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VARIATIONS OF BPS STRUCTURE AND A LARGE RANK LIMIT

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Abstract We study a class of flat bundles, of finite rank N, which arise naturally from the Donaldson–Thomas theory of a Calabi–Yau threefold X via the notion of a variation of BPS structure. We prove that in a large N limit their flat sections converge to the solutions to certain infinite-dimensional Riemann–Hilbert problems recently found by Bridgeland. In particular this implies an expression for the positive degree, genus 0 Gopakumar–Vafa contribution to the Gromov–Witten partition function of X in terms of solutions to confluent hypergeometric differential equations.

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1. Introduction and main results

In this Introduction we describe the circle of ideas and main results of this paper. All definitions and proofs are given in the following sections.

Let X be a complex projective Calabi–Yau threefold. Write Γ for its numerical Grothendieck group endowed with the skew-symmetric bilinear Euler form $\langle -, - \rangle$.

Some of the aims of (generalised, unrefined) Donaldson–Thomas theory (see [22, 23]) are

(1) to define deformation invariants $\mathrm{DT}(\alpha, Z) \in \mathbb{Q}$, virtually enumerating objects in $D^b(X)$ which have prescribed class $\alpha \in \Gamma$ and which are semistable with respect to a numerical Bridgeland stability condition, locally described by a *central charge* $Z \in \mathrm{Hom}(\Gamma, \mathbb{C})$;

(2) to define underlying *Bogomol'nyi-Prasad-Sommerfield* (*BPS*) invariants $\Omega(\alpha, Z) \in \mathbb{Q}$ via a known, universal multi-cover formula, and to prove that in fact they take values in $\mathbb{Z}$ (at least for sufficiently general Z);

(3) to prove that the variation of $\mathrm{DT}(\alpha, Z)$ (equivalently $\Omega(\alpha, Z)$) when we deform the stability condition Z is given by a known, universal expression, the JS/KS wall-crossing formula (due to Joyce–Song and Kontsevich–Soibelman).

Thanks to the work of several authors these aims have now been achieved in some special but highly nontrivial cases (see in particular [3, 27]). A much simpler example is discussed at the end of this Introduction.

This general theory leads to formulate the abstract notions of a *BPS structure* (Γ, Z, Ω) on a lattice Γ with a form $\langle -, - \rangle$, and of its *variation*, which simply describe the outcome of (1)–(2) above for a fixed Z, respectively (3) above when varying Z. (In general one allows Γ to be a nontrivial local system along a variation, but in the present paper we will only need to consider *framed* variations, for which the local system is in fact trivial).

So Z is an element of $\mathrm{Hom}(\Gamma, \mathbb{C})$ and Ω a map of sets $\Gamma \to \mathbb{Q}$ (or $\Gamma \to \mathbb{Z}$ in the integral case), satisfying certain constraints, including the JS/KS formula when Z varies. The function DT is then defined from Ω by inverting the multi-cover formula.

This idea is due to Kontsevich and Soibelman ([23, § 2], [24, § 2]). It is somewhat analogous to introducing the abstract notion of a (variation of) Hodge structure starting from the case of (a family of) Kaehler manifolds. In this analogy the JS/KS formula may be compared to Griffiths transversality: it is the most nontrivial constraint on a variation. The terminology adopted in the present paper was introduced by Bridgeland in [5] in order to single out a special case of Kontsevich and Soibelman's more general notions of *stability data* and *wall-crossing structures*. Important motivation for this abstract approach comes from the fact that variations of BPS structure appear naturally in other contexts, notably in symplectic geometry (see e.g. [8, 24, 25]) and in the Gross–Siebert programme for mirror symmetry (see e.g. [4, 17, 18]).

One of the main aims of the present paper is to show how some very special but interesting variations of BPS structure (which correspond roughly to the case of torsion coherent sheaves on X supported in dimension at most 1) can be described effectively in terms of classical objects, namely *linear complex differential equations of hypergeometric type*. At the same time we relate this description to recent work of Bridgeland [5]. As an application we find an expression for the positive degree, genus 0 Gopakumar–Vafa contribution to the Gromov–Witten partition function of a Calabi–Yau threefold X in terms of solutions to confluent hypergeometric differential equations.

We will follow two closely related approaches, based respectively on Riemann–Hilbert factorisation problems (RH problems) and on flat bundles (of Frobenius type). In our

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loose analogy with variations of Hodge structure the latter correspond to the Gauss–Manin connection, the former to the inverse problem of reconstructing the Gauss–Manin connection from its monodromy.

RH problems are a special type of boundary value problems for holomorphic functions and form a classical topic in complex analysis and mathematical physics (see e.g. [13]). A BPS structure (Γ, Z, Ω) induces in a very natural way various RH problems, formulated for maps from $\mathbb{C}^*$ to an affine algebraic torus $\mathbb{T}$, given by characters of Γ twisted by the form $\langle -, - \rangle$. The BPS invariants Ω prescribe the boundary behaviour of the maps along certain rays in $\mathbb{C}^*$. Unlike the classical case the corresponding structure group is always infinite-dimensional, and for the purposes of the present paper it is a subgroup of $\mathsf{Bir}(\mathbb{T})$. This idea is due to Gaiotto, Moore and Neitzke (see [15]) and was studied e.g. in [5, 12, 20].

Let us recall a recent result in this connection, concerning the case of *finite, uncoupled BPS structures*. These are the simplest objects in the theory, and are defined by the condition that the Euler pairing vanishes when restricted to the locus where $\Omega \neq 0$, i.e. to *active* classes (which are finitely many, in the *finite* case). In particular we will see that the function Ω is in fact *constant* along a variation of uncoupled BPS structure. Geometrically such structures correspond to the case of torsion coherent sheaves on X supported in dimension at most 1, as discussed at the end of this Introduction. It is convenient to introduce a special multi-valued meromorphic function on $\mathbb{C}^*$, given by

$$\Lambda(w) = \frac{e^w \Gamma(w)}{\sqrt{2\pi}\, w^{w-\frac{1}{2}}},$$

where $\Gamma(w)$ is the classical gamma function (see e.g. [11, Ch. I]). Given a ray $\ell \subset \mathbb{C}^*$ emanating from $0 \in \mathbb{C}$ we also introduce the half-plane

$$\mathbb{H}_\ell = \{z \in \mathbb{C}^* | z = uv \text{ with } u \in \ell \text{ and } \Im(v) > 0\} \subset \mathbb{C}^*.$$

Theorem 1 (Bridgeland [5, Theorem 5.3]). *Let (Γ, Z, Ω) be a finite, integral, uncoupled BPS structure. Suppose $\xi \in \mathbb{T}$ is such that $\xi(\gamma) = 1$ when $\Omega(\gamma) \neq 0$. Then the infinite-dimensional, birational RH problem with values in $\mathbb{T}$ attached to (Γ, Z, Ω), with $t \to 0$ asymptotics prescribed by ξ, admits a unique solution $\Psi(t)$. Its component along $\beta \in \Gamma$ is given explicitly by the collection of functions*

$$\Psi_{\mathbb{H}_\ell, \beta}(t) = \prod_{\gamma | \Omega(\gamma) \neq 0, Z(\gamma) \in \mathbb{H}_\ell} \Lambda\left(\frac{Z(\gamma)}{t}\right)^{\Omega(\gamma)\langle \beta, \gamma \rangle}, \quad t \in \mathbb{H}_\ell$$

for generic $\ell \subset \mathbb{C}^*$.

We will relate this infinite-dimensional result to large rank limits of classical, finite-dimensional flat bundles (i.e. systems of linear complex ordinary differential equations [ODEs]).

A central notion for us is that of a *Frobenius bundle*, introduced by Hertling (following Dubrovin [10]) in his study of geometric structures on unfolding spaces of singularities (see [19, § 5.2]). A Frobenius bundle K is a holomorphic bundle over a complex manifold M

with additional data, including a flat connection ∇^r, a Higgs field C and a holomorphic quadratic form g (the 'metric'). Barbieri and the second author (see [1]) showed that under some conditions there is a correspondence between variations of BPS structure and Frobenius bundles of a special form. The main ingredient is a holomorphic generating function $f(Z)$ for the invariants $\mathrm{DT}(\alpha, Z)$ introduced by Joyce (see [21]).

Proposition 2 (see Theorem 34 and Proposition 35). *There is a correspondence between*

(1) *framed variations of BPS structure* (Γ, Z, Ω) *over a complex manifold* M*, which are either uncoupled or satisfy suitable conditions; and*

(2) *Frobenius bundle structures* K *on the trivial bundle over* M *with fibre the group algebra* $\mathbb{C}[\Gamma]$*, with values in formal power series, such that the Higgs field* C *equals* $-dZ$ *and the flat connection* ∇^r *is given by the adjoint action of the holomorphic generating function* $f(Z)$*.*

The correspondence is not canonical but depends on a suitable choice of a basis for Γ. Note that the bundle K is infinite-dimensional, generated by the global sections x_α, $\alpha \in \Gamma$ corresponding to the generators of the group algebra. For all finite subsets $\Delta = \{\alpha_i\} \subset \Gamma$ there is a finite-dimensional subbundle $K_\Delta \subset K$ spanned by $\{x_{\alpha_i}\}$, and the metric g gives a canonical projection $K \to K_\Delta$. Our first result in this paper characterises uncoupled variations of BPS structure in terms of these finite-dimensional subbundles.

Theorem 3. *Let* (Γ, Z, Ω) *be a framed variation of BPS structure over a complex manifold as in Proposition 2,* K *the corresponding Frobenius bundle. The following are equivalent.*

(1) *The BPS structures in* (Γ, Z, Ω) *are uncoupled.*

(2) *For all* Δ *the canonical projection* $K \to K_\Delta$ *induces a Frobenius bundle structure on the finite-dimensional subbundle* $K_\Delta \subset K$*.*

Remark 4. We will see that in the uncoupled case the Frobenius bundles K, K_Δ actually fit in 1-parameter families $K_\hbar$, $K_{\Delta,\hbar}$ induced by rescaling the form

$$\langle -, - \rangle \mapsto i\hbar \langle -, - \rangle \tag{1.1}$$

for $\hbar \in \mathbb{R}_{>0}$. This is a special case of a more general construction, which extends to the coupled case, see Remark 36.

Fix an uncoupled variation of BPS structure (Γ, Z, Ω) as above. The simplest nontrivial example of a Frobenius subbundle $K_\Delta \subset K$ has rank 2 and is obtained by choosing $\Delta = \{m\gamma + m\beta, m\beta\}$ where γ is an active class, $\langle \gamma, \beta \rangle \neq 0$ and $m > 0$. We take into account the extra parameter $\hbar$ of the rescaling (1.1) and call this Frobenius bundle $K_{\Delta,\hbar}$ the *simple oscillator spanned by* γ*,* β *with frequency* m. We will see that this Frobenius bundle is determined by classical objects, namely $\mathrm{GL}(2, \mathbb{C})$ fundamental solutions $Y^{(m)}_\hbar(t)$ to the system of complex linear differential equations

$$\frac{\partial}{\partial t} Y^{(m)}_\hbar(t) = (-t^{-2} U^{(m)} + t^{-1} V^{(m)}_\hbar) Y^{(m)}_\hbar(t) \tag{1.2}$$

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where

$$U^{(m)} = \begin{pmatrix} mZ(\gamma+\beta) & 0 \\ 0 & mZ(\beta) \end{pmatrix},$$

$$V^{(m)}_{\hbar} = \frac{\langle\gamma,\beta\rangle\hbar}{2\pi}\Omega(\gamma)\begin{pmatrix} 0 & (-1)^{m\langle\gamma,\beta\rangle} \\ -(-1)^{m\langle\gamma,\beta\rangle} & 0 \end{pmatrix}.$$

Turning the system into a single ODE in a standard way shows that $K_{\Delta,\hbar}$ is given by fundamental solutions to the confluent hypergeometric differential equation

$$u''(z) + \left(\frac{1}{z} - z_1 - z_2\right)u'(z) + \left(\frac{\mu^2}{z^2} - \frac{z_1}{z} + z_1 z_2\right)u(z) = 0 \tag{1.3}$$

with the choice of parameters

$$z = t^{-1}, \quad z_1 = mZ(\gamma+\beta), \quad z_2 = mZ(\beta), \quad \mu = -(-1)^{m\langle\gamma,\beta\rangle}\frac{\langle\gamma,\beta\rangle\hbar}{2\pi}\Omega(\gamma).$$

Remark 5. By a slight abuse of notation we will also refer to the standard normalisation $\Psi^{(m)}_{\hbar}(t)$ of the $\mathrm{GL}(2,\mathbb{C})$ fundamental solution $Y^{(m)}_{\hbar}(t)$ (determined by the asymptotic condition $\Psi^{(m)}_{\hbar}(t) \to I$ for $t \to 0$) as a simple oscillator. We will show that $\Psi^{(m)}_{\hbar}(t) = I + O(\hbar)$ and $\log \Psi^{(m)}_{\hbar}(t) \in M_2(\mathbb{C})$ is off-diagonal modulo $\hbar^2$ for all t.

In view of Theorem 3 it seems natural to ask if the solution of the infinite-dimensional RH problem $\Psi_{\mathbb{H}_\ell,\beta}(t)$ of Theorem 1 can be recovered in a large rank limit, i.e. as the limiting behaviour along an infinite increasing sequence of Frobenius subbundles $K_\Delta \subset K$. One of our main results confirms this expectation.

Theorem 6. *Let (Γ, Z, Ω) be a framed variation of uncoupled BPS structure. Fix a basis $\{\beta_j\}$ for Γ and let $\{\gamma_i\}$ be any finite collection of active classes. Let $\hat{\xi}$ denote the vector $(1,1)^{\mathrm{T}} \in \mathbb{C}^2$ and Π be the linear function on $\mathbb{C}^2$ given by $\Pi(w_1, w_2)^{\mathrm{T}} = w_1 + w_2$.*

(1) *For all $N > 0$, the Frobenius bundle K attached to (Γ, Z, Ω) contains a canonical, finite-dimensional Frobenius subbundle isomorphic to the direct sum of all the simple oscillators spanned by γ_i, β_j with frequency $m = 1, \ldots, N$.*

(2) *Suppose now (Γ, Z, Ω) is finite and $\{\gamma_i\}$ is a maximal set of active classes such that all the $Z(\gamma_i)$ lie in a half-plane $\mathbb{H}_\ell$. Let $\Psi^{(m),ij}_{\hbar}$ denote the simple oscillator spanned by γ_i, β_j with frequency m. Then we have an expansion*

$$\exp\left(\frac{1}{\hbar}\sum_{m=1}^{\infty}\sum_{i|\langle\gamma_i,\beta_j\rangle\neq 0}\frac{(-1)^{m\langle\gamma_i,\beta_j\rangle}}{m}\Pi\log\Psi^{(m),ij}_{\hbar}((2\pi)^{-1}\sqrt{-1}t)\hat{\xi}\right)$$

$$= \prod_i \Lambda\left(\frac{Z(\gamma_i)}{t}\right)^{\Omega(\gamma)\langle\beta_j,\gamma_i\rangle} + O(\hbar)$$

for all $t \in \mathbb{C}^*$ such that $\Re(Z(\gamma_i)/t) > 0$ for all i. Integrality is not required. If (Γ, Z, Ω) is also integral the latter product equals the function $\Psi_{\mathbb{H}_\ell,\beta_j}(t)$ appearing in Theorem 1.

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Thus the solution to the infinite-dimensional, birational RH problem attached to (Γ, Z, Ω), with asymptotics prescribed by $\xi \in \mathbb{T}$, turns out to be the leading order term in the $\hbar \to 0$, $N \to \infty$ limit of a sum of simple oscillators, at least in a nonempty open sector of $\mathbb{H}_\ell$.

Remark 7. Evaluating at $\hat{\xi}$ (more precisely at $\bigoplus_{i,m} \hat{\xi}$) is the finite-dimensional analogue of evaluating at a special point $\xi \in \mathbb{T}$ as in Theorem 1. Similarly the linear functional $\bigoplus_{i,m} \frac{(-1)^{m\langle\gamma_i,\beta_j\rangle}}{m}\Pi$ is the finite-dimensional analogue of the torus character projecting along the β_j component as in Theorem 1. In terms of matrix entries we have

$$\Pi \log \Psi^{(m),ij}_{\hbar}\left((2\pi)^{-1}\sqrt{-1}t\right)\hat{\xi} = \log \Psi^{(m),ij}_{\hbar}\left((2\pi)^{-1}\sqrt{-1}t\right)_{(12)} + O(\hbar^2)$$

where for a matrix A we write $A_{(kl)} = A_{kl} + A_{lk}$. We will see that in fact there is an explicit formula

$$\begin{aligned}\Pi \log \Psi^{(m),ij}_{\hbar}(t)\hat{\xi} &= -(-1)^{m\langle\gamma_i,\beta_j\rangle} m\langle\gamma_i, \beta_j\rangle\hbar\Omega(\gamma_i)\\ &\quad\times\frac{1}{\pi}\int_0^\infty \arctan\left(\left(\frac{Z(\gamma_i)}{t}\right)^{-1} s\right) e^{-ms}\, ds + O(\hbar^2).\end{aligned}$$

Remark 8. Both Theorem 1 and the proof of Theorem 6 are very much inspired by a calculation of Gaiotto (see [14, § 3.1]). We note that the idea of looking at large rank, weak coupling limits of the form $\hbar \to 0$, $N \to \infty$ is familiar from the 'large N limit' in the theory of matrix models, with the standard notation $g_s = 1/N$, $N \to \infty$ (see e.g. [26, Ch. I, § 1.1]). It seems interesting to ask if the higher order terms in the $\hbar$ expansion of Theorem 6(2) also have a natural interpretation.

We consider now the case when (Γ, Z, Ω) is a *miniversal* variation of finite, integral BPS structure. This means that fixing a basis $\{\beta_j\}$ one can use the central charges $Z(\beta_j)$ as local coordinates on the base. If $v^j(t, Z)$ is a vector function of t, Z with vector index j, we follow [5, § 3.4] and define a *tau function* τ_v for v as a solution to

$$\frac{\partial}{\partial t}\log v^j = \sum_p \langle\beta_j, \beta_p\rangle \frac{\partial}{\partial Z(\beta_p)}\log \tau_v, \tag{1.4}$$

for all j, which is invariant under a common rescaling of t and all $Z(\beta_j)$. Define a multi-valued meromorphic function on $\mathbb{C}^*$ by

$$\Upsilon(w) = \frac{-\zeta'(-1)e^{\frac{3}{4}w^2}G(w+1)}{(2\pi)^{w/2}w^{w^2/2}},$$

where $G(w)$ is the Barnes G-function (see [28, p. 264]).

Theorem 9 (Bridgeland [5, Theorem 3.4]). *Let* (Γ, Z, Ω) *be a miniversal variation of finite, integral, uncoupled BPS structure. Then the vector function* $\Psi_{\mathbb{H}_\ell,\beta_j}$ *(vector index* j*) admits the tau function*

$$\tau_\ell(t, Z) = \prod_{\gamma|\Omega(\gamma)\neq 0, Z(\gamma)\in\mathbb{H}_\ell} \Upsilon\left(\frac{Z(\gamma)}{t}\right)^{\Omega(\gamma)}, \quad t \in \mathbb{H}_\ell. \tag{1.5}$$

The tau function $\tau_\ell(t, Z)$ plays an important role because it can be related more directly to Gromov–Witten partition functions, as we explain below. We can prove an analogue of Theorem 6 for tau functions. Write $\{\gamma_i\}$ for the active classes as above. Introduce the scalar functions

$$\log \tau_\hbar^{(m),i}(\sqrt{-1}t) = \frac{\Omega(\gamma_i)}{2\pi}\hbar \int_0^\infty s \log\left(s^2 + \left(\frac{Z(\gamma_i)}{t}\right)^2\right) e^{-ms}\, ds \tag{1.6}$$

(compare to the explicit formula in Remark 7).

Theorem 10. *Let (Γ, Z, Ω) be a miniversal variation of finite, uncoupled BPS structure. Let notation and assumptions be as in Theorem 6.*

(1) *The vector function (vector index j)*

$$\exp\left(\frac{1}{\hbar}\sum_{m=1}^{\infty}\sum_{i|\langle\gamma_i,\beta_j\rangle\neq 0}\frac{(-1)^{m\langle\gamma_i,\beta_j\rangle}}{m}\Pi \log \Psi_\hbar^{(m),ij}((2\pi)^{-1}\sqrt{-1}t)\hat{\xi}\right)$$

admits the tau function

$$\exp\left(\frac{1}{\hbar}\sum_{m=1}^{\infty}\sum_{i|\Omega(\gamma_i)\neq 0}\log \tau_\hbar^{(m),i}((2\pi)^{-1}\sqrt{-1}t)\right)$$

modulo $\hbar$, i.e. this solves (1.4) up to $O(\hbar)$.

(2) *In the integral case the latter function equals the tau function $\tau_\ell(t, Z)$ given by (1.5).*

By Theorem 6(2) this implies Theorem 9, i.e. the tau function $\tau_\ell(t, Z)$ of the infinite-dimensional, birational RH problem attached to (Γ, Z, Ω) is the tau function for the leading order term in the $\hbar \to 0$, $N \to \infty$ expansion of a sum of simple oscillators (at least in a nonempty, open sector).

Let us return to the geometric case of a Calabi–Yau threefold X. Theorem 10 can be used in conjunction with results from [5] to show that a certain (Gopakumar–Vafa) contribution to the Gromov–Witten partition function of X can be expressed in terms of solutions to the confluent hypergeometric equation (1.3), i.e. in terms of a sum of simple oscillators.

To explain this we recall that Bridgeland [5, § 6] constructs a miniversal variation of uncoupled BPS structure where $\Gamma = H_{2*}(X, \mathbb{Z})$ (modulo torsion), $\langle -, - \rangle$ is the intersection pairing, and $\Omega(\alpha)$ vanishes except when $\alpha = (n, \beta, 0, 0)$, when it is the BPS invariant enumerating coherent sheaves on X supported in dimension $\leqslant 1$ and with Chern character dual to α (see [22, § 6]). Central charges of active classes are specified by $Z(n, \beta, 0, 0) = \int_\beta \omega_{\mathbb{C}} - n$, $\omega_{\mathbb{C}}$ denoting a complexified Kähler class. Note that these BPS structures are not finite. Their formal tau function is given by the right hand side of (1.5), regarded as a formal infinite product.

Proposition 11 (Bridgeland–Iwaki [5, § 6.3]). *Consider the positive degree, genus* 0 *Gopakumar–Vafa contribution to the Gromov–Witten partition function of X, given*

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explicitly by

$$\chi(X)\sum_{g\geqslant 2}\frac{(-1)^{g-1}B_{2g}B_{2g-2}}{4g(2g-2)(2g-2)!}\lambda^{2g-2}+\sum_{g\geqslant 2}\sum_{\beta\in H_2(X,\mathbb{Z})}\mathrm{GV}(0,\beta)\frac{(-1)^{g-1}B_{2g}}{2g(2g-2)!}\,\mathrm{Li}_{3-2g}(x^{\beta})\lambda^{2g-2}. \tag{1.7}$$

Assuming the conjectural relation $\Omega(n,\beta,0,0)=\mathrm{GV}(0,\beta)$ *for all* n*, with* β *a positive curve class (see [22, Conjecture 6.20]), the change of variables*

$$\lambda=2\pi t,\quad x^{\beta}=\exp(2\pi i v_{\beta}),\quad v_{\beta}=\int_{\beta}\omega_{\mathbb{C}} \tag{1.8}$$

gives the logarithm of the formal tau function for sheaves on X *supported in dimension* $\leqslant 1$*, i.e. the logarithm of the right hand side of (1.5) regarded as a formal infinite product.*

The following result thus follows immediately from Theorem 10.

Corollary 12. *Assume the conjectural relation* $\Omega(n,\beta,0,0)=\mathrm{GV}(0,\beta)$ *as above. Then, after the change of variables (1.8), the positive degree, genus* 0 *Gopakumar–Vafa contribution to the Gromov–Witten partition function of* X *(1.7) can be written as a sum of simple oscillator tau functions*

$$\frac{1}{\hbar}\sum_{m=1}^{\infty}\sum_{\beta,n}\log\tau_{\hbar}^{(m),(n,\beta,0,0)}((2\pi)^{-1}\sqrt{-1}t)$$

regarded as a formal power series in t*,* v_{β}*, where* $\log\tau_{\hbar}^{(m),(n,\beta,0,0)}$ *is given by setting* $\gamma_i=(n,\beta,0,0)$ *in the right hand side of (1.6).*

Remark 13. Bridgeland [6] has shown how to extend Theorems 1 and 9 to the variation of BPS structure of sheaves on X with dimension $\leqslant 1$ when X is the resolved conifold. The corresponding tau function turns out to be another classical special (double sine) function. We expect that this function can be recovered from sums of simple oscillators as in Theorem 10.

Plan of the paper

Section 2 contains the required background on BPS structures, their variations, and the associated Frobenius bundles. Sections 3–5 discuss and prove Theorem 6 for the special case of rank 2 BPS structures, i.e. when $\mathrm{rk}(\Gamma)=2$. Section 6 completes the proof for arbitrary rank of Γ. Given the results of the previous sections this is mostly a matter of notation. Section 7 proves Theorem 10.

2. BPS structures and Frobenius bundles

In this section we introduce BPS structures, their variations, and the corresponding Frobenius bundles. Since many references for this material are already available we will be quite brief.

Remark 14. Definitions 21, 26 and the wall-crossing identity (2.1) below are only given for the sake of motivation, in incomplete form. They are never used in the present paper. However we will point out the main difficulties involved and give references which contain a fully rigorous treatment.

Definition 15 ([5, § 2.1], [23, § 2]). A *BPS structure* comprises a finite rank lattice Γ (*charge lattice*), endowed with a skew-symmetric integral bilinear form $\langle -, - \rangle$ (*intersection form*), an element $Z \in \mathrm{Hom}(\Gamma, \mathbb{C})$ (*central charge*) and a map of sets $\Omega\colon \Gamma \to \mathbb{Q}$ (*BPS spectrum*), with constraints given by $\Omega(\alpha) = \Omega(-\alpha)$ (*symmetry*) and the property that there is a fixed $C > 0$ such that $\Omega(\gamma) \neq 0$ implies

$$|Z(\gamma)| > C\|\gamma\|$$

for some fixed choice of norm on $\Gamma \otimes \mathbb{R}$ (*support property*). The *rank* of a BPS structure is the rank of Γ. We say that a BPS structure is *integral* if Ω takes values in $\mathbb{Z}$.

Note that the required symmetry models the shift functor [1] acting on $D^b(X)$.

Definition 16 ([5, § 2.2], [23, § 2.5]). Let (Γ, Z, Ω) be a BPS structure. The corresponding *DT spectrum* is the map of sets $\mathrm{DT}\colon \Gamma \to \mathbb{Q}$ defined by

$$\mathrm{DT}(\alpha) = \sum_{k>0 | k^{-1}\alpha \in \Gamma} \frac{\Omega(\alpha/k)}{k^2}.$$

The maps Ω, DT are equivalent data (by virtue of the Möbius inversion formula).

Definition 17 [5, § 2.1]. An element $\gamma \in \Gamma$ is called an *active class* if $\Omega(\gamma) \neq 0$. An *active ray* $\ell \subset \mathbb{C}^*$ is a ray of the form $\mathbb{R}_{>0} Z(\gamma)$ where γ is an active class. We say ℓ is *generic* if it is not active. A BPS structure is *finite* if there are finitely many active classes.

The following definition is central to this paper.

Definition 18 ([5, Definition 2.3], [15, § 4]). We say that a BPS structure (Γ, Z, Ω) is *uncoupled* if we have $\langle \gamma_i, \gamma_j \rangle = 0$ for all active classes γ_i.

To formulate the correct notion of a variation we need some further ingredients.

Definition 19. In this paper we always denote by $\mathbb{C}[\Gamma]$ the group algebra of Γ endowed with the twist of the usual associative, commutative product by the form $\langle -, - \rangle$,

$$x_\alpha x_\beta = (-1)^{\langle \alpha, \beta \rangle} x_{\alpha+\beta}.$$

The *torus of twisted characters* is the affine algebraic torus

$$\mathbb{T} = \mathrm{Spec}\, \mathbb{C}[\Gamma].$$

We write $\mathbb{T}_+$ for the usual affine algebraic torus $\mathrm{Spec}\, \mathbb{C}[\Gamma]_*$, where $\mathbb{C}[\Gamma]_*$ denotes the usual group algebra with untwisted commutative product. Then $\mathbb{T}$ is a torsor for $\mathbb{T}_+$ (see [5, § 2.4], [23, § 2.5]).

Note that one can think of $x_\alpha \in \mathbb{C}[\Gamma]$ as a map $\mathbb{T} \to \mathbb{C}^*$ (a twisted character), and similarly of $y_\alpha \in \mathbb{C}[\Gamma]_*$ as a usual character $\mathbb{T}_+ \to \mathbb{C}^*$.

Lemma 20 ([5, § 2.4], [23, § 2.5]). *The pairing*

$$[x_\alpha, x_\beta] = (-1)^{\langle \alpha, \beta \rangle} \langle \alpha, \beta \rangle x_{\alpha+\beta}$$

defines a Poisson bracket on the commutative algebra $\mathbb{C}[\Gamma]$.

Proof. This is a straightforward computation. □

Definition 21 ([5, § 2.5], [23, § 2.5]). Given a ray ℓ we define

$$\mathrm{DT}(\ell) = \sum_{\gamma \in \Gamma | Z(\gamma) \in \ell} \mathrm{DT}(\gamma) x_\gamma.$$

The *BPS automorphism* attached to an active ray ℓ is

$$S(\ell) = \exp([\mathrm{DT}(\ell), -]) \in \mathrm{Aut}(\mathbb{C}[\Gamma]).$$

Remark 22. The sum defining $\mathrm{DT}(\ell)$ is either empty or infinite, and the vector field $[\mathrm{DT}(\ell), -]$ may be ill-defined. It turns out that one can always make sense of $S(\ell)$ as a formal automorphism, and when the BPS structure is finite and integral $S(\ell)$ is in fact an element of $\mathrm{Bir}(\mathbb{T})$, the group of birational automorphism of $\mathbb{T}$ (see [5, § 2.7], [23, § 2.5]).

Definition 23 ([5, § 3.3], [23, § 2.3]). A *variation of BPS structure* is a family of BPS structures $(\Gamma_p, \Omega_p, Z_p)$ as above, parametrised by points p of a complex manifold M, where Γ_p fit together in a local system, Z_p are holomorphic sections of $\mathrm{Hom}(\Gamma_p, \mathbb{C})$ (the central charges), and $\Omega(\alpha_p, Z_p)$ satisfy the *JS/KS wall-crossing formula*. This means that the product

$$\prod_{\ell \subset V} S_p(\ell) \in \mathrm{Aut}(\mathbb{T}_p) \tag{2.1}$$

is locally constant, where $\mathbb{T}_p$ is the local system of algebraic affine tori $\mathrm{Spec}(\mathbb{C}[\Gamma_p])$, $V \subset \mathbb{C}^*$ is the interior of a convex sector, and $\prod_{\ell \subset V}$ is computed writing the ensuing automorphisms from left to right according to the clockwise ordering of rays ℓ. (The crucial point is that, in any fixed local trivialisation of the local system, the relative order of active rays depends on $p \in M$). A variation is called *framed* if the local system Γ_p is trivial. A framed variation is called *miniversal* if fixing a basis β_j of Γ induces local coordinates $Z(\beta_j)$ on M.

Remark 24. In general one regards (2.1) as a formal automorphism and only imposes local constancy modulo a sequence of powers of a maximal ideal (see [5, Appendix A], [23, § 2]). When the BPS structures are finite and integral this is not necessary and one simply requires that (2.1) is a locally constant section of $\mathrm{Bir}(\mathbb{T}_p)$. When the BPS structures are uncoupled the condition that (2.1) is locally constant always holds automatically, since $S_p(\ell)$ commute (this is clear from Definition 21).

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We briefly introduce RH problems, a classical topic in complex analysis and mathematical physics, appearing quite naturally in the context of BPS structures.

Definition 25 [13, Ch. II, § 1]. Let G be a Lie group acting holomorphically on a complex manifold X, $\Sigma \subset \mathbb{C}^*$ the support of an oriented path, $J\colon \Sigma \to G$ a map. An RH problem with values in X defined by J consists of finding a map $\Phi(t)\colon \mathbb{C}^* \setminus \Sigma \to X$ with the following properties:

(1) Φ is analytic in $\mathbb{C}^* \setminus \Sigma$;

(2) the limits $\Phi_-(t)$ of Φ from the minus side of Σ and the limit $\Phi_+(t)$ from the plus side of Σ exist for all $t \in \Sigma$ and are related by

$$\Phi_+(t) = J(t) \cdot \Phi_-(t);$$

(3) $\Phi(t)$ has prescribed asymptotic behaviour as $t \to 0$.

A BPS structure (Γ, Z, Ω) induces in a very natural way various RH problems, with values in $\mathbb{T}$ and in $\mathbf{Aut}(\mathbb{T})$.

Definition 26 ([5, § 3.1], [15, § 5.1], [12, § 3.2]). The *RH problem of a BPS structure* (Γ, Z, Ω) *with values in* $\mathrm{Aut}(\mathbb{T})$ is obtained with the choices $\Sigma = \bigcup_{\gamma|\Omega(\gamma)\neq 0} \ell_\gamma$ and $J|_\ell = S_\ell$ for all rays $\ell \subset \Sigma$. The $t \to 0$ asymptotics imposed on a solution $\tilde{\Phi}(t)$ are $e^{Z/t}\tilde{\Phi}(t) \to I$, where Z is regarded naturally as a vector field on $\mathbb{T}$ and $I \in \mathrm{Aut}(\mathbb{T})$ is the identity. We define the corresponding *RH problem with values in* $\mathbb{T}$ *and* $t \to 0$ *asymptotics* ξ by using the natural action of $\mathbf{Aut}(\mathbb{T})$ on $\mathbb{T}$ and evaluating $\tilde{\Phi}$ at a point $\xi \in \mathbb{T}$. The $t \to 0$ asymptotics imposed on a solution $\Phi(t)$ are then $\Phi(t) = (e^{Z/t}\tilde{\Phi}(t))(\xi) \to \xi$.

Remark 27. The main difficulty with this general definition is that $\Sigma \subset \mathbb{C}^*$ might be dense. This does not happen in the finite integral case of course, and in that case J takes values in $\mathbf{Bir}(\mathbb{T})$. In that case one needs to make sure that ξ does not lie in the indeterminacy locus.

Composing with twisted characters we define the components

$$\Phi_\alpha(t) = x_\alpha \circ \Phi.$$

Definition 28 [5, Problem 3.1]. The *birational RH problem* of a finite, integral BPS structure (Γ, Z, Ω) (as in Theorem 1) with $t \to 0$ asymptotics $\xi \in \mathbb{T}$ is the RH problem in the sense of Definition 26, with values in $\mathbb{T}$ and where J takes values in $\mathbf{Bir}(\mathbb{T})$, with the additional constraint that for some $k > 0$ we have for all $\alpha \in \Gamma$

$$|t|^{-k} < |\Phi_\alpha(t)| < |t^k|, \quad |t| \gg 0.$$

Definition 29 [5, Equation (12)]. Suppose Φ is a solution to the birational RH problem (Γ, Z, Ω). We define a map $\Psi\colon \mathbb{C}^* \setminus \Sigma \to \mathbb{T}_+$ (as in Theorem 1), using the simply transitive action of $\mathbb{T}_+$ on $\mathbb{T}$, by

$$e^{Z/t}\Phi = \Psi \cdot \xi.$$

We write $\Psi_\alpha = y_\alpha \circ \Psi$ for its components. Clearly Φ and Ψ are equivalent data, and we still call Ψ a solution to the birational RH problem.

Remark 30. The functions $\Psi_{\ell,\beta}$ appearing in Theorem 1 denote the unique analytic continuation to the half-plane $\mathbb{H}_\ell$ of the restriction of Ψ_β to a sector between active rays containing the generic ray ℓ.

Next we turn to Frobenius bundles, modelled on Dubrovin's Frobenius manifolds [10]. Let M be a complex manifold.

Definition 31 [19, Definition 5.6]. A *Frobenius bundle* is a holomorphic vector bundle $K \to M$ endowed with data $(\nabla^r, C, \mathcal{U}, \mathcal{V}, g)$, in the holomorphic category, with values in the bundle K, where

- ∇^r is a flat connection,
- C is a Higgs field, that is a 1-form with values in endomorphisms, with $C \wedge C = 0$,
- $\mathcal{U}, \mathcal{V}$ are endomorphisms,
- g is a nondegenerate symmetric bilinear form (called the holomorphic metric, although it is not positive definite),

satisfying the conditions

$$\begin{aligned} \nabla^r(C) &= 0,\\ [C, \mathcal{U}] &= 0,\\ \nabla^r(\mathcal{V}) &= 0,\\ \nabla^r(\mathcal{U}) - [C, \mathcal{V}] + C &= 0 \end{aligned} \tag{2.2}$$

and the conditions on the metric g

$$\begin{aligned} \nabla^r(g) &= 0,\\ g(C_X a, b) &= g(a, C_X b),\\ g(\mathcal{U}a, b) &= g(a, \mathcal{U}b),\\ g(\mathcal{V}a, b) &= -g(a, \mathcal{V}b). \end{aligned} \tag{2.3}$$

Remark 32. The conditions (2.2) are in fact equivalent to the flatness of a suitable meromorphic connection on the pullback of K to the product $M \times \mathbb{P}^1$, as discussed in [19, § 5.2].

Remark 33. For our purposes we will need to work over a ring of formal power series. In other words we will consider a situation in which all the objects involved in Definition 31 are in fact formal power series in some auxiliary variables, and the conditions (2.2), (2.3) are satisfied in the sense of formal power series with respect to these variables. This extension is straightforward, and it is commonplace in the theory of Frobenius manifolds.

It turns out that, under some conditions on the family, variations of BPS structure are equivalent to certain Frobenius bundles. This construction uses the holomorphic generating function for DT invariants introduced by Joyce [21]. We consider the class of framed variations of BPS structure (Γ, Z, Ω), over a complex manifold M, satisfying the following conditions:

(C1) There exists a fixed basis $\{\beta_i\}$ for Γ such that Ω is always supported in the double cone $\mathbb{Z}_{\geqslant 0}\{\beta_i\} \cup \mathbb{Z}_{\leqslant 0}\{\beta_i\}$.

(C2) For some fixed ray $\ell \subset \mathbb{C}^*$ we have $Z(\{\beta_i\}) \subset \mathbb{H}_\ell$ along the variation. (One may assume $\ell = \mathbb{R}_{>0}$, $\mathbb{H}_\ell = \mathbb{H}$ without loss of generality.)

Conditions C1, C2 are quite restrictive, but they are satisfied in several concrete examples, discussed in detail in [1, 2]. Given a variation satisfying C1, C2, with a fixed basis $\{\beta_i\}$, we introduce a vector of formal parameters $\mathbf{s}$, with components s_i, corresponding to the basis elements β_i. Writing $\alpha \in \Gamma$ as $\alpha = \sum_j a_j \beta_j$ we set $\mathbf{s}^{|\alpha|} = \prod_j s_j^{|\alpha_j|}$. Moreover we define combinatorial coefficients $c(\alpha_1, \dots, \alpha_k) \in \mathbb{Q}$, given by a sum over connected trees T with vertices labelled by $\{1, \dots, k\}$, endowed with an orientation compatible with the labelling (i.e. such that $i \to j$ implies $i < j$),

$$c(\alpha_1, \dots, \alpha_k) = \sum_T \frac{1}{2^{k-1}} \prod_{\{i \to j\} \subset T} (-1)^{\langle \alpha_i, \alpha_j \rangle} \langle \alpha_i, \alpha_j \rangle.$$

Theorem 34 (Joyce [21, Theorem 3.7], [1, Proposition 3.17]). *Suppose* (Γ, Z, Ω) *is a framed variation of BPS structure over a complex manifold* M*, satisfying the conditions* C1, C2*, with a fixed basis* $\{\beta_i\}$*. Then there exist unique multi-valued holomorphic functions* $J_k \colon (\mathbb{C}^*)^k \to \mathbb{C}^*$*, satisfying* $J_1 \equiv \frac{1}{2\pi i}$ *and suitable growth conditions (see [21, § 3]), such that*

$$f^\alpha(Z) = \sum_{\alpha_1 + \dots + \alpha_k = \alpha, Z(\alpha_i) \neq 0} c(\alpha_1, \dots, \alpha_k) J_k(Z(\alpha_1), \dots, Z(\alpha_k)) \prod_i \mathbf{s}^{|\alpha_i|} \operatorname{DT}(\alpha_i, Z)$$

is a well-defined formal power series in $\mathbf{s}$*, whose coefficients are holomorphic functions of* Z*. When* (Γ, Z, Ω) *is uncoupled the result holds without assuming the conditions* C1, C2*.*

In the uncoupled case the result follows at once from the explicit formula for $f^\alpha(Z)$ we prove in Lemma 60.

We define the corresponding Joyce holomorphic generating function as the well-defined formal power series in $\mathbf{s}$ with coefficients in $\mathbb{C}[\Gamma]$ given by

$$f(Z) = \sum_{\alpha \neq 0} f^\alpha(Z) x_\alpha.$$

Proposition 35 [1, Proposition 3.17]. *Let* (Γ, Z, Ω) *be a framed variation of BPS structure as in Theorem 34. Let* $K \to M$ *be the trivial infinite-dimensional bundle with fibre* $\mathbb{C}[\Gamma]$*. Then the choices*

$$\nabla^r = d + \sum_{\alpha \neq 0} [f^\alpha(Z) x_\alpha, -] \frac{dZ(\alpha)}{Z(\alpha)},$$

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$$\begin{aligned} C &= -dZ, \\ \mathcal{U} &= Z, \quad \mathcal{V} = [f(Z), -], \\ g(x_\alpha, x_\beta) &= \delta_{\alpha\beta} \end{aligned}$$

satisfy the Frobenius bundle conditions (2.2), (2.3) in the sense of formal power series in the variables $\mathbf{s}$.

Remark 36. We may deform the Poisson bracket on $\mathbb{C}[\Gamma]$ by

$$[x_\alpha, x_\beta]_\hbar = (i\hbar)[x_\alpha, x_\beta] = (-1)^{\langle\alpha,\beta\rangle}(i\hbar)\langle\alpha, \beta\rangle x_{\alpha+\beta},$$

and the combinatorial coefficients by

$$c_\hbar(\alpha_1, \dots, \alpha_k) = \sum_T \frac{1}{2^{k-1}} \prod_{\{i\to j\}\subset T} (-1)^{\langle\alpha_i,\alpha_j\rangle}(i\hbar\langle\alpha_i, \alpha_j\rangle).$$

Under the assumptions of Theorem 34 it is possible to find a lift $\mathrm{DT}_\hbar \colon \Gamma \to \mathbb{Q}[\hbar]$ (with $\mathrm{DT} = \mathrm{DT}_\hbar\,|_{\hbar=1}$) such that

$$f^\alpha_\hbar(Z) = \sum_{\alpha_1+\cdots+\alpha_k=\alpha, Z(\alpha_i)\neq 0} c_\hbar(\alpha_1, \dots, \alpha_k) J_k(Z(\alpha_1), \dots, Z(\alpha_k)) \prod_i \mathbf{s}^{|\alpha_i|}\,\mathrm{DT}_\hbar(\alpha_i, Z)$$

is a well-defined formal power series in $\mathbf{s}$, $\hbar$ whose coefficients are holomorphic functions of Z. This may be proved as in [1, Proposition 3.17]. The lift is not canonical but depends on the choice of an initial point Z_0. The 1-parameter family $K_\hbar$ of (1.1) is then given by the deformations

$$\begin{aligned} \nabla^r_\hbar &= d + \sum_{\alpha\neq 0} [f^\alpha_\hbar(Z)x_\alpha, -]_\hbar \frac{dZ(\alpha)}{Z(\alpha)}, \\ \mathcal{V} &= [f_\hbar(Z), -]_\hbar, \end{aligned}$$

where $f_\hbar(Z) = \sum_{\alpha\neq 0} f^\alpha_\hbar(Z) x_\alpha$.

Clearly in the uncoupled case we have $f^\alpha_\hbar(Z) = f^\alpha(Z)$ so only the Poisson bracket is deformed as above.

An advantage of working with Frobenius bundles is that the holomorphic data $(\nabla^r, C, \mathcal{U}, \mathcal{V})$ can be canonically projected to a subbundle $K' \subset K$ using the metric g. This seems especially useful if K' is finite-dimensional. However in general the resulting bundle is no longer Frobenius, i.e. the connection ∇^r is not flat. This construction is studied in detail in [2]. We will see that requiring flatness for all such projections in fact characterises uncoupled BPS structures.

3. A_1 Frobenius bundles

In this section we study a general uncoupled variation of BPS structure $(\Gamma, \mathbb{Z}, \Omega)$ of rank 2, i.e. with $\mathrm{rk}(\Gamma) = 2$.

In order to make contact with the material of [5, § 5.1] we write the charge lattice Γ as $\mathbb{Z}\gamma \oplus \mathbb{Z}\gamma^\vee$ and refer to the rank 2 uncoupled case as the *(double)* A_1 *case*. However for us the pairing $\langle\gamma, \gamma^{\vee}\rangle$ is arbitrary (while it is fixed to -1 in loc. cit.). The BPS spectrum is constant in $Z \in \mathrm{Hom}(\Gamma, \mathbb{C})$ and vanishes except for $\Omega(\pm\gamma) = \Omega$. It follows that the DT spectrum vanishes except for

$$\mathrm{DT}(\pm k\gamma) = \frac{\Omega}{k^2}.$$

Let $f(Z)$ denote the Joyce holomorphic generating function. In general, as we explained in the previous section, this is a Laurent series $f(Z) = \sum_{\alpha \neq 0} f^{\alpha}(Z) x_{\alpha}$ with coefficients in $\mathbb{C}[\![\mathbf{s}]\!]$ (and in fact an element of $\mathbb{C}[\Gamma][\![\mathbf{s}]\!]$). In the present double A_1 case we have formal parameters

$$s = s_1 = s_{\gamma}, \quad s_2 = s_{\gamma^{\vee}}.$$

Lemma 37. *For the double A_1 we have for $k \in \mathbb{Z} \setminus \{0\}$*

$$f^{k\gamma}(Z) = \frac{1}{2\pi i} \frac{\Omega}{k^2} s^k$$

(a constant, independent of Z) while all the other $f^{\alpha}(Z)$ vanish identically. In particular we have the symmetry $f^{\alpha} = f^{-\alpha}$.

Proof. The formal power series $f^{\alpha}(Z)$ can be written as a sum over trees T with vertices labelled by charges α_i. The contribution of T is weighted by factors of $\prod_i \mathrm{DT}(\alpha_i)$ and $\prod_{i \to j} \langle\alpha_i, \alpha_j\rangle$. In the present double A_1 case the first factor vanishes unless all the vertices of T are labelled by integral multiples of γ. But for such T the second factor vanishes unless there is only a single vertex, labelled by $k\gamma$. The contribution for this T is the constant $\mathrm{DT}(k\gamma) = \frac{\Omega}{k^2}$, multiplied by $J_1 \equiv \frac{1}{2\pi i}$. □

Corollary 38. *For the Frobenius type structure of the double A_1 we have*

$$\nabla^r = d + \sum_{k \neq 0} \frac{\Omega}{2\pi i k^2} s^k [x_{k\gamma}, -] \frac{dZ(k\gamma)}{Z(k\gamma)},$$

$$\mathcal{V} = \sum_{k \neq 0} \frac{\Omega}{2\pi i k^2} s^k [x_{k\gamma}, -].$$

Proof. This follows at once from Lemma 37 and the general formulae for ∇^r, $\mathcal{V}$ of Proposition 35. □

Fix a finite subset $\Delta = \{\alpha_i\} \subset \Gamma$, $i = 1, \ldots, N$.

Definition 39. We denote by $K_{\Delta} \subset K$ the rank N subbundle spanned by $\{x_{\alpha_i}\}$, $i = 1, \ldots, N$. We write $\pi : K \to K_{\Delta}$ for the orthogonal projection with respect to g.

Note that $K_{\Delta} \subset K$ is preserved by the endomorphism $\mathcal{U}$ and Higgs field C.

Lemma 40. *The collection of holomorphic objects $(\pi\nabla^r, C, \mathcal{U}, \pi\mathcal{V}, g)$ is a Frobenius type structure on K_{Δ}.*

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Proof. Let us first check that the connection $\pi\nabla^r$ is flat. Fixing $i = 1, \ldots, N$ we compute

$$\begin{aligned}\pi\nabla^r(x_{\alpha_i}) &= \sum_{\alpha\neq 0}\pi(f^{\alpha}(-1)^{\langle\alpha,\alpha_i\rangle}\langle\alpha,\alpha_i\rangle x_{\alpha+\alpha_i})\, d\log Z(\alpha)\\ &= \sum_{j=1}^{N}(-1)^{\langle\alpha_j,\alpha_i\rangle}\langle\alpha_j,\alpha_i\rangle f^{\alpha_j-\alpha_i}x_{\alpha_j}\, d\log Z(\alpha_j-\alpha_i).\end{aligned}$$

So writing $\pi\nabla^r = d + A$ in the frame x_{α_i} we have

$$A_{ji} = (-1)^{\langle\alpha_j,\alpha_i\rangle}\langle\alpha_j,\alpha_i\rangle f^{\alpha_j-\alpha_i}\, d\log Z(\alpha_j-\alpha_i). \tag{3.1}$$

By Lemma 37 f^{α} is constant in Z, so the curvature 2-form $F(A) = dA + A\wedge A$ of $\pi\nabla^r$ is given by

$$\begin{aligned}F(A)_{ji} &= \sum_{k=1}^{N}(-1)^{\langle\alpha_j,\alpha_k\rangle+\langle\alpha_k,\alpha_i\rangle}\langle\alpha_j,\alpha_k\rangle\langle\alpha_k,\alpha_i\rangle\\ &\quad\times f^{\alpha_j-\alpha_k}f^{\alpha_k-\alpha_i}\, d\log Z(\alpha_j-\alpha_k)\wedge d\log Z(\alpha_k-\alpha_i).\end{aligned} \tag{3.2}$$

By Lemma 37 the product $f^{\alpha_j-\alpha_k}f^{\alpha_k-\alpha_i}$ vanishes unless the classes $\alpha_j-\alpha_k$, $\alpha_k-\alpha_i$ are both multiples of γ. But in that case the 2-form $d\log Z(\alpha_j-\alpha_k)\wedge d\log Z(\alpha_k-\alpha_i)$ vanishes.

Similarly we check that $\pi\mathcal{V}$ is flat with respect to $\pi\nabla^r$. Fixing $i = 1, \ldots, N$ we compute

$$\begin{aligned}\pi\mathcal{V}(x_{\alpha_i}) &= \sum_{\alpha\neq 0}\pi(f^{\alpha}(-1)^{\langle\alpha,\alpha_i\rangle}\langle\alpha,\alpha_i\rangle x_{\alpha+\alpha_i})\\ &= \sum_{j=1}^{N}(-1)^{\langle\alpha_j,\alpha_i\rangle}\langle\alpha_j,\alpha_i\rangle f^{\alpha_j-\alpha_i}x_{\alpha_j}.\end{aligned}$$

So the matrix V representing $\pi\mathcal{V}$ in the frame x_{α_i} is

$$V_{ji} = (-1)^{\langle\alpha_j,\alpha_i\rangle}\langle\alpha_j,\alpha_i\rangle f^{\alpha_j-\alpha_i}. \tag{3.3}$$

In particular, by Lemma 37, V is constant in Z, so in the frame x_{α_i} we have

$$\pi\nabla^r(\pi\mathcal{V}) = [A, V].$$

Using (3.1) and (3.3) we compute

$$\begin{aligned}[A, V]_{kl} &= \sum_{p=1}^{N}(-1)^{\langle\alpha_k,\alpha_p\rangle+\langle\alpha_p,\alpha_l\rangle}\langle\alpha_k,\alpha_p\rangle\langle\alpha_p,\alpha_l\rangle f^{\alpha_k-\alpha_p}f^{\alpha_p-\alpha_l}\\ &\quad\times(d\log Z(\alpha_k-\alpha_p) - d\log Z(\alpha_p-\alpha_l)).\end{aligned}$$

By Lemma 1, the product $f^{\alpha_k-\alpha_p}f^{\alpha_p-\alpha_l}$ vanishes unless $\alpha_k-\alpha_p$, $\alpha_p-\alpha_l$ are both multiples of γ. But in that case we have

$$d\log Z(\alpha_k-\alpha_p) = d\log Z(\alpha_p-\alpha_l) = d\log Z(\gamma).$$

So $[A, V]$ vanishes identically. Checking the other conditions for a Frobenius type structure is straightforward. □

Definition 41. In the following we call the structure $(\pi\nabla^r, C, \mathcal{U}, \pi\mathcal{V}, g)$ evaluated at the natural point $s = 1$ the Frobenius type structure on K_{Δ}.

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To the Frobenius type structure on K_Δ we can associate a family of meromorphic connections on the trivial rank N holomorphic bundle over $\mathbb{P}^1$, parametrised by $Z \in M$. This is given by

$$\nabla(Z) = d + \left(\frac{U(Z)}{t^2} - \frac{V}{t}\right) dt \tag{3.4}$$

where U, V are the $N \times N$ matrices representing $\mathcal{U}$, $\pi\mathcal{V}$ with respect to the frame x_{α_i}. In particular V is a constant skew-symmetric matrix, independent of Z.

Definition 42 [19, Definition 5.6 and Theorem 5.7]. The meromorphic connections $\nabla(Z)$ of the Frobenius type structure K_Δ are the meromorphic connections (3.4), depending on Z.

Lemma 43. *We have*

$$U(Z)_{ij} = Z(\alpha_i)\delta_{ij}, \qquad V_{ij} = (-1)^{\langle\alpha_i,\alpha_j\rangle}\langle\alpha_i,\alpha_j\rangle f^{\alpha_i-\alpha_j}.$$

In particular $U(Z)$ is diagonal and V is skew-symmetric.

Proof. The expression for $U(Z)$ follows at once from $\mathcal{U}(Z) = Z$. The expression for V is (3.3). □

The simplest nontrivial Frobenius bundle K_Δ contained in K has rank $N = 2$ and is given by the following example.

Example 44. Let $\Delta^1 = \{\alpha_1, \alpha_2\} = \{\gamma + \gamma^\vee, \gamma^\vee\}$. Then we have

$$\nabla(Z) = d + \left(\frac{1}{t^2}\begin{pmatrix} Z(\gamma+\gamma^\vee) & 0 \\ 0 & Z(\gamma^\vee)\end{pmatrix} - \frac{1}{t}\frac{\langle\gamma,\gamma^\vee\rangle}{2\pi i}\Omega\begin{pmatrix} 0 & (-1)^{\langle\gamma,\gamma^\vee\rangle} \\ -(-1)^{\langle\gamma,\gamma^\vee\rangle} & 0\end{pmatrix}\right) dt.$$

Remark 45. Note that since $\Omega(\gamma - \gamma^\vee) = 0$ the more obvious choice $\Delta = \{\gamma, \gamma^\vee\}$ yields the essentially trivial connection

$$\nabla(Z) = d + \frac{1}{t^2}\begin{pmatrix} Z(\gamma) & 0 \\ 0 & Z(\gamma^\vee)\end{pmatrix} dt.$$

The previous example can be immediately generalised.

Lemma 46. *For all $k \geqslant 1$ choose $\Delta^k = \{\alpha_1, \ldots, \alpha_{2k}\}$ with $\{\alpha_{2i-1}, \alpha_{2i}\} = \{i(\gamma+\gamma^\vee), i\gamma^\vee\}$. Then the meromorphic connection of the Frobenius type structure on K_{Δ^k} has U, V block diagonal, with blocks*

$$U^{(i)} = \begin{pmatrix} iZ(\gamma+\gamma^\vee) & 0 \\ 0 & iZ(\gamma^\vee)\end{pmatrix},$$

$$V^{(i)} = \frac{\langle\gamma,\gamma^\vee\rangle}{2\pi\sqrt{-1}}\Omega\begin{pmatrix} 0 & (-1)^{i\langle\gamma,\gamma^\vee\rangle} \\ -(-1)^{i\langle\gamma,\gamma^\vee\rangle} & 0\end{pmatrix}$$

for $i = 1, \ldots, k$. The rank of K_{Δ^k} is $N = 2k$.

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Proof. We only need to check that V is block diagonal with blocks $V^{(i)}$ as above, for $i = 1, \dots, k$. According to Lemma 43 for all k, l we have $V_{kl} = (-1)^{\langle \alpha_k, \alpha_l \rangle} \langle \alpha_k, \alpha_l \rangle f^{\alpha_k - \alpha_l}$. We compute

$$\begin{aligned} f^{\alpha_{2i} - \alpha_{2j}} &= f^{(i-j)\gamma^\vee} = 0, \quad f^{\alpha_{2i-1} - \alpha_{2j}} = f^{i\gamma + (i-j)\gamma^\vee} = \frac{1}{2\pi\sqrt{-1} i^2} \delta_{ij} \Omega, \\ f^{\alpha_{2i-1} - \alpha_{2j-i}} &= f^{(i-j)(\gamma + \gamma^\vee)} = 0. \end{aligned}$$

It follows that V_{kl} vanishes except for

$$\begin{aligned} V_{(2i-1)(2i)} &= (-1)^{\langle i(\gamma+\gamma^\vee), i\gamma^\vee \rangle} \langle i(\gamma + \gamma^\vee), i\gamma^\vee \rangle f^{i\gamma + (i-j)\gamma^\vee} \\ &= (-1)^{i\langle \gamma, \gamma^\vee \rangle} \frac{\langle \gamma, \gamma^\vee \rangle}{2\pi\sqrt{-1}} \Omega, \\ V_{(2i)(2i-1)} &= -V_{(2i-1)(2i)} = -(-1)^{i\langle \gamma, \gamma^\vee \rangle} \frac{\langle \gamma, \gamma^\vee \rangle}{2\pi\sqrt{-1}} \Omega. \end{aligned}$$ □

The bundle K_{Δ^k} of the previous lemma is the simplest nontrivial rank $N = 2k$ Frobenius subbundle of K.

Definition 47. For all even $N > 0$ we define the A_1 *simple oscillator of rank* N to be the Frobenius type structure on $K_{\Delta^{N/2}} \subset K$ constructed in Lemma 46.

We will study $K_{\Delta^{N/2}}$ in more detail in the next section. Let us go back to a general Frobenius type structure $K_\Delta \subset K$.

Lemma 48. *The generalised monodromy of the meromorphic connections $\nabla(Z)$ of K_Δ is constant in Z.*

Proof. This is a standard result for the family of meromorphic connections underlying a Frobenius type structure, see e.g. [7, § 3.3] and [19, § 5.2] (based on Dubrovin [10]). □

We close this section by giving a standard formula for the generalised monodromy of $\nabla(Z)$ (i.e. its Stokes factors, see e.g. [7, § 2]). In particular this shows explicitly that the Stokes factors are constant in Z.

There is a classical formula for the Stokes factors of a linear connection of the form

$$d - \left(\frac{\Lambda}{t^2} + \frac{f}{t} \right) dt,$$

where Λ is diagonal and f is off-diagonal, in terms of periods, see e.g. [7, Theorem 4.5]. Periods appear here in the guise of multilogarithms, i.e. the iterated integrals

$$M_n(w_1, \dots, w_n) = (-2\pi i)^n \int_{[0, w_1 + \dots + w_n]} \frac{dt}{t - w_1} \circ \dots \circ \frac{dt}{t - (w_1 + \dots + w_{n-1})}$$

(see e.g. [7, § 7]).

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Remark 49. The functions $M_n(w_1, \dots, w_n)$ are also known as hyperlogarithms, see e.g. [16, § 2], where these are defined as the multi-valued functions

$$I(a_1 : \dots : a_{m+1}) = \int_0^{a_{m+1}} \frac{dt}{t-a_1} \circ \cdots \circ \frac{dt}{t-a_m}.$$

In particular we have

$$M_n(w_1, \dots, w_n) = (-2\pi i)^n I(w_1 : w_1 + w_2 : \dots : w_1 + \cdots + w_n).$$

According to loc. cit. $I(a_1 : \dots : a_{m+1})$ is invariant under the affine transformations $a_i \mapsto \lambda a_i + \beta$, so in particular we have

$$M_n(\lambda w_1, \dots, \lambda w_n) = M_n(w_1, \dots, w_n).$$

We apply the classical formula to the connection $\nabla(Z)$, of the form

$$d - \left(\frac{-U(Z)}{t^2} + \frac{V}{t}\right) dt.$$

Note that according to Lemma 43 $-U(Z)$ is diagonal, with ordered eigenvalues $-Z(\alpha_i)$, and V is off-diagonal.

Definition 50. We introduce a function $m\colon \Delta \times \Delta \to \mathbb{Z}$ such that $m(\alpha_i, \alpha_j)$ equals m if $\alpha_i - \alpha_j = m\gamma$ for $m \in \mathbb{Z}$, while $m(\alpha_i, \alpha_j) = 0$ if $\alpha_i - \alpha_j$ is not a multiple of γ.

In the following we write E_{ij} to denote the elementary matrix with $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$ and I for the identity matrix.

Lemma 51. *Let $\ell = \pm\mathbb{R}_{>0}Z(\gamma)$. Consider all sequences $1 \leqslant i_1 \neq i_2 \neq \cdots \neq i_{n+1} \leqslant N$ with $Z(\alpha_{i_{n+1}} - \alpha_1) \in \ell$ and $n \geqslant 0$. Then the Stokes factor S_ℓ for the connection $\nabla(Z)$ of (3.4) is the sum of all products of the form*

$$\begin{aligned}
&M_n(m(\alpha_{i_2}, \alpha_{i_1}), m(\alpha_{i_3}, \alpha_{i_2}), \dots, m(\alpha_{i_{n+1}}, \alpha_{i_n}))\\
&(-1)^{m(\alpha_{i_1},\alpha_{i_2})\langle\gamma,\alpha_{i_2}\rangle} m(\alpha_{i_1}, \alpha_{i_2})\langle\gamma, \alpha_{i_2}\rangle f^{m(\alpha_{i_1},\alpha_{i_2})\gamma}\\
&(-1)^{m(\alpha_{i_2},\alpha_{i_3})\langle\gamma,\alpha_{i_3}\rangle} m(\alpha_{i_2}, \alpha_{i_3})\langle\gamma, \alpha_{i_3}\rangle f^{m(\alpha_{i_2},\alpha_{i_3})\gamma}\\
&\dots\\
&(-1)^{m(\alpha_{i_n},\alpha_{i_{n+1}})\langle\gamma,\alpha_{i_{n+1}}\rangle} m(\alpha_{i_n}, \alpha_{i_{n+1}})\langle\gamma, \alpha_{i_{n+1}}\rangle f^{m(\alpha_{i_n},\alpha_{i_{n+1}})\gamma} E_{i_1 i_{n+1}}
\end{aligned}$$

where the empty product corresponding to $n = 0$ conventionally equals I. All the other Stokes factors are trivial. In particular the Stokes factors of $\nabla(Z)$ are constant in Z.

Proof. Let $\ell = \mathbb{R}_{>0}Z(\alpha_j - \alpha_i)$ be any potential Stokes ray, i.e. the ray spanned by a difference of eigenvalues of $-U(Z)$. The formula discussed in [7, § 1.2] shows that the Stokes factor attached to ℓ is a sum of contributions

$$M_n(Z(\alpha_{i_2} - \alpha_{i_1}), \dots, Z(\alpha_{i_{n+1}} - \alpha_{i_n}))V_{i_1 i_2} \cdots V_{i_n i_{n+1}} E_{i_1 i_{n+1}}, \tag{3.5}$$

for each sequence $1 \leqslant i_1 \neq i_2 \neq \cdots \neq i_{n+1} \leqslant N$ with $Z(\alpha_{i_{n+1}} - \alpha_1) \in \ell$. Here $n \geqslant 0$ is arbitrary, and the term corresponding to $n = 0$ conventionally equals I. Lemma 43 shows

$$V_{i_k i_{k+1}} = (-1)^{\langle\alpha_{i_k},\alpha_{i_{k+1}}\rangle}\langle\alpha_{i_k}, \alpha_{i_{k+1}}\rangle f^{\alpha_{i_k} - \alpha_{i_{k+1}}},$$

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and according to Lemma 37 this vanishes unless $\alpha_{i_k} - \alpha_{i_{k+1}}$ is a multiple of γ. It follows that the contribution (3.5) to S_ℓ can be written as

$$M_n(m(\alpha_{i_2}, \alpha_{i_1})Z(\gamma), \ldots, m(\alpha_{i_{n+1}}, \alpha_{i_n})Z(\gamma))$$
$$(-1)^{m(\alpha_{i_1}, \alpha_{i_2})\langle\gamma, \alpha_{i_2}\rangle} m(\alpha_{i_1}, \alpha_{i_2})\langle\gamma, \alpha_{i_2}\rangle f^{\alpha_{i_1} - \alpha_{i_2}} \ldots$$
$$(-1)^{m(\alpha_{i_n}, \alpha_{i_{n+1}})\langle\gamma, \alpha_{i_{n+1}}\rangle} m(\alpha_{i_n}, \alpha_{i_{n+1}})\langle\gamma, \alpha_{i_{n+1}}\rangle f^{\alpha_{i_n} - \alpha_{i_{n+1}}} E_{i_1 i_{n+1}}.$$

By Remark 49 we have

$$M_n(m(\alpha_{i_2}, \alpha_{i_1})Z(\gamma), \ldots, m(\alpha_{i_{n+1}}, \alpha_{i_n})Z(\gamma)) = M_n(m(\alpha_{i_2}, \alpha_{i_1}), \ldots, m(\alpha_{i_{n+1}}, \alpha_{i_n})).$$

By Definition 50 and Lemma 37 we have

$$f^{\alpha_{i_n} - \alpha_{i_{n+1}}} = f^{m(\alpha_{i_n}, \alpha_{i_{n+1}})\gamma}.$$

Finally we see that the general contribution (3.5) to S_ℓ vanishes unless all $\alpha_{i_k} - \alpha_{i_{k+1}}$ are (nonzero) multiples of γ. But then $\alpha_{i_1} - \alpha_{i_{n+1}}$ is also a multiple of γ, i.e. ℓ must be one of the rays $\pm Z(\gamma)$. The Lemma follows. □

4. A_1 simple oscillators

In the present section we collect some (rather standard) computations for the rank N Frobenius bundles $K_{\Delta^{N/2}} \subset K$ contained in the double A_1 infinite-dimensional Frobenius type structure, i.e. our A_1 simple oscillators. Recall from Lemma 46 that the meromorphic connection ∇ of $K_{\Delta^{N/2}}$ is a direct sum

$$\nabla = \bigoplus_{m=1}^{N/2} \nabla^{(m)} = \bigoplus_{m=1}^{N/2} d + (t^{-2}U^{(m)} - t^{-1}V^{(m)})\,dt,$$

where

$$U^{(m)} = \begin{pmatrix} mZ(\gamma + \gamma^\vee) & 0 \\ 0 & mZ(\gamma^\vee) \end{pmatrix},$$
$$V^{(m)} = \frac{\langle\gamma, \gamma^\vee\rangle}{2\pi i}\Omega \begin{pmatrix} 0 & (-1)^{m\langle\gamma,\gamma^\vee\rangle} \\ -(-1)^{m\langle\gamma,\gamma^\vee\rangle} & 0 \end{pmatrix}.$$

Lemma 52. *The Stokes rays of $\nabla^{(m)}$ are $\pm\ell_\gamma$. The corresponding Stokes factors are given by*

$$S_{\ell_\gamma} = \begin{pmatrix} 1 & 2i\sinh\left(-\frac{(-1)^{m\langle\gamma,\gamma^\vee\rangle}}{2}\langle\gamma, \gamma^\vee\rangle\Omega\right) \\ 0 & 1 \end{pmatrix},$$
$$S_{-\ell_\gamma} = \begin{pmatrix} 1 & 0 \\ -2i\sinh\left(-\frac{(-1)^{m\langle\gamma,\gamma^\vee\rangle}}{2}\langle\gamma, \gamma^\vee\rangle\Omega\right) & 1 \end{pmatrix}.$$

Proof. We use standard Fourier–Laplace methods, see e.g. [7, § 8]. The Fourier–Laplace transform of $\nabla^{(m)}$ is the Fuchsian connection with simple poles at z_1, z_2

$$\widehat{\nabla} = d - \left(\frac{A_1}{z - z_1} + \frac{A_2}{z - z_2}\right) dz$$

where we set $z_1 = -mZ(\gamma + \gamma^\vee)$, $z_2 = -mZ(\gamma^\vee)$, and the nilpotent residues are given by

$$\begin{aligned} A_1 &= -\frac{(-1)^{m\langle\gamma,\gamma^\vee\rangle}\langle\gamma,\gamma^\vee\rangle}{2\pi i}\Omega\begin{pmatrix}0 & -1\\ 0 & 0\end{pmatrix},\\ A_2 &= -\frac{(-1)^{m\langle\gamma,\gamma^\vee\rangle}\langle\gamma,\gamma^\vee\rangle}{2\pi i}\Omega\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}. \end{aligned}$$

Suppose $\phi(z) = \binom{u}{v}$ is a horizontal section of $\widehat{\nabla}$. Then

$$\tilde{\phi}(z) = \phi((z_2 - z_1)z + z_1) = \begin{pmatrix}\tilde{u}(z)\\ \tilde{v}(z)\end{pmatrix}$$

solves $\frac{d}{dz}\binom{\tilde{u}}{\tilde{v}} = -\frac{(-1)^{m\langle\gamma,\gamma^\vee\rangle}\langle\gamma,\gamma^\vee\rangle}{2\pi i}\Omega\begin{pmatrix}-\frac{1}{z}\tilde{v}\\ \frac{1}{z-1}\tilde{u}\end{pmatrix}$ and so we have

$$z(1-z)\frac{d^2}{dz^2}\tilde{u} + (1-z)\frac{d}{dz}\tilde{u} + \left(\frac{\langle\gamma,\gamma^\vee\rangle}{2\pi}\right)^2\Omega^2\tilde{u} = 0,$$

a standard hypergeometric equation

$$z(1-z)\frac{d^2}{dz^2}\tilde{u} + (c - (a+b+1)z)\frac{d}{dz}\tilde{u} - (ab)\tilde{u} = 0$$

with parameters

$$\begin{aligned} a &= iV_{21} = -(-1)^{m\langle\gamma,\gamma^\vee\rangle}\frac{\langle\gamma,\gamma^\vee\rangle}{2\pi}\Omega,\\ b &= -a = iV_{12},\\ c &= 1. \end{aligned}$$

So the unique solution $\tilde{\phi}^{(0)}(z)$ at $z = 0$ with $\tilde{\phi}^{(0)}(0) = e_1 = \binom{1}{0}$ is given in terms of Gauss hypergeometric functions as

$$\tilde{\phi}^{(0)}(z) = \begin{pmatrix} {}_2F_1(-a, a, 1; z)\\ \frac{2\pi i}{(-1)^{m\langle\gamma,\gamma^\vee\rangle}\langle\gamma,\gamma^\vee\rangle}\frac{1}{\Omega}z\frac{d}{dz}{}_2F_1(-a, a, 1; z)\end{pmatrix} \tag{4.1}$$

and similarly the unique solution $\tilde{\phi}^{(1)}(z)$ at $z = 1$ with $\tilde{\phi}^{(1)}(1) = e_2 = \binom{0}{1}$ is given by

$$\tilde{\phi}^{(1)}(z) = \begin{pmatrix} -\frac{(-1)^{m\langle\gamma,\gamma^\vee\rangle}\langle\gamma,\gamma^\vee\rangle}{2\pi i}\Omega(1-z), {}_2F_1(1-a, 1+a, 2; 1-z)\\ -z\frac{d}{dz}(1-z){}_2F_1(1-a, 1+a, 2; 1-z)\end{pmatrix}$$

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(see [11, Ch. II, § 2.1]). It is well known that the Fourier–Laplace transform allows to express Stokes factors for $\nabla^{(m)}$ in terms of the analytic continuation of solutions to $\widehat{\nabla}$, see e.g. [7, § 9]. In particular applying the formulae in loc. cit. § 9.2 we find

$$S_{\ell_\gamma} = \begin{pmatrix} 1 & 2\pi i V_{21}(\tilde{\phi}^{(0)}(1))_1 \\ 0 & 1 \end{pmatrix}, \quad S_{-\ell_\gamma} = \begin{pmatrix} 1 & 0 \\ 2\pi i V_{12}(\tilde{\phi}^{(1)}(0))_2 & 1 \end{pmatrix}.$$

On the other hand we have

$$\begin{aligned} \tilde{\phi}^{(0)}(1) &= {}_2F_1(-a, a, 1; z) \text{ by (4.1)} \\ &= \frac{1}{\Gamma(1-a)\Gamma(1+a)} \text{ (see [28, Ch. XIV, p. 282])} \\ &= \frac{\sin(\pi a)}{\pi a} \text{ (by Euler reflection [11, Ch. I, § 1.2, Equation (8)]).} \end{aligned}$$

Using the relation $a = iV_{21}$ gives the result for S_{ℓ_γ}. The computation for $S_{-\ell_\gamma}$ is completely analogous. □

Let $Y^{(m)}(t) = Y^{(m)}_{ij}(t)$ be the $\mathrm{GL}(2, \mathbb{C})$ fundamental solution to $\nabla^{(m)}$. Define

$$\Psi^{(m)}_{ij}(t) = e^{Z_j/t} Y^{(m)}_{ij}(t) \tag{4.2}$$

where $Z_1 = mZ(\gamma + \gamma^\vee)$, $Z_2 = mZ(\gamma^\vee)$. Recall $Y^{(m)}(t)$ is characterised by the asymptotics $\Psi^{(m)}(t) \to I$ as $t \to 0$ in a sector.

Lemma 53. *The functions $\Psi^{(m)}_{ij}(t)$ satisfy the integral equations*

$$\begin{aligned} \Psi^{(m)}_{11}(t) &= 1 - \eta \int_{-\ell_\gamma} \frac{dt'}{t'} \frac{t}{t'-t} \Psi^{(m)}_{12}(t') e^{mZ(\gamma)/t'}, \\ \Psi^{(m)}_{12}(t) &= \eta \int_{\ell_\gamma} \frac{dt'}{t'} \frac{t}{t'-t} \Psi^{(m)}_{11}(t') e^{-mZ(\gamma)/t'}, \\ \Psi^{(m)}_{21}(t) &= -\eta \int_{-\ell_\gamma} \frac{dt'}{t'} \frac{t}{t'-t} \Psi^{(m)}_{22}(t') e^{mZ(\gamma)/t'}, \\ \Psi^{(m)}_{22}(t) &= 1 + \eta \int_{\ell_\gamma} \frac{dt'}{t'} \frac{t}{t'-t} \Psi^{(m)}_{21}(t') e^{-mZ(\gamma)/t'} \end{aligned}$$

where

$$\eta = \frac{1}{2\pi i}(S_{\ell_\gamma})_{12} = -\frac{1}{2\pi i}(S_{-\ell_\gamma})_{21} = \frac{1}{\pi} \sinh\left(-\frac{(-1)^{m\langle\gamma,\gamma^\vee\rangle}}{2}\langle\gamma, \gamma^\vee\rangle\Omega\right).$$

Proof. The function $\Psi^{(m)}(t)$ is uniquely characterised as the solution to a RH problem with contour given by the rays $\pm\ell_\gamma$, jumps $S_{\pm\ell_\gamma}$ as in Lemma 52, asymptotics $\Psi^{(m)}(t) \to I$ as $t \to 0$ in a sector, and polynomial growth as $t \to \infty$. Standard results allow to recast this RH problem in terms of integral equations as claimed, see e.g. [13, Ch. 3, § 1]. Our present application is in fact a limiting case of [9, Proposition 2.2]. A reference which is very close to our notation is [15, Appendix C]. Indeed our function $\Psi(t)$ is precisely the function $\Phi(x)$ appearing in loc. cit. Equation (6), evaluated at $x=\beta t^{-1}$ and in the limit $\beta\to 0$, with parameters $\mu_{12}=-\mu_{21}=\eta$. Note that the change of variable $y=\beta t'^{-1}$, $x=\beta t^{-1}$ turns the integral kernel $dy(y-x)^{-1}$ appearing in loc. cit. Equation (6) into our kernel $t\,dt'(t'(t'-t))^{-1}$. □

5. A_1 large N limit

We continue our study of the rank N simple oscillator $K_{\Delta^{N/2}}\subset K$. We regard the Frobenius bundle structure on $K_{\Delta^{N/2}}$ as depending on the free parameter $\langle\gamma,\gamma^\vee\rangle$ via the formulae of Lemma 46.

Definition 54. Let $\hbar\in\mathbb{R}_{>0}$. The rescaled simple oscillator $K_{\Delta^{N/2},\hbar}$ is obtained by replacing

$$\langle\gamma,\gamma^\vee\rangle\Omega\mapsto(\sqrt{-1}\hbar)\langle\gamma,\gamma^\vee\rangle\Omega \tag{5.1}$$

in the formulae of Lemma 46.

In other words $K_{\Delta^{N/2},\hbar}$ is the projection of the deformed bundle $K_\hbar$ discussed in Remark 36. By Definition 54 the meromorphic connection $\nabla_\hbar$ of $K_{\Delta^{N/2},\hbar}$ splits just as before

$$\nabla_\hbar=\bigoplus_{m=1}^{N/2}\nabla_\hbar^{(m)}=\bigoplus_{m=1}^{N/2}d+(t^{-2}U^{(m)}-t^{-1}V_\hbar^{(m)})\,dt,$$

where

$$V_\hbar^{(m)}=\frac{\langle\gamma,\gamma^\vee\rangle\hbar}{2\pi}\Omega\begin{pmatrix}0 & (-1)^{m\langle\gamma,\gamma^\vee\rangle}\\ -(-1)^{m\langle\gamma,\gamma^\vee\rangle} & 0\end{pmatrix}.$$

Lemma 55. *The Stokes rays of $\nabla_\hbar^{(m)}$ are $\pm\ell_\gamma$. The corresponding Stokes factors are given by*

$$\begin{aligned}S_{\ell_\gamma,\hbar}&=\begin{pmatrix}1 & 2i\sinh\left(-\dfrac{(-1)^{m\langle\gamma,\gamma^\vee\rangle}}{2}\langle\gamma,\gamma^\vee\rangle i\hbar\Omega\right)\\ 0 & 1\end{pmatrix}\\
&=\begin{pmatrix}1 & (-1)^{m\langle\gamma,\gamma^\vee\rangle}\langle\gamma,\gamma^\vee\rangle\hbar\Omega\\ 0 & 1\end{pmatrix}+O(\hbar^2),\\
S_{-\ell_\gamma,\hbar}&=\begin{pmatrix}1 & 0\\ -2i\sinh\left(-\dfrac{(-1)^{m\langle\gamma,\gamma^\vee\rangle}}{2}\langle\gamma,\gamma^\vee\rangle i\hbar\Omega\right) & 1\end{pmatrix}\\
&=\begin{pmatrix}1 & 0\\ -(-1)^{m\langle\gamma,\gamma^\vee\rangle}\langle\gamma,\gamma^\vee\rangle\hbar\Omega & 1\end{pmatrix}+O(\hbar^2).\end{aligned}$$

Proof. The result follows at once from Lemma 52. □

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Let $Y^{(m)}_{\hbar}(t) = Y^{(m)}_{\hbar,ij}(t)$ be the $\mathrm{GL}(2,\mathbb{C})$ fundamental solution to $\nabla^{(m)}_{\hbar}$. Define

$$\Psi^{(m)}_{\hbar,ij}(t) = e^{Z_j/t} Y^{(m)}_{\hbar,ij}(t) \tag{5.2}$$

where $Z_1 = mZ(\gamma+\gamma^{\vee})$, $Z_2 = mZ(\gamma^{\vee})$.

Lemma 56. *The functions $\Psi^{(m)}_{\hbar,ij}(t)$ satisfy*

$$\begin{aligned}
\Psi^{(m)}_{\hbar,11}(t) &= 1 + O(\hbar^2),\\
\Psi^{(m)}_{\hbar,12}(t) &= \frac{1}{2\pi i}(-1)^{m\langle\gamma,\gamma^{\vee}\rangle}\langle\gamma,\gamma^{\vee}\rangle\hbar\Omega\int_{\ell_\gamma}\frac{dt'}{t'}\frac{t}{t'-t}e^{-mZ(\gamma)/t'} + O(\hbar^2),\\
\Psi^{(m)}_{\hbar,21}(t) &= -\frac{1}{2\pi i}(-1)^{m\langle\gamma,\gamma^{\vee}\rangle}\langle\gamma,\gamma^{\vee}\rangle\hbar\Omega\int_{-\ell_\gamma}\frac{dt'}{t'}\frac{t}{t'-t}e^{mZ(\gamma)/t'} + O(\hbar^2),\\
\Psi^{(m)}_{\hbar,22}(t) &= 1 + O(\hbar^2).
\end{aligned}$$

Proof. By Lemma 55 the functions $\Psi^{(m)}_{ij,\hbar}(t)$ satisfy equations identical to those of Lemma 53, with η replaced by

$$\frac{1}{2\pi i}(S_{\ell_\gamma,\hbar})_{12} = -\frac{1}{2\pi i}(S_{-\ell_\gamma,\hbar})_{21} = \frac{1}{\pi}\sinh\left(-\frac{(-1)^{m\langle\gamma,\gamma^{\vee}\rangle}}{2}\langle\gamma,\gamma^{\vee}\rangle i\hbar\Omega\right).$$

Expanding around $\hbar = 0$ gives the result. □

Corollary 57. *We have*

$$\log\Psi^{(m)}_{\hbar} = \begin{pmatrix} 0 & \delta^{(m)}(t)\\ -\delta^{(m)}(-t) & 0\end{pmatrix} + O(\hbar^2)$$

where

$$\delta^{(m)}(t) = \frac{1}{2\pi i}(-1)^{m\langle\gamma,\gamma^{\vee}\rangle}\langle\gamma,\gamma^{\vee}\rangle\hbar\Omega\int_{\ell_\gamma}\frac{dt'}{t'}\frac{t}{t'-t}e^{-mZ(\gamma)/t'}.$$

Proof. The result follows from Lemma 56, by making the change of variable $t' \mapsto -t'$ in the integral for $\Psi^{(m)}_{21,\hbar}(t)$. □

Corollary 58. *We have*

$$\log\Psi^{(m)}_{\hbar}(t)_{(12)} = (-1)^{m\langle\gamma,\gamma^{\vee}\rangle}\langle\gamma,\gamma^{\vee}\rangle\hbar\Omega\frac{t}{\pi i}\int_{\ell_\gamma}\frac{dt'}{(t')^2-t^2}e^{-mZ(\gamma)/t'} + O(\hbar^2).$$

Proof. Following the notation of the previous lemma we have

$$(\log\Psi^{(m)}_{\hbar}(t))_{12} + (\log\Psi^{(m)}_{\hbar}(t))_{21} = \delta^{(m)}(t) - \delta^{(m)}(-t).$$

So the claim follows from a straightforward calculation. □

Let $\hat{\xi}$ denote the vector $(1,1)^{\mathrm{T}} \in \mathbb{C}^2$ and Π be the linear function on $\mathbb{C}^2$ given by $\Pi(w_1,w_2)^{\mathrm{T}} = w_1 + w_2$.

Proposition 59. *We have*

$$\exp\left(\frac{1}{\hbar}\sum_{m=1}^{\infty}\frac{(-1)^{m\langle\gamma,\gamma^\vee\rangle}}{m}\Pi\log\Psi_{\hbar}^{(m)}((2\pi)^{-1}it)\hat{\xi}\right)=\Psi_{\gamma^\vee}(t)+O(\hbar).$$

Proof. In this proof we write $Z=Z(\gamma)$ for brevity. Note that we have $\Pi\log\Psi_{\hbar}^{(m)}((2\pi)^{-1}it)\hat{\xi}=\log\Psi_{\hbar}^{(m)}((2\pi)^{-1}it)_{(12)}+O(\hbar^2)$. By the previous lemma we have

$$\begin{aligned}(\log\Psi_{\hbar}^{(m)}(it))_{12}+(\log\Psi_{\hbar}^{(m)}(it))_{21} &= (-1)^{m\langle\gamma,\gamma^\vee\rangle}\langle\gamma,\gamma^\vee\rangle\hbar\Omega\\ &\quad\times\frac{t}{\pi}\int_{\ell_\gamma}\frac{dt'}{(t')^2+t^2}e^{-mZ/t'}+O(\hbar^2).\end{aligned}$$

By the definition of ℓ_γ we are integrating over $t'=Zs$, $s>0$, so we have

$$\begin{aligned}\frac{t}{\pi}\int_{\ell_\gamma}\frac{dt'}{(t')^2+t^2}e^{-mZ(\gamma)/t'} &= \frac{1}{\pi}\int_0^\infty\left(\frac{Z}{t}\right)\frac{ds}{\left(\frac{Z}{t}\right)^2s^2+1}e^{-m/s}\\ &= \frac{1}{\pi}\int_0^\infty\left(\frac{Z}{t}\right)^{-1}\frac{ds}{\left(\frac{Z}{t}\right)^{-2}s^2+1}e^{-ms}\end{aligned}$$

(using the change of variable $s\mapsto s^{-1}$). The right hand side can be rewritten as

$$\frac{1}{\pi}\int_0^\infty e^{-ms}\frac{d}{ds}\arctan\left(\left(\frac{Z}{t}\right)^{-1}s\right)ds$$

and so integrating by parts as

$$-m\frac{1}{\pi}\int_0^\infty\arctan\left(\left(\frac{Z}{t}\right)^{-1}s\right)e^{-ms}\,ds.$$

By these identities we can rewrite the series

$$\frac{1}{\hbar}\sum_{m=1}^{\infty}\frac{(-1)^{m\langle\gamma,\gamma^\vee\rangle}}{m}((\log\Psi_{\hbar}^{(m)}(it))_{12}+(\log\Psi_{\hbar}^{(m)}(it))_{21})$$

as

$$\begin{aligned}&-\langle\gamma,\gamma^\vee\rangle\Omega\frac{1}{\pi}\int_0^\infty\arctan\left(\left(\frac{Z}{t}\right)^{-1}s\right)\sum_{m=1}^{\infty}e^{-ms}\,ds\\ &\quad=\langle\gamma^\vee,\gamma\rangle\Omega\frac{1}{\pi}\int_0^\infty\arctan\left(\left(\frac{Z}{t}\right)^{-1}s\right)\frac{1}{e^s-1}\,ds+O(\hbar).\end{aligned}$$

Binet's formula for the log gamma function is the identity

$$\log\Gamma(z)=\left(z-\frac{1}{2}\right)\log z-z+\frac{1}{2}\log(2\pi)+\frac{1}{\pi}\int_0^\infty\frac{\arctan(s/(2\pi z))}{e^s-1}ds$$

valid for $\Re(z)>0$ (see [11, p. 22, Equation (9)]). Applying this identity shows

$$\begin{aligned}\frac{1}{\hbar}\sum_{m=1}^{\infty}\frac{(-1)^{m\langle\gamma,\gamma^\vee\rangle}}{m}\log\Psi_{\hbar}^{(m)}((2\pi)^{-1}it)_{(12)} &= \langle\gamma^\vee,\gamma\rangle\Omega\log\Lambda\left(\frac{Z(\gamma)}{t}\right)+O(\hbar)\\ &= \log\Psi_{\gamma^\vee}(t)+O(\hbar)\end{aligned}$$

as required. $\square$

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6. Finite uncoupled case

In this section we spell out how to extend our results from the A_1 case to a finite, uncoupled variation of BPS structure. This is mostly a matter of notation.

In this case there is a finite subset $\{\gamma_i\} \subset \Gamma$ such that $\Omega(\pm\gamma_i)$ is nonvanishing, and we have $\langle \gamma_i, \gamma_j \rangle = 0$ for all i, j. We also fix a reference basis $\{\beta_i\}$ for Γ.

Lemma 60. *For a finite uncoupled variation of BPS structure we have for $k \in \mathbb{Z} \setminus \{0\}$*

$$f^{k\gamma_j}(Z) = \frac{\Omega(\gamma_j)}{2\pi i k^2} \mathbf{s}^{k\gamma_j}$$

(a constant, independent of Z) while all the other $f^{\alpha}(Z)$ vanish identically. In particular we have the symmetry $f^{\alpha} = f^{-\alpha}$.

Proof. The proof is the same as that of Lemma 37. □

Let us still denote by $(K, \nabla, C, \mathcal{U}, \mathcal{V}, g)$ the Frobenius type structure underlying a finite, uncoupled variation of BPS structure.

Corollary 61. *For a finite uncoupled variation of BPS structure we have*

$$\begin{aligned}
\nabla^r &= d + \sum_{i,k\neq 0} \frac{\Omega(\gamma_i)}{2\pi i k^2} \mathbf{s}^{k\gamma_i} [x_{k\gamma_i}, -] \frac{dZ(k\gamma_i)}{Z(k\gamma_i)},\\
\mathcal{V} &= \sum_{i,k\neq 0} \frac{\Omega(\gamma_i)}{2\pi i k^2} \mathbf{s}^{k\gamma_i} [x_{k\gamma_i}, -].
\end{aligned}$$

Proof. The proof is the same as that of Corollary 38. □

Just as in the A_1 case we write $K_\Delta \subset K$ for the finite-dimensional subbundle spanned by the sections x_{α_i}, where $\Delta = \{\alpha_i\} \subset \Gamma$ is a subset with N elements.

Theorem 62. *Let (Γ, Z, Ω) be a framed variation of BPS structure. For all finite $\Delta \subset \Gamma$ write $K_\Delta \subset K$ for the subbundle spanned by x_α, $\alpha \in \Delta$, endowed with the structure induced by the canonical projection $K \to K_\Delta$. Then K_Δ is Frobenius if and only if (Γ, Z, Ω) is uncoupled.*

Proof. In one direction the proof is the same as that of Lemma 40. The converse is established in [2, Lemma 20]. More precisely choose $\Delta = \{\alpha_i, \alpha_j, \alpha_k\}$ such that $\alpha_j - \alpha_k$, $\alpha_k - \alpha_i$ are active classes with $\langle \alpha_j - \alpha_k, \alpha_k - \alpha_i \rangle \neq 0$ and

$$\langle \alpha_j, \alpha_i \rangle \langle \alpha_j - \alpha_k, \alpha_k - \alpha_i \rangle \neq \langle \alpha_j, \alpha_k \rangle \langle \alpha_k, \alpha_i \rangle.$$

This is always possible if (Γ, Z, Ω) is not uncoupled. Then it is shown in loc. cit. that the projection of ∇^r to K_Δ is not flat. □

As usual once we project to a finite-dimensional subbundle K_Δ we always evaluate at the geometric point $s_i = 1$, $i = 1, \ldots, N$, and we consider the meromorphic connections of K_Δ

$$\nabla(Z) = d + \left(\frac{U(Z)}{t^2} - \frac{V}{t} \right) dt.$$

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As in Lemma 43 we have

$$\begin{aligned} U(Z)_{ij} &= Z(\alpha_i)\delta_{ij}, \\ V_{ij} &= (-1)^{\langle \alpha_i, \alpha_j \rangle} \langle \alpha_i, \alpha_j \rangle f^{\alpha_i - \alpha_j}. \end{aligned}$$

Definition 63. Fix an active class γ_i and a basis element β_j with $\langle \gamma_i, \beta_j \rangle \neq 0$. For all $k \geqslant 1$ choose $\Delta^k_{ij} = \{\alpha_1, \dots \alpha_{2k}\} \subset \Gamma$ with $(\alpha_{2m-1}, \alpha_{2m}) = (m(\gamma_i + \beta_j), m\beta_j)$. We define the (even) rank N simple oscillator between γ_i, β_j as the Frobenius bundle $K_{\Delta^{N/2}_{ij}}$.

Lemma 64. *The meromorphic connections of the Frobenius bundle* $K_{\Delta^{N/2}_{ij}}$ *have* U, V *block diagonal, with blocks*

$$\begin{aligned} U^{(m),ij} &= \begin{pmatrix} mZ(\gamma_i + \beta_j) & 0 \\ 0 & mZ(\beta_j) \end{pmatrix}, \\ V^{(m),ij} &= \frac{\langle \gamma_i, \beta_j \rangle}{2\pi\sqrt{-1}} \Omega(\gamma_i) \begin{pmatrix} 0 & (-1)^{m\langle \gamma_i, \beta_j \rangle} \\ -(-1)^{m\langle \gamma_i, \beta_j \rangle} & 0 \end{pmatrix} \end{aligned}$$

for $m = 1, \dots, N/2$.

Proof. The proof is the same as that of Lemma 46. □

Definition 65. The *rank N simple oscillator* of a finite, uncoupled variation of BPS structure with respect to a basis element β_j is the Frobenius bundle

$$K_{\beta_j}(N) = \bigoplus_{i \mid \langle \gamma_i, \beta_j \rangle \neq 0} K_{\Delta^{N/2}_{ij}}.$$

In the following we denote the meromorphic connections of $K_{\beta_j}(N)$ by ∇_{β_j}.

By construction ∇_{β_j} splits as a direct sum

$$\nabla_{\beta_j} = \bigoplus_{m,i \mid \langle \gamma_i, \beta_j \rangle \neq 0} \nabla^{(m),ij} = \bigoplus_{m,i \mid \langle \gamma_i, \beta_j \rangle \neq 0} d + (t^{-2} U^{(m),ij} - t^{-1} V^{(m),ij})\, dt.$$

In particular we have the rescaling

$$\langle \gamma_i, \beta_j \rangle \mapsto \langle \gamma_i, \beta_j \rangle \sqrt{-1}\hbar$$

acting on all our structures. For the meromorphic connections we have

$$\nabla_{\hbar,\beta_j} = \bigoplus_{m,i \mid \langle \gamma_i, \beta_j \rangle \neq 0} \nabla^{(m),ij}_{\hbar} = \bigoplus_{m,i \mid \langle \gamma_i, \beta_j \rangle \neq 0} d + (t^{-2} U^{(m),ij} - t^{-1} V^{(m),ij}_{\hbar})\, dt,$$

where

$$V^{(m),ij}_{\hbar} = \frac{\langle \gamma_i, \beta_j \rangle \hbar}{2\pi} \Omega(\gamma_i) \begin{pmatrix} 0 & (-1)^{m\langle \gamma_i, \beta_j \rangle} \\ -(-1)^{m\langle \gamma_i, \beta_j \rangle} & 0 \end{pmatrix}.$$

Let $Y^{(m),ij}_{\hbar}(t) = Y^{(m)}_{\hbar,pq}(t)$ be the $\mathrm{GL}(2,\mathbb{C})$ fundamental solution to $\nabla^{(m),ij}_{\hbar}$. Define

$$\Psi^{(m),ij}_{\hbar,pq}(t) = e^{Z_q/t} Y^{(m),ij}_{\hbar,pq}(t) \tag{6.1}$$

where $Z_1 = mZ(\gamma_i + \beta_j)$, $Z_2 = mZ(\beta_j)$. Write $\hat{\xi} \in \mathbb{C}^2$, $\Pi \in \mathrm{Hom}(\mathbb{C}^2, \mathbb{C})$ for the usual vector and linear functional.

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Theorem 66. *We have*

$$\exp\left(\frac{1}{\hbar}\sum_{m=1}^{\infty}\sum_{i\,|\,\langle\gamma_i,\beta_j\rangle\neq 0}\frac{(-1)^{m\langle\gamma_i,\beta_j\rangle}}{m}\Pi\log\Psi^{(m),ij}_{\hbar}((2\pi)^{-1}\sqrt{-1}t)\hat{\xi}\right)$$
$$= \Psi_{\beta_j}(t) + O(\hbar).$$

Proof. As in Lemma 55 one proves that the Stokes rays of $\nabla^{(m),ij}_{\hbar}$ are $\pm\ell_{\gamma_i}$ and the corresponding Stokes factors are given by

$$\begin{aligned}
S^{ij}_{\ell_\gamma,\hbar} &= \begin{pmatrix} 1 & 2\sqrt{-1}\sinh\left(-\frac{(-1)^{m\langle\gamma_i,\beta_j\rangle}}{2}\langle\gamma_i,\beta_j\rangle\sqrt{-1}\hbar\Omega(\gamma_i)\right)\\ 0 & 1\end{pmatrix}\\
&= \begin{pmatrix} 1 & (-1)^{m\langle\gamma_i,\beta_j\rangle}\langle\gamma_i,\beta_j\rangle\hbar\Omega(\gamma_i)\\ 0 & 1\end{pmatrix} + O(\hbar^2),\\
S^{ij}_{-\ell_\gamma} &= \begin{pmatrix} 1 & 0\\ -2\sqrt{-1}\sinh\left(-\frac{(-1)^{m\langle\gamma_i,\beta_j\rangle}}{2}\langle\gamma_i,\beta_j\rangle\sqrt{-1}\hbar\Omega(\gamma_i)\right) & 1\end{pmatrix}\\
&= \begin{pmatrix} 1 & 0\\ -(-1)^{m\langle\gamma_i,\beta_j\rangle}\langle\gamma_i,\beta_j\rangle\hbar\Omega(\gamma_i) & 1\end{pmatrix} + O(\hbar^2).
\end{aligned}$$

As in Lemma 56 this implies the identities

$$\begin{aligned}
\Psi^{(m),ij}_{\hbar,11}(t) &= 1 + O(\hbar^2),\\
\Psi^{(m),ij}_{\hbar,12}(t) &= \frac{1}{2\pi\sqrt{-1}}(-1)^{m\langle\gamma_i,\beta_j\rangle}\langle\gamma_i,\beta_j\rangle\hbar\Omega(\gamma_j)\int_{\ell_{\gamma_i}}\frac{dt'}{t'}\frac{t}{t'-t}e^{-mZ(\gamma_i)/t'} + O(\hbar^2),\\
\Psi^{(m),ij}_{\hbar,21}(t) &= -\frac{1}{2\pi\sqrt{-1}}(-1)^{m\langle\gamma_i,\beta_j\rangle}\langle\gamma_i,\beta_j\rangle\hbar\Omega(\gamma_j)\int_{-\ell_{\gamma_i}}\frac{dt'}{t'}\frac{t}{t'-t}e^{mZ(\gamma_i)/t'} + O(\hbar^2),\\
\Psi^{(m),ij}_{\hbar,22}(t) &= 1 + O(\hbar^2).
\end{aligned}$$

From here we can proceed as in the proof of Proposition 59. □

7. Tau functions

Suppose $f_\hbar(t, Z(\gamma))$ is a scalar function depending on the parameter $\hbar$, admitting a formal power series expansion in $\hbar$ around $\hbar = 0$.

Definition 67. A first order tau function for $\exp(f_\hbar(t, Z(\gamma)))$ is a function $\tau_\hbar(t, Z)$ which is invariant under common rescaling of t, $Z(\gamma)$, admits a formal power series expansion in $\hbar$ around $\hbar = 0$, and such that the first nonzero terms in the expansions in $\hbar$ of the quantities

$$\frac{\partial}{\partial t}f_\hbar(t, Z(\gamma)),\ \langle\gamma^\vee,\gamma\rangle\frac{\partial}{\partial Z(\gamma)}\log\tau_\hbar(t, Z(\gamma))$$

around $\hbar = 0$ are the same.

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Lemma 68. *The function given by*

$$\log \tau_{\hbar}^{(m)}(it) = \frac{\Omega}{2\pi}\hbar \int_0^{\infty} s \log\left(s^2 + \left(\frac{Z(\gamma)}{t}\right)^2\right) e^{-ms}\, ds$$

is a first order tau function for $\exp\left(\frac{(-1)^{m\langle\gamma,\gamma^{\vee}\rangle}}{m}(\log \Psi_{\hbar}^{(m)}(it))_{(12)}\right)$.

Proof. In the rest of the proof we write $Z = Z(\gamma)$ and suppress $O(\hbar^2)$ terms. According to the proof of Proposition 59 we have

$$\frac{(-1)^{m\langle\gamma,\gamma^{\vee}\rangle}}{m}(\log \Psi_{\hbar}^{(m)}(it))_{(12)} = \langle\gamma^{\vee},\gamma\rangle F\left(\frac{Z}{t}\right) \tag{7.1}$$

where the function $F(w)$ is given by

$$F(w) = \hbar\Omega\frac{1}{\pi}\int_0^{\infty} \arctan\left(\frac{s}{w}\right) e^{-ms}\, ds.$$

Suppose the function $H(w)$ satisfies $H'(w) = wF'(w)$. Then we have

$$\frac{\partial}{\partial Z} H\left(\frac{Z}{t}\right) = \frac{1}{t}H'\left(\frac{Z}{t}\right) = \frac{Z}{t^2}F'\left(\frac{Z}{t}\right).$$

From the general form (7.1) we get

$$\frac{(-1)^{m\langle\gamma,\gamma^{\vee}\rangle}}{m}\frac{\partial}{\partial t}(\log \Psi_{\hbar}^{(m)})_{(12)} = \langle\gamma^{\vee},\gamma\rangle\frac{\partial}{\partial t}F\left(\frac{Z}{t}\right) = -\langle\gamma^{\vee},\gamma\rangle\frac{Z}{t^2}F'\left(\frac{Z}{t}\right).$$

So e^{-H} gives a tau function for $\exp\left(\frac{(-1)^{m\langle\gamma,\gamma^{\vee}\rangle}}{m}(\log \Psi_{\hbar}^{(m)}(it))_{(12)}\right)$. A solution $H(w)$ is given by choosing the primitive

$$\hbar\Omega\frac{1}{\pi}\int w\frac{\partial}{\partial w}\arctan\left(\frac{s}{w}\right) dw = -\hbar\Omega\frac{1}{\pi}\frac{1}{2}s\log(s^2+w^2)$$

and integrating in $e^{-ms}\, ds$. □

We can now prove a large rank limit in the A_1 case.

Corollary 69. *The function*

$$\exp\left(\frac{1}{\hbar}\sum_{m=1}^{\infty}\frac{(-1)^{m\langle\gamma,\gamma^{\vee}\rangle}}{m}\log \Psi_{\hbar}^{(m)}((2\pi)^{-1}it)_{(12)}\right)$$

(which equals $\Psi_{\gamma^{\vee}}(t) + O(\hbar)$ *by Proposition 59) admits the first order tau function*

$$\exp\left(\frac{1}{\hbar}\sum_{m=1}^{\infty}\log \tau_{\hbar}^{(m)}((2\pi)^{-1}it)\right),$$

and the latter equals the tau function $\tau_{\ell}(t, Z(\gamma))$ *of (1.5). In particular this implies Theorem 9 in the rank 2 case.*

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Proof. The claim that the second exponential is a first order tau function follows from Lemma 68 by summing over all frequencies and multiplying by $\hbar^{-1}$ throughout. To prove the second exponential equals $\tau_\ell(t, Z(\gamma))$ recall from the proof of Lemma 68 that

$$\frac{1}{\hbar}\sum_{m=1}^{\infty}\log\tau_{\hbar}^{(m)}((2\pi)^{-1}it) = -\sum_{m=1}^{\infty}\frac{\Omega}{\pi}\int_0^{\infty} w\frac{d}{dw}\arctan\left(\frac{s}{w}\right)|_{w=(2\pi)^{-1}\frac{Z}{t}}e^{-ms}\,ds.$$

By the proof of Proposition 59 the right hand side equals

$$\Omega w\frac{d}{dw}\log\Lambda(w)|_{w=\frac{Z}{t}}.$$

Now we use the identity

$$w\frac{d}{dw}\log\Lambda(w) = \frac{d}{dw}\log\Upsilon(w)$$

(see [5, Lemma 5.4]), which follows at once from the identity for the Barnes G-function

$$\frac{d}{dw}\log G(w+1) = \frac{1}{2}\log(2\pi) + \frac{1}{2} - w + w\frac{d}{dw}\log\Gamma(w)$$

(see [28, p. 268, Equation (50)]). The upshot is the required identity

$$\frac{1}{\hbar}\sum_{m=1}^{\infty}\log\tau_{\hbar}^{(m)}((2\pi)^{-1}it) = \Omega\log\Upsilon\left(\frac{Z}{t}\right) = \log\tau_\ell(t, Z).$$

The last claim that $\tau_\ell(t, Z)$ is a tau function for $\Psi(t)$ now follows from the fact that both functions are independent of $Z(\gamma^\vee)$. □

We consider now the case of a finite, uncoupled variation of BPS structure, and follow the notation of § 6. In particular we have a basis $\{\beta_j\}$, yielding local coordinates $Z(\beta_j)$. The active classes are $\{\gamma_i\}$, and we write

$$\gamma_i = \sum_p c_{ip}\beta_p.$$

Recall we have elementary simple oscillators $\nabla_{\hbar}^{(m),ij}$, or equivalently in terms of solutions the functions $\Psi_{\hbar}^{(m),ij}$.

Lemma 70. *Fix i, j. The function given by*

$$\log\tau_{\hbar}^{(m),i}(\sqrt{-1}t) = \frac{\Omega(\gamma_i)}{2\pi}\hbar\int_0^{\infty} s\log\left(s^2 + \left(\frac{Z(\gamma_i)}{t}\right)^2\right)e^{-ms}\,ds$$

is a first order tau function for the scalar

$$\exp\left(\frac{(-1)^{m\langle\gamma_i,\beta_j\rangle}}{m}(\log\Psi_{\hbar}^{(m),ij}(\sqrt{-1}t))_{(12)}\right).$$

Proof. The proof is the same as that of Lemma 68. □

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Suppose $v^j_\hbar(t, Z)$ is a vector function of the local coordinates $Z(\beta_k)$, with one component for each β_j, depending on the additional parameter $\hbar$.

Definition 71. A first order tau function for the vector $\exp(v^j_\hbar(t, Z))$ is a scalar function $\tau_\hbar(t, Z)$ which is invariant under common rescaling of t, Z and satisfies

$$\frac{\partial}{\partial t} v^j_\hbar = \sum_p \langle \beta_j, \beta_p\rangle \frac{\partial}{\partial Z(\beta_p)} \log \tau_\hbar$$

for all j.

Theorem 72. *Fix a finite, uncoupled variation of BPS structure as above. The vector function*

$$\exp\left(\frac{1}{\hbar}\sum_{m=1}^{\infty}\sum_{i\,|\,\langle\gamma_i,\beta_j\rangle\neq 0}\frac{(-1)^{m\langle\gamma_i,\beta_j\rangle}}{m}\log\Psi^{(m),ij}_\hbar((2\pi)^{-1}\sqrt{-1}t)_{(12)}\right)$$

(which equals the vector $\Psi_{\beta_j}(t) + O(\hbar)$ by Theorem 66) admits the first order tau function

$$\exp\left(\frac{1}{\hbar}\sum_{m=1}^{\infty}\sum_{i}\log\tau^{(m),i}_\hbar((2\pi)^{-1}it)_{(12)}\right), \tag{7.2}$$

and the latter equals $\tau_\ell(t, Z)$. This implies in particular Theorem 9.

Proof. Fix i, j, and evaluate at $(2\pi)^{-1}t$. We have

$$\begin{aligned}
\frac{\partial}{\partial t}(\log\Psi^{(m),ij}_\hbar)_{(12)} &= \langle\gamma_i,\beta_j\rangle\frac{\partial}{\partial Z(\gamma_i)}\log\tau^{(m),i}_\hbar\\
&= \sum_p\langle\beta_p,\beta_j\rangle c_{ip}\frac{\partial}{\partial Z(\gamma_i)}\log\tau^{(m),i}_\hbar\\
&= \sum_p\langle\beta_p,\beta_j\rangle\frac{\partial Z(\gamma_i)}{\partial Z(\beta_p)}\frac{\partial}{\partial Z(\gamma_i)}\log\tau^{(m),i}_\hbar\\
&= \sum_p\langle\beta_p,\beta_j\rangle\frac{\partial}{\partial Z(\beta_p)}\log\tau^{(m),i}_\hbar.
\end{aligned}$$

To prove the first part of the claim sum over all i and note that the right hand side vanishes when $\langle\gamma_i, \beta_j\rangle = 0$. Arguing as in Corollary 69 shows that the function 7.2 equals $\tau_\ell(t, Z) = \prod_i \Upsilon^{\Omega(\gamma_i)}\left(\frac{Z(\gamma_i)}{t}\right)$ as required. The last claim that $\tau_\ell(t, Z)$ is a tau function for $\Psi(t)$ follows at once. □

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