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A NONDEGENERATE EXCHANGE MOVE ALWAYS PRODUCES INFINITELY MANY NONCONJUGATE BRAIDS

TETSUYA ITO

Abstract. We show that if a link L has a closed n-braid representative admitting a nondegenerate exchange move, an exchange move that does not obviously preserve the conjugacy class, L has infinitely many nonconjugate closed n-braid representatives.

Let B_n be the braid group with standard generators $\sigma_1, \ldots, \sigma_{n-1}$. We denote the closure of a braid $\beta \in B_n$ by $\widehat{\beta} \subset S^3$. For an oriented link $L \subset S^3$, let $\mathbf{Br}_n(L) = \{\beta \in B_n \mid \widehat{\beta} = L\}$ be the set of n-braids whose closures are L.

The set $\mathbf{Br}_n(L)$ may contain infinitely many mutually nonconjugate braids. However, Birman and Menasco proved a remarkable (non)finiteness theorem [BM]: $\mathbf{Br}_n(L)$ modulo the *exchange move* $\alpha\sigma_{n-1}^{-1}\beta\sigma_{n-1} \leftrightarrow \alpha\sigma_{n-1}\beta\sigma_{n-1}^{-1}$ ($\alpha, \beta \in B_{n-1}$; see Figure 1(i)) has only finitely many conjugacy classes. In particular, when $\mathbf{Br}_n(L)$ does not contain a braid admitting an exchange move, $\mathbf{Br}_n(L)$ contains only finitely many conjugacy classes.

We ask the converse[1]: *Does* $\mathbf{Br}_n(L)$ *contain infinitely many mutually nonconjugate braids if* $\mathbf{Br}_n(L)$ *contains a braid admitting an exchange move?*

This question was studied in [SS, Sh, St1, St2] where it was shown that under some additional and technical assumptions, iterations of exchange moves indeed produce infinitely many nonconjugate braids.

In this note, we give a simpler and shorter proof of infiniteness under the weakest assumption. We use a formulation of iterations of exchange moves following [SS].

Definition 1. We say that an n-braid β *admits an exchange move* if one can write $\beta = AB$ for $A \in \langle \sigma_1^{\pm 1}, \ldots, \sigma_{n-2}^{\pm 1}\rangle$ and $B \in \langle \sigma_2^{\pm 1}, \ldots, \sigma_{n-1}^{\pm 1}\rangle$.

For $k \in \mathbb{Z}$ and an n-braid $\beta = AB$ admitting an exchange move, the *k-iterated exchange move* of $\beta = AB$ is the braid $\mathrm{ex}^k(\beta) = \tau^k A \tau^{-k} B$, where $\tau = (\sigma_2 \cdots \sigma_{n-2})^{n-2} \in B_n$. We say that an (iterated) exchange move is *degenerate* if $A\tau = \tau A$ or $B\tau = \tau B$. Otherwise, an (iterated) exchange move is *nondegenerate*.

A k-iterated exchange move is attained by repeating an exchange move $|k|$ times (see Figure 1(ii)) so the closures of β and $\mathrm{ex}^k(\beta)$ represent the same link. A degenerate exchange move obviously preserves the conjugacy classes. Our main theorem shows that, except in trivial cases, iterated exchange moves *always* produce infinitely many mutually nonconjugate braids.

We identify the braid group B_n with the mapping class group $MCG(D_n)$ of the n-punctured disk D_n. Let $\mathrm{ent}(\beta)$ be the topological entropy of β, the infimum of the topological entropy of homeomorphisms that represent β.

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[1] We need to be slightly careful in formulating the problem since some exchange moves are "trivial" in the sense that they obviously yield conjugate braids.

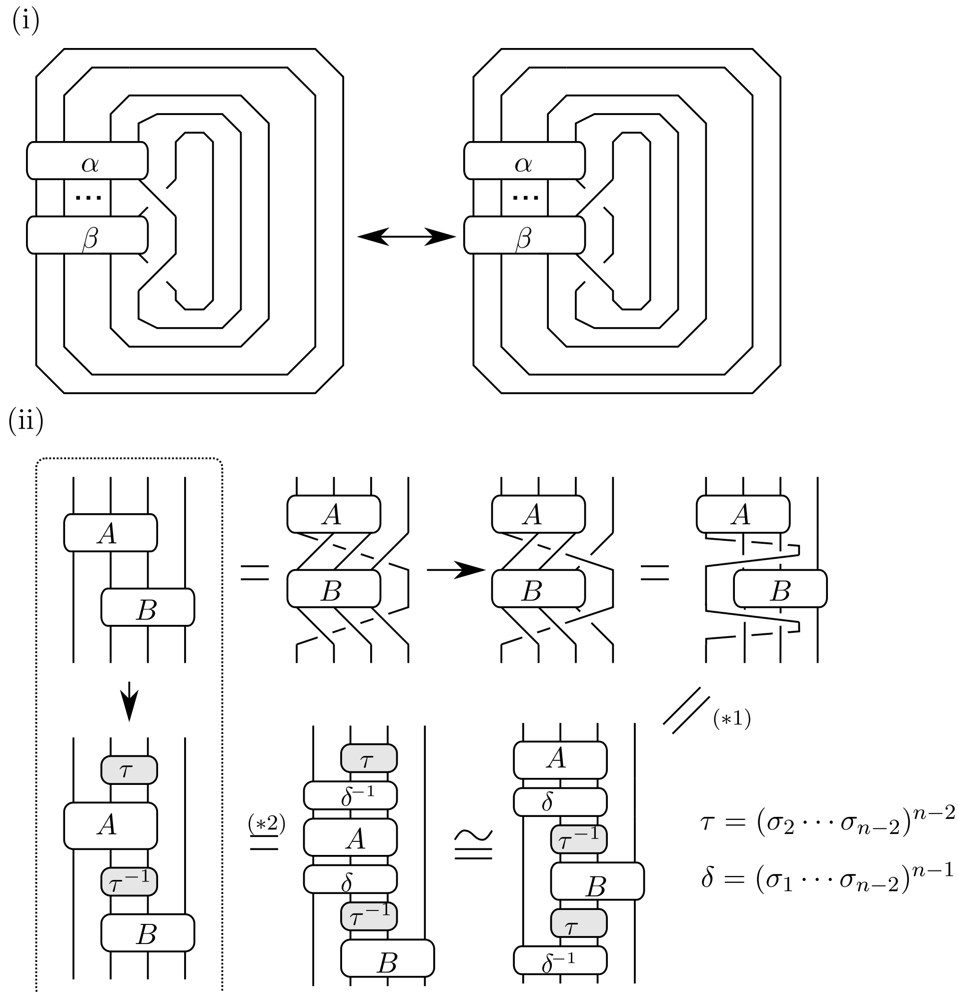

Figure 1.
(i) Exchange move for (closed) braids. (ii) Realization of the move $\beta = AB \to \mathrm{ex}^1(\beta) = \tau A \tau^{-1} B$ by exchange move and conjugation. (The figure illustrates the $n = 4$ case.) Here $=$ and $\cong$ denote the identity and conjugation in the braid group, respectively. At $(*1)$, we use the relation $(\sigma_1\sigma_2 \cdots \sigma_{n-2})(\sigma_{n-2} \cdots \sigma_2\sigma_1) = \delta\tau^{-1}$ and at $(*2)$, we use the relation $A\delta = \delta A$ since δ is the full twist of $(n-1)$ strands.

THEOREM 2. *If $\beta \in \mathbf{Br}_n(L)$ admits a nondegenerate exchange move, then the set $\{\mathrm{ent}(\mathrm{ex}^k(\beta)) \mid k \in \mathbb{Z}\}$ is unbounded. In particular, the set $\{\mathrm{ex}^k(\beta) \mid k \in \mathbb{Z}\} \subset \mathbf{Br}_n(L)$ contains infinitely many distinct conjugacy classes.*

Let S be a closed orientable surface minus finitely many open disks and puncture points in its interior. A simple closed curve c in S is *essential* if c is neither boundary parallel nor surrounds a single puncture. We denote by T_c the Dehn twist along c. A family of essential simple closed curves $\{c_1, \ldots, c_N\}$ *fills* S if $\max i(c, c_i) \neq 0$ for any essential simple closed curve c, where $i(c, c')$ denotes the geometric intersection number. Our proof is based on the following theorem of Fathi [Fa, Theorem 7.9].

THEOREM 3. *Let $f \in MCG(S)$ and $c_1, \ldots, c_N$ be essential simple closed curves in S. Assume the following.*

(i) *The set of curves $\{c_1, \ldots, c_N\}$ fills S.*
(ii) *$i(c_i, c_{i+1}) \neq 0$ for $i = 1, \ldots, N-1$ and $i(c_N, c_1) \neq 0$.*

Then for given $R > 0$, there is $k = k(R) > 0$ such that $T_{c_1}^{n_1} T_{c_2}^{n_2} \cdots T_{c_N}^{n_N} f$ is pseudo-Anosov whose dilatation is $>R$ whenever $|n_i| > k$ for all i.

REMARK. Although the statements and assumptions of Theorem 3 are slightly different, Theorem 3 follows from the proof of [Fa, Theorem 7.9]. First, assumptions (i) and (ii) allow us to apply an interpolation inequality [Fa, Theorem 7.4]. Second, to get the dilatation bound, we take a choice of $\varepsilon > 0$ on page 149 of the proof of [Fa, Theorem 7.9] as $\varepsilon = (K^2 R^{2l-1})^{-1}$ instead of $(2K^2)^{-1}$ as in the original argument. Then the same argument gives the desired dilatation bound.

Proof of Theorem 2. The braid τ in the iterated exchange move corresponds to the Dehn twist T_c along the simple closed curve c surrounding punctures 2 through $n-2$. The nondegeneracy assumption is equivalent to saying that $A(c) \neq c$ and $B(c) \neq c$.

For $i > 0$, let $c_{2i} = \beta^{i-1}(A(c)) = (AB)^{i-1}A(c)$ and $c_{2i-1} = \beta^{i-1}(c) = (AB)^{i-1}(c)$. Thus $\{c_1, c_2, c_3, c_4, \ldots\} = \{c, A(c), \beta(c), \beta(A(c)), \ldots\}$. Since $fT_C = T_{f(C)}f$ for $f \in MCG(D_n)$, for $N > 0$ we have

$$\begin{aligned}
\mathrm{ex}^k(\beta)^N &= (T_c^k A T_c^{-k} B) \cdots (T_c^k A T_c^{-k} B)(T_c^k A T_c^{-k} B) \\
&= (T_c^k T_{A(c)}^{-k} AB) \cdots (T_c^k T_{A(c)}^{-k} AB)(T_c^k T_{A(c)}^{-k} AB) \\
&= (T_c^k T_{A(c)}^{-k} AB) \cdots (T_c^k T_{A(c)}^{-k} T_{AB(c)}^k T_{ABA(c)}^{-k} (AB)^2) \\
&= \cdots \\
&= T_{c_1}^k T_{c_2}^{-k} \cdots T_{c_{2N-1}}^k T_{c_{2N}}^{-k} \beta^N.
\end{aligned}$$

We equip a complete hyperbolic metric (of finite area) on D_n and take c_i as a closed geodesic. Let S be the smallest complete geodesic subsurface of D_n that contains $\{c_1, c_2, c_3, c_4, \ldots\}$. Then for sufficiently large M, the set of curves $\{c_1, c_2, \ldots, c_M\}$ fills S. By the nondegeneracy assumption, S contains ∂D_n in its boundary.

Claim 1. β preserves the subsurface S setwise. In particular, the restriction $\mathrm{ex}^k(\beta)|_S \in MCG(S)$ is well defined for all k.

Proof of Claim 1. If $S = D_n$, then it is obvious, and so we assume that $S \neq D_n$. Then $C = \partial S \setminus \partial D_n$ is a nonempty multicurve. If $\beta(S) \neq S$, then $\beta^{-1}(S) \neq S$ and $i(\beta^{-1}(C), C) \neq 0$. Since $\{c_1, c_2, \ldots, c_M\}$ fills S, this means $i(\beta^{-1}(C), c_i) = i(C, \beta(c_i)) = i(C, c_{i+2}) \neq 0$ for some i. This is a contradiction since $i(C, c_{i+2}) = 0$ by definition. ∎

Claim 2. There exists $N \geqslant M$ such that $i(c_1, c_{2N}) \neq 0$.

Proof of Claim 2. Assume to the contrary that $i(c_1, c_{2N}) = 0$ for all $N \geqslant M$. Let S' be the smallest geodesic subsurface of S that contains $\{c_{2M}, c_{2(M+1)}, \ldots\} = \{c_{2M}, \beta(c_{2M}), \beta^2(c_{2M}), \ldots\}$. By the same argument as Claim 1, β preserves S', and $i(c, c') = 0$ for every curve $c' \subset S'$. Then $\beta^{-2(M-1)}(c_{2M}) = A(c) \subset S'$, so $i(c, A(c)) = 0$. This contradicts the nondegeneracy assumption. ∎

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By the nondegeneracy assumption, $i(c_i, c_{i+1}) \neq 0$ for every $i > 0$. Thus by Claim 2, there is $N > 0$ such that

$$\mathrm{ex}^k(\beta)^N|_S = T_{c_1}^k T_{c_2}^{-k} \cdots T_{c_{2N-1}}^k T_{c_{2N}}^{-k} \beta^N|_S \in MCG(S)$$

satisfies the assumptions of Theorem 3. Hence for any given $R > 0$, whenever $|k|$ is sufficiently large, $\mathrm{ex}^k(\beta)^N|_S$ is pseudo-Anosov whose dilatation $\lambda(\mathrm{ex}^k(\beta)^N|_S)$ is $>R$. Since

$$\mathrm{ent}(\mathrm{ex}^k(\beta)) = \frac{\mathrm{ent}(\mathrm{ex}^k(\beta)^N)}{N} \geqslant \frac{\mathrm{ent}(\mathrm{ex}^k(\beta)^N|_S)}{N} = \frac{\log \lambda(\mathrm{ex}^k(\beta)^N|_S)}{N} \geqslant \frac{\log R}{N},$$

the set $\{\mathrm{ent}(\mathrm{ex}^k(\beta)) \mid k \in \mathbb{Z}\}$ is unbounded. ∎

Our proof shows that as k increases, a nondegenerate k-iterated exchange move increases the entropy, the complexity of dynamics, as long as k is sufficiently large. Since the core of Birman–Menasco's proof of the (non)finiteness theorem is to reduce the complexity (the number of singular points) of a braid foliation corresponding to a Seifert surface, it is natural to expect relations between the entropy and braid foliation.

QUESTION 1. If a braid β is obtained from β' by an exchange move reducing the complexity of braid foliation, then is $\mathrm{ent}(\beta) \leqslant \mathrm{ent}(\beta')$?

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Department of Mathematics, Kyoto University, Kyoto 606-8502, Japan
tetitoh@math.kyoto-u.ac.jp