations of $\psi_I, I = \{i_1, i_2, \cdots, i_n\} \subseteq \mathbb{N}$. It follows that $\widetilde{\Gamma_q(\mathcal{H})} \subset D(\epsilon)$ since $D(\epsilon)$ is a subspace of $L^2(\Gamma_q(\mathcal{H}), \tau_q)$ and $\widetilde{\Gamma_q(\mathcal{H})}$ is dense in $D(\epsilon)$ with respect to the graph norm. On the other hand, since $\widetilde{\Gamma_q(\mathcal{H})}$ is dense in $\Gamma_q(\mathcal{H})$ with respect to the operator norm, so that $A \cap D(\epsilon)(\supset \widetilde{\Gamma_q(\mathcal{H})})$ is dense in $\Gamma_q(\mathcal{H})$ with operator norm. The above proof shows that the Dirichlet form $\epsilon[,]$ is regular.

Finally, in order to prove $\epsilon[|J(x)|] = \epsilon[|x|]$ for all $x \in D(\epsilon)$, it is only need to verify $\epsilon[|(\psi_I)^*|] = \epsilon[|\psi_I|]$ for all $\psi_I = \psi(e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_n}) \in \Gamma_q(\mathcal{H}) \cap D(\epsilon)$. Since

$$< \psi_I\Omega, \psi_I\Omega >_q = \tau_q((\psi_I)^*\psi_I) = \tau_q(\psi_I(\psi_I)^*) =< (\psi_I)^*\Omega, (\psi_I)^*\Omega >_q,$$

then there exists a unitary operator $U : L^2(\Gamma_q(\mathcal{H}), \tau_q) \to L^2(\Gamma_q(\mathcal{H}), \tau_q)$, such that $U|\psi_I| = |(\psi_I)^*|$, here we regard ψ_I and $\psi_I\Omega$ as the same. Furthermore, it is easy to check that $N^qU = UN^q$, hence

$$\begin{aligned}\epsilon[|(\psi_I)^*|] &=< |(\psi_I)^*|, N^q|(\psi_I)^*| >_q =_q \\ &=_q =< |\psi_I|, N^q(|\psi_I| >_q = \epsilon[|\psi_I|].\end{aligned}$$

□

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