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SPEED OF EXTINCTION FOR CONTINUOUS-STATE BRANCHING PROCESSES IN A WEAKLY SUBCRITICAL LÉVY ENVIRONMENT

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Abstract

We continue with the systematic study of the speed of extinction of continuous-state branching processes in Lévy environments under more general branching mechanisms. Here, we deal with the weakly subcritical regime under the assumption that the branching mechanism is regularly varying. We extend recent results of Li and Xu (2018) and Palau et al. (2016), where it is assumed that the branching mechanism is stable, and complement the recent articles of Bansaye et al. (2021) and Cardona-Tobón and Pardo (2021), where the critical and the strongly and intermediate subcritical cases were treated, respectively. Our methodology combines a path analysis of the branching process together with its Lévy environment, fluctuation theory for Lévy processes, and the asymptotic behaviour of exponential functionals of Lévy processes. Our approach is inspired by the last two previously cited papers, and by Afanasyev et al. (2012), where the analogue was obtained.

Keywords: Continuous-state branching processes; Lévy process; Lévy process conditioned to stay positive; random environment; long-term behaviour; extinction

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1. Introduction and main results

We are interested in continuous-state branching processes in random environments, in particular when the environment is driven by a Lévy process. This family of processes is known as *continuous-state branching processes in a Lévy environment* (or CBLEs for short) and they have been constructed independently in [11, 16] as the unique non-negative strong solution of a stochastic differential equation whose linear term is driven by a Lévy process.

Classification of the asymptotic behaviour of rare events of CBLEs, such as the survival probability, depends on the long-term behaviour of the environment. In other words, an auxiliary Lévy process, which is associated to the environment, leads to the usual classification for the long-term behaviour of branching processes. To be more precise, the CBLE is called *supercritical*, *critical*, or *subcritical* according as the auxiliary Lévy process drifts to ∞, oscillates, or drifts to $-\infty$. Furthermore, in the subcritical regime another phase transition arises which depends on whether the Lévy process drifts to ∞, oscillates, or drifts to $-\infty$ under a

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suitable exponential change of measure. These regimes are known in the literature as *strongly*, *intermediate*, and *weakly subcritical* regimes, respectively.

The study of the long-term behaviour of CBLEs has attracted considerable attention in the last decade; see, for instance, [3, 6, 11, 14–17, 19]. All the aforementioned studies deal with the case when the branching mechanism is associated to a stable jump structure or a Brownian component on the branching term. For simplicity of exposition we call such branching mechanisms *stable*. [3] determined the long-term behaviour for stable CBLEs when the random environment is driven by a Lévy process with bounded variation paths. [15] studied the case when the random environment is driven by a Brownian motion with drift. Then, [14, 16] independently extended this result to the case when the environment is driven by a general Lévy process. More recently, [19] provided an exact description for the speed of the extinction probability for CBLEs with a stable branching mechanism and where the Lévy environment is heavy-tailed. It is important to note that all these manuscripts exploited explicit knowledge of the survival probability, which is given in terms of exponential functionals of Lévy processes.

Much less is known about the long-term behaviour of CBLEs when the associated branching mechanism is more general. To our knowledge, the only studies in this direction are [2, 7], where the speed of extinction for more general branching mechanisms is studied. More precisely, [2] focused on the critical case (oscillating Lévy environments satisfying the so-called Spitzer condition at ∞) and relaxed the assumption that the branching mechanism is stable. Shortly afterwards, [7] studied the speed of extinction of CBLEs in the strongly and intermediate subcritical regimes. Their methodology combines a path analysis of the branching process together with its Lévy environment, fluctuation theory for Lévy processes, and the asymptotic behaviour of exponential functionals of Lévy processes.

Here we continue with such systematic study on the asymptotic behaviour of the survival probability for CBLEs under more general branching mechanisms, but now in the weakly subcritical regime. It is important to note that extending such asymptotic behaviour to more general branching mechanisms is not as easy as we might think since we are required to control a functional of the associated Lévy process to the environment, which is somehow quite involved. Moreover, contrary to the discrete case, the state 0 can be polar and the process might be very close to 0 but never reach this point. To focus on the absorption event, we use Grey's condition which guarantees that 0 is accessible.

Our main contribution is to provide its precise asymptotic behaviour under some assumptions on the auxiliary Lévy process and the branching mechanism. In particular, we obtain that the speed of the survival probability decays exponentially with a polynomial factor of order $3/2$ (up to a multiplicative constant which is computed explicitly and depends on the limiting behaviour of the survival probability, given favorable environments). In particular, for the stable case we recover the results of [14] where the limiting constant is given in terms of the exponential functional of the Lévy process. In order to deduce such asymptotic behaviour, we combine the approach developed in [1], for the discrete-time setting, with fluctuation theory of Lévy processes and a similar strategy developed in [2]. A key point in our arguments is to rewrite the probability of survival under a suitable change of measure which is associated to an exponential martingale of the Lévy environment. In order to do so, the existence of some exponential moments for the Lévy environment is required. Under this exponential change of measure the Lévy environment now oscillates and we can apply a similar strategy developed in [2] to study the extinction rate for CBLEs in the critical regime. More precisely, under this new measure, we split the event of survival into two parts, i.e. when the running infimum is either negative or positive, and then we show that only paths of the Lévy process with a positive running infimum give a substantial contribution to the speed of survival. In this regime, we assume that the branching mechanism is regularly varying and a lower bound for the branching mechanism, which allows us to control the events of survival under favourable environments and unfavourable environments, respectively. Our results complement those in [2, 7].

1.1. Main results

Let $(\Omega^{(\mathrm{b})}, \mathcal{F}^{(\mathrm{b})}, (\mathcal{F}_t^{(\mathrm{b})})_{t\geq 0}, \mathbb{P}^{(\mathrm{b})})$ be a filtered probability space satisfying the usual hypothesis on which we may construct the demographic (branching) term of the model that we are interested in. We suppose that $(B_t^{(\mathrm{b})}, t \geq 0)$ is an $(\mathcal{F}_t^{(\mathrm{b})})_{t\geq 0}$-adapted standard Brownian motion, and $N^{(\mathrm{b})}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)$ is an $(\mathcal{F}_t^{(\mathrm{b})})_{t\geq 0}$-adapted Poisson random measure on $\mathbb{R}_+^3$ with intensity $\mathrm{d}s\, \mu(\mathrm{d}z)\, \mathrm{d}u$, where μ satisfies

$$\int_{(0,\infty)} (z \wedge z^2)\, \mu(\mathrm{d}z) < \infty. \tag{1}$$

We denote by $\widetilde{N}^{(\mathrm{b})}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)$ the compensated version of $N^{(\mathrm{b})}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)$. Further, we also introduce the so-called branching mechanism ψ, a convex function with the Lévy–Khintchine representation

$$\psi(\lambda) = \psi'(0+)\lambda + \varrho^2\lambda^2 + \int_{(0,\infty)} (\mathrm{e}^{-\lambda x} - 1 + \lambda x)\, \mu(\mathrm{d}x), \qquad \lambda \geq 0,$$

where $\varrho \geq 0$. Observe that the term $\psi'(0+)$ is well defined (finite) since condition (1) holds. Moreover, the function ψ describes the stochastic dynamics of the population.

On the other hand, for the environmental term we consider another filtered probability space $(\Omega^{(\mathrm{e})}, \mathcal{F}^{(\mathrm{e})}, (\mathcal{F}_t^{(\mathrm{e})})_{t\geq 0}, \mathbb{P}^{(\mathrm{e})})$ satisfying the usual hypotheses. Let us consider $\sigma \geq 0$ and α real constants; and π a measure concentrated on $\mathbb{R} \setminus \{0\}$ such that $\int_{\mathbb{R}} (1 \wedge z^2)\pi(\mathrm{d}z) < \infty$. Suppose that $(B_t^{(\mathrm{e})}, t \geq 0)$ is an $(\mathcal{F}_t^{(\mathrm{e})})_{t\geq 0}$-adapted standard Brownian motion, $N^{(\mathrm{e})}(\mathrm{d}s, \mathrm{d}z)$ is an $(\mathcal{F}_t^{(\mathrm{e})})_{t\geq 0}$-Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}$ with intensity $\mathrm{d}s\, \pi(\mathrm{d}z)$, and $\widetilde{N}^{(\mathrm{e})}(\mathrm{d}s, \mathrm{d}z)$ its compensated version. We denote by $S = (S_t, t \geq 0)$ a Lévy process, i.e. a process with càdlàg paths and stationary and independent increments, with the Lévy–Itô decomposition

$$S_t = \alpha t + \sigma B_t^{(\mathrm{e})} + \int_0^t \int_{(-1,1)} (\mathrm{e}^z - 1)\, \widetilde{N}^{(\mathrm{e})}(\mathrm{d}s, \mathrm{d}z) + \int_0^t \int_{(-1,1)^c} (\mathrm{e}^z - 1)\, N^{(\mathrm{e})}(\mathrm{d}s, \mathrm{d}z).$$

Note that S is a Lévy process with no jumps smaller than or equal to -1.

In our setting, we consider independent processes for the demographic and environmental terms. More precisely, we work now on the space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, the direct product of the two probability spaces defined above, i.e. $\Omega := \Omega^{(\mathrm{e})} \times \Omega^{(\mathrm{b})}$, $\mathcal{F} := \mathcal{F}^{(\mathrm{e})} \otimes \mathcal{F}^{(\mathrm{b})}$, $\mathcal{F}_t := \mathcal{F}_t^{(\mathrm{e})} \otimes \mathcal{F}_t^{(\mathrm{b})}$ for $t \geq 0$, $\mathbb{P} := \mathbb{P}^{(\mathrm{e})} \otimes \mathbb{P}^{(\mathrm{b})}$. Therefore, $(Z_t, t \geq 0)$, the *continuous-state branching process in the Lévy environment* $(S_t, t \geq 0)$ is defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ as the unique non-negative strong solution of the stochastic differential equation

$$\begin{aligned} Z_t = Z_0 - \psi'(0+)\int_0^t Z_s\, \mathrm{d}s + \int_0^t \sqrt{2\varrho^2 Z_s}\, \mathrm{d}B_s^{(\mathrm{b})} \\ + \int_0^t \int_{(0,\infty)} \int_0^{Z_{s-}} z\, \widetilde{N}^{(\mathrm{b})}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) + \int_0^t Z_{s-}\, \mathrm{d}S_s. \end{aligned} \tag{2}$$

According to [11, Theorem 3.1] or [16, Theorem 1], the equation has a unique non-negative strong solution which is not explosive. An important property satisfied by Z is that, given the

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environment, it inherits the branching property of the underlying continuous-state branching process. We denote by $\mathbb{P}_z$ its law starting from $z \geq 0$, and by $\mathbb{E}_z$ its expectation.

The analysis of the process Z is deeply related to the behaviour and fluctuations of the Lévy process $\xi = (\xi_t, t \geq 0)$, defined as

$$\xi_t = \overline{\alpha}t + \sigma B_t^{(e)} + \int_0^t \int_{(-1,1)} z\,\widetilde{N}^{(e)}(ds, dz) + \int_0^t \int_{(-1,1)^c} z\,N^{(e)}(ds, dz),$$

where

$$\overline{\alpha} := \alpha - \psi'(0+) - \frac{\sigma^2}{2} - \int_{(-1,1)} (e^z - 1 - z)\,\pi(dz).$$

Note that both processes S and ξ generate the same natural filtration. Indeed, the process ξ is obtained from S, changing only the drift and jump sizes. In addition, we see that the drift term $\overline{\alpha}$ provides the interaction between the demographic and environmental parameters. We denote by $\mathbb{P}_x^{(e)}$ (respectively $\mathbb{E}_x^{(e)}$ for its expectation) the law of the process ξ starting from $x \in \mathbb{R}$, and when $x = 0$ we use the notation $\mathbb{P}^{(e)}$ for $\mathbb{P}_0^{(e)}$ (respectively $\mathbb{E}^{(e)}$ for its expectation). We also denote by $\mathbb{P}_{(z,x)}$ the law of the process (Z, ξ) starting at (z,x), and $\mathbb{E}_{(z,x)}$ for its expectation.

Further, under condition (1), the process $(Z_t e^{-\xi_t}, t \geq 0)$ is a quenched martingale, implying that, for any $t \geq 0$ and $z \geq 0$,

$$\mathbb{E}_z[Z_t \mid S] = z e^{\xi_t} \quad \mathbb{P}_z\text{-a.s;} \tag{3}$$

see [2]. In other words, the process ξ plays an analogous role to the random walk associated to the logarithm of the mean of the offsprings in the discrete-time framework and leads to the usual classification for the long-term behaviour of branching processes. More precisely, we say that the process Z is subcritical, critical, or supercritical according as ξ drifts to $-\infty$, oscillates, or drifts to $+\infty$.

In addition, under condition (1), there is another quenched martingale associated to $(Z_t e^{-\xi_t}, t \geq 0)$ which allows us to compute its Laplace transform; see, for instance, [16, Proposition 2] or [11, Theorem 3.4]. In order to compute the Laplace transform of $Z_t e^{-\xi_t}$, we first introduce the unique positive solution $(v_t(s, \lambda, \xi), s \in [0, t])$ of the backward differential equation

$$\frac{\partial}{\partial s} v_t(s, \lambda, \xi) = e^{\xi_s} \psi_0(v_t(s, \lambda, \xi) e^{-\xi_s}), \qquad v_t(t, \lambda, \xi) = \lambda, \tag{4}$$

where $\psi_0(\lambda) = \psi(\lambda) - \lambda\psi'(0+) = \varrho^2\lambda^2 + \int_{(0,\infty)} (e^{-\lambda x} - 1 + \lambda x)\,\mu(dx)$. Then the process $(\exp\{-v_t(s, \lambda, \xi) Z_s e^{-\xi_s}\},\ 0 \leq s \leq t)$ is a quenched martingale, implying that, for any $\lambda \geq 0$ and $t \geq s \geq 0$,

$$\mathbb{E}_{(z,x)}[\exp\{-\lambda Z_t e^{-\xi_t}\} \mid S, \mathcal{F}_s^{(b)}] = \exp\{-Z_s e^{-\xi_s} v_t(s, \lambda e^{-x}, \xi)\}. \tag{5}$$

We may think of $v_t(\cdot, \cdot, \xi)$ as an inhomogeneous cumulant semigroup determined by the time-dependent branching mechanism $(s, \theta) \mapsto e^{\xi_s}\psi_0(\theta e^{-\xi_s})$. The functional $v_t(\cdot, \cdot, \xi)$ is quite involved, except for a few cases (stable and Neveu cases), due to the stochasticity coming from the time-dependent branching mechanism which makes it not so easy to control.

In what follows, we assume that ξ is not a compound Poisson process to avoid the possibility that the process visits the same maxima or minima at distinct times, which could make our analysis more involved. Moreover, we also require the following exponential moment condition:

$$\text{there exists } \vartheta > 1 \text{ such that } \int_{\{|x|>1\}} e^{\lambda x}\,\pi(dx) < \infty \text{ for all } \lambda \in [0, \vartheta), \tag{H1}$$

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which is equivalent to the existence of the Laplace transform on $[0, \vartheta)$, i.e. $\mathbb{E}^{(e)}[e^{\lambda\xi_1}]$ is finite for $\lambda \in [0, \vartheta)$ (see, for instance, [18, Lemma 26.4]). The latter implies that we can introduce the Laplace exponent of ξ as $\Phi_\xi(\lambda) := \log \mathbb{E}^{(e)}[e^{\lambda\xi_1}]$ for $\lambda \in [0, \vartheta)$. Again from [18, Lemma 26.4], we also have $\Phi_\xi(\lambda) \in C^\infty$ and $\Phi''_\xi(\lambda) > 0$ for $\lambda \in (0, \vartheta)$.

Another object which will be relevant in our analysis is the so-called exponential martingale associated to the Lévy process ξ, i.e. $M_t^{(\lambda)} = \exp\{\lambda\xi_t - t\Phi_\xi(\lambda)\}$, $t \geq 0$, which is well defined for $\lambda \in [0, \vartheta)$ under the assumption (H1). It is well known that $(M_t^{(\lambda)}, t \geq 0)$ is an $(\mathcal{F}_t^{(e)})_{t\geq 0}$-martingale and that it induces a change of measure which is known as the Esscher transform:

$$\mathbb{P}^{(e,\lambda)}(\Lambda) := \mathbb{E}^{(e)}[M_t^{(\lambda)}\mathbf{1}_\Lambda] \qquad \text{for } \Lambda \in \mathcal{F}_t^{(e)}. \tag{6}$$

Under the probability $\mathbb{P}^{(e,\lambda)}$, the process ξ is still a Lévy process whose characteristic triplet can be computed explicitly, see, for instance, [13, Theorem 3.9]. We introduce $\mathbb{P}_x^{(e,\lambda)}$ for the law of ξ starting at x, under $\mathbb{P}^{(e,\lambda)}$. Their respective expectations are denoted by $\mathbb{E}_x^{(e,\lambda)}$ and $\mathbb{E}^{(e,\lambda)}$. Similarly, we may introduce $\mathbb{P}_{(z,x)}^{(\lambda)}$, the measure induced by the Esscher transform $M^{(\lambda)}$ under the measure $\mathbb{P}_{(z,x)}$ (with respect to $(\mathcal{F}_t)_{t\geq 0}$), and its associated expectation $\mathbb{E}_{(z,x)}^{(\lambda)}$. It is important to note that such a transform only affects the environmental terms and none of the demographic terms.

Another important object in our analysis is the so-called dual process, which is defined as $\widehat{\xi} = -\xi$ and turns out to also be a Lévy process satisfying that, for any fixed time $t > 0$, the processes $(\xi_{(t-s)^-} - \xi_t, 0 \leq s \leq t)$ and $(\widehat{\xi}_s, 0 \leq s \leq t)$ have the same law, with the convention that $\xi_{0^-} = \xi_0$ (see, for instance, [13, Lemma 3.4]). For every $x \in \mathbb{R}$, let $\widehat{\mathbb{P}}_x^{(e)}$ be the law of $x + \xi$ under $\widehat{\mathbb{P}}^{(e)}$, i.e. the law of $\widehat{\xi}$ under $\mathbb{P}_{-x}^{(e)}$. We also introduce the running infimum and supremum of ξ by $\underline{\xi}_t = \inf_{0\leq s\leq t} \xi_s$ and $\overline{\xi}_t = \sup_{0\leq s\leq t} \xi_s$ for $t \geq 0$. Similarly to the critical case studied in [2], the asymptotic analysis of the weakly subcritical regime requires the notion of the renewal functions $U^{(\lambda)}$ and $\widehat{U}^{(\lambda)}$ under $\mathbb{P}^{(e,\lambda)}$, which are associated to the supremum and infimum of ξ, respectively. See Section 2.1 for a proper definition (or the references therein).

For our purposes, we also require the notion of conditioned Lévy processes and continuous-state branching processes in a conditioned Lévy environment. According to [9, Lemma 1], under the assumption that ξ does not drift towards $-\infty$, we have that the renewal function $\widehat{U} := \widehat{U}^{(0)}$ is invariant for the process that is killed when it first enters $(-\infty, 0)$. In other words, for all $x > 0$ and $t \geq 0$, $\mathbb{E}_x^{(e)}\big[\widehat{U}(\xi_t)\mathbf{1}_{\{\underline{\xi}_t > 0\}}\big] = \widehat{U}(x)$. Hence, from the Markov property, we deduce that $(\widehat{U}(\xi_t)\mathbf{1}_{\{\underline{\xi}_t>0\}}, t \geq 0)$ is a martingale with respect to $(\mathcal{F}_t^{(e)})_{t\geq 0}$. We may now use this martingale to define a change of measure corresponding to the law of ξ *conditioned to stay positive* as a Doob-h transform. Under the assumption that ξ does not drift towards $-\infty$, the law of the process ξ conditioned to stay positive is defined as follows, for $\Lambda \in \mathcal{F}_t^{(e)}$ and $x > 0$:

$$\mathbb{P}_x^{(e),\uparrow}(\Lambda) := \frac{1}{\widehat{U}(x)}\mathbb{E}_x^{(e)}\big[\widehat{U}(\xi_t)\mathbf{1}_{\{\underline{\xi}_t>0\}}\mathbf{1}_\Lambda\big].$$

We denote by $\mathbb{E}_x^{(e),\uparrow}$ its associated expectation.

On the other hand, by duality, under the assumption that ξ does not drift towards ∞, the law of the process ξ *conditioned to stay negative* is defined for $x < 0$ as

$$\mathbb{P}_x^{(e),\downarrow}(\Lambda) := \frac{1}{U(-x)}\mathbb{E}_x^{(e)}\big[U(-\xi_t)\mathbf{1}_{\{\overline{\xi}_t<0\}}\mathbf{1}_\Lambda\big],$$

where $U := U^{(0)}$. The associated expectation is denoted by $\mathbb{E}_x^{(e),\downarrow}$.

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As above, for $\lambda \in [0, \vartheta)$ we can also introduce the probability measures $\mathbb{P}_x^{(e,\lambda),\uparrow}$ and $\mathbb{P}_x^{(e,\lambda),\downarrow}$ using the renewal functions $\widehat{U}^{(\lambda)}$ and $U^{(\lambda)}$, respectively, and under the probability measure $\mathbb{P}_x^{(e,\lambda)}$. Their respective expectations are defined by $\mathbb{E}_x^{(e,\lambda),\uparrow}$ and $\mathbb{E}_x^{(e,\lambda),\downarrow}$.

Lévy processes conditioned to stay positive (and negative) are well-studied objects. For a complete overview of this theory the reader is referred to [4, 8, 9] and references therein.

Similarly to the definition of Lévy processes conditioned to stay positive (and negative) given above, we may introduce a continuous-state branching processes in a Lévy environment conditioned to stay positive as a Doob-h transform. The aforementioned process was first investigated in [2] with the aim of studying the survival event in a critical Lévy environment. In other words, [2] proved the following result.

Lemma 1. ([2].) *Let us assume that* $z, x > 0$. *Under the law* $\mathbb{P}_{(z,x)}$, *the process* $(\widehat{U}(\xi_t)\mathbf{1}_{\{\underline{\xi}_t>0\}}, t \geq 0)$ *is a martingale with respect to* $(\mathcal{F}_t)_{t\geq 0}$. *Moreover, the following Doob-h transform holds for* $\Lambda \in \mathcal{F}_t$:

$$\mathbb{P}^{\uparrow}_{(z,x)}(\Lambda) := \frac{1}{\widehat{U}(x)}\mathbb{E}_{(z,x)}\big[\widehat{U}(\xi_t)\mathbf{1}_{\{\underline{\xi}_t>0\}}\mathbf{1}_\Lambda\big],$$

and defines a continuous-state branching process in a Lévy environment ξ *conditioned to stay positive.*

Furthermore, appealing to duality and Lemma 1, we may deduce that, under $\mathbb{P}_{(z,x)}$ with $z > 0$ and $x < 0$, the process $(U(-\xi_t)\mathbf{1}_{\{\overline{\xi}_t<0\}}, t \geq 0)$ is a martingale with respect to $(\mathcal{F}_t)_{t\geq 0}$. Hence, the law of *continuous-state branching processes in a Lévy environment* ξ *conditioned to stay negative* is defined as follows: for $z > 0$, $x < 0$, and $\Lambda \in \mathcal{F}_t$,

$$\mathbb{P}^{\downarrow}_{(z,x)}(\Lambda) := \frac{1}{U(-x)}\mathbb{E}_{(z,x)}\big[U(-\xi_t)\mathbf{1}_{\{\overline{\xi}_t<0\}}\mathbf{1}_\Lambda\big]. \tag{7}$$

We denote by $\mathbb{E}^{\uparrow}_{(z,x)}$ and $\mathbb{E}^{\downarrow}_{(z,x)}$ their respective expectation operators.

Observe that, for $\lambda \in [0, \vartheta)$, we may also introduce the probability measures $\mathbb{P}^{(\lambda),\uparrow}_{(z,x)}$ and $\mathbb{P}^{(\lambda),\downarrow}_{(z,x)}$, similarly to Lemma 1 and (7), using the renewal functions $\widehat{U}^{(\lambda)}$ and $U^{(\lambda)}$, respectively, and under the probability measure $\mathbb{P}^{(\lambda)}_{(z,x)}$. Their respective expectation operators are defined by $\mathbb{E}^{(\lambda),\uparrow}_{(z,x)}$ and $\mathbb{E}^{(\lambda),\downarrow}_{(z,x)}$.

Recall that we are interested in the probability of survival under the *weakly subcritical regime*. More precisely, we say that Z is weakly subcritical if (H1) is satisfied and the Laplace exponent of ξ is such that $\Phi'_\xi(0) < 0 < \Phi'_\xi(1)$ and there exists $\gamma \in (0, 1)$ that solves $\Phi'_\xi(\gamma) = 0$. In other words, the Lévy process ξ drifts to $-\infty$ a.s. under $\mathbb{P}^{(e)}$, oscillates a.s. under $\mathbb{P}^{(e,\gamma)}$, and drifts to $+\infty$ a.s. under $\mathbb{P}^{(e,1)}$. In the remainder of this manuscript, we will always assume that the process Z is in the weakly subcritical regime.

Our first main result requires that the branching mechanism ψ_0 is regularly varying at 0, i.e. there exists $\beta \in (0, 1]$ such that

$$\psi_0(\lambda) = \lambda^{1+\beta}\ell(\lambda), \tag{H2}$$

where ℓ is a slowly varying function at 0. See [5] for a proper definition.

For simplicity of exposition, for $\lambda \in [0, \vartheta)$ we introduce the function $\kappa^{(\lambda)}(0, \theta)$ as

$$\int_0^\infty e^{-\theta y} U^{(\lambda)}(y)\, dy = \frac{1}{\theta\kappa^{(\lambda)}(0, \theta)}, \qquad \theta > 0.$$

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Theorem 1. *Let $x, z > 0$. Assume that Z is weakly subcritical and that condition (H2) holds. Then the random variable $\mathcal{U}_t := Z_t e^{-\xi_t}$ converges in distribution to some random variable Q with values in $[0, \infty)$ as $t \to \infty$ under $\mathbb{P}_{(z,x)}(\cdot \mid \underline{\xi}_t > 0)$. Moreover,*

$$\mathfrak{b}(z, x) := \lim_{t\to\infty} \mathbb{P}_{(z,x)}\big(Z_t > 0 \mid \underline{\xi}_t > 0\big) > 0, \tag{8}$$

where

$$\mathfrak{b}(z, x) = 1 - \lim_{\lambda\to\infty} \lim_{s\to\infty} \int_0^\infty \int_0^1 \int_0^\infty w^{\mathbf{u}} \mathbb{P}^{(\gamma),\uparrow}_{(z,x)}(\mathcal{U}_s \in \mathrm{d}\mathbf{u}) \mathbb{P}^{(e,\gamma),\downarrow}_{-y}\big(\widehat{W}_s(\lambda) \in \mathrm{d}w\big)\, \mu_\gamma(\mathrm{d}y),$$

with $\widehat{W}_s(\lambda) := \exp\{-v_s(0, \lambda, \widehat{\xi})\}$ and $\mu_\gamma(\mathrm{d}y) := \gamma\kappa^{(\gamma)}(0, \gamma)e^{-\gamma y}U^{(\gamma)}(y)\mathbf{1}_{\{y>0\}}\,\mathrm{d}y$.

It is important to note that, in general, it seems difficult to explicitly compute the constant $\mathfrak{b}(z, x)$ except for the stable case. In the stable case, we observe that the constant $\mathfrak{b}(z, x)$ is given in terms of two independent exponential functionals of conditioned Lévy processes. Denote by $\mathtt{I}_{s,t}(\beta\xi)$ the exponential functional of the Lévy process $\beta\xi$, i.e.

$$\mathtt{I}_{s,t}(\beta\xi) := \int_s^t e^{-\beta\xi_u}\,\mathrm{d}u, \qquad 0 \le s \le t. \tag{9}$$

Hence, when $\psi_0(\lambda) = C\lambda^{1+\beta}$ with $C > 0$ and $\beta \in (0, 1)$, we have

$$\mathfrak{b}(z, x) = \gamma\kappa^{(\gamma)}(0, \gamma) \int_0^\infty e^{-\gamma y}U^{(\gamma)}(y)G_{z,x}(y)\,\mathrm{d}y,$$

where

$$G_{z,x}(y) := \int_0^\infty \int_0^\infty (1 - \exp\{-ze^{-x}(\beta Cw + \beta Cu)^{-1/\beta}\}) \times \mathbb{P}^{(\gamma),\uparrow}_{(z,x)}(\mathtt{I}_{0,\infty}(\beta\xi) \in \mathrm{d}w)\mathbb{P}^{(e,\gamma),\downarrow}_{-y}(\mathtt{I}_{0,\infty}(\beta\widehat{\xi}) \in \mathrm{d}u). \tag{10}$$

We refer to Section 2.4 for further details about the computation of this constant.

Under the assumption that Z is weakly subcritical, the running infimum of the auxiliary process ξ satisfies the following asymptotic behaviour: for $x > 0$,

$$\mathbb{P}^{(e)}_x(\underline{\xi}_t > 0) \sim \frac{A_\gamma}{\gamma\kappa^{(\gamma)}(0, \gamma)} e^{\gamma x}\widehat{U}^{(\gamma)}(x)t^{-3/2}e^{\Phi_\xi(\gamma)t} \qquad \text{as } t \to \infty, \tag{11}$$

where

$$A_\gamma := \frac{1}{\sqrt{2\pi\Phi''_\xi(\gamma)}} \exp\left\{\int_0^\infty (e^{-t} - 1)t^{-1}e^{-t\Phi_\xi(\gamma)}\mathbb{P}^{(e)}(\xi_t = 0)\,\mathrm{d}t\right\}; \tag{12}$$

see, for instance, [12, Lemma A] (see also [14, Proposition 4.1]). Such an asymptotic turns out to be the leading term in the asymptotic behaviour of the probability of survival as stated below.

Theorem 2. (Weakly subcritical regime.) *Let $z > 0$. Assume that Z is weakly subcritical and that the slowly varying function in (H2) satisfies that there exists a constant $C > 0$ such that $\ell(\lambda) > C$. Then $\lim_{t\to\infty} t^{3/2}e^{-\Phi_\xi(\gamma)t}\mathbb{P}_z(Z_t > 0) = \mathfrak{B}(z)$, with*

$$\mathfrak{B}(z) := \frac{A_\gamma}{\gamma\kappa^{(\gamma)}(0, \gamma)} \lim_{x\to\infty} \mathfrak{b}(z, x)e^{\gamma x}\widehat{U}^{(\gamma)}(x) \in (0, \infty),$$

where $\mathfrak{b}(z, x)$ and A_γ are the constants defined in (8) and (12), respectively.

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It is important to note that in the stable case, the constant $\mathfrak{B}(z)$ coincides with the constant that appears in [14, Theorem 5.1], i.e.

$$\mathfrak{B}(z) = A_\gamma \lim_{x\to\infty} \mathrm{e}^{\gamma x}\widehat{U}^{(\gamma)}(x) \int_0^\infty \mathrm{e}^{-\gamma y} U^{(\gamma)}(y) G_{z,x}(y)\,\mathrm{d}y,$$

where $G_{z,x}$ is defined in (10).

Remark 1. Note that our assumption (H2) clearly implies that

$$\int^\infty x \log^2(x)\,\mu(\mathrm{d}x) < \infty. \tag{13}$$

The latter condition was used before in [2, Proposition 3.4] to control the effect of a favourable environment on the event of survival. Unlike the critical case, in the weakly subcritical regime the slightly stronger condition (H2) is required to guarantee the convergence in Theorem 1, which allows us to have good control on the event of survival given favourable environments. A crucial ingredient in Theorem 1 is an extension of a sort of functional limit theorem for conditioned Lévy and CBLE processes (see Proposition 1). More precisely, we would require the asymptotic independence of the processes $((Z_u, \xi_u), 0 \le u \le r \mid \underline{\xi}_t > 0)$ and $(\xi_{(t-u)^-}, 0 \le u \le \delta t \mid \underline{\xi}_t > 0)$ as t goes to ∞, for every $r, t \ge 0$ and $\delta \in (0, 1)$. We claim that this result must be true in full generality (in particular Theorem 1 under (13)) since it holds for random walks (see [1, Theorem 2.7]), but it seems not so easy to deduce. Meanwhile in the discrete setting the result follows directly from duality, in the Lévy case the convergence will depend on a much deeper analysis on the asymptotic behaviour for bridges of Lévy processes and their conditioned version. It seems that a better understanding of conditioned Lévy bridges is required.

Remark 2. The condition that the slowly varying function ℓ is bounded from below is required to control the absorption event under unfavourable environments (see Lemma 7) and to a.s. guarantee absorption. Indeed, under Grey's condition,

$$\int^\infty \frac{1}{\psi_0(\lambda)}\,\mathrm{d}\lambda < \infty,$$

and (5), we deduce that, for $z, x > 0$ and $y \ge 0$,

$$\mathbb{P}_{(z,x)}(Z_t > 0,\ \underline{\xi}_t \le -y) = \mathbb{E}^{(\mathrm{e})}\big[(1 - \mathrm{e}^{-z v_t(0,\infty,\xi)})\mathbf{1}_{\{\underline{\xi}_t \le -y-x\}}\big], \tag{14}$$

where $v_t(0, \infty, \xi)$ is $\mathbb{P}^{(\mathrm{e})}$-a.s. finite for all $t \ge 0$, (see [11, Theorem 4.1 and Corollary 4.4]) but perhaps equals 0. We note that (13) (and implicitly (H2)) guarantees that $v_t(0, \infty, \xi) > 0$, $\mathbb{P}^{(\mathrm{e})}$-a.s. for all $t > 0$ (see, for instance, [16, Proposition 3]). Since the functional $v_t(0, \infty, \xi)$ depends strongly on the environment, it seems difficult to estimate the right-hand side of (14). Actually, it seems not so easy to obtain sharp control of (14). Condition (H2) implies that Grey's condition is fulfilled, and the assumption that ℓ is bounded from below allows us to upper bound (14) in terms of the exponential functional of ξ.

Finally, we point out that in the discrete setting such probability can be estimated directly in terms of the infimum of the environment since the event of survival is equal to the event that the current population is larger than or equal to one, something that cannot be performed in our setting.

The remainder of this paper is devoted to the proofs of the main results.

2. Proofs

This section is devoted to the proofs of our main results and the computation of the constant $\mathfrak{b}(z, x)$ in the stable case. We start with some preliminaries on Lévy processes.

2.1. Lévy processes

Recall that $\mathbb{P}_x^{(\mathrm{e})}$ denotes the law of the Lévy process ξ starting from $x \in \mathbb{R}$, and when $x = 0$ we use the notation $\mathbb{P}^{(\mathrm{e})}$ for $\mathbb{P}_0^{(\mathrm{e})}$. We also recall that $\widehat{\xi} = -\xi$ denotes the dual process and denote by $\widehat{\mathbb{P}}_x^{(\mathrm{e})}$ its law starting at $x \in \mathbb{R}$.

In what follows, we require the notion of the reflected processes $\xi - \underline{\xi}$ and $\overline{\xi} - \xi$ which are Markov processes with respect to the filtration $(\mathcal{F}_t^{(\mathrm{e})})_{t\geq 0}$ and whose semigroups satisfy the Feller property (see, for instance, [4, Proposition VI.1]). We denote by $L = (L_t, t \geq 0)$ and $\widehat{L} = (\widehat{L}_t, t \geq 0)$ the local times of $\overline{\xi} - \xi$ and $\xi - \underline{\xi}$ at 0, respectively, in the sense of [4, Chapter IV]. If 0 is regular for $(-\infty, 0)$ or regular downwards, i.e. $\mathbb{P}^{(\mathrm{e})}(\tau_0^- = 0) = 1$, where $\tau_0^- = \inf\{s > 0 \colon \xi_s \leq 0\}$, then 0 is regular for the reflected process $\xi - \underline{\xi}$ and then, up to a multiplicative constant, $\widehat{L}$ is the unique additive functional of the reflected process whose set of increasing points is $\{t \colon \xi_t = \underline{\xi}_t\}$. If 0 is not regular downwards then the set $\{t \colon \xi_t = \underline{\xi}_t\}$ is discrete and we define the local time $\widehat{L}$ as the counting process of this set. The same properties hold for L by duality.

Let us denote by L^{-1} and $\widehat{L}^{-1}$ the right-continuous inverse of L and $\widehat{L}$, respectively. The range of the inverse local times L^{-1} and $\widehat{L}^{-1}$ correspond to the sets of real times at which new maxima and new minima occur, respectively. Next, we introduce the so-called increasing ladder height process by $H_t = \overline{\xi}_{L_t^{-1}}, t \geq 0$. The pair (L^{-1}, H) is a bivariate subordinator, as is the pair $(\widehat{L}^{-1}, \widehat{H})$, with $\widehat{H}_t = -\underline{\xi}_{\widehat{L}_t^{-1}}, t \geq 0$. The range of the process H (resp. $\widehat{H}$) corresponds to the set of new maxima (resp. new minima). The pairs are known as descending and ascending ladder processes, respectively.

We also recall that $U^{(\lambda)}$ and $\widehat{U}^{(\lambda)}$ denote the renewal functions under $\mathbb{P}^{(\mathrm{e},\lambda)}$. Such functions are defined, for all $x > 0$, as

$$U^{(\lambda)}(x) := \mathbb{E}^{(\mathrm{e},\lambda)}\left[\int_{[0,\infty)} \mathbf{1}_{\{\overline{\xi}_t \leq x\}}\, \mathrm{d}L_t\right], \qquad \widehat{U}^{(\lambda)}(x) := \mathbb{E}^{(\mathrm{e},\lambda)}\left[\int_{[0,\infty)} \mathbf{1}_{\{\underline{\xi}_t \geq -x\}}\, \mathrm{d}\widehat{L}_t\right].$$

The renewal functions $U^{(\lambda)}$ and $\widehat{U}^{(\lambda)}$ are finite, subadditive, continuous, and increasing. Moreover, they are identically 0 on $(-\infty, 0]$, strictly positive on $(0, \infty)$, and satisfy $U^{(\lambda)}(x) \leq C_1 x$ and $\widehat{U}^{(\lambda)}(x) \leq C_2 x$ for any $x \geq 0$, where C_1, C_2 are finite constants (see, for instance, [10, Lemma 6.4 and Section 8.2]). Moreover, $U^{(\lambda)}(0) = 0$ and $\widehat{U}^{(\lambda)}(0) = 0$ if 0 is regular upwards; $U^{(\lambda)}(0) = 1$ and $\widehat{U}^{(\lambda)}(0) = 1$ otherwise.

Furthermore, it is important to note that by a simple change of variables, we can rewrite the renewal functions $U^{(\lambda)}$ and $\widehat{U}^{(\lambda)}$ in terms of the ascending and descending ladder height processes. Indeed, the measures induced by $U^{(\lambda)}$ and $\widehat{U}^{(\lambda)}$ can be rewritten as

$$U^{(\lambda)}(x) = \mathbb{E}^{(\mathrm{e},\lambda)}\left[\int_0^\infty \mathbf{1}_{\{H_t \leq x\}}\, \mathrm{d}t\right], \qquad \widehat{U}^{(\lambda)}(x) = \mathbb{E}^{(\mathrm{e},\lambda)}\left[\int_0^\infty \mathbf{1}_{\{\widehat{H}_t \leq x\}}\, \mathrm{d}t\right].$$

Roughly speaking, the renewal function $U^{(\lambda)}(x)$ (resp. $\widehat{U}^{(\lambda)}(x)$) “measures” the amount of time that the ascending (resp. descending) ladder height process spends on the interval $[0, x]$, and in particular induces a measure on $[0, \infty)$ which is known as the renewal measure. The latter

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implies that

$$\int_{[0,\infty)} \mathrm{e}^{-\theta x} U^{(\lambda)}(x)\,\mathrm{d}x = \frac{1}{\theta\kappa^{(\lambda)}(0,\theta)}, \qquad \theta > 0, \tag{15}$$

where $\kappa^{(\lambda)}(\cdot,\cdot)$ is the bivariate Laplace exponent of the ascending ladder process (L^{-1}, H) under $\mathbb{P}^{(\mathrm{e},\lambda)}$ (see, for instance, [4, 10, 13]).

2.2. Proof of Theorem 1

Our argument follows a similar strategy to [1], where the discrete setting is considered, although considering continuous time leads to significant changes, such as that 0 might be polar. Our first proposition is the continuous analogue of [1, Proposition 2.5] and in some sense is a generalisation of [12, Theorem 2(a)] (see also [14, Proposition 4.2]). In particular, the result tells us that, for every $r, s \geq 0$ and $s \leq t$, the conditional processes $((Z_u, \xi_u), 0 \leq u \leq r \mid \underline{\xi}_t > 0)$ and $(\xi_{(t-u)^-}, 0 \leq u \leq s \mid \underline{\xi}_t > 0)$ are asymptotically independent as $t \to \infty$.

Before we state our first result in this subsection, we recall that $\mathbb{D}([0,t])$ denotes the space of càdlàg real-valued functions on $[0,t]$ equipped with the Skorokhod topology.

Proposition 1. *Let f and g be continuous functionals on $\mathbb{D}([0,t])$. We also set $\mathcal{U}_r := g((Z_u,\xi_u), 0 \leq u \leq r)$, and, for $s \leq t$, $\widehat{W}_s := f(-\xi_u, 0 \leq u \leq s)$ and $\widetilde{W}_{t-s,t} := f(\xi_{(t-u)^-}, 0 \leq u \leq s)$. Then, for any bounded continuous function $\varphi\colon \mathbb{R}^3 \to \mathbb{R}$ and $x > 0$,*

$$\lim_{t\to\infty} \frac{\mathbb{E}^{(\gamma)}_{(z,x)}\big[\varphi(\mathcal{U}_r, \widetilde{W}_{t-s,t}, \xi_t)\mathrm{e}^{-\gamma\xi_t}\mathbf{1}_{\{\underline{\xi}_t>0\}}\big]}{\mathbb{E}^{(\mathrm{e},\gamma)}_x\big[\mathrm{e}^{-\gamma\xi_t}\mathbf{1}_{\{\underline{\xi}_t>0\}}\big]} = \int_0^\infty\int_0^\infty\int_0^\infty \varphi(u,v,y)\mathbb{P}^{(\gamma),\uparrow}_{(z,x)}(\mathcal{U}_r \in \mathrm{d}u)\mathbb{P}^{(\mathrm{e},\gamma),\downarrow}_{-y}\big(\widehat{W}_s \in \mathrm{d}v\big)\,\mu_\gamma(\mathrm{d}y),$$

with $\mu_\gamma(\mathrm{d}y) := \gamma\kappa^{(\gamma)}(0,\gamma)\mathrm{e}^{-\gamma y}U^{(\gamma)}(y)\mathbf{1}_{\{y>0\}}\,\mathrm{d}y$.

Proof. By a monotone class argument, it is enough to show the result for continuous bounded functions of the form $\varphi(u,v,y) = \varphi_1(u)\varphi_2(v)\varphi_3(y)$, where $\varphi_i\colon \mathbb{R}\to\mathbb{R}$ are bounded and continuous functions for $i = 1, 2, 3$. That is, we show that, for $z, x > 0$,

$$\lim_{t\to\infty} \frac{\mathbb{E}^{(\gamma)}_{(z,x)}\big[\varphi_1(\mathcal{U}_r)\varphi_2(\widetilde{W}_{t-s,t})\varphi_3(\xi_t)\mathrm{e}^{-\gamma\xi_t}\mathbf{1}_{\{\underline{\xi}_t>0\}}\big]}{\mathbb{E}^{(\mathrm{e},\gamma)}_x\big[\mathrm{e}^{-\gamma\xi_t}\mathbf{1}_{\{\underline{\xi}_t>0\}}\big]} = \mathbb{E}^{(\gamma),\uparrow}_{(z,x)}[\varphi_1(\mathcal{U}_r)]\mathbb{E}^{(\mathrm{e},\gamma),\downarrow}_{\mu_\gamma}[\varphi_2(\widehat{W}_s)\varphi_3(-\xi_0)],$$

where $\mathbb{E}^{(\mathrm{e},\gamma),\downarrow}_{\mu_\gamma}[\varphi_2(\widehat{W}_s)\varphi_3(-\xi_0)] = \int_0^\infty \mathbb{E}^{(\mathrm{e},\gamma),\downarrow}_{-y}[\varphi_2(\widehat{W}_s)\varphi_3(-\xi_0)]\,\mu_\gamma(\mathrm{d}y)$. For simplicity of exposition, we assume $0 \leq \varphi_i \leq 1$ for $i = 1, 2, 3$. We first observe from the Markov property that, for $t \geq r + s$,

$$\mathbb{E}^{(\gamma)}_{(z,x)}\big[\varphi_1(\mathcal{U}_r)\varphi_2(\widetilde{W}_{t-s,t})\varphi_3(\xi_t)\mathrm{e}^{-\gamma\xi_t}\mathbf{1}_{\{\underline{\xi}_t>0\}}\big] = \mathbb{E}^{(\gamma)}_{(z,x)}\big[\varphi_1(\mathcal{U}_r)\Phi_{t-r}(\xi_r)\mathbf{1}_{\{\underline{\xi}_r>0\}}\big], \tag{16}$$

where $\Phi_u(y) := \mathbb{E}^{(\mathrm{e},\gamma)}_y\big[\varphi_2(\widetilde{W}_{u-s,u})\varphi_3(\xi_u)\mathrm{e}^{-\gamma\xi_u}\mathbf{1}_{\{\underline{\xi}_u>0\}}\big]$, $u \geq s$, $y > 0$. Using the last definition and the Markov property again, we deduce the following identity:

$$\Phi_{t-r}(y) = \mathbb{E}^{(\mathrm{e},\gamma)}_y\big[\Phi_s(\xi_{t-r-s})\mathbf{1}_{\{\underline{\xi}_{t-r-s}>0\}}\big], \qquad y > 0. \tag{17}$$

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On the other hand, by [12, Lemma 1], we know that, for $\delta > 0$ and $t \geq v$,

$$\lim_{t\to\infty} \frac{\mathbb{E}_y^{(\mathrm{e},\gamma)}\big[\mathrm{e}^{-(\delta+\gamma)\xi_{t-v}}\mathbf{1}_{\{\underline{\xi}_{t-v}>0\}}\big]}{\mathbb{E}_x^{(\mathrm{e},\gamma)}\big[\mathrm{e}^{-\gamma\xi_t}\mathbf{1}_{\{\underline{\xi}_t>0\}}\big]} = \frac{\widehat{U}^{(\gamma)}(y)}{\widehat{U}^{(\gamma)}(x)} \frac{\int_0^\infty \mathrm{e}^{-(\delta+\gamma)z}U^{(\gamma)}(z)\,\mathrm{d}z}{\int_0^\infty \mathrm{e}^{-\gamma z}U^{(\gamma)}(z)\,\mathrm{d}z}.$$

Then, by the continuity theorem for the Laplace transform and using identity (15), for h bounded and continuous, μ_γ-a.s., it follows that

$$\lim_{t\to\infty} \frac{\mathbb{E}_y^{(\mathrm{e},\gamma)}\big[h(\xi_{t-v})\mathrm{e}^{-\gamma\xi_{t-v}}\mathbf{1}_{\{\underline{\xi}_{t-v}>0\}}\big]}{\mathbb{E}_x^{(\mathrm{e},\gamma)}\big[\mathrm{e}^{-\gamma\xi_t}\mathbf{1}_{\{\underline{\xi}_t>0\}}\big]} = \frac{\widehat{U}^{(\gamma)}(y)}{\widehat{U}^{(\gamma)}(x)} \int_0^\infty h(z)\,\mu_\gamma(\mathrm{d}z). \tag{18}$$

If h is positive and continuous but not bounded, we can truncate the function h, i.e. fix $n \in \mathbb{N}$ and define $h_n(x) := h(x)\mathbf{1}_{\{h(x)\leq n\}}$. Then, by (18),

$$\begin{aligned}\liminf_{t\to\infty} \frac{\mathbb{E}_y^{(\mathrm{e},\gamma)}\big[h(\xi_{t-v})\mathrm{e}^{-\gamma\xi_{t-v}}\mathbf{1}_{\{\underline{\xi}_{t-v}>0\}}\big]}{\mathbb{E}_x^{(\mathrm{e},\gamma)}\big[\mathrm{e}^{-\gamma\xi_t}\mathbf{1}_{\{\underline{\xi}_t>0\}}\big]} &\geq \liminf_{t\to\infty} \frac{\mathbb{E}_y^{(\mathrm{e},\gamma)}\big[h_n(\xi_{t-v})\mathrm{e}^{-\gamma\xi_{t-v}}\mathbf{1}_{\{\underline{\xi}_{t-v}>0\}}\big]}{\mathbb{E}_x^{(\mathrm{e},\gamma)}\big[\mathrm{e}^{-\gamma\xi_t}\mathbf{1}_{\{\underline{\xi}_t>0\}}\big]} \\ &= \frac{\widehat{U}^{(\gamma)}(y)}{\widehat{U}^{(\gamma)}(x)} \int_0^\infty h_n(z)\,\mu_\gamma(\mathrm{d}z).\end{aligned}$$

On the other hand, since $h_n(x) \to h(x)$ as $n \to \infty$, by Fatou's lemma,

$$\liminf_{n\to\infty} \int_0^\infty h_n(z)\,\mu_\gamma(\mathrm{d}z) \geq \int_0^\infty h(z)\,\mu_\gamma(\mathrm{d}z).$$

Thus, putting both pieces together, we get

$$\liminf_{t\to\infty} \frac{\mathbb{E}_y^{(\mathrm{e},\gamma)}\big[h(\xi_{t-v})\mathrm{e}^{-\gamma\xi_{t-v}}\mathbf{1}_{\{\underline{\xi}_{t-v}>0\}}\big]}{\mathbb{E}_x^{(\mathrm{e},\gamma)}\big[\mathrm{e}^{-\gamma\xi_t}\mathbf{1}_{\{\underline{\xi}_t>0\}}\big]} \geq \frac{\widehat{U}^{(\gamma)}(y)}{\widehat{U}^{(\gamma)}(x)} \int_0^\infty h(z)\,\mu_\gamma(\mathrm{d}z). \tag{19}$$

We want to apply the previous inequality to the function $h(x) = \Phi_s(x)\mathrm{e}^{\gamma x}$. To do so, we need to verify that $\Phi_s(\cdot)$ is a positive and μ_γ-a.s.-continuous function. First, we observe that since φ_2 and φ_3 are continuous functions, the discontinuities of $\Phi_s(\cdot)$ correspond to discontinuities of the map $\mathtt{e}\colon y \mapsto \mathbb{P}^{(\mathrm{e},\gamma)}(\underline{\xi}_t > -y)$. Since $\mathtt{e}(\cdot)$ is bounded and monotone, it has at most a countable number of discontinuities. Thus, the same holds for the function $\Phi_s(\cdot)$, which in turn implies that $\Phi_s(\cdot)$ is continuous almost everywhere with respect to the Lebesgue measure and therefore μ_γ-a.s. □

Now, from (17) and (19) with $v = r + s$ and $h(x) = \Phi_s(x)\mathrm{e}^{\gamma x}$, we have

$$\begin{aligned}\liminf_{t\to\infty} \frac{\Phi_{t-r}(y)}{\mathbb{E}_x^{(\mathrm{e},\gamma)}\big[\mathrm{e}^{-\gamma\xi_t}\mathbf{1}_{\{\underline{\xi}_t>0\}}\big]} &= \liminf_{t\to\infty} \frac{\mathbb{E}_y^{(\mathrm{e},\gamma)}\big[\Phi_s(\xi_{t-v})\mathrm{e}^{\gamma\xi_{t-v}}\mathrm{e}^{-\gamma\xi_{t-v}}\mathbf{1}_{\{\underline{\xi}_{t-v}>0\}}\big]}{\mathbb{E}_x^{(\mathrm{e},\gamma)}\big[\mathrm{e}^{-\gamma\xi_t}\mathbf{1}_{\{\underline{\xi}_t>0\}}\big]} \\ &= \liminf_{t\to\infty} \frac{\mathbb{E}_y^{(\mathrm{e},\gamma)}\big[h(\xi_{t-v})\mathrm{e}^{-\gamma\xi_{t-v}}\mathbf{1}_{\{\underline{\xi}_{t-v}>0\}}\big]}{\mathbb{E}_x^{(\mathrm{e},\gamma)}\big[\mathrm{e}^{-\gamma\xi_t}\mathbf{1}_{\{\underline{\xi}_t>0\}}\big]} \\ &\geq \frac{\widehat{U}^{(\gamma)}(y)}{\widehat{U}^{(\gamma)}(x)} \int_0^\infty \Phi_s(z)\mathrm{e}^{\gamma z}\,\mu_\gamma(\mathrm{d}z).\end{aligned}$$

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In view of identity (16) and the above inequality, replacing y by ξ_r, we get, from Fatou's lemma,

$$\begin{aligned}\liminf_{t\to\infty}&\frac{\mathbb{E}^{(\gamma)}_{(z,x)}\big[\varphi_1(\mathcal{U}_r)\varphi_2(\widetilde{W}_{t-s,t})\varphi_3(\xi_t)\mathrm{e}^{-\gamma\xi_t}\mathbf{1}_{\{\underline{\xi}_t>0\}}\big]}{\mathbb{E}^{(\mathrm{e},\gamma)}_{x}\big[\mathrm{e}^{-\gamma\xi_t}\mathbf{1}_{\{\underline{\xi}_t>0\}}\big]}\\ &=\liminf_{t\to\infty}\frac{\mathbb{E}^{(\gamma)}_{(z,x)}\big[\varphi_1(\mathcal{U}_r)\Phi_{t-r}(\xi_r)\mathbf{1}_{\{\underline{\xi}_r>0\}}\big]}{\mathbb{E}^{(\mathrm{e},\gamma)}_{x}\big[\mathrm{e}^{-\gamma\xi_t}\mathbf{1}_{\{\underline{\xi}_t>0\}}\big]}\\ &\geq\frac{\mathbb{E}^{(\gamma)}_{(z,x)}\big[\varphi_1(\mathcal{U}_r)\widehat{U}^{(\gamma)}(\xi_r)\mathbf{1}_{\{\underline{\xi}_r>0\}}\big]}{\widehat{U}^{(\gamma)}(x)}\int_0^\infty\Phi_s(u)\mathrm{e}^{\gamma u}\,\mu_\gamma(\mathrm{d}u)\\ &=\mathbb{E}^{(\gamma),\uparrow}_{(z,x)}[\varphi_1(\mathcal{U}_r)]\int_0^\infty\Phi_s(u)\mathrm{e}^{\gamma u}\,\mu_\gamma(\mathrm{d}u).\end{aligned}\tag{20}$$

Now we use the duality relationship, with respect to the Lebesgue measure, between ξ and $\widehat{\xi}$ (see, for instance, [12, Lemma 3]) to get

$$\begin{aligned}\int_0^\infty\Phi_s(z)\mathrm{e}^{\gamma z}\mathrm{e}^{-\gamma z}U^{(\gamma)}(z)\,\mathrm{d}z&=\int_0^\infty\mathbb{E}^{(\mathrm{e},\gamma)}_{z}\big[\varphi_2(\widetilde{W}_{0,s})\varphi_3(\xi_s)\mathrm{e}^{-\gamma\xi_s}\mathbf{1}_{\{\underline{\xi}_s>0\}}\big]U^{(\gamma)}(z)\,\mathrm{d}z\\ &=\int_0^\infty\mathbb{E}^{(\mathrm{e},\gamma)}_{-z}\big[\varphi_2(\widehat{W}_s)U^{(\gamma)}(-\xi_s)\mathbf{1}_{\{\overline{\xi}_s<0\}}\big]\varphi_3(z)\mathrm{e}^{-\gamma z}\,\mathrm{d}z\\ &=\int_0^\infty\mathbb{E}^{(\mathrm{e},\gamma),\downarrow}_{-z}\big[\varphi_2(\widehat{W}_s)\big]U^{(\gamma)}(z)\varphi_3(z)\mathrm{e}^{-\gamma z}\,\mathrm{d}z\\ &=\int_0^\infty\mathbb{E}^{(\mathrm{e},\gamma),\downarrow}_{-z}\big[\varphi_2(\widehat{W}_s)\varphi_3(-\xi_0)\big]\mathrm{e}^{-\gamma z}U^{(\gamma)}(z)\,\mathrm{d}z.\end{aligned}$$

Using this equality in (20), we obtain

$$\begin{aligned}\liminf_{t\to\infty}&\frac{\mathbb{E}^{(\gamma)}_{(z,x)}\big[\varphi_1(\mathcal{U}_r)\varphi_2(\widetilde{W}_{t-s,t})\varphi_3(\xi_t)\mathrm{e}^{-\gamma\xi_t}\mathbf{1}_{\{\underline{\xi}_t>0\}}\big]}{\mathbb{E}^{(\mathrm{e},\gamma)}_{x}\big[\mathrm{e}^{-\gamma\xi_t}\mathbf{1}_{\{\underline{\xi}_t>0\}}\big]}\\ &\geq\mathbb{E}^{(\gamma),\uparrow}_{(z,x)}[\varphi_1(\mathcal{U}_r)]\mathbb{E}^{(\mathrm{e},\gamma),\downarrow}_{\mu_\gamma}[\varphi_2(\widehat{W}_s)\varphi_3(-\xi_0)].\end{aligned}\tag{21}$$

On the other hand, by taking $y=x$, $v=0$, and $h(z)=\varphi_3(z)$ in (18), we deduce that

$$\lim_{t\to\infty}\frac{\mathbb{E}^{(\mathrm{e},\gamma)}_{x}\big[\varphi_3(\xi_t)\mathrm{e}^{-\gamma\xi_t}\mathbf{1}_{\{\underline{\xi}_t>0\}}\big]}{\mathbb{E}^{(\mathrm{e},\gamma)}_{x}\big[\mathrm{e}^{-\gamma\xi_t}\mathbf{1}_{\{\underline{\xi}_t>0\}}\big]}=\int_0^\infty\varphi_3(z)\,\mu_\gamma(\mathrm{d}z)=\mathbb{E}^{(\mathrm{e},\gamma),\downarrow}_{\mu_\gamma}[\varphi_3(-\xi_0)].$$

Using this last identity and replacing $\varphi_1(\mathcal{U}_r)$ by $1-\varphi_1(\mathcal{U}_r)$ and $\varphi_2\equiv 1$ in (21), we get

$$\begin{aligned}&\mathbb{E}^{(\gamma),\uparrow}_{(z,x)}\big[1-\varphi_1(\mathcal{U}_r)\big]\mathbb{E}^{(\mathrm{e},\gamma),\downarrow}_{\mu_\gamma}[\varphi_3(-\xi_0)]\\ &\qquad\leq\liminf_{t\to\infty}\frac{\mathbb{E}^{(\gamma)}_{(z,x)}\big[(1-\varphi_1(\mathcal{U}_r))\varphi_3(\xi_t)\mathrm{e}^{-\gamma\xi_t}\mathbf{1}_{\{\underline{\xi}_t>0\}}\big]}{\mathbb{E}^{(\mathrm{e},\gamma)}_{x}\big[\mathrm{e}^{-\gamma\xi_t}\mathbf{1}_{\{\underline{\xi}_t>0\}}\big]}\\ &\qquad=\mathbb{E}^{(\mathrm{e},\gamma),\downarrow}_{\mu_\gamma}[\varphi_3(-\xi_0)]-\limsup_{t\to\infty}\frac{\mathbb{E}^{(\gamma)}_{(z,x)}\big[\varphi_1(\mathcal{U}_r)\varphi_3(\xi_t)\mathrm{e}^{-\gamma\xi_t}\mathbf{1}_{\{\underline{\xi}_t>0\}}\big]}{\mathbb{E}^{(\mathrm{e},\gamma)}_{x}\big[\mathrm{e}^{-\gamma\xi_t}\mathbf{1}_{\{\underline{\xi}_t>0\}}\big]}.\end{aligned}$$

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Therefore,

$$\limsup_{t\to\infty} \frac{\mathbb{E}^{(\gamma)}_{(z,x)}\big[\varphi_1(\mathcal{U}_r)\varphi_3(\xi_t)\mathrm{e}^{-\gamma\xi_t}\mathbf{1}_{\{\underline{\xi}_t>0\}}\big]}{\mathbb{E}^{(\mathrm{e},\gamma)}_{x}\big[\mathrm{e}^{-\gamma\xi_t}\mathbf{1}_{\{\underline{\xi}_t>0\}}\big]} \le \mathbb{E}^{(\gamma),\uparrow}_{(z,x)}[\varphi_1(\mathcal{U}_r)]\mathbb{E}^{(\mathrm{e},\gamma),\downarrow}_{\mu_\gamma}[\varphi_3(-\xi_0)].$$

In other words, by taking $\varphi_2\equiv 1$ in (21) and the above inequality, we obtain the identity

$$\lim_{t\to\infty} \frac{\mathbb{E}^{(\gamma)}_{(z,x)}\big[\varphi_1(\mathcal{U}_r)\varphi_3(\xi_t)\mathrm{e}^{-\gamma\xi_t}\mathbf{1}_{\{\underline{\xi}_t>0\}}\big]}{\mathbb{E}^{(\mathrm{e},\gamma)}_{x}\big[\mathrm{e}^{-\gamma\xi_t}\mathbf{1}_{\{\underline{\xi}_t>0\}}\big]} = \mathbb{E}^{(\gamma),\uparrow}_{(z,x)}[\varphi_1(\mathcal{U}_r)]\mathbb{E}^{(\mathrm{e},\gamma),\downarrow}_{\mu_\gamma}[\varphi_3(-\xi_0)].$$

Finally, we pursue the same strategy as before, i.e. we replace $\varphi_2(\widetilde{W}_{t-s,t})$ by $1-\varphi_2(\widetilde{W}_{t-s,t})$ in (21) to obtain

$$\liminf_{t\to\infty} \frac{\mathbb{E}^{(\gamma)}_{(z,x)}\big[\varphi_1(\mathcal{U}_r)(1-\varphi_2(\widetilde{W}_{t-s,t}))\varphi_3(\xi_t)\mathrm{e}^{-\gamma\xi_t}\mathbf{1}_{\{\underline{\xi}_t>0\}}\big]}{\mathbb{E}^{(\mathrm{e},\gamma)}_{x}\big[\mathrm{e}^{-\gamma\xi_t}\mathbf{1}_{\{\underline{\xi}_t>0\}}\big]}$$
$$\ge \mathbb{E}^{(\gamma),\uparrow}_{(z,x)}[\varphi_1(\mathcal{U}_r)]\mathbb{E}^{(\mathrm{e},\gamma),\downarrow}_{\mu_\gamma}[(1-\varphi_2(\widehat{W}_s))\varphi_3(-\xi_0)].$$

Then, it follows that

$$\limsup_{t\to\infty} \frac{\mathbb{E}^{(\gamma)}_{(z,x)}\big[\varphi_1(\mathcal{U}_r)\varphi_2(\widetilde{W}_{t-s,t})\varphi_3(\xi_t)\mathrm{e}^{-\gamma\xi_t}\mathbf{1}_{\{\underline{\xi}_t>0\}}\big]}{\mathbb{E}^{(\mathrm{e},\gamma)}_{x}\big[\mathrm{e}^{-\gamma\xi_t}\mathbf{1}_{\{\underline{\xi}_t>0\}}\big]}$$
$$\le \mathbb{E}^{(\gamma),\uparrow}_{(z,x)}[\varphi_1(\mathcal{U}_r)]\mathbb{E}^{(\mathrm{e},\gamma),\downarrow}_{\mu_\gamma}[\varphi_2(\widehat{W}_s)\varphi_3(-\xi_0)].$$

Finally, putting all the pieces together, we conclude that

$$\lim_{t\to\infty} \frac{\mathbb{E}^{(\gamma)}_{(z,x)}\big[\varphi_1(\mathcal{U}_r)\varphi_2(\widetilde{W}_{t-s,t})\varphi_3(\xi_t)\mathrm{e}^{-\gamma\xi_t}\mathbf{1}_{\{\underline{\xi}_t>0\}}\big]}{\mathbb{E}^{(\mathrm{e},\gamma)}_{x}\big[\mathrm{e}^{-\gamma\xi_t}\mathbf{1}_{\{\underline{\xi}_t>0\}}\big]}$$
$$= \mathbb{E}^{(\gamma),\uparrow}_{(z,x)}[\varphi_1(\mathcal{U}_r)]\mathbb{E}^{(\mathrm{e},\gamma),\downarrow}_{\mu_\gamma}[\varphi_2(\widehat{W}_s)\varphi_3(-\xi_0)],$$

as expected. □

The following lemmas are preparatory results for the proof of Theorem 1. We first observe from the Wiener–Hopf factorisation that there exists a non-decreasing function Ψ_0 satisfying $\psi_0(\lambda)=\lambda\Psi_0(\lambda)$ for $\lambda\ge 0$, where Ψ_0 is the Laplace exponent of a subordinator and takes the form

$$\Psi_0(\lambda)=\varrho^2\lambda+\int_{(0,\infty)}(1-\mathrm{e}^{-\lambda x})\mu(x,\infty)\,\mathrm{d}x.$$

From (H2), it follows that $\Psi_0(\lambda)$ is regularly varying at 0 with index β, and implicitly the term ϱ equals 0 when $\beta\in(0,1)$.

Lemma 2. *Let $x,\lambda>0$, and assume that (H2) holds; then*

$$\lim_{s\to\infty}\lim_{t\to\infty}\mathrm{e}^{-t\Phi_\xi(\gamma)}t^{3/2}\int_s^{t-s}\mathbb{E}^{(\mathrm{e})}_x\big[\Psi_0(\lambda\mathrm{e}^{-\xi_u})\mathbf{1}_{\{\underline{\xi}_t>0\}}\big]\,\mathrm{d}u=0.$$

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Proof. Let $x > 0$ and $\lambda > 0$. From the Markov property, we observe that

$$\mathbb{E}_x^{(\mathrm{e})}\big[\Psi_0(\lambda \mathrm{e}^{-\xi_u})\mathbf{1}_{\{\underline{\xi}_t>0\}}\big] = \mathbb{E}_x^{(\mathrm{e})}\big[\Psi_0(\lambda \mathrm{e}^{-\xi_u})\mathbf{1}_{\{\underline{\xi}_u>0\}}\mathbb{P}_{\xi_u}^{(\mathrm{e})}\big(\underline{\xi}_{t-u} > 0\big)\big].$$

Next, we take $x_0 > x$ and, from the monotonicity of $z \mapsto \mathbb{P}_z^{(\mathrm{e})}(\underline{\xi}_{t-u} > 0)$, we obtain

$$\begin{aligned}\mathbb{E}_x^{(\mathrm{e})}\big[\Psi_0(\lambda \mathrm{e}^{-\xi_u})\mathbf{1}_{\{\underline{\xi}_t>0\}}\big] &\le \mathbb{E}_x^{(\mathrm{e})}\big[\Psi_0(\lambda \mathrm{e}^{-\xi_u})\mathbf{1}_{\{\underline{\xi}_u>0\}}\mathbb{P}_{\xi_u}^{(\mathrm{e})}\big(\underline{\xi}_{t-u} > 0\big)\mathbf{1}_{\{\xi_u>x_0\}}\big] \\ &\quad + \mathbb{E}_x^{(\mathrm{e})}\big[\Psi_0(\lambda \mathrm{e}^{-\xi_u})\mathbf{1}_{\{\underline{\xi}_u>0\}}\mathbf{1}_{\{\xi_u\le x_0\}}\big]\mathbb{P}_{x_0+x}^{(\mathrm{e})}\big(\underline{\xi}_{t-u} > 0\big).\end{aligned}$$

Now, using the asymptotic behaviour given in (11) and the Esscher transform (6), for t large enough,

$$\begin{aligned}&\mathbb{E}_x^{(\mathrm{e})}\big[\Psi_0(\lambda \mathrm{e}^{-\xi_u})\mathbf{1}_{\{\underline{\xi}_t>0\}}\big] \\ &\quad \le C_\gamma \mathbb{E}_x^{(\mathrm{e})}\big[\Psi_0(\lambda \mathrm{e}^{-\xi_u})\mathbf{1}_{\{\underline{\xi}_u>0\}}\mathbf{1}_{\{\xi_u>x_0\}}\mathrm{e}^{\gamma\xi_u}\widehat{U}^{(\gamma)}(\xi_u)\big](t-u)^{-3/2}\mathrm{e}^{\Phi_\xi(\gamma)(t-u)} \\ &\qquad + C_{\gamma,x+x_0}\mathbb{E}_x^{(\mathrm{e})}\big[\Psi_0(\lambda \mathrm{e}^{-\xi_u})\mathbf{1}_{\{\underline{\xi}_u>0\}}\mathbf{1}_{\{\xi_u\le x_0\}}\big](t-u)^{-3/2}\mathrm{e}^{\Phi_\xi(\gamma)(t-u)} \\ &\quad \le C_\gamma \mathbb{E}_x^{(\mathrm{e},\gamma)}\big[\Psi_0(\lambda \mathrm{e}^{-\xi_u})\mathbf{1}_{\{\underline{\xi}_u>0\}}\mathbf{1}_{\{\xi_u>x_0\}}\widehat{U}^{(\gamma)}(\xi_u)\big](t-u)^{-3/2}\mathrm{e}^{\Phi_\xi(\gamma)t} \\ &\qquad + C_{\gamma,x+x_0}\mathbb{E}_x^{(\mathrm{e})}\big[\Psi_0(\lambda \mathrm{e}^{-\xi_u})\mathbf{1}_{\{\underline{\xi}_u>0\}}\mathbf{1}_{\{\xi_u\le x_0\}}\big](t-u)^{-3/2}\mathrm{e}^{\Phi_\xi(\gamma)(t-u)},\end{aligned} \tag{22}$$

where C_γ and $C_{\gamma,x+x_0}$ are strictly positive constants.

First, we deal with the first expectation on the right-hand side of (22). Recalling that $\Phi_\xi''(\gamma) < \infty$, we get from [13, Corollary 5.3] that

$$y^{-1}\widehat{U}^{(\gamma)}(y) \to \frac{1}{\widehat{\mathbb{E}}^{(\mathrm{e},\gamma)}[H_1]} \qquad \text{as } y \to \infty.$$

Furthermore, since $\widehat{U}^{(\gamma)}$ is increasing, the map $y \mapsto \mathrm{e}^{-(\varsigma/2)y}\widehat{U}^{(\gamma)}(y)$ is bounded for any $\varsigma \in (0, \beta)$, and from (H2) we also deduce that the map $y \mapsto \mathrm{e}^{-(\varsigma/2)y}\ell(\lambda \mathrm{e}^{-y})$ is also bounded. With these observations in mind, it follows that, for u large enough,

$$\mathbb{E}_x^{(\mathrm{e},\gamma)}\big[\Psi_0(\lambda \mathrm{e}^{-\xi_u})\mathbf{1}_{\{\underline{\xi}_u>0\}}\mathbf{1}_{\{\xi_u>x_0\}}\widehat{U}^{(\gamma)}(\xi_u)\big] \le C_\lambda \mathbb{E}_x^{(\mathrm{e},\gamma)}\big[\mathrm{e}^{-(\beta-(\varsigma/2))\xi_u}\mathbf{1}_{\{\underline{\xi}_u>0\}}\big],$$

where C_λ is a strictly positive constant. According to [12, Lemma 1], there exists $C_{\lambda,\beta,x} > 0$ such that, for u sufficiently large,

$$\mathbb{E}_x^{(\mathrm{e},\gamma)}\big[\Psi_0(\lambda \mathrm{e}^{-\xi_u})\mathbf{1}_{\{\underline{\xi}_u>0\}}\mathbf{1}_{\{\xi_u>x_0\}}\widehat{U}^{(\gamma)}(\xi_u)\big] \le C_{\lambda,\beta,x}u^{-3/2}.$$

For the second expectation in (22) we use the monotonicity of Ψ_0 to get

$$\mathbb{E}_x^{(\mathrm{e})}\big[\Psi_0(\lambda \mathrm{e}^{-\xi_u})\mathbf{1}_{\{\underline{\xi}_u>0\}}\mathbf{1}_{\{\xi_u\le x_0\}}\big] \le \Psi_0(\lambda)\mathbb{P}_x^{(\mathrm{e})}\big(\underline{\xi}_u > 0\big) \le \widehat{C}_{\gamma,x,\lambda}u^{-3/2}\mathrm{e}^{\Phi_\xi(\gamma)u},$$

where $\widehat{C}_{\gamma,x,\lambda}$ is a positive constant. The last inequality follows from (11). Putting all the pieces together in (22), we deduce that, for t large enough,

$$\mathbb{E}_x^{(\mathrm{e})}\big[\Psi_0(\lambda \mathrm{e}^{-\xi_u})\mathbf{1}_{\{\underline{\xi}_t>0\}}\big] \le C_{\lambda,\beta,x,\gamma}u^{-3/2}(t-u)^{-3/2}\mathrm{e}^{\Phi_\xi(\gamma)t},$$

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where $C_{\lambda,\beta,x,\gamma} > 0$. Finally, observe that, for t large enough,

$$\mathrm{e}^{-t\Phi_\xi(\gamma)} t^{3/2} \int_s^{t-s} \mathbb{E}_x^{(\mathrm{e})}\big[\Psi_0(\lambda \mathrm{e}^{-\xi_u})\mathbf{1}_{\{\underline{\xi}_t>0\}}\big]\,\mathrm{d}u \le C_{\lambda,\beta,x,\gamma} t^{3/2} \int_s^{t-s} (t-u)^{-3/2} u^{-3/2}\,\mathrm{d}u$$
$$\le 2C_{\lambda,\beta,x,\gamma} t^{3/2} \left(\frac{t}{2}\right)^{-3/2} \int_s^{\infty} u^{-3/2}\,\mathrm{d}u$$
$$\le 2C_{\lambda,\beta,x,\gamma} s^{-1/2}.$$

The result now follows by taking $t\to\infty$ and then $s\to\infty$. □

Lemma 3. *Let $z, x > 0$ and assume that (H2) holds. Then*

$$\lim_{s\to\infty}\lim_{t\to\infty} t^{3/2}\mathrm{e}^{-t\Phi_\xi(\gamma)}\mathbb{E}_{(z,x)}\big[|\exp\{-Z_s\mathrm{e}^{-\xi_s}v_t(s,\lambda,\xi)\} - \exp\{-Z_s\mathrm{e}^{-\xi_s}v_t(t-s,\lambda,\xi)\}|\mathbf{1}_{\{\underline{\xi}_t>0\}}\big] = 0.$$

Proof. Fix $z, x > 0$ and take $t \ge 2s$. We begin by observing that since $f(y) = \mathrm{e}^{-y}$, $y \ge 0$, it is Lipschitz and hence there exists a positive constant C_1 such that

$$\mathbb{E}_{(z,x)}\big[|\exp\{-Z_s\mathrm{e}^{-\xi_s}v_t(s,\lambda,\xi)\} - \exp\{-Z_s\mathrm{e}^{-\xi_s}v_t(t-s,\lambda,\xi)\}|\mathbf{1}_{\{\underline{\xi}_t>0\}}\big]$$
$$\le C_1\mathbb{E}_{(z,x)}\big[Z_s\mathrm{e}^{-\xi_s}|v_t(s,\lambda,\xi) - v_t(t-s,\lambda,\xi)\big|\mathbf{1}_{\{\underline{\xi}_t>0\}}\big]$$
$$= C_1 z^{-1}\mathbb{E}_x^{(\mathrm{e})}\big[|v_t(s,\lambda,\xi) - v_t(t-s,\lambda,\xi)|\mathbf{1}_{\{\underline{\xi}_t>0\}}\big],$$

where in the last identity we conditioned on the environment and used (3). Since ψ_0 is positive, from (4) we have that $s \mapsto v_t(s,\lambda,\xi)$ is an increasing function. This, together with the facts that ψ_0 is a non-decreasing function and $v_t(t,\lambda,\xi) = \lambda$, mean that $\psi_0\big(v_t(u,\lambda,\xi)\mathrm{e}^{-\xi_u}\big) \le \psi_0(\lambda\mathrm{e}^{-\xi_u})$ for $u \le t$. Hence, we obtain

$$v_t(s,\lambda,\xi) - v_t(t-s,\lambda,\xi) = \int_s^{t-s} \mathrm{e}^{\xi_u}\psi_0(v_t(u,\lambda,\xi)\mathrm{e}^{-\xi_u})\,\mathrm{d}u$$
$$\le \int_s^{t-s} \mathrm{e}^{\xi_u}\psi_0(\lambda\mathrm{e}^{-\xi_u})\,\mathrm{d}u = \int_s^{t-s} \lambda\Psi_0(\lambda\mathrm{e}^{-\xi_u})\,\mathrm{d}u.$$

In other words, we have deduced that

$$\mathbb{E}_x^{(\mathrm{e})}\big[|v_t(s,\lambda,\xi) - v_t(t-s,\lambda,\xi)|\mathbf{1}_{\{\underline{\xi}_t>0\}}\big] \le \lambda \int_s^{t-s} \mathbb{E}_x^{(\mathrm{e})}\big[\Psi_0(\lambda\mathrm{e}^{-\xi_u})\mathbf{1}_{\{\underline{\xi}_t>0\}}\big]\,\mathrm{d}u.$$

Appealing to Lemma 2, we conclude that

$$\lim_{s\to\infty}\lim_{t\to\infty} t^{3/2}\mathrm{e}^{-t\Phi_\xi(\gamma)}\mathbb{E}_{(z,x)}\big[|\exp\{-Z_s\mathrm{e}^{-\xi_s}v_t(s,\lambda,\xi)\} - \exp\{-Z_s\mathrm{e}^{-\xi_s}v_t(t-s,\lambda,\xi)\}|\mathbf{1}_{\{\underline{\xi}_t>0\}}\big]$$
$$\le C_1 z^{-1}\lambda \lim_{s\to\infty}\lim_{t\to\infty} t^{3/2}\mathrm{e}^{-t\Phi_\xi(\gamma)} \int_s^{t-s} \mathbb{E}_x^{(\mathrm{e})}\big[\Psi_0(\lambda\mathrm{e}^{-\xi_u})\mathbf{1}_{\{\underline{\xi}_t>0\}}\big]\,\mathrm{d}u = 0,$$

as required. □

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The following lemma states that, with respect to the measure $\mathbb{P}^{(\gamma),\uparrow}_{(z,x)}$ with $z, x > 0$, the reweighted process $(Z_t e^{-\xi_t}, t \geq 0)$ is a martingale that converges towards a strictly positive random variable under $\mathbb{P}^{(\gamma),\uparrow}_{(z,x)}$. This is another preparatory lemma for the proof of Theorem 1.

Lemma 4. *Let $z, x > 0$ and assume that (H2) holds. Then the process $(Z_t e^{-\xi_t}, t \geq 0)$ is a martingale with respect to $(\mathcal{F}_t)_{t\geq 0}$ under $\mathbb{P}^{(\gamma),\uparrow}_{(z,x)}$. Moreover, as $t \to \infty$, $Z_t e^{-\xi_t} \to \mathcal{U}_\infty$ $\mathbb{P}^{(\gamma),\uparrow}_{(z,x)}$-a.s., where the random variable $\mathcal{U}_\infty$ is finite and satisfies $\mathbb{P}^{(\gamma),\uparrow}_{(z,x)}(\mathcal{U}_\infty > 0) > 0$.*

In order to prove this result, we require the following lemma.

Lemma 5. ([2, Proposition 3.4].) *Let $z, x > 0$ and assume that the environment ξ is critical under $\mathbb{P}_{(z,x)}$, and that (13) is fulfilled. Then $\lim_{t\to\infty} \mathbb{P}^{\uparrow}_{(z,x)}(Z_t > 0) > 0$.*

We recall that (H2) implies the $x \log^2(x)$-moment condition (13).

Proof of Lemma 4. From [2, Proposition 1.1], which we may apply here with respect to the measure $\mathbb{P}^{(\gamma)}_{(z,x)}$, we have that the process $(Z_t e^{-\xi_t}, t \geq 0)$ is a quenched martingale with respect to the environment. We assume that $s \leq t$ and take $A \in \mathcal{F}_s$. In order to deduce the first claim of this lemma, we first show that

$$\mathbb{E}^{(\gamma)}_{(z,x)}\big[Z_t e^{-\xi_t} \mathbf{1}_A \widehat{U}^{(\gamma)}(\xi_t) \mathbf{1}_{\{\underline{\xi}_t > 0\}}\big] = \mathbb{E}^{(\gamma)}_{(z,x)}\big[Z_s e^{-\xi_s} \mathbf{1}_A \widehat{U}^{(\gamma)}(\xi_s) \mathbf{1}_{\{\underline{\xi}_s > 0\}}\big].$$

First, conditioning on the environment, we deduce that

$$\begin{aligned}\mathbb{E}^{(\gamma)}_{(z,x)}\big[Z_t e^{-\xi_t} \mathbf{1}_A \widehat{U}^{(\gamma)}(\xi_t) \mathbf{1}_{\{\underline{\xi}_t > 0\}}\big] &= \mathbb{E}^{(\gamma)}_{(z,x)}\big[\mathbb{E}^{(\gamma)}_{(z,x)}[Z_t e^{-\xi_t} \mathbf{1}_A \mid \xi] \widehat{U}^{(\gamma)}(\xi_t) \mathbf{1}_{\{\underline{\xi}_t > 0\}}\big] \\ &= \mathbb{E}^{(\gamma)}_{(z,x)}\big[\mathbb{E}^{(\gamma)}_{(z,x)}[Z_s e^{-\xi_s} \mathbf{1}_A \mid \xi] \widehat{U}^{(\gamma)}(\xi_t) \mathbf{1}_{\{\underline{\xi}_t > 0\}}\big].\end{aligned}$$

We can see that the random variable $\mathbb{E}^{(\gamma)}_{(z,x)}[Z_s e^{-\xi_s} \mathbf{1}_A \mid \xi]$ is $\mathcal{F}_s$-measurable. Thus, conditioning on $\mathcal{F}_s$, we have

$$\mathbb{E}^{(\gamma)}_{(z,x)}\big[Z_t e^{-\xi_t} \mathbf{1}_A \widehat{U}^{(\gamma)}(\xi_t) \mathbf{1}_{\{\underline{\xi}_t > 0\}}\big] = \mathbb{E}^{(\gamma)}_{(z,x)}\big[\mathbb{E}^{(\gamma)}_{(z,x)}[Z_s e^{-\xi_s} \mathbf{1}_A \mid \xi] \mathbb{E}^{(\gamma)}_{(z,x)}[\widehat{U}^{(\gamma)}(\xi_t) \mathbf{1}_{\{\underline{\xi}_t > 0\}} \mid \mathcal{F}_s]\big].$$

Further, by [2, Lemma 3.1], which we can apply here under the measure $\mathbb{P}^{(\gamma)}_{(z,x)}$, the process $(\widehat{U}^{(\gamma)}(\xi_t) \mathbf{1}_{\{\underline{\xi}_t > 0\}}, t \geq 0)$ is a martingale with respect to $(\mathcal{F}_t)_{t\geq 0}$ under $\mathbb{P}^{(\gamma)}_{(z,x)}$. Hence,

$$\begin{aligned}\mathbb{E}^{(\gamma)}_{(z,x)}\big[Z_t e^{-\xi_t} \mathbf{1}_A \widehat{U}^{(\gamma)}(\xi_t) \mathbf{1}_{\{\underline{\xi}_t > 0\}}\big] &= \mathbb{E}^{(\gamma)}_{(z,x)}\big[\mathbb{E}^{(\gamma)}_{(z,x)}[Z_s e^{-\xi_s} \mathbf{1}_A \mid \xi] \widehat{U}^{(\gamma)}(\xi_s) \mathbf{1}_{\{\underline{\xi}_s > 0\}}\big] \\ &= \mathbb{E}^{(\gamma)}_{(z,x)}\big[Z_s e^{-\xi_s} \mathbf{1}_A \widehat{U}^{(\gamma)}(\xi_s) \mathbf{1}_{\{\underline{\xi}_s > 0\}}\big].\end{aligned}$$

Therefore, by the definition of the measure $\mathbb{P}^{(\gamma),\uparrow}_{(z,x)}$, we see that

$$\begin{aligned}\mathbb{E}^{(\gamma),\uparrow}_{(z,x)}\big[Z_t e^{-\xi_t} \mathbf{1}_A\big] &= \frac{1}{\widehat{U}(x)} \mathbb{E}^{(\gamma)}_{(z,x)}\big[Z_t e^{-\xi_t} \mathbf{1}_A \widehat{U}^{(\gamma)}(\xi_t) \mathbf{1}_{\{\underline{\xi}_t > 0\}}\big] \\ &= \frac{1}{\widehat{U}(x)} \mathbb{E}^{(\gamma)}_{(z,x)}\big[Z_s e^{-\xi_s} \mathbf{1}_A \widehat{U}^{(\gamma)}(\xi_s) \mathbf{1}_{\{\underline{\xi}_s > 0\}}\big] = \mathbb{E}^{(\gamma),\uparrow}_{(z,x)}\big[Z_s e^{-\xi_s} \mathbf{1}_A\big],\end{aligned}$$

which allows us to conclude that the process $(Z_t e^{-\xi_t}, t \geq 0)$ is a martingale with respect to $(\mathcal{F}_t)_{t\geq 0}$ under $\mathbb{P}^{(\gamma),\uparrow}_{(z,x)}$. Moreover, by Doob's convergence theorem, there is a non-negative finite

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random variable $\mathcal{U}_\infty$ such that, as $t \to \infty$, $Z_t e^{-\xi_t} \to \mathcal{U}_\infty$ $\mathbb{P}^{(\gamma),\uparrow}_{(z,x)}$-a.s. Next, by the dominated convergence theorem, we have

$$\mathbb{P}^{(\gamma),\uparrow}_{(z,x)}(\mathcal{U}_\infty > 0) = \lim_{t\to\infty} \mathbb{P}^{(\gamma),\uparrow}_{(z,x)}(Z_t e^{-\xi_t} > 0).$$

The proof is thus completed as soon as we can show that

$$\lim_{t\to\infty} \mathbb{P}^{(\gamma),\uparrow}_{(z,x)}(Z_t e^{-\xi_t} > 0) > 0. \tag{23}$$

In order to do so, we first observe that the following identity holds:

$$\mathbb{P}^{(\gamma),\uparrow}_{(z,x)}(Z_t e^{-\xi_t} = 0) = \mathbb{P}^{(\gamma),\uparrow}_{(z,x)}(Z_t = 0);$$

then, by noting that under $\mathbb{P}^{(\gamma)}_{(z,x)}$ the Lévy process ξ oscillates (since $\Phi'_\xi(\gamma) = 0$), we can apply Lemma 5 to deduce (23). □

With Proposition 1 and Lemmas 3 and 4 in hand, we may now proceed to prove Theorem 1 following ideas similar to those used in [1, Lemma 3.4], although we might consider that the continuous setting leads to significant changes since an extension of Proposition 1 seems difficult to deduce, unlike in the discrete case (see [1, Theorem 2.7]). Indeed, it seems that such an extension will depend on a much deeper analysis of the asymptotic behaviour for bridges of Lévy processes and their conditioned version.

Proof of Theorem 1. Fix $x, z > 0$ and recall that the process $(\mathcal{U}_s, s \geq 0)$ is defined as $\mathcal{U}_s := Z_s e^{-\xi_s}$. For any $\lambda \geq 0$, we shall prove the convergence of the following Laplace transform as $t \to \infty$: $\mathbb{E}_{(z,x)}\big[\exp\{-\lambda Z_t e^{-\xi_t}\} \mid \underline{\xi}_t > 0\big]$.

First, we rewrite the latter expression in a form which allows us to use Proposition 1 and Lemma 3. We begin by recalling from (5) that, for any $\lambda \geq 0$ and $t \geq s \geq 0$,

$$\mathbb{E}_{(z,x)}[\exp\{-\lambda Z_t e^{-\xi_t}\} \mid \xi, \mathcal{F}^{(b)}_s] = \exp\{-Z_s e^{-\xi_s} v_t(s, \lambda, \xi)\}.$$

Thus,

$$\begin{aligned}
\mathbb{E}_{(z,x)}\big[\exp\{-\lambda Z_t e^{-\xi_t}\}\mathbf{1}_{\{\underline{\xi}_t>0\}}\big] &= \mathbb{E}_{(z,x)}[\mathbb{E}_{(z,x)}[\exp\{-\lambda Z_t e^{-\xi_t}\} \mid \xi, \mathcal{F}^{(b)}_s]\mathbf{1}_{\{\underline{\xi}_t>0\}}] \\
&= \mathbb{E}_{(z,x)}\big[\exp\{-Z_s e^{-\xi_s} v_t(s, \lambda, \xi)\}\mathbf{1}_{\{\underline{\xi}_t>0\}}\big] \\
&= \mathbb{E}_{(z,x)}\big[\exp\{-Z_s e^{-\xi_s} v_t(t-s, \lambda, \xi)\}\mathbf{1}_{\{\underline{\xi}_t>0\}}\big] \\
&\quad + \mathbb{E}_{(z,x)}\big[(\exp\{-Z_s e^{-\xi_s} v_t(s, \lambda, \xi)\} \\
&\qquad - \exp\{-Z_s e^{-\xi_s} v_t(t-s, \lambda, \xi)\})\mathbf{1}_{\{\underline{\xi}_t>0\}}\big].
\end{aligned}$$

Now, using the same notation as in Proposition 1, we note that, for any $s \leq t$,

$$\exp\{-Z_s e^{-\xi_s} v_t(t-s, \lambda, \xi)\} = \varphi(\mathcal{U}_s, \widetilde{W}_{t-s,t}, \xi_t),$$

where $(\widehat{W}_s(\lambda), s \geq 0)$ and $(\widetilde{W}_{t-s,t}, s \leq t)$ are defined by

$$\widehat{W}_s(\lambda) := \exp\{-v_s(0, \lambda, \widehat{\xi})\}, \qquad \widetilde{W}_{t-s,t} := \exp\{-v_t(t-s, \lambda, \xi)\},$$

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and φ is the bounded and continuous function $\varphi(\mathbf{u}, w, y) := w^{\mathbf{u}}$, $0 \le w \le 1$, $\mathbf{u} \ge 0$, $y \in \mathbb{R}$. Hence, appealing to Proposition 1, Lemma 3 and (11), for $z, x > 0$, we see that

$$\begin{aligned}
&\lim_{t\to\infty} \mathbb{E}_{(z,x)}\big[\exp\{-\lambda Z_t e^{-\xi_t}\} \mid \underline{\xi}_t > 0\big] \\
&= \lim_{s\to\infty}\lim_{t\to\infty} \mathbb{E}_{(z,x)}\big[\varphi(\mathcal{U}_s, \widetilde{W}_{t-s,t}, \xi_t) \mid \underline{\xi}_t > 0\big] \\
&\quad + \lim_{s\to\infty}\lim_{t\to\infty} \mathbb{E}_{(z,x)}\big[|\exp\{-Z_s e^{-\xi_s} v_t(s,\lambda,\xi)\} - \exp\{-Z_s e^{-\xi_s} v_t(t-s,\lambda,\xi)\}| \mid \underline{\xi}_t > 0\big] \\
&= \lim_{s\to\infty}\lim_{t\to\infty} \frac{\mathbb{E}^{(\gamma)}_{(z,x)}\big[\varphi(\mathcal{U}_s, \widetilde{W}_{t-s,t}, \xi_t) e^{-\gamma\xi_t}\mathbf{1}_{\{\underline{\xi}_t>0\}}\big]}{\mathbb{E}^{(e,\gamma)}_x\big[e^{-\gamma\xi_t}\mathbf{1}_{\{\underline{\xi}_t>0\}}\big]} = \lim_{s\to\infty} \Upsilon_{z,x}(\lambda, s),
\end{aligned}$$

where

$$\Upsilon_{z,x}(\lambda, s) := \int_0^\infty \int_0^1 \int_0^\infty \varphi(\mathbf{u}, w, y)\mathbb{P}^{(\gamma),\uparrow}_{(z,x)}(\mathcal{U}_s \in d\mathbf{u})\mathbb{P}^{(e,\gamma),\downarrow}_{-y}(\widehat{W}_s(\lambda) \in dw)\, \mu_\gamma(dy).$$

On the other hand, from Lemma 4, we recall that, under $\mathbb{P}^{(\gamma),\uparrow}_{(z,x)}$, the process $(\mathcal{U}_s, s \ge 0)$ is a non-negative martingale with respect to $(\mathcal{F}_t)_{t\ge0}$ that converges towards the non-negative and finite random variable $\mathcal{U}_\infty$. Next, we observe from [11, Proposition 2.3] that the mapping $s \mapsto v_s(0, \lambda, \widehat{\xi})$ is decreasing, implying that $s \mapsto \widehat{W}_s(\lambda)$ is increasing $\mathbb{P}^{(e,\gamma),\downarrow}_{-y}$-a.s. for $y > 0$. Further, since $v_s(0, \lambda, \widehat{\xi}) \le \lambda$, the process $(\widehat{W}_s(\lambda), s \ge 0)$ is bounded below, i.e., for any $\lambda \ge 0$, $0 < e^{-\lambda} \le \widehat{W}_s(\lambda) \le 1$. Therefore, it follows that, for any $\lambda \ge 0$ and $y > 0$, $\widehat{W}_s(\lambda) \xrightarrow[s\to\infty]{} \widehat{W}_\infty(\lambda)$ $\mathbb{P}^{(e,\gamma),\downarrow}_{-y}$-a.s., where $\widehat{W}_\infty(\lambda)$ is a strictly positive random variable. These observations, together with the dominated convergence theorem, imply that

$$\begin{aligned}
\lim_{s\to\infty} \Upsilon_{z,x}(\lambda, s) &= \int_0^\infty \int_0^1 \int_0^\infty \varphi(\mathbf{u}, w, y)\mathbb{P}^{(\gamma),\uparrow}_{(z,x)}(\mathcal{U}_\infty \in d\mathbf{u})\mathbb{P}^{(e,\gamma),\downarrow}_{-y}(\widehat{W}_\infty(\lambda) \in dw)\, \mu_\gamma(dy) \\
&:= \Upsilon_{z,x}(\lambda).
\end{aligned}$$

In other words, $\mathcal{U}_t = Z_t e^{-\xi_t}$ converges weakly, under $\mathbb{P}_{(z,x)}\big(\cdot \mid \underline{\xi}_t > 0\big)$, towards some positive and finite random variable whose Laplace transform is given by $\Upsilon_{z,x}(\lambda)$. For simplicity of exposition we denote by Q such a limiting random variable, and we denote its law by $\mathbf{P}$.

Next, we observe that $\mathbf{P}(Q > 0)$ is strictly positive. The latter is equivalent to showing that $\Upsilon_{z,x}(\lambda) < 1$ for all $\lambda > 0$. In other words, from the definition of $\varphi(\mathbf{u}, w, y)$, it is enough to show that $\mathbb{P}^{(\gamma),\uparrow}_{(z,x)}(\mathcal{U}_\infty > 0) > 0$ and $\mathbb{P}^{(e,\gamma),\downarrow}_{-y}(\widehat{W}_\infty(\lambda) < 1) = 1$ for all $\lambda > 0$. The first claim has been proved in Lemma 4. For the second claim, we observe that, for any $\lambda > 0$,

$$\mathbb{P}^{(e,\gamma),\downarrow}_{-y}(\widehat{W}_\infty(\lambda) < 1) = \mathbb{P}^{(e,\gamma),\downarrow}_{-y}(v_\infty(0, \lambda, \widehat{\xi}) > 0).$$

By the proof of [2, Proposition 3.4], we have

$$v_\infty(0, \lambda, \xi) \ge \lambda \exp\left\{-\int_0^\infty \Psi_0(\lambda e^{-\xi_u})\, du\right\}.$$

Moreover, from the same reference and under assumption (H2), it follows that

$$\mathbb{E}^{(e,\gamma),\uparrow}_y\left[\int_0^\infty \Psi_0(\lambda e^{-\xi_u})\, du\right] < \infty,$$

which implies that $\mathbb{P}^{(e,\gamma),\downarrow}_{-y}(v_\infty(0, \lambda, \widehat{\xi}) > 0) = 1$ for all $\lambda \ge 0$. In other words, $\mathbf{P}(Q > 0) > 0$, which implies that $\lim_{t\to\infty} \mathbb{P}_{(z,x)}\big(Z_t e^{-\xi_t} > 0 \mid \underline{\xi}_t > 0\big) > 0$. This completes the proof. □

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2.3. Proof of Theorem 2

The proof of this theorem follows a similar strategy to the proof of [2, Theorem 1.2] for the critical regime where the assumption that $\ell(\lambda) > C$ for $C > 0$ and the asymptotic behaviour of exponential functionals of Lévy processes are crucial. We also recall that Z is in the weakly subcritical regime.

For simplicity of exposition, we split the proof of Theorem 2 into two lemmas. The first lemma is a direct consequence of Theorem 1.

Lemma 6. *Suppose that* (H2) *holds. Then, for any $z, x > 0$ we have, as $t \to \infty$,*

$$\mathbb{P}_{(z,x)}\big(Z_t > 0,\ \underline{\xi}_t > 0\big) \sim \mathfrak{b}(z, x)\mathbb{P}_x^{(e)}\big(\underline{\xi}_t > 0\big) \\ \sim \mathfrak{b}(z, x)\frac{A_\gamma}{\gamma\kappa^{(\gamma)}(0, \gamma)}\mathrm{e}^{\gamma x}\widehat{U}^{(\gamma)}(x)t^{-3/2}\mathrm{e}^{\Phi_\xi(\gamma)t},$$

where the constant A_γ is defined in (12).

Proof. We begin by recalling from Theorem 1 that

$$\lim_{t\to\infty} \mathbb{P}_{(z,x)}\big(Z_t > 0 \mid \underline{\xi}_t > 0\big) = \mathfrak{b}(z, x) > 0.$$

Thus, appealing to (11) we obtain

$$\mathbb{P}_{(z,x)}\big(Z_t > 0,\ \underline{\xi}_t > 0\big) = \mathbb{P}_{(z,x)}\big(Z_t > 0 \mid \underline{\xi}_t > 0\big)\mathbb{P}_x^{(e)}\big(\underline{\xi}_t > 0\big) \\ \sim \mathfrak{b}(z, x)\frac{A_\gamma}{\gamma\kappa^{(\gamma)}(0, \gamma)}\mathrm{e}^{\gamma x}\widehat{U}^{(\gamma)}(x)t^{-3/2}\mathrm{e}^{\Phi_\xi(\gamma)t}$$

as $t \to \infty$, which yields the desired result. □

The following lemma tells us that, under the condition that $\ell(\lambda) > C$ for $C > 0$, only a Lévy random environment with a high infimum contributes substantially to the non-extinction probability, and moreover that the mapping $x \mapsto \mathfrak{b}(z, x)\mathrm{e}^{\gamma x}\widehat{U}^{(\gamma)}(x)$ on $(0, \infty)$ is increasing, strictly positive, and bounded.

Lemma 7. *Suppose that $\ell(\lambda) > C$ for $C > 0$. Then, for $\delta \in (0, 1)$ and $z, x > 0$,*

$$\lim_{y\to\infty} \limsup_{t\to\infty} t^{3/2}\mathrm{e}^{-t\Phi_\xi(\gamma)}\mathbb{P}_{(z,x)}\big(Z_t > 0,\ \underline{\xi}_{t-\delta} \le -y\big) = 0.$$

Furthermore, for each $z > 0$ the mapping $x \mapsto \mathfrak{b}(z, x)\mathrm{e}^{\gamma x}\widehat{U}^{(\gamma)}(x)$ on $(0, \infty)$ is increasing, strictly positive, and bounded.

Proof. The proof of this lemma follows similar arguments to those used in the proofs of [2, Lemma 6] and [14, Lemma 4.4]. More precisely, from (5) we deduce the following identity, which holds for all $t > 0$:

$$\mathbb{P}_{(z,x)}\big(Z_t > 0 \mid \xi\big) = 1 - \exp\{-zv_t(0, \infty, \xi - \xi_0)\}.$$

Similarly to [2, Lemma 6], since $\ell(\lambda) > C$ we can bound the functional $v_t(0, \infty, \xi - \xi_0)$ in terms of the exponential functional of the Lévy process ξ, i.e.

$$v_t(0, \infty, \xi - \xi_0) \le (\beta C \mathtt{I}_{0,t}(\beta(\xi - \xi_0)))^{-1/\beta},$$

where we recall that $\mathtt{I}_{s,t}(\beta(\xi-\xi_0)) := \int_s^t \mathrm{e}^{-\beta(\xi_u-\xi_0)}\,\mathrm{d}u$ for $t \geq s \geq 0$. In other words, for $0 < \delta < t$, we deduce that

$$\mathbb{P}_{(z,x)}\big(Z_t > 0,\ \underline{\xi}_{t-\delta} \leq -y\big) \leq C(z)\mathbb{E}_x^{(\mathrm{e})}\big[F(\mathtt{I}_{0,t}(\beta(\xi-\xi_0)));\underline{\xi}_{t-\delta} \leq -y\big]$$
$$= C(z)\mathbb{E}^{(\mathrm{e})}[F(\mathtt{I}_{0,t}(\beta\xi));\tau^-_{-\tilde{y}} \leq t-\delta], \tag{24}$$

where $\tilde{y} = y + x$, $\tau^-_{-\tilde{y}} = \inf\{t \geq 0\colon\ \xi_t \leq -\tilde{y}\}$, $C(z) = z(\beta C)^{-1/\beta} \vee 1$, and $F(w) = 1 - \exp\{-z(\beta Cw)^{-1/\beta}\}$.

To upper bound the right-hand side of (24), we recall from [14, Lemma 4.4] that there exists a positive constant $\tilde{C}$ such that

$$\limsup_{t\to\infty} t^{3/2}\mathrm{e}^{-t\Phi_\xi(\gamma)}\mathbb{E}^{(\mathrm{e})}[F(\mathtt{I}_{0,t}(\beta\xi));\tau^-_{-\tilde{y}} \leq t-\delta] \leq \tilde{C}\mathrm{e}^{-\tilde{y}} + \tilde{C}\mathrm{e}^{-(1-\gamma)\tilde{y}}\widehat{U}^{(\gamma)}(\tilde{y}),$$

which clearly goes to 0 as y increases, since $\gamma \in (0, 1)$ and $\widehat{U}^{(\gamma)}(y) = \mathcal{O}(y)$ as y goes to ∞. Putting all pieces together allows us to deduce the first claim.

For the second claim, we begin by recalling from Section 2.1 that the renewal function is finite and strictly positive on $(0, \infty)$. With this in hand, together with Theorem 1, we obtain that the mapping $x \mapsto \mathfrak{b}(z, x)\mathrm{e}^{\gamma x}\widehat{U}^{(\gamma)}(x)$ is strictly positive. Now, by Lemma 6, we have

$$\lim_{t\to\infty} t^{3/2}\mathrm{e}^{-\Phi_\xi(\gamma)t}\mathbb{P}_{(z,0)}(Z_t > 0,\ \tau^-_{-x} > t) = \lim_{t\to\infty} t^{3/2}\mathrm{e}^{-\Phi_\xi(\gamma)t}\mathbb{P}_{(z,x)}\big(Z_t > 0,\ \underline{\xi}_t > 0\big)$$
$$= \mathfrak{b}(z, x)\frac{A_\gamma}{\gamma\kappa^{(\gamma)}(0, \gamma)}\mathrm{e}^{\gamma x}\widehat{U}^{(\gamma)}(x),$$

which implies that the mapping $x \mapsto \mathfrak{b}(z, x)\mathrm{e}^{\gamma x}\widehat{U}^{(\gamma)}(x)$ is increasing since the left-hand side of the previous equality is increasing in $x > 0$. It remains to prove that the function is bounded. In order to do so we first observe, from inequality (24) but with $\delta = 0$, that

$$\mathbb{P}_{(z,x)}\big(Z_t > 0,\ \underline{\xi}_t > 0\big) \leq C(z)\mathbb{E}^{(\mathrm{e})}[F(\mathtt{I}_{0,t}(\beta\xi));\tau^-_{-x} > t] \leq C(z)\mathbb{E}^{(\mathrm{e})}[F(\mathtt{I}_{0,t}(\beta\xi))].$$

Since $F(w) \leq C_0 w^{-\alpha_0/\beta}$ for some $C_0 > 0$ and $\alpha_0 > 0$, we use [14, Lemma 4.6] to deduce that there exits a constant $C_1 > 0$ such that

$$\limsup_{t\to\infty} t^{3/2}\mathrm{e}^{-t\Phi_\xi(\gamma)}\mathbb{E}^{(\mathrm{e})}[F(\mathtt{I}_{0,t}(\beta\xi))] \leq C_1,$$

which allows us to conclude that the mapping $x \mapsto \mathfrak{b}(z, x)\mathrm{e}^{\gamma x}\widehat{U}^{(\gamma)}(x)$ is a bounded function. This completes the proof. □

We are now ready to deduce our second main result. The proof of Theorem 2 follows the same arguments as used in the proof of [2, Theorem 1.2]; we provide the proof for the sake of completeness.

Proof of Theorem 2. Let $z, x, \varepsilon > 0$. We begin by noting from (2) that $\mathbb{P}_{(z,x)}(Z_t > 0)$ does not depend on the initial value x of the Lévy process ξ. From Lemmas 6 and 7, we deduce that we may choose $y > 0$ such that, for t sufficiently large,

$$\mathbb{P}_{(z,x)}\big(Z_t > 0,\ \underline{\xi}_{t-\delta} \leq -y\big) \leq \varepsilon\mathbb{P}_{(z,x)}\big(Z_t > 0,\ \underline{\xi}_{t-\delta} > -y\big).$$

Further, since $\{Z_t > 0\} \subset \{Z_{t-\delta} > 0\}$ for t large, we deduce that

$$\mathbb{P}_z(Z_t > 0) = \mathbb{P}_{(z,x)}\big(Z_t > 0,\ \underline{\xi}_{t-\delta} > -y\big) + \mathbb{P}_{(z,x)}\big(Z_t > 0,\ \underline{\xi}_{t-\delta} \leq -y\big)$$
$$\leq (1+\varepsilon)\mathbb{P}_{(z,x+y)}\big(Z_{t-\delta} > 0,\ \underline{\xi}_{t-\delta} > 0\big).$$

In other words, for every $\varepsilon > 0$ there exists $y' > 0$ such that

$$(1-\varepsilon)t^{3/2}\mathrm{e}^{-\Phi_\xi(\gamma)t}\mathbb{P}_{(z,y')}\big(Z_t > 0,\ \underline{\xi}_t > 0\big) \le t^{3/2}\mathrm{e}^{-\Phi_\xi(\gamma)t}\mathbb{P}_z(Z_t > 0)$$
$$\le (1+\varepsilon)t^{3/2}\mathrm{e}^{-\Phi_\xi(\gamma)t}\mathbb{P}_{(z,y')}\big(Z_{t-\delta} > 0,\ \overline{\xi}_{t-\delta} > 0\big).$$

Now, appealing to Lemma 6, we have

$$\lim_{t\to\infty} t^{3/2}\mathrm{e}^{-\Phi_\xi(\gamma)t}\mathbb{P}_{(z,y')}\big(Z_t > 0,\ \underline{\xi}_t > 0\big) = \mathfrak{b}(z, y')\frac{A_\gamma}{\gamma\kappa^{(\gamma)}(0, \gamma)}\mathrm{e}^{\gamma y'}\widehat{U}^{(\gamma)}(y').$$

Hence, we obtain

$$(1-\varepsilon)\frac{A_\gamma}{\gamma\kappa^{(\gamma)}(0, \gamma)}\mathfrak{b}(z, y')\mathrm{e}^{\gamma y'}\widehat{U}^{(\gamma)}(y') \le \liminf_{t\to\infty} t^{3/2}\mathrm{e}^{-t\Phi_\xi(\gamma)}\mathbb{P}_z(Z_t > 0)$$
$$\le (1+\varepsilon)\frac{A_\gamma}{\gamma\kappa^{(\gamma)}(0, \gamma)}\mathfrak{b}(z, y')\mathrm{e}^{\gamma y'}\widehat{U}^{(\gamma)}(y')\mathrm{e}^{-\Phi_\xi(\gamma)\delta},$$

where y' may depend on ε and z. Next, we choose y' in such a way that it goes to infinity as ε goes to 0. In other words, for any $y' = y_\varepsilon(z)$ which goes to ∞ as ε goes to 0, we have

$$\begin{aligned} 0 &< (1-\varepsilon)\frac{A_\gamma}{\gamma\kappa^{(\gamma)}(0, \gamma)}\mathfrak{b}(z, y_\varepsilon(z))\mathrm{e}^{\gamma y_\varepsilon(z)}\widehat{U}^{(\gamma)}(y_\varepsilon(z)) \\ &\le \liminf_{t\to\infty} t^{3/2}\mathrm{e}^{-\Phi_\xi(\gamma)t}\mathbb{P}_z(Z_t > 0) \\ &\le (1+\varepsilon)\frac{A_\gamma}{\gamma\kappa^{(\gamma)}(0, \gamma)}\mathfrak{b}(z, y_\varepsilon(z))\mathrm{e}^{\gamma y_\varepsilon(z)}\widehat{U}^{(\gamma)}(y_\varepsilon(z))\mathrm{e}^{-\Phi_\xi(\gamma)\delta} < \infty, \end{aligned} \tag{25}$$

where the strict positivity and finiteness in the previous inequality follows from Lemma 7. Now, letting $\varepsilon \to 0$, we get

$$\begin{aligned} 0 &< \limsup_{\varepsilon\to 0}(1-\varepsilon)\frac{A_\gamma}{\gamma\kappa^{(\gamma)}(0, \gamma)}\mathfrak{b}(z, y_\varepsilon(z))\mathrm{e}^{\gamma y_\varepsilon(z)}\widehat{U}^{(\gamma)}(y_\varepsilon(z)) \\ &\le \liminf_{t\to\infty} t^{3/2}\mathrm{e}^{-\Phi_\xi(\gamma)t}\mathbb{P}_z(Z_t > 0). \end{aligned}$$

Similarly, by first taking δ to 0 and then ε tending to 0 in (25), we obtain

$$\liminf_{t\to\infty} t^{3/2}\mathrm{e}^{-\Phi_\xi(\gamma)t}\mathbb{P}_z(Z_t > 0) \le \liminf_{\varepsilon\to 0}(1+\varepsilon)\frac{A_\gamma}{\gamma\kappa^{(\gamma)}(0, \gamma)}\mathfrak{b}(z, y_\varepsilon(z))\mathrm{e}^{\gamma y_\varepsilon(z)}U^{(\gamma)}(y_\varepsilon(z)) < \infty,$$

where in the last inequality we used that $x \mapsto \mathfrak{b}(z, x)\mathrm{e}^{\gamma x}U^{(\gamma)}(x)$ is bounded, see Lemma 7. Therefore, putting all pieces together, we see that both the inferior and superior limits coincide. In other words, the following limit exists:

$$\mathfrak{B}(z) := \frac{A_\gamma}{\gamma\kappa^{(\gamma)}(0, \gamma)}\lim_{\varepsilon\to 0}\mathfrak{b}(z, y_\varepsilon(z))\mathrm{e}^{\gamma y_\varepsilon(z)}\widehat{U}^{(\gamma)}(y_\varepsilon(z)) \in (0, \infty).$$

Moreover, using this together with (25), we get $\lim_{t\to\infty} t^{3/2}\mathrm{e}^{-\Phi_\xi(\gamma)t}\mathbb{P}_z(Z_t > 0) = \mathfrak{B}(z)$, which concludes the proof. □

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2.4. The stable case

Here, we compute the constant $\mathfrak{B}(z)$ in the stable case and verify that it coincides with the constant that appears in [14, Theorem 5.1]. To this end, we recall that in the stable case we have $\psi_0(\lambda) = C\lambda^{1+\beta}$ with $\beta \in (0, 1)$ and $C > 0$. Moreover, the backward differential equation (4) can be solved explicitly (see, e.g., [11, Section 5]), i.e., for any $\lambda \geq 0$ and $s \in [0, t]$,

$$v_t(s, \lambda, \xi) = (\lambda^{-\beta} + \beta C \mathtt{I}_{s,t}(\beta\xi))^{-1/\beta}, \tag{26}$$

where $\mathtt{I}_{s,t}(\beta\xi)$ denotes the exponential functional of the Lévy process $\beta\xi$, see (9).

Next, we observe that, for any $z, x > 0$, the constant $\mathfrak{b}(z, x)$ defined in Theorem 1 can be rewritten as

$$\mathfrak{b}(z, x) = 1 - \lim_{\lambda\to\infty} \lim_{s\to\infty} \gamma\kappa^{(\gamma)}(0, \gamma) \int_0^\infty \mathrm{e}^{-\gamma y} U^{(\gamma)}(y) R_{s,\lambda}(z, x, y)\, \mathrm{d}y,$$

where

$$R_{s,\lambda}(z, x, y) := \int_0^1 \int_0^\infty w^{\mathbf{u}} \mathbb{P}^{(\gamma),\uparrow}_{(z,x)}(\mathcal{U}_s \in \mathrm{d}\mathbf{u}) \mathbb{P}^{(\mathrm{e},\gamma),\downarrow}_{-y}(\widehat{W}_s(\lambda) \in \mathrm{d}w).$$

In order to find an explicit expression for the previous double integral we use [2, Proposition 3.3], which claims that, for any $z, x > 0$ and $\theta \geq 0$,

$$\mathbb{E}^{(\gamma),\uparrow}_{(z,x)}[\exp\{-\theta Z_s \mathrm{e}^{-\xi_s}\}] = \mathbb{E}^{(\mathrm{e},\gamma),\uparrow}_{x}[\exp\{-z v_s(0, \theta \mathrm{e}^{-x}, \xi - x)\}].$$

It then follows that

$$\begin{aligned}
R_{s,\lambda}(z, x, y) &= \int_0^1 \mathbb{E}^{(\gamma),\uparrow}_{(z,x)}\big[w^{\mathcal{U}_s}\big] \mathbb{P}^{(\mathrm{e},\gamma),\downarrow}_{-y}\big(\widehat{W}_s(\lambda) \in \mathrm{d}w\big) \\
&= \int_0^1 \mathbb{E}^{(\gamma),\uparrow}_{(z,x)}[\exp\{\log(w) Z_s \mathrm{e}^{-\xi_s}\}] \mathbb{P}^{(\mathrm{e},\gamma),\downarrow}_{-y}\big(\widehat{W}_s(\lambda) \in \mathrm{d}w\big) \\
&= \int_0^1 \mathbb{E}^{(\mathrm{e},\gamma),\uparrow}_{x}[\exp\{-z v_s(0, -\log(w)\mathrm{e}^{-x}, \xi - x)\}] \mathbb{P}^{(\mathrm{e},\gamma),\downarrow}_{-y}\big(\widehat{W}_s(\lambda) \in \mathrm{d}w\big) \\
&= \int_0^\infty \int_0^\infty \exp\{-z\mathrm{e}^{-x}(\beta C w + \beta C u)^{-1/\beta}\} \mathbb{P}^{(\gamma),\uparrow}_{(z,x)}(\mathtt{I}_{0,\infty}(\beta\xi) \in \mathrm{d}w) \\
&\qquad\qquad \mathbb{P}^{(\mathrm{e},\gamma),\downarrow}_{-y}(\mathtt{I}_{0,\infty}(\beta\widehat{\xi}) \in \mathrm{d}u),
\end{aligned}$$

where in the last equality we used (26). Thus, putting all the pieces together and appealing to the dominated convergence theorem, we deduce that

$$\begin{aligned}
\mathfrak{b}(z, x) &= 1 - \gamma\kappa^{(\gamma)}(0, \gamma) \int_0^\infty \mathrm{e}^{-\gamma y} U^{(\gamma)}(y) \lim_{\lambda\to\infty} \lim_{s\to\infty} R_{s,\lambda}(z, x, y)\, \mathrm{d}y \\
&= \gamma\kappa^{(\gamma)}(0, \gamma) \int_0^\infty \mathrm{e}^{-\gamma y} U^{(\gamma)}(y) G_{z,x}(y)\, \mathrm{d}y,
\end{aligned}$$

where $G_{z,x}(\cdot)$ is as in (10). Therefore, we have that the limiting constant in the stable case is given by

$$\mathfrak{B}(z) := \frac{A_\gamma}{\gamma\kappa^{(\gamma)}(0, \gamma)} \lim_{x\to\infty} \mathfrak{b}(z, x) \mathrm{e}^{\gamma x} \widehat{U}^{(\gamma)}(x) = A_\gamma \lim_{x\to\infty} \mathrm{e}^{\gamma x} \widehat{U}^{(\gamma)}(x) \int_0^\infty \mathrm{e}^{-\gamma y} U^{(\gamma)}(y) G_{z,x}(y)\, \mathrm{d}y,$$

as expected.

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