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REPRESENTATIONS OF LIE GROUPS BY CONTACT TRANSFORMATIONS, II: NON-COMPACT SIMPLE GROUPS

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ABSTRACT If a Lie group acts faithfully as a transitive group of contact transformations of a compact manifold it is either compact with centre of dimension at most 1 or non-compact simple The latter case is described

RESUME Si un groupe de Lie se présente comme groupe transitif de transformations de contact de variété compacte, alors il est ou compact de centre de dimension au plus un ou non-compact simple de centre fini On décrit ce qui se passe dans le second cas

0. **Introduction.** Certain Lie algebras have interesting presentations as infinitesimal contact transformations. For example, the real normal forms of the exceptional Lie algebras G_2, F_4, and E_8 have minimal presentations in this form; the same is true for the normal form of C_n, $n \geq 1$. For the case of G_2 this was done by [Engel 1893] and [Cartan 1893]. The result is valid globally with the corresponding Lie groups having minimal presentations as transitive groups of contact transformations of an appropriate compact manifold. Heisenberg groups act on themselves as contact transformations by group multiplication, but there is no compact model. In fact, one has

THEOREM 1. *Let* $\mathbf{G}$ *be a connected Lie group acting faithfully as a transitive group of contact transformations of a compact manifold. There is an essential dichotomy: either*

(i) $\mathbf{G}$ *is compact with centre of dimension at most* 1, *or*

(ii) $\mathbf{G}$ *is non-compact simple with finite centre.*

In the first case one can state

ADDENDUM THEOREM 1(i). *A connected compact Lie group has a faithful presentation as a transitive group of contact transformations of a compact manifold iff it has a faithful irreducible representation.*

NOTE. Proposition 1.8 of [Herz 1991] is mis-stated. The correct statement is as above.

In case (ii) the possibilities are more limited. For each simple Lie algebra, $\mathfrak{g}$, there will be a unique minimum model for the adjoint group, $\mathrm{Int}(\mathfrak{g})$, but there are other cases. Theorem 4 below gives the full description. The basic idea is that if $\mathbf{G}$ is as in case (ii)

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with $\mathbf{G} = \mathbf{KAN}$ an Iwasawa decomposition and Z is a non-trivial element of the centre of the Lie algebra of $\mathbf{N}$ then the elements of $\mathbf{AN}$ act on Z via the adjoint representation of $\mathbf{G}$ by positive scalar multiplication. Put Ω for the adjoint orbit through Z, and put $\Delta = \Omega/\mathbb{R}_+$. Then Δ is homogeneous space of the maximal compact subgroup $\mathbf{K}$; hence it is compact. There is a contact structure on Δ arising from the Kirillov-Kostant-Souriau symplectic structure of Ω. A point $z \in \Delta$ corresponds to a ray through $Z \in \Omega$, *i.e.* the set of positive scalar multiples of Z, and $\Delta \cong \mathbf{K}/\mathbf{K}(z)$. When $\mathbf{G}$ is the adjoint group of $\mathfrak{g}$ the action of $\mathbf{G}$ on Δ is always faithful. The question of what covering groups of the adjoint group can occur with what covering spaces of Δ to give faithful presentations is rather complicated (as usual the infinitesimal situation is much easier to handle than the global one). This analysis is quite lengthy, and we are forced to examine various cases using the results about contact representations of compact groups and the classification of real simple Lie algebras.

It is time to give the basic definitions.

Let Δ be a connected $C^{(\infty)}$-manifold of dimension $2m-1$, and let $\Omega \xrightarrow{\pi} \Delta$ be a smooth principal $\mathbb{R}_+$ bundle. We write the action of $\mathbb{R}_+$ as multiplication on the right. All such bundles are trivial: there exist maps $r\colon \Omega \longrightarrow \mathbb{R}_+$ which are homogeneous of degree 1, *i.e.* $r(Zc) = r(Z)c$ for $Z \in \Omega$, $c > 0$, such that $\Omega \xrightarrow{\pi\times r} \Delta \times \mathbb{R}_+$ is a diffeomorphism. The trivialization, r, is determined up to multiplication by $f \circ \pi$ where $f \in C^{(\infty)}(\Delta, \mathbb{R}_+)$. Let θ be a smooth 1-form on Δ such that $\theta \wedge d\theta^{\wedge(m-1)}$ vanishes nowhere. The same will be true of $g\theta$ for any $g \in C^{(\infty)}(\Delta, \mathbb{R}_+)$. Put $\kappa = r\theta \circ T(\pi)$. Then $\omega = d\kappa$ gives a symplectic structure to Ω. Let E be the vector field on Ω which is the infinitesimal generator of the $\mathbb{R}_+$-action. To say that a form is homogeneous of degree 1 is to say that it is equal to its Lie derivative with respect to E. Thus, a closed 2-form ω is homogeneous of degree 1 iff $\omega = d\kappa$ where $\kappa(\cdot) = \omega(E, \cdot)$. It is clear that if one starts with a symplectic form ω on Ω which is homogeneous of degree 1 then it arises from a 1-form on Δ in the manner prescribed. Let S be a smooth diffeomorphism of Δ and write $T(S)$ for the induced map of the tangent bundle $T(\Delta)$. Suppose S has the property that $\theta \circ T(S) = \chi_S^{-1}\theta$ where $\chi_S \in C^{(\infty)}(\Delta, \mathbb{R}_+)$. There is a unique extension of S to a diffeomorphism of Ω which commutes with the action of $\mathbb{R}_+$ such that $r \circ S = (\chi_S \circ \pi)r$. We use the same notation for this extension and note that $\kappa \circ T(S) = \kappa$. This amounts to saying that S is a symplectic transformation of (Ω, ω) which commutes with the action of $\mathbb{R}_+$.

A *contact structure* for a manifold Δ, assumed to be connected and necessarily odd dimensional, consists of a principal $\mathbb{R}_+$-bundle $\Omega \longrightarrow \Delta$ together with a closed 2-form, ω, on Ω which is homogeneous of degree 1 for the action of $\mathbb{R}_+$ such that (Ω, ω) is a symplectic manifold. A *contact transformation* of Δ is a symplectic automorphism of (Ω, ω) which commutes with the action of $\mathbb{R}_+$. We shall regard the 1-form $\kappa = \omega(E, \cdot)$ as the fundamental object. The structure which arises by replacing κ with $(g \circ \pi)\kappa$ where $g \in C^{(\infty)}(\Delta, \mathbb{R}_+)$ is *equivalent* to the original one. If $\dim \Delta = 2m-1$ with m odd, changing the sign of κ reverses the orientation of Δ; when m is even, more subtle distinctions are required in order to distinguish $+\kappa$ and $-\kappa$. At any rate, we shall say that a pair of contact manifolds are *isomorphic* if they are described by (Ω_1, κ_1), (Ω_2, κ_2) and there exists a smooth homeomorphism $\phi\colon \Omega_1 \longrightarrow \Omega_2$ which commutes with the action of $\mathbb{R}_+$ such that $\kappa_2 \circ T(\phi)$ is equivalent to $\pm\kappa_1$.

Let $\mathbf{G}$ be a (connected) Lie group acting as a transitive group of smooth contact transformations of Δ. To say that $\mathbf{G}$ acts transitively on Δ is to say that the orbits in Ω project onto Δ. A consequence of the transitivity is that the contact structure is now fixed within its equivalence class up to multiplication by a positive constant.

The standard example of a contact manifold is a cotangent sphere bundle. Let Γ be a connected manifold and $T^*(\Gamma) \xrightarrow{\psi} \Gamma$ its cotangent bundle. The group $\mathbb{R}_+$ acts on $T^*(\Gamma)$ by scalar multiplication in the fibres. There is a canonical horizontal 1-form κ on $T^*(\Gamma)$ which is homogeneous of degree 1 and provides a canonical contact structure for $S^*(\Gamma) = \Psi/\mathbb{R}_+$ where Ψ is $T^*(\Gamma)$ with the 0-section removed if $\dim\Gamma > 1$ or one of its two components if $\dim\Gamma = 1$. Now suppose that Γ is a homogeneous space of a connected Lie group $\mathbf{G}$ and that $\mathbf{G}$ acts faithfully. Put $\mathfrak{g}$ for the Lie algebra of $\mathbf{G}$ and let Γ_X designate the vector field on Γ corresponding to the infinitesimal action of $X \in \mathfrak{g}$. We write $\mathfrak{g}^*$ for the dual vector space of $\mathfrak{g}$; the adjoint action of $\mathbf{G}$ on $\mathfrak{g}$ is written on the left and the dual action on $\mathfrak{g}^*$ on the right. (We shall consistently use a notation where if V is a vector space and V^* its dual then the pairing of $v \in V$ with $f \in V^*$ is fv, and is S is a linear transformation acting on the left in V it acts on the right on V^* by associativity of fSv.) One has a canonical map

$$(0.01) \qquad \Psi \longrightarrow \mathfrak{g}^* \setminus \{0\}, \quad Z \longmapsto \hat{Z}, \quad \hat{Z}X = Z\Gamma_X(\psi Z).$$

We regard an element $Z \in T_p^*(\Gamma)$ as a linear functional on the tangent space $T_p(\Gamma)$. A vector field V on Γ has a canonical extension to a vector field V^* on $T^*(\Gamma)$, and one has

$$\kappa\Big(V^*(Z)\Big) = ZV(\psi Z).$$

A diffeomorphism S of Γ has a canonical extension to a covering diffeomorphism of $T^*(\Gamma)$; by abuse of notation we use the same letter for both. One then has

$$\widehat{SZ} = \hat{Z}(\operatorname{ad} S)^{-1}.$$

The cotangent sphere bundle $S^*(\Gamma)$ has the contact structure given by (Ψ, κ) and $\mathbf{G}$ acts faithfully on $S^*(\Gamma)$ as a group of contact transformations. The action is transitive only in very special cases. Indeed, there is no reason to expect a $\mathbf{G}$-orbit in $S^*(\Gamma)$ to be a regular submanifold. Suppose however we can find a closed orbit $\Delta \subset S^*(\Gamma)$. Let Ω be the inverse image of Δ in Ψ; it will be a closed regular submanifold. If Δ is a contact manifold under the contact structure inherited from $S^*(\Gamma)$ then we have the situation being investigated.

If θ is a non-vanishing 1-form on Γ then it gives a section of $S^*(\Gamma)$. The image will be a contact manifold under the contact structure of the cotangent sphere bundle iff θ already gives a contact structure for Γ which is necessarily in the equivalence class of (Ω, κ).

We shall begin in §2 by supposing that Δ is a compact contact manifold on which $\mathbf{G}$ acts as a transitive group of contact transformations and consider the above construction in the cotangent bundle $T^*(\Delta)$. This amounts to a special case of the Kirillov-Kostant-Souriau theory. Once one has proved the dichotomy of Theorem 1 the two cases can be separated. We have already examined the compact case: the Addendum to Theorem 1(i) applies. Therefore we may suppose that we have at hand a real simple Lie algebra $\mathfrak{g}$ of non-compact type. For this case, each adjoint orbit in $\mathfrak{g}$ has a canonical symplectic structure given by the Kostant-Souriau form. This situation is analyzed in §2.

A *boundary* of a semi-simple Lie algebra $\mathfrak{g}$ is a homogeneous space of the form $\Gamma \cong \mathbf{G}/\mathbf{P}$ where $\mathbf{G} = \mathrm{Int}(\mathfrak{g})$ and $\mathbf{P}$ is a parabolic subgroup. We shall say that $\mathfrak{g}$ is of *holomorphic type* if the maximal compact subalgebras have centres of positive dimension.

THEOREM 2. *Each real simple Lie algebra* $\mathfrak{g}$ *of holomorphic type has a uniquely specified boundary* Γ *which has a contact structure on which the adjoint group* $\mathrm{Int}(\mathfrak{g})$ *acts as a faithful transitive group of contact transformations. In this case there are exactly two adjoint orbits in* $\mathfrak{g}$, Ω *and* $-\Omega$, *such that* (Ω, ω) *gives the contact structure where* ω *is the Kostant-Souriau form.*

THEOREM 3. *Each real simple Lie algebra* $\mathfrak{g}$ *of non-holomorphic type has a unique adjoint orbit* Ω *such that* (Ω, ω) *gives rise to a compact contact manifold,* Δ. *Let d be the multiplicity of the highest root of* $\mathfrak{g}$. *Then there is a uniquely specified boundary* Γ *such that* Δ *is a sub-bundle of* $S^*(\Gamma)$ *whose fibres are spheres* S^{d-1} *with the action of* $\mathrm{Int}(\mathfrak{g})$ *on* Ω *agreeing with the canonical lifting to the cotangent bundle of the action of* $\mathrm{Int}(\mathfrak{g})$ *on* Γ.

In all of the cases above the boundary Γ is minimal (rank 1) except when the (restricted) root system of $\mathfrak{g}$ is of type A_n, $n > 1$, when it is a rank 2 boundary. The boundary in question leads to an interesting classification of real simple Lie algebras. The connectivity of the parabolic subgroup $\mathbf{P}$ associated with this boundary is determined in §3 and §4.

In §4 we arrive at a complete classification of the compact contact manifolds which arise for the various simple Lie algebras, but this is based on ad hoc methods, and it would be desirable to have more general arguments.

THEOREM 4. *The simple Lie groups of non-compact type with finite centre which arise as transitive groups of contact transformations of a compact manifold are, up to isomorphism of groups and equivalence of contact structures:*

(i) *For* $\mathfrak{g} = \mathfrak{so}(1,2)$ *each Lie group,* $\mathbf{G}$, *with finite centre having Lie algebra* $\mathfrak{g}$ *acts faithfully as a transitive group of contact transformations of a compact manifold* $\Delta(\mathbf{G})$.

(ii) *If* $\mathbf{G} = \mathrm{SL}(3,\mathbb{R})$ *and* $\tilde{\mathbf{G}}$ *is its universal covering group then* $\tilde{\mathbf{G}}$ *acts faithfully by contact transformations of* $\tilde{\Delta} \cong S^3$. *Here* $\tilde{\Delta}$ *is a double covering of* $\Delta_2 \cong \mathrm{Proj}(3,\mathbb{R})$ *on which* $\mathrm{SL}(3,\mathbb{R})$ *acts faithfully and a cyclic covering of order 4 of the minimal* Δ *on which* $\mathrm{Int}(\mathfrak{g}) = \mathrm{SL}(3,\mathbb{R})$ *also acts faithfully.*

(iii) *If* $\mathbf{G} = \mathrm{SL}(n+1, \mathbb{R})$ *with* $n > 2$, *then, when n is even,* $\mathbf{G}$ *acts as a transitive group of contact transformations of both the minimal* Δ *and a double covering* $\tilde{\Delta}$, *and, when n is odd,* $\mathrm{Int}(\mathfrak{g})$ *acts faithfully on* Δ *and* $\mathbf{G}$ *on a double covering* $\tilde{\Delta}$.

(iv) *For* $\mathbf{G} = \mathrm{Sp}(n, \mathbb{R})$, $n > 1$, $\mathbf{G}$ *acts faithfully by contact transformations of* $\tilde{\Delta} \cong S^{2n-1}$ *while* $\mathrm{Int}(\mathfrak{g})$ *acts faithfully on* $\Delta \cong \mathrm{Proj}(2n-1, \mathbb{R})$. *For* $\mathbf{G} = \mathrm{Sp}(n, \mathbb{C})$, *all n,* $\mathbf{G}$ *acts faithfully by contact transformations of* $\tilde{\Delta} \cong S^{4n-1}$ *while* $\mathrm{Int}(\mathfrak{g})$ *acts faithfully on* $\Delta \cong \mathrm{Proj}(4n-1, \mathbb{R})$.

(v) *For all real simple Lie algebras* $\mathfrak{g}$ *not isomorphic to* $\mathfrak{sp}(n, \mathbb{R})$, $\mathfrak{sp}(n, \mathbb{C})$, *or* $\mathfrak{sl}(n+1, \mathbb{R})$, *only* $\mathrm{Int}(\mathfrak{g})$ *acts faithfully as a transitive group of contact transformations of a compact manifold and this manifold is unique.*

The starting point for the analysis in the non-compact case involves the study of certain nilpotent orbits. Many authors have examined these, in particular [Wolf 1978]; but the concerns here are very specialized. For simple groups, the Killing form allows an identification of adjoint orbits with co-adjoint orbits; this is of interest in representation theory, but we have not examined that aspect of the subject. The boundaries, Γ, which occur in Theorems 2 and 3 have some geometric interest. They are discussed in §5 where we give some concrete illustrations of Theorem 4.

1. **Generalities.** Define

$$\Omega \longrightarrow \mathfrak{g}^*, \quad Z \longmapsto \hat{Z}, \quad \hat{Z}X = \kappa\big(\Omega_X(Z)\big), \quad X \in \mathfrak{g}.$$

One then gets

$$\widehat{SZ} = \hat{Z}\,\mathrm{ad}\, S^{-1}$$

and

$$\omega\big(\Omega_X(Z), \Omega_Y(Z)\big) = -\hat{Z}[X, Y].$$

Thus, if $\hat{\Omega}$ is the image of Ω in $\mathfrak{g}^*$, the symplectic form ω on Ω corresponds to the Kostant-Souriau form on the co-adjoint orbits in $\hat{\Omega}$. The $\mathbb{R}_+$-action on Ω transforms to scalar multiplication.

Put $\mathbf{Q}(z)$ for the subgroup of $\mathbf{G}$ leaving $z \in \Delta$ fixed and $\mathfrak{q}(z)$ for its Lie algebra.

LEMMA 1.01. *$X \in \mathfrak{q}(z)$ iff $\hat{Z}X = 0$ and $\hat{Z}\,\mathrm{Ad}\,X \in \mathbb{R}\hat{Z}$ for all $Z \in \Omega$ such that $z = \pi Z$.*

PROOF. It is clear from the set-up that $X \in \mathfrak{q}(z)$ iff $\Omega_X(Z)$ is a vertical vector for each Z above z. This is to say that $\Omega_X(Z) = E(Z)c$ for some $c \in \mathbb{R}$. In view of the fact that $\kappa = \omega(E, \cdot)$ we get the desired conclusion.

Put $\mathfrak{c}(z) = \{X \in \mathfrak{g} : Z\,\mathrm{Ad}\,X = 0\}$ and put $\mathfrak{c}$ for the centre of $\mathfrak{g}$. Obviously $\mathfrak{c} \subset \mathfrak{c}(z)$ for all $z \in \Delta$. On the other hand $\mathfrak{c} \cap \mathfrak{q}(z)$ is independent of $z \in \Delta$; since $\mathfrak{g}$ is assumed to act effectively on Δ we conclude that $\mathfrak{c} \cap \mathfrak{q}(z) = 0$ for all z. Since $\mathfrak{r}(z) = \mathfrak{c}(z) \cap \mathfrak{q}(z)$ is of co-dimension at most 1 in $\mathfrak{c}(z)$ by Lemma 1.01, we conclude that $\dim \mathfrak{c} \leq 1$.

The next uses the compactness of Δ in an essential way.

LEMMA 1.02. *Let $\mathfrak{n} \subset \mathfrak{g}$ be a unipotent subalgebra. Then there exists $z \in \Delta$ such that $\hat{Z}\operatorname{Ad} N = 0$ for all $Z \in \pi^{-1}(z)$ and $N \in \mathfrak{n}$.*

PROOF. Let $\mathfrak{n}$ be a subalgebra of $\mathfrak{g}$ such that $\operatorname{Ad} N$ is nilpotent for each $N \in \mathfrak{n}$. Then, for each $z \in \Delta$ there exists an integer $n = n(z, N)$ such that $\hat{Z}(\operatorname{Ad} N)^n \neq 0$, $\hat{Z}(\operatorname{Ad} N)^{n+1} = 0$. Fix $Z_0 \in \pi^{-1}(z)$, and, for $t \geq 0$, consider $Z(t) = \exp(-tN)Z_0$. Then we have

$$\hat{Z}(t) = \sum_1^n (k!)^{-1} t^k \hat{Z}_0(\operatorname{Ad} N)^k.$$

We can choose a continuous function $f\colon [0, \infty) \longrightarrow (0, \infty)$ with $f(0) = 1$ so that $f(t)Z(t)$ stays in a compact portion of $\Omega \cong \Delta \times \mathbb{R}_+$. Let $t \longrightarrow \infty$ through a sequence such that $f(t)Z(t) \longrightarrow Z$. By continuity we must have $\hat{Z} = c\hat{Z}_0(\operatorname{Ad} N)^n$ for some $c \in \mathbb{R}_+$. Since $\mathbf{G}Z_0$ projects onto Δ, by changing the constant c we can assert that there exists $Z \in \mathbf{G}Z_0$ of the form described. Given $N_1, \ldots, N_k \in \mathfrak{n}$ we obtain $Z_1 \in \mathbf{G}Z_0$ with $\hat{Z}_1 = c_1\hat{Z}_0(\operatorname{Ad} N_1)^{n_1}$ and, recursively,

$$Z_i \in \mathbf{G}Z_{i-1} \text{ with } \hat{Z}_i = c_i\hat{Z}_{i-1}(\operatorname{Ad} N_i)^{n_i}$$

and $n_i > 0$ if $\hat{Z}_{i-1} \operatorname{Ad} N_i \neq 0$. Note that

$$\hat{Z}_k = c_1 \cdots c_k \hat{Z}_0(\operatorname{Ad} N_1)^{n_1} \cdots (\operatorname{Ad} N_k)^{n_k}.$$

By Engel's Theorem this must come to a halt for some k if all $n_i > 0$. Thus $\hat{Z}_k \operatorname{Ad} N = 0$ for all $N \in \mathfrak{n}$; this gives the assertion.

LEMMA 1.03. *Every nilpotent ideal of $\mathfrak{g}$ is contained in the centre $\mathfrak{c}$.*

PROOF. If $\mathfrak{n}$ is a nilpotent ideal, the above Lemma gives that $\mathfrak{n} \subset \mathfrak{c}(z)$ for all $z \in \Delta$. It follows that $[\mathfrak{g}, \mathfrak{n}] \subset \mathfrak{q}(z)$ for all $z \in \Delta$. The faithfulness of the action of $\mathfrak{g}$ on Δ gives the assertion.

We may summarize much of what has been obtained thus far in

THEOREM 1.04. *Let $\mathbf{G}$ be a connected Lie group acting faithfully as a transitive group of contact transformations of a compact manifold. Then either $\mathfrak{g}$ is a compact Lie algebra with centre of dimension ≤ 1 and G acts as a group of restricted contact transformations of Δ or $\mathfrak{g}$ is simple non-compact and $\mathbf{G}$ acts transitively on Ω.*

PROOF. Consider the Levi decomposition gives $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{s}$ where $\mathfrak{g}_1$ is a semi-simple subalgebra and $\mathfrak{s}$ is a solvable ideal. By Lemma 1.03, $\mathfrak{s} = \mathfrak{c}$. Since $\dim \mathfrak{c} \leq 1$, $\mathfrak{g}_1$ is an ideal. This proves that $\mathfrak{g}$ is reductive. Consider first the case $\dim \mathfrak{c} = 1$. As we have seen, for each z, $\mathfrak{c}(z) = \mathfrak{c} \oplus \mathfrak{r}(z)$ where $\mathfrak{r}(z) = \mathfrak{c}(z) \cap \mathfrak{q}(z)$. By Lemma 1.01 we must have $\hat{Z}C \neq 0$ for each $C \in \mathfrak{c} \setminus \{0\}$. Fix $\mathbb{C}_0 \in \mathfrak{c} \setminus \{0\}$, and define $r\colon \Omega \longrightarrow \mathbb{R}_+$ by $r(Z) = \hat{Z}C_0$. Then $r^{-1}\kappa = \theta \circ T(\pi)$ where θ is a contact form on Δ which is invariant under the action of $\mathbf{G}$. This says that $\mathbf{G}$ acts as restricted contact transformations of (Δ, θ). The fact that $\mathfrak{g}$ is compact follows from [Herz 1991, Lemma (2.4)]. Now consider the case in which $\mathfrak{g}$ is semi-simple. Since the Killing form is non-degenerate we may write

$$\hat{Z} = -(2c)^{-1}\operatorname{Kill}(Z^\circ, \cdot) \text{ where } c \in \mathbb{R}_*.$$

For each $k \in \mathbb{Z}_+$ define a map

$$r_k \colon \Omega \longrightarrow [0,\infty), \quad r_k^k(Z) = \Big|\mathrm{tr}\Big(\big(\mathrm{Ad}\, Z^\circ\big)^k\Big)\Big|.$$

Obviously r_k is homogeneous of degree 1 and $\mathbf{G}$-invariant. If for some k, r_k is not identically 0 we have a $\mathbf{G}$-invariant trivialization; so $\mathbf{G}$ acts as restricted contact transformations. As just noted, this implies that $\mathfrak{g}$ is compact. Excluding this case, we conclude that $\mathrm{Ad}\, Z^\circ$ is nilpotent. This allows us to exclude compact simple factors from $\mathfrak{g}$. Now let $\mathfrak{n}$ be some maximal unipotent subalgebra containing Z° and $\mathfrak{a}$ an abelian subalgebra in an Iwasawa decomposition of $\mathfrak{g}$ such that, for some ordering of the (restricted) roots,

$$\mathfrak{n} = \sum_{\alpha>0} \mathfrak{g}(\alpha)$$

where $\mathfrak{g}(\alpha)$ is the root space corresponding to the root α. Let $Z^\circ = \sum_\alpha X_\alpha$ be the decomposition with each $X_\alpha \in \mathfrak{g}(\alpha)$. We can find an $H \in \mathfrak{a}$ such that all the roots $\alpha(H)$ are distinct and at least one is positive. Let β be the root such that $\beta(H) = b > \alpha(H)$ for all positive roots $\alpha \neq \beta$. By the argument used in the proof of Lemma 1.02 applied to $f(t)\exp(tH)Z$ we find that there is some $W \in \Omega$ such that $W^\circ \in \mathfrak{g}(\beta)$. This shows that W° lies in a single simple factor of $\mathfrak{g}$, from which we conclude that $\mathfrak{g}$ is simple. Moreover we have found an $H \in \mathfrak{g}$ such that $[H, W^\circ] = Wb$ with $b \neq 0$. There is no harm in assuming $b = 1$. Thus we have $\Omega_H(W) = E(W)$. Therefore the tangent vectors to the $\mathbf{G}$-orbit through W fill out the entire tangent space to Ω. Since Ω is connected it is a single orbit.

The above result provides the essential dichotomy for the representation of Lie groups as transitive groups of contact transformations of a compact manifold.

In the case where $\mathfrak{g}$ is compact, if the centre $\mathfrak{c}$ is non-trivial, then it is not contained in any $\mathfrak{q}(z)$. Since $\Delta \cong \mathbf{G}/\mathbf{Q}(z)$ is compact, we conclude that the centre of $\mathbf{G}$ is compact. Thus

PROPOSITION 1.05. *If* $\mathbf{G}$ *has compact Lie algebra and acts faithfully as transitive group of contact transformations of a compact manifold then* $\mathbf{G}$ *is compact.*

We have just seen that in the simple, non-compact, case, the adjoint orbit Ω° is a cone, *i.e.* invariant under multiplication by positive scalars. It follows that if we put $\Delta^\circ = \Omega^\circ/\mathbb{R}_+$ then we get a smooth covering $\Delta \longrightarrow \Delta^\circ$ where the covering group is the centre of $\mathbf{G}$. This proves that $\mathbf{G}$ has finite centre. Let us take an Iwasawa decomposition $\mathbf{G} = \mathbf{KAN}$ and take $z \in \Delta$ such that $\mathfrak{n} \subset \mathfrak{q}(z)$ where $\mathfrak{n}$ is the Lie algebra of $\mathbf{N}$ as prescribed by Lemma 1.01. In the proof of Theorem 1.04 we saw that Z° was a root vector. Thus $\mathbf{AN} \subset \mathbf{Q}(z)$. Therefore $\mathbf{K}$ acts transitively on Δ. We record this as

THEOREM 1.06. *If* $\mathbf{G}$ *has non-compact Lie algebra it is simple with finite centre and its maximal compact subgroups act as transitive groups of restricted contact transformations of* Δ.

In the simple non-compact case $\mathrm{Int}(\mathfrak{g})$ acts as a faithful transitive group of contact transformations of Δ°.

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We shall see that in the simple non-compact case the algebras $\mathfrak{q}(z)$ are determined up to inner automorphism. Put $\mathbf{Q}_0(z)$ for the connected subgroup with this Lie algebra. Since $\mathbf{Q}_0(z)$ is the component of the identity in $\mathbf{Q}(z)$, it is closed. We get a faithful representation of $\mathbf{G}$ by contact transformations iff $\mathbf{Q}_0(z)$ meets the centre of $\mathbf{G}$ only in the identity. In the case $\mathbf{G} = \mathrm{Int}(\mathfrak{g})$ the connected subgroup with Lie algebra $\mathfrak{q}(z)$ meets the requirements, but there are cases where one has disconnected closed subgroups $\mathbf{Q}(z)$ meeting the requirements. If $\mathbf{G}$ has non-trivial centre then there is also the problem of determining whether $\mathbf{Q}_0(z)$ meets the centre non-trivially.

2. **Contact actions of the adjoint group of a non-compact simple algebra.** Here $\mathfrak{g}$ is a simple Lie algebra of non-compact type. We continue with the investigations of the previous section.

If $\mathbf{G}$ is any connected Lie group with finite centre having Lie algebra $\mathfrak{g}$ which acts as a transitive group of contact transformations on Δ for the structure (Ω, ω) then $\Omega \longrightarrow \hat{\Omega}$ is a covering map onto a coadjoint orbit with ω a constant multiple of the Kostant-Souriau form. Since there is no essential distinction between adjoint and coadjoint orbits for simple Lie algebras, we shall switch back and forth as suits the convenience of the moment. For the rest of this section we shall assume that Ω is an adjoint orbit and $\mathbf{G} = \mathrm{Int}(\mathfrak{g})$.

LEMMA 2.01. *A non-trivial adjoint orbit Ω is an $\mathbb{R}_+$-bundle over a compact manifold Δ iff it contains an element in the centre of a maximal unipotent subalgebra.*

PROOF. An orbit Ω which is an $\mathbb{R}_+$-bundle over a compact set is a manifold of the type we have been considering. From the proof of Theorem 1.04 we know that the elements of $Z \in \Omega$ are ad-nilpotent. Suppose $Z \in \mathfrak{n}$ where $\mathfrak{n}$ is a maximal unipotent subalgebra. Either $Z \in \mathrm{centre}(\mathfrak{n})$ or there exists $N \in \mathfrak{n}$ such that $[N, Z]$ is a non-trivial element of the centre. In the second case

$$\lim_{t \longrightarrow \infty} t^{-1} \exp(t \,\mathrm{Ad}\, N) Z = [N, Z].$$

If Δ is compact we get $[N, Z] \in \Omega$. Conversely, if Z is in the centre of the maximal unipotent subalgebra $\mathfrak{n}$ and $\mathrm{Int}(\mathfrak{g}) = \mathbf{KAN}$ is an Iwasawa decomposition where N has Lie algebra $\mathfrak{n}$, then AN leaves $Z\mathbb{R}_+$ fixed. Thus Δ is a homogeneous space of the compact group $\mathbf{K}$.

From the above and the considerations of §1, we see that the minimal compact contact manifolds Δ on which $\mathrm{Int}(\mathfrak{g})$ acts faithfully as a transitive group of contact transformations are equivalent to the manifolds $\Omega/\mathbb{R}_+$ where Ω is the adjoint orbit of a non-trivial element of the centre of a maximal unipotent subalgebra with the symplectic form ω given by

$$\omega\big(\Omega_X(Z), \Omega_Y(Z)\big) = -(2c)^{-1}\mathrm{Kill}([X, Y], Z), \quad c \in \mathbb{R}_+. \tag{2.02}$$

In order to classify these minimal contact manifolds we have to do some work. Consider the subalgebras $\mathfrak{z} \in \mathfrak{g}$ which are the centres of some maximal unipotent subalgebra.

We shall call them *corner* subalgebras. Their non-trivial elements will be called "corner elements", and a *ray* through a corner element, *i.e.* a set of the form $z = Z\mathbb{R}_+$ where Z is a corner element, will be called a "corner ray".

PROPOSITION 2.03. *Let $\mathfrak{z}$ be a corner subalgebra and $\mathbf{P}(\mathfrak{z})$ its stabilizer in* $\mathrm{Int}(\mathfrak{g})$. *Then $\mathbf{P}(\mathfrak{z})$ is a parabolic subgroup. If $\mathfrak{w}$ is another corner subalgebra then $\mathbf{P}(\mathfrak{w})$ is conjugate to $\mathbf{P}(\mathfrak{z})$. If $z \subset \mathfrak{z}$ is a ray and $\mathbf{Q}(z)$ its stabilizer in* $\mathrm{Int}(\mathfrak{g})$, *then $\mathbf{Q}(z) \subset \mathbf{P}(\mathfrak{z})$.*

PROOF. Let $\mathfrak{n}$ be a maximal unipotent subalgebra of $\mathfrak{g}$ with centre $\mathfrak{z}$ and take $\mathbf{F}$ to be the normalizer of $\mathfrak{n}$ in $\mathrm{Int}(\mathfrak{g})$. Then $\mathbf{F}$ is a minimal parabolic subgroup contained in $\mathbf{P}(\mathfrak{z})$, and the latter is a parabolic subgroup. Since all the algebras $\mathfrak{n}$ are conjugate, the same is true of the $\mathfrak{z}$. Put $\mathfrak{f}$ for the Lie algebra of $\mathbf{F}$. Let σ be a Cartan involution of $\mathfrak{g}$ and put $\mathfrak{a} = \mathfrak{f} \cap \mathfrak{g}^{-\sigma}$. Then $\mathfrak{a}$ is abelian with root spaces $\mathfrak{g}(\xi)$, and there is an ordering of the roots of $\mathfrak{a}$ in $\mathfrak{g}$ so that $\mathfrak{n} = \sum_{\xi>0} \mathfrak{g}(\xi)$. The highest root is ζ where $\mathfrak{z} = \mathfrak{g}(\zeta)$. It is clear that $\mathbf{AN} \subset \mathbf{Q}(z)$ where $\mathbf{A}$ and $\mathbf{N}$ are the closed connected subgroups with Lie algebras $\mathfrak{a}$ and $\mathfrak{n}$ respectively. Put $\mathbf{K}$ for the maximal compact subgroup constituted by the elements of $\mathrm{Int}(\mathfrak{g})$ which commute with σ. From the Iwasawa decomposition, $\mathrm{Int}(\mathfrak{g}) = \mathbf{KAN}$, it suffices to prove that $\mathbf{K}(z) \subset \mathbf{L}(\mathfrak{z})$ where the former is the centralizer in $\mathbf{K}$ of Z and the latter is the stabilizer in $\mathbf{K}$ of $\mathfrak{z}$. Thus we may suppose that $SZ = Z$ and S commutes with σ. We have $[\sigma Z, Z] = bH_\zeta$ where $b \neq 0$ and $H_\zeta \in \mathfrak{a}$ is the the co-root vector corresponding to the highest root ζ. The elements $W \in \mathfrak{z}$ are characterized by the equation $[H_\zeta, W] = 2W$. If $S \in \mathbf{Q}(z)$ then $SH_\zeta = H_\zeta$ which implies $S \in \mathbf{P}(\mathfrak{z})$.

We may go one step further.

PROPOSITION 2.04. *Let $\mathfrak{z}$ be a corner subalgebra. Then for each $Z \in \mathfrak{z} \setminus \{0\}$ we have $[\mathfrak{g}(0), Z] = \mathfrak{z}$ where $\mathfrak{g}(0)$ is the centralizer of some $\mathfrak{a}$, as above, in $\mathfrak{g}$. Moreover, z uniquely determines $\mathfrak{z}$, and if $\mathfrak{q}(z)$ and $\mathfrak{p}(\mathfrak{z})$ are the respective Lie algebras, their nil radicals are identical.*

PROOF. It is clear that $[\mathfrak{g}(0), Z]$ is a subspace of $\mathfrak{z}$. Choose $W \in \mathfrak{z}$ and put $X = [\sigma Z, W]$; then $X \in \mathfrak{g}(0)$. We have

$$[X, Z] = \big[[\sigma Z, Z], W\big] = b[H_\zeta, W] = 2bW$$

with notation of the proof of Proposition 2.03. The root space decomposition of $\mathfrak{n}(\mathfrak{z})$ shows that the two nil radicals are identical. Moreover $\mathfrak{z}$ is uniquely specified as the centre of $\mathfrak{n}(\mathfrak{z})$.

Let us remark that there may be corner subalgebras, $\mathfrak{w}$, contained in $\mathfrak{n}(\mathfrak{z}) \setminus \mathfrak{z}$. In this case $\mathfrak{n}(\mathfrak{w}) \neq \mathfrak{n}(\mathfrak{z})$. We may state

THEOREM 2.05. *Either the corner elements form a single adjoint orbit Ω or there are two such orbits, Ω and $-\Omega$. In the latter case $\mathfrak{z}$ must be 1-dimensional.*

PROOF. We have seen that all corner subalgebras are conjugate. Proposition 2.04 shows that the intersection of a corner subalgebra with the orbit through one of its non-trivial elements has dimension $d = \dim \mathfrak{z}$. It is an open submanifold of $\mathfrak{z} \setminus \{0\}$. According

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to Theorem 1.06 a maximal compact subgroup produces the same orbit of corner rays as the full group does; therefore we may conclude that $\Omega \cap (\mathfrak{z} \setminus \{0\})$ has no boundary points. The space $\mathfrak{z} \setminus \{0\}$ is disconnected iff $d = 1$.

Let $\Gamma \cong \mathrm{Int}(\mathfrak{g})/\mathbf{P}(\mathfrak{z})$ be the manifold of corner subalgebras of $\mathfrak{g}$. Let $T^*(\Gamma)$ be its cotangent bundle. Then we have

PROPOSITION 2.06. *There is a canonical imbedding*

$$\Omega \longrightarrow T^*(\Gamma)$$

in which the canonical 1*-form* λ *on the cotangent bundle pulls back to the form* κ *defining the contact structure.*

PROOF. Given $X \in \mathfrak{g}$ put Γ_X for the vector field representing its infinitesimal action on Γ. Given $Z \in \Omega$, let $\mathfrak{z}$ be the unique corner subalgebra containing it. Let us prove that if $\Gamma_X(\mathfrak{z}) = 0$ then $\kappa\big(\Omega_X(Z)\big) = 0$. Note that $\Gamma_X(\mathfrak{z}) = 0$ is equivalent to $X \in \mathfrak{p}(\mathfrak{z})$. We have seen that there exists $H \in \mathfrak{q}(\mathfrak{z})$ such that

$$[H, Z] = 2Z; \text{ therefore } E(Z) = \tfrac{1}{2}\Omega_H(Z).$$

Using (2.02) and the relation between ω and κ we get

$$\kappa\big(\Omega_X(Z)\big) = -(2c)^{-1}\mathrm{Kill}(Z, X).$$

If $X \in \mathfrak{p}(\mathfrak{z})$ and $[X, Z] \neq 0$ then $X \in \mathfrak{g}(0)$ which shows that X is orthogonal to Z for the Killing form. This shows that we have a well defined map $\psi \colon \Omega \longrightarrow T^*(\Gamma)$ given by

$$\langle \psi Z, \Gamma_X(\mathfrak{z})\rangle = \kappa\big(\Omega_X(Z)\big).$$

Combining Proposition 2.06 with Theorem 2.05 we get

COROLLARY 2.07. *If the corner subalgebras of* $\mathfrak{g}$ *have dimension d and* $d > 1$ *then* Δ *is a smooth* S^{d-1}*-bundle over the boundary* Γ. *If* $d = 1$ *either* $\Delta = \Gamma$ *or* Δ *is a double covering of* Γ.

The proof of Proposition 2.06 can easily be extended to show that one may identify $T^*_{\mathfrak{z}}(\Gamma)$ with $\mathfrak{n}(\mathfrak{z})$ in such a way that the natural extension of the action to the cotangent bundle agrees with the adjoint action.

Put

$$\mathbf{P}(\mathfrak{z}) = \mathbf{M}(\mathfrak{z})\mathbf{A}(\mathfrak{z})\mathbf{N}(\mathfrak{z}) \tag{2.08}$$

for the Langlands decomposition of the parabolic subgroup $\mathbf{P}(\mathfrak{z})$ assuming a given choice σ for the Cartan involution. We may state

PROPOSITION 2.09. *The reductive (not necessarily connected) Lie group* $\mathbf{M}(\mathfrak{z})$ *acts irreducibly on* $\mathfrak{z}$ *and faithfully on* $\mathfrak{n}(\mathfrak{z})$ *via the adjoint action on* $\mathfrak{g}$.

PROOF. Proposition 2.04 shows that the action on $\mathfrak{z}$ is irreducible. An element of $\mathbf{M}(\mathfrak{z})$ which acts trivially on $\mathfrak{n}(\mathfrak{z})$ also acts trivially on the tangent space $T_{\mathfrak{z}}(\Gamma)$. It therefore leaves a neighborhood of $\mathfrak{z}$ in Γ fixed. By the connectedness of Γ it acts trivially on all of Γ which proves that it is the identity of $\mathrm{Int}(\mathfrak{g})$.

It is relatively easy to determine the Lie algebras $\mathfrak{p}(\mathfrak{z})$ and $\mathfrak{q}(z)$ as subalgebras of $\mathfrak{g}$. Once one has this, the determination of, say $\mathbf{Q}(\mathfrak{z})$, is carried out in two stages. The component of the identity of $\mathbf{M}(\mathfrak{z})$ is the closed subgroup of $\mathrm{Int}(\mathfrak{g})$ corresponding to the Lie algebra $\mathfrak{m}(\mathfrak{z})$. The full determination of $\mathbf{M}(\mathfrak{z})$ then depends on its connectivity. From Corollary 2.07 we see that, for $d > 1$, $\mathbf{M}(\mathfrak{z})$ is connected iff $\mathbf{Q}(z)$ is connected. Therefore the question of whether $\mathbf{Q}(z)$ is connected is of great importance. Put

$$\mathbf{K}(z) = \mathbf{K} \cap \mathbf{Q}(z)$$

where $\mathbf{K}$ is a maximal compact subgroup of $\mathrm{Int}(\mathfrak{g})$. Clearly $\Delta \cong \mathbf{K}/\mathbf{K}(z)$, and $\mathbf{Q}(z)$ is connected iff $\mathbf{K}(z)$ is. Indeed, one has

$$\mathbf{Q}(z) = \mathbf{K}(z)\mathbf{A}\mathbf{N}.$$

If H_ζ is the root vector corresponding to the highest root ζ then $\exp(\pi i \,\mathrm{Ad}\, H_\zeta) \in \mathbf{K}(z)$. It will turn out that $\mathbf{K}(z)$ has one or two components according to whether this element is in the component of the identity or not. In order to treat this question we need to look more closely at the structure of the Lie algebra $\mathfrak{g}$.

3. **The structure of $\mathbf{Q}(z)$.** Write $\mathfrak{g}^\sigma$ and $\mathfrak{g}^{-\sigma}$ for the respective +1 and -1 eigenspaces of the Cartan involution σ. Choose a maximal unipotent subalgebra $\mathfrak{n}$ containing $\mathfrak{n}(\mathfrak{z})$ and let $\mathfrak{a}$ be the intersection of the normalizer in $\mathfrak{g}$ of $\mathfrak{n}$ with $\mathfrak{g}^{-\sigma}$. Then $\mathfrak{a}$ is a maximal abelian subalgebra of $\mathfrak{g}^{-\sigma}$. Let Σ be a system of positive (restricted) roots for $\mathfrak{a}$; so

$$\mathfrak{n} = \sum_{\xi\in\Sigma} \mathfrak{g}(\xi).$$

Write $\Sigma_1 = \{\xi \in \Sigma : \frac{1}{2}\xi \notin \Sigma\}$. For $\xi \in \Sigma$ we put H_ξ for the corresponding co-root. This is the element of $\mathfrak{a}$ defined by

$$\xi H = 2\mathrm{Kill}(H_\xi, H)/\mathrm{Kill}(H_\xi, H_\xi).$$

Let ζ be the highest root. We put

$$\Phi = \{\xi \in \Sigma : \xi H_\zeta > 0\}, \quad \Phi_1 = \Phi \setminus \{\zeta\}.$$

For the nil-radical $\mathfrak{n}(\mathfrak{z})$ of $\mathfrak{p}(\mathfrak{z})$ we have

$$\mathfrak{n}(\mathfrak{z}) = \sum_{\xi\in\Phi} \mathfrak{g}(\xi).$$

We write $m(\xi)$ for the multiplicity of a root $\xi \in \Sigma$; we single out $d = m(\zeta)$. There is an essential distinction between the cases $d = 1$ and $d > 1$. Let $\mathfrak{h}$ be a σ-stable Cartan subalgebra of $\mathfrak{g}$ containing $\mathfrak{a}$. Let τ be the complex conjugation of $\mathbb{C} \otimes \mathfrak{g}$ which defines $\mathfrak{g}$ and commutes with σ—the Cartan involution of $\mathfrak{g}$ extended as a compact complex conjugation of $\mathbb{C} \otimes \mathfrak{g}$. The Killing form in $\mathbb{C} \otimes \mathfrak{g}$ is taken to be complex so that the Killing form of $\mathfrak{g}$ is the real part of the Killing form of $\mathbb{C} \otimes \mathfrak{g}$. If α is a root of $\mathbb{C} \otimes \mathfrak{h}$ then $\overline{\alpha\tau}$ defined by

$$\overline{\alpha\tau}H = \overline{\alpha\tau H} \text{ for } H \in \mathbb{C} \otimes \mathfrak{h}$$

is also a root. Note that all roots satisfy $\alpha = -\overline{\alpha\sigma}$. For the root vectors in $\mathbb{C} \otimes \mathfrak{h}$ it remains true that $\sigma H_\alpha = -H_\alpha$. If we put $\upsilon = \sigma\tau$ we get a complex automorphism of $\mathbb{C} \otimes \mathfrak{g}$. The restricted roots of $\mathfrak{a}$ are precisely the complex linear functionals on $\mathfrak{h}$ of the form

$$\xi = \tfrac{1}{2}(\alpha - \alpha\upsilon),$$

where α is a root of $\mathbb{C} \otimes \mathfrak{h}$, restricted to $\mathfrak{a}$. The roots may be ordered so that, if $\alpha > 0$ then either α is pure imaginary, *i.e.* $\alpha = \alpha\upsilon$, or $-\alpha\upsilon > 0$. This ordering may be chosen to be consistent with that given by $\mathfrak{n}$. The corresponding co-roots are given by

$$H_\xi = (1 - \tfrac{1}{2}\alpha\upsilon H_\alpha)(H_\alpha - \upsilon H_\alpha), \text{ with } \alpha\upsilon H_\alpha = -2, 0, \text{ or } 1.$$

Let $x \subset \mathfrak{g}(\xi)$ be a ray in a root space. Then there is a unique element $X \in x$ such that $[\sigma X, X] = H_\xi$, the co-root. Having chosen this X we put

$$K_x = X + \sigma X \in \mathfrak{g}^\sigma.$$

LEMMA 3.01. *The eigenvalues of* $-i \operatorname{Ad} K_x$ *are the same (counting multiplicity) as those of* $\operatorname{Ad} H_\xi$; $\exp(\pi \operatorname{Ad} K_x) = \exp(\pi i \operatorname{Ad} H_\xi)$.

PROOF. It suffices to consider the adjoint representation of $\mathfrak{g}$ restricted to the 3-dimensional simple algebra generated by X and σX. The irreducible constituents extend to holomorphic representations of their complexifications in which $\operatorname{Ad} H_\xi$ and $-i \operatorname{Ad} K_x$ are conjugate. Each irreducible constituent is a subspace of some $\sum_j \mathfrak{g}(\eta + j\xi)$ where η is a restricted root. Here the eigenvalues of $\operatorname{Ad} H_\xi$ are all congruent to $\eta H_\xi \bmod 2$.

The elements K_x have interesting properties, but we shall concentrate on K_z where $z \subset \mathfrak{g}(\zeta)$ is a contact ray. A useful formula is

$$\exp(t \operatorname{Ad} K_z)Z = Z\cos^2 t + \sigma Z \sin^2 t + H_z \sin 2t. \tag{3.02}$$

PROPOSITION 3.03. *Put* $\mathbf{U}(z)$ *for the centralizer in* $\mathbf{K}$ *of* K_z *and put* $\mathbf{K}(z) = \mathbf{K} \cap \mathbf{Q}(z)$. *Then* $\mathbf{U}(z)$ *is connected, and*

$$\mathbf{U}(z) = \exp(\mathbb{R}K_z) \times \mathbf{K}(z), \quad \exp(\mathbb{R}K_z) \cap \mathbf{K}(z) = \{I, \exp(\pi i H_\zeta)\}.$$

Therefore $\mathbf{K}(z)$ *(and accordingly* $\mathbf{Q}(z)$*) is connected iff* $\exp(\pi i H_\zeta)$ *belongs to the identity component of* $\mathbf{K}(z)$.

PROOF. The only thing that is not straightforward is that if $X \in \mathfrak{g}^\sigma$ and $X \perp K_z$ then $X \in \mathfrak{l}(z)$. To see this it suffices to note that, for $X \in \mathfrak{g}^\sigma$

$$\kappa\big(\Omega_X(Z)\big) = -(2c)^{-1}\operatorname{Kill}(K_z, X).$$

This shows that if $X \perp K_z$ then $\kappa\big(\Omega_X(Z)\big) = 0$ and, for all $Y \in \mathfrak{g}^\sigma$ (2.02) gives

$$\omega\big(\Omega_X(Z), \Omega_Y(Z)\big) = \kappa\big([\Omega_X(Z), \Omega_Y(Z)\big) = -(2c)^{-1}\mathrm{Kill}([K_z, X], Y) = 0.$$

Since $\mathbf{K}$ acts transitively on Ω this gives the assertion once one observes that (3.02) shows that $\exp t \operatorname{Ad} K_z \in \mathbf{K}(z)$ iff t is an integral multiple of π.

Choose the constant c in (2.02) so that $r = \frac{1}{2}\kappa\big(\Omega_{H_{\sigma Z}}(Z)\big)$ has the value 1 when Z is normalized. Then $r\colon \Omega \longrightarrow \mathbb{R}_+$ is a $\mathbf{K}$-invariant function homogeneous of degree 1. Consider the vector field Θ on Ω specified by

$$\omega(\cdot, \Theta) = dr.$$

We then have

$$\Theta = \tfrac{1}{2}\Omega_{K_z}.$$

If we identify Δ with the locus $\{r = 1\} \subset \Omega$, we see that Θ has period 2π and gives a fibration of Δ as a circle bundle over a manifold $\mathcal{H}$ which is a homogeneous space of $\mathbf{K}$ whose stability group at a point below $z \in \Delta$ is $\mathbf{U}(z)$. The manifold $\mathcal{H}$ is of interest when we view $\mathbf{K}$ faithfully represented as a transitive group of contact transformations of Δ.

Now assume that $\mathbb{C} \otimes \mathfrak{g}$ is simple. Take μ to be the highest root of $\mathbb{C} \otimes \mathfrak{h}$. Then we have $\zeta = \frac{1}{2}(\mu - \mu\upsilon)$. The case $\mu\upsilon H_\mu = -2$ occurs only if $\mu\upsilon = -\mu$, *i.e.* $\mu = \zeta$ is already a real root. This is the case $d = 1$ where we may take $H_\zeta = H_\mu$. When $d > 1$ the only possibility is that $-\mu\upsilon$ is a positive root different from μ with $\mu\upsilon H_\mu = 0$, and one has that $H_\zeta = H_\mu - \upsilon H_\mu$. Now consider $L \equiv i(H_\mu + \upsilon H_\mu) \in \mathfrak{k} \cap \mathfrak{h}$. For a root $\alpha > 0$ with $\alpha \neq \mu$ we have $\alpha H_\mu = 0$, or 1 because μ is the highest root. This says that $\alpha L = 0$, or $\pm i$. We can now show

LEMMA 3.04. *If $d > 2$ then $\exp(\pi i \operatorname{Ad} H_\zeta)$ belongs to the identity component of* $\mathbf{Q}(z)$.

PROOF. It is clear that $\operatorname{Ad} H_\zeta$ is 0 on all root spaces of $\mathfrak{g}$ of the form $\mathfrak{g}(\eta)$ where $\pm\eta \notin \mathbf{\Phi}$. The value is 2 on $\mathfrak{g}(\zeta)$ and 1 on $\mathfrak{g}(\xi)$ with $\xi \in \mathbf{\Phi}_1$. The unipotent subalgebra $\mathfrak{n}$ is the projection by $\frac{1}{2}(I+\tau)$ of the sum of the root spaces $\mathfrak{g}(\alpha)$ in $\mathbb{C}\otimes\mathfrak{g}$ with $\alpha > 0$ and $\alpha\upsilon < 0$. $\mathfrak{n}(\mathfrak{z})$ is the projection of the sum of those $\mathfrak{g}(\alpha)$ above where $\alpha H_\mu > 0$ or $\alpha\upsilon H_\mu < 0$. We have a decomposition $\mathfrak{n}(\mathfrak{z}) = \mathfrak{n}_1(\mathfrak{z}) \oplus \mathfrak{z}$ where $\mathfrak{n}_1(\mathfrak{z})$ is the projection of the sum of those $\mathfrak{g}(\alpha)$ for which $\alpha H_\mu = 1$ and $\alpha\upsilon H_\mu = 0$ or $\alpha H_\mu = 0$ and $\alpha\upsilon H_\mu = -1$. $\exp(\pi \operatorname{Ad} L) = -I$ on $\mathfrak{n}_1(\mathfrak{z})$ while $\exp(\pi \operatorname{Ad} L) = +I$ on $\mathfrak{z}$. The result is that $\exp(\pi \operatorname{Ad} L) = \exp(\pi i \operatorname{Ad} H_\zeta)$. Now, since $d > 2$ (which can only occur if $\mathbb{C} \otimes \mathfrak{g}$ is simple) there exists a positive root $\alpha \neq \mu, -\mu\upsilon$ such that $\zeta = \frac{1}{2}(\alpha - \alpha\upsilon)$. In this case one has necessarily that $\alpha H_\mu = 1$ and $\alpha\upsilon H_\mu = -1$. For a suitable choice of $X \in \mathfrak{g}(\alpha)$ we shall have that $\frac{1}{2}(I+\tau)X \in \mathfrak{z} \setminus \{0\}$, and we may suppose that this represents $Z \in z$. One sees that $[L, Z] = 0$ which shows that $\exp(t \operatorname{Ad} L) \in \mathbf{Q}(z)$ for all $t \in \mathbb{R}$.

LEMMA 3.05. *If $d = 2$ then $\exp(\pi i \operatorname{Ad} H_\zeta)$ belongs to the identity component of $\mathbf{Q}(z)$.*

PROOF. This situation can only arise when $\mathfrak{g}$ is a complex simple Lie algebra. There are two cases. If there is a root δ such that $\zeta H_\delta = 1$ put $L = i(H_\zeta - 2H_\delta)$. The only case for which this fails is $\mathfrak{g} = \mathfrak{sp}(n, \mathbb{C}) = C_n$. For the C_n case we use the fact that the co-root lattice has a basis $H_1, \ldots, H_n$ with $H_\zeta = H_1$; here we put $L = i(H_2 + \cdots + H_n)$. (Note that if $n = 1$ we have $L = 0$.) In either case $\operatorname{Ad} L$ is trivial on $\mathfrak{z}$ and $\exp(\pi i \operatorname{Ad} H_\zeta) = \exp(\pi \operatorname{Ad} L)$.

LEMMA 3.06. *If $\zeta = 2\lambda$ where λ is a (restricted) root of $\mathfrak{a}$ then $\exp(\pi i \operatorname{Ad} H_\zeta)$ belongs to the identity component of $\mathbf{Q}(z)$.*

PROOF. To say that $\zeta = 2\lambda$ is to say that there is a root α of $\mathbb{C} \times \mathfrak{h}$ such that $\zeta = \alpha - \alpha v$. Put $K = i(H_\alpha + vH_\alpha)$. Then $K \in \mathfrak{g}^\sigma \cap \mathfrak{h}$ represents an element of the co-root lattice with $\zeta K = 0$. Since $H_\zeta = H_\alpha - vH_\alpha$ it is clear that $\exp(\pi \operatorname{Ad} K) = \exp(\pi i \operatorname{Ad} H)$.

The foregoing three lemmas will settle the connectivity of $\mathbf{Q}(z)$ when $d > 1$, but when $d = 1$ Lemma 3.06 handles only a small part of the problem, and the validity of the conclusion varies according to cases.

We define the *altimeter* by

$$A = \tfrac{1}{2} \sum_{\xi \in \Sigma_1} H_\xi.$$

Any restricted root of $\mathfrak{a}$ is an integral linear combination of primitive roots. The sum of the coefficients is called the *height*. A standard argument using Weyl reflections shows that the height of a root ξ is exactly ξA.

LEMMA 3.07. *When the highest root ζ has multiplicity 1 and is not twice a root, $\exp(\pi i \operatorname{Ad} H_\zeta)$ belongs to the identity component of $\mathbf{Q}(z)$ iff the height of ζ is odd.*

PROOF. Let h be the height of ζ and put $B = 2A - hH_\zeta$. If $\xi \in \Sigma_1$ then H_ξ is an element of the co-root lattice, *i.e.* it is an integral linear combination of the H_β where β is a primitive co-root. Thus $2A$ is an element of the co-root lattice. The same is true of H_ζ when ζ is not twice a root. Therefore, under the present hypotheses, B is an element of the co-root lattice. If ζ is the only positive root the assertion is trivial, and we may dismiss this case. Suppose that there is only one primitive root $\delta \in \Phi_1$. This implies $\zeta H_\delta > 0$ and $\zeta H_\beta = 0$ for all primitive roots $\beta \neq \delta$. Note that $\zeta B = 0$. Thus, when B is expanded in terms of primitive co-roots, H_δ does not occur. Therefore we have $B = \sum c_\beta H_\beta$ and

$$\exp(\pi i \operatorname{Ad} B) = \prod \exp(\pi i c_\beta \operatorname{Ad} H_\beta) = \prod \exp(\pi c_\beta \operatorname{Ad} K_b)$$

where β ranges over the primitive roots $\beta \notin \Phi$ and $b \subset \mathfrak{g}(\beta)$. Thus $K_b \in \mathfrak{q}(z)$ for each b which occurs and we have that $\exp(\pi i \operatorname{Ad} B)$ is in the identity component of $\mathbf{Q}(z)$. Obviously $\exp(2\pi i \operatorname{Ad} A) = I$; so $\exp(\pi i \operatorname{Ad} B) = \exp(\pi i \operatorname{Ad} H_\zeta)$ precisely when h is odd. It is a fact that the height of the highest root in all reduced irreducible root systems is odd except for A_n with n even. The Lie algebras $\mathfrak{sl}(n+1, \mathbb{R}), n \geq 2$, arise in our considerations when there are exactly two primitive roots $\delta, \delta' \in \Phi_1$. (There cannot be more than two.) Here n is the height of ζ, and the previous argument works when n is odd, for δ and δ' are transformed to one another by the reflection of the Dynkin diagram. Thus they enter with the same coefficients in the expansion of B. When n is even $\mathbf{Q}(z)$ is disconnected as one will see later.

The above proof is somewhat unsatisfactory in that it uses details of structure theory. This can be avoided to some extent, but the formulation of Lemma 3.07 seems to be the right one.

Summarizing the sequence of lemmas and Proposition 3.03 we have

THEOREM 3.08. *The group* $\mathbf{Q}(z)$ *is connected except when* $\mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{R})$, *$n$ even, in which case there are two components.*

This proves Theorem 4 as far as the classification of the representations of $\mathrm{Int}(\mathfrak{g})$ by contact transformations is concerned.

4. **The action of a maximal compact subgroup.**

We take a fixed Cartan involution σ and take $\mathbf{K}$ to be the maximal compact subgroup of $\mathrm{Int}(\mathfrak{g})$ with Lie algebra $\mathfrak{k} = \mathfrak{g}^{\sigma}$.

PROPOSITION 4.01. *Under the restriction of the adjoint representation of* $\mathrm{Int}(\mathfrak{g})$, $\mathbf{K}$ *acts real-irreducibly and faithfully in* $\mathfrak{g}^{-\sigma}$. *The representation is complex iff* $\mathfrak{g}$ *is of holomorphic type. Otherwise the representation is real; it can never be quaternionic.*

PROOF. $\mathbf{K}$ acts irreducibly in $\mathfrak{g}^{-\sigma}$ because $U+[U, U]$ is an ideal in $\mathfrak{g}$ whenever U is a $\mathbf{K}$-invariant subspace of $\mathfrak{g}^{-\sigma}$, and $\mathfrak{g}$ is assumed simple. The action of $\mathbf{K}$ in $\mathfrak{g}^{-\sigma}$ is faithful by the definition of $\mathbf{K}$ as a maximal compact subgroup of $\mathrm{Int}(\mathfrak{g})$. If the representation is complex then there is a linear transformation j of $\mathfrak{g}^{-\sigma}$ commuting with the action of $\mathbf{K}$ such that $j^2 = -I$ and j is skew-symmetric for inner product $-\langle \sigma\cdot, \cdot\rangle$. Extend j to a linear transformation of $\mathfrak{g}$ which is trivial on $\mathfrak{g}^{\sigma}$. We assert that j is a derivation of $\mathfrak{g}$. To verify this it suffices to prove that $[jX, Y] + [X, jY] = 0$ whenever $X, Y \in \mathfrak{g}^{-\sigma}$. Take $K \in \mathfrak{g}^{-\sigma}$ and consider

$$\begin{aligned}\mathrm{Kill}(K, [jX, Y]) &= \mathrm{Kill}([K, jX], Y) = \mathrm{Kill}(j[K, X], Y)\\ &= \mathrm{Kill}([K, X], jY) = -\mathrm{Kill}(K, [X, jY]).\end{aligned}$$

This proves that $[jX, Y] + [X, jY]$ is orthogonal to $\mathfrak{g}^{-\sigma}$, hence 0. Since $\mathfrak{g}$ is a simple Lie algebra, j must be inner, *i.e.* $j = \mathrm{Ad}\, J$ where $J \in \mathfrak{g} = \mathfrak{k} + \mathfrak{g}^{-\sigma}$. On the other hand $\mathrm{Ad}\,\mathfrak{k}$ has no fixed vectors in $\mathfrak{g}^{-\sigma}$. Therefore J is a non-trivial element of the centre of $\mathfrak{k}$ and $\mathfrak{g}$ is of holomorphic type. Were the representation quaternionic we should have a linear transformation k of the same type as j with $jk = -kj$ which leads to a contradiction.

With $Z \in z$ a normalized element, so $H_\zeta = [\sigma Z, Z]$, put $T(z) = Z - \sigma Z$. We consider the complexified representation of $\mathbf{K}$ in $\mathbb{C} \otimes \mathfrak{g}^{-\sigma}$. When the representation is real put $V = \mathbb{C} \otimes \mathfrak{g}^{-\sigma}$ and when it is complex let V be the $+i$ eigenspace of the j considered above. Then the representation on V is complex-irreducible. On V we get a Hilbert space inner product given by

$$2c\langle X_1 + iX_2, Y_1 + iY_2\rangle = \mathrm{Kill}(X_1, Y_1) + \mathrm{Kill}(X_2, Y_2) + i\mathrm{Kill}(X_1, Y_2) - i\mathrm{Kill}(X_2, Y_1)$$

where c is the constant in (2.02). This makes the representation unitary. Put

$$e = \tfrac{1}{2}\big(H_\zeta + iT(z)\big).$$

A little calculation using (2.02) gives, for $X \in \mathfrak{k}$,

$$\kappa\big(\Omega_X(Z)\big) = \Im\langle e, \operatorname{Ad} Xe\rangle. \tag{4.02}$$

Now e is an eigenvector of $\operatorname{Ad} K_z$ for the eigenvalue $2i$. We need

LEMMA 4.03. *The multiplicity of the eigenvalue* $2i$ *of* $\operatorname{Ad} K_z$ *on* $\mathbb{C} \otimes \mathfrak{g}^{-\sigma}$ *is exactly* 1.

PROOF. Let $w \subset \mathfrak{z}$ be a ray orthogonal to σz for the Killing form. If $W \in w$ is the normalized element we have

$$[K_z, T(w)] = [\sigma Z, W] - \sigma[\sigma Z, W] \in \mathfrak{a}.$$

For all $H \in \mathfrak{a}$ we get

$$\operatorname{Kill}(H, [\sigma Z, W]) = (\zeta H)\operatorname{Kill}(\sigma Z, W) = 0.$$

Let us note that the eigenvalue $2i$ for $\operatorname{Ad} K_z$ on $\mathfrak{g}^\sigma$ occurs with multiplicity $d-1$ where $d = \dim \mathfrak{z}$.

From Lemma 4.03 we conclude that e may be used as a highest weight vector for the representation, and this puts us in the situation used to describe the representations of compact Lie groups by contact transformations according to [Herz 1991].

PROPOSITION 4.04. *A necessary and sufficient condition that the representation in Proposition 4.01 be complex is that* $\dim \mathfrak{z} = 1$ *and* ζH *be even for all* H *in the co-root lattice where* ζ *is the highest root.*

PROOF. The representation is complex iff it is not equivalent to its complex conjugate. Therefore $H_\zeta - iT(z)$ cannot be the image under an element of $\mathbf{K}$ of $H_\zeta + iT(z)$ in the complex case, but for H in the co-root lattice we have $\exp(\pi i \operatorname{Ad} H) \in \mathbf{K}$ and

$$\exp(\pi i \operatorname{Ad} H)\big(H_\zeta + iT(z)\big) = H_\zeta + i(-1)^h T(z) \text{ where } h = \zeta H.$$

It follows that ζH is even for all H in the co-root lattice if the representation is complex. By a well-known result [Warner 1972; see 3.3.1.1], the adjoint representation has a $\mathbf{K}$-fixed vector iff ζH is even for all H in the co-root lattice and $d = \dim \mathfrak{z} = 1$. Such a vector must be in the centre of $\mathfrak{k}$. It is obvious that the representation of $\mathbf{K}$ in $\mathfrak{g}^{-\sigma}$ is complex if $\mathfrak{k}$ has a non-trivial centre. We already know from Corollary 2.07 that if $d > 1$ there is an element of $\mathbf{K}$ which maps z to $-z$ and, hence, $H_\zeta + iT(z)$ to $H_\zeta - iT(z)$; so the representation cannot be complex. When $d = 1$ and ζH is odd for some H in the co-root lattice we get the same result.

The above result combined with Corollary 2.07 completes the proof of Theorems 2 and 3.

PROPOSITION 4.05. *Suppose that $\mathfrak{g}$ is of non-holomorphic type. Let λ the highest weight of the representation of $\mathfrak{k}$ on V as described above. Then there is a non-trivial covering compact contact manifold $\tilde{\Delta}$ on which a covering group $\mathbf{G}$ of* $\mathrm{Int}(\mathfrak{g})$ *acts faithfully by contact transformations iff $\lambda = k\varpi$ where ϖ is an indivisible weight of $\mathfrak{k}$ and $k = 1, 2$ or 4. The group of covering transformations of $\tilde{\Delta}$ over Δ is necessarily cyclic of order a divisor of k, and there is a one-to-one correspondance between the $\tilde{\Delta}$ which can occur and these divisors.*

PROOF. The contact form for $\tilde{\Delta}$ can only differ from κ by a constant normalizing factor. Put $\tilde{\mathbf{K}}$ for the inverse image of $\mathbf{K}$ in $\tilde{\mathbf{G}}$. It is necessarily connected, and the contact form must correspond to a faithful irreducible representation whose highest weight is then a multiple of that for the representation of $\mathfrak{k}$ on V. Since we are dealing with coverings, the weights which arise are integral divisors of λ. We may assume that $\lambda = k\varpi$ where ϖ is an indivisible weight corresponding to an irreducible representation. Let $\tilde{\mathbf{K}}$ be the covering group of $\mathbf{K}$ defined by ϖ. Let $\tilde{\mathbf{K}}(z)$ be the inverse image under the projection of $\tilde{\mathbf{K}}$ onto $\mathbf{K}$ of the stability subgroup $\mathbf{K}(z)$ for a point $z \in \Delta$, and let $\tilde{\mathbf{K}}_0(z)$ be its identity component. Thus $\tilde{\Delta} \cong \tilde{\mathbf{K}}/\tilde{\mathbf{K}}_0(z)$ is a covering space of Δ with covering group isomorphic to $\tilde{\mathbf{K}}(z)/\tilde{\mathbf{K}}_0(z)$. It is clear from Proposition 3.03 and its proof that

$$\tilde{\mathbf{K}}(z) = \mathbf{C}\tilde{\mathbf{K}}_0(z) \tag{4.06}$$

where $\mathbf{C}$ is the cyclic subgroup of $\tilde{\mathbf{K}}$ generated by $\exp(\pi K_z)$. Note that $\alpha(K_z) \in i\mathbb{Z}$ for all roots, α, of $\mathfrak{k}$. Hence, if ϖ' is any weight of the representation corresponding to ϖ we have $\varpi'(K_z) - (2i/k) \in i\mathbb{Z}$ since $\varpi(K_z) = \lambda(K_z)/k = 2i/k$. In the case at hand, the representation corresponding to λ is real; so $\varpi' = -\varpi$ occurs as a weight and we get $4/k \in \mathbb{Z}$. For $k > 1$ we see that $\exp(k\pi K_z) = I$ while $\exp(j\pi K_z) \notin \tilde{\mathbf{K}}(z)$ for $0 < j < k$. Therefore $\mathbf{C}$ is cyclic of order k and the product in (4.06) is direct.

For the holomorphic case the result is even simpler

PROPOSITION 4.07. *Suppose that $\mathfrak{g}$ is of holomorphic type and $\mathfrak{k}$ is non-abelian. Let λ be the highest weight of the representation of $\mathfrak{k}$ on V as described above. Then there is a non-trivial covering compact contact manifold $\tilde{\Delta}$ on which a covering group $\mathbf{G}$ of* $\mathrm{Int}(\mathfrak{g})$ *acts faithfully by contact transformations iff $\lambda = k\varpi$ where ϖ is an indivisible weight of $\mathfrak{k}$. A value $k > 1$ occurs iff $\mathbb{C} \otimes \mathfrak{g} \cong C_n$, $n > 1$, and $k = 2$.*

PROOF. We proceed as in the preceding proof, but this time we know that $\mathfrak{g}$ has a compact Cartan subalgebra. This is to say that the representation of $\mathfrak{k}$ on V is a restriction of the adjoint representation of $\mathbb{C} \otimes \mathfrak{g}$. This means that we may regard λ as the restriction to $\mathfrak{k}$ of the highest weight μ. If $\lambda = k\varpi$ then ϖ maps a Cartan subalgebra of $\mathbb{C} \otimes \mathfrak{g}$ to $i\mathbb{Z}$. Therefore, ϖ corresponds to a weight of $\mathbb{C} \otimes \mathfrak{g}$ for which $\mu = k\varpi$. This only occurs with $k > 1$ when $\mathbb{C} \otimes \mathfrak{g}$ is of type C_n. We have assumed $\mathfrak{k}$ non-abelian; so $n > 1$.

The situation not yet treated is

PROPOSITION 4.08. *If $\mathfrak{k}$ is abelian then for each Lie group $\mathbf{G}$ with finite centre having Lie algebra $\mathfrak{g}$ there is a compact contact manifold $\tilde{\Delta} \cong \mathbf{G}/\mathbf{AN}$, $\mathbf{G} = \mathbf{KAN}$ being the*

Iwasawa decomposition, on which **G** *acts faithfully by contact transformations. This situation arises only when* $\mathfrak{g} \cong \mathfrak{so}(1,2)$. *Each* $\tilde{\Delta}$ *is a circle.*

PROOF. Here we know that $\mathfrak{k}$ is 1-dimensional and $\mathbf{K} \cong \mathbb{T}$, for this is the only abelian group with a faithful representation by contact transformations. In the converse direction, it is obvious that $\mathbf{G}/\mathbf{AN}$ has a natural structure as a contact manifold.

To complete the proof of Theorem 4 one has to look at cases. If $\mathfrak{g}$ is a complex simple algebra then Proposition 4.05 applies, and, as previously remarked, the only candidate for a non-trivial covering is $\mathfrak{g} = \mathfrak{sp}(n, \mathbb{C})$. When $\mathfrak{g}$ is of holomorphic type we have $\mathfrak{g} = \mathfrak{sp}(n, \mathbb{R})$ with $n > 1$ coming under Proposition 4.07 and $n = 1$ under Proposition 4.08. What remains are the real simple Lie algebras of non-holomorphic type; Proposition 4.05 applies. Here we must appeal to detailed structure theory: a complete listing of the representations of **K** on $\mathfrak{g}^{-\sigma}$ is given in [Freudenthal & deVries 1969, § 52 and 53]. One sees that $k = 4$ occurs for $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{R})$ and $k = 2$ for $\mathfrak{sl}(n+1, \mathbb{R})$, $n > 2$. Explicit geometric constructions are given in the next section.

The reader may wonder what happens in a case such as $\mathfrak{g} = \mathfrak{sp}(n, n+m)$ where the proof of Proposition 4.07 would seem to apply. In terms of the notation there it is indeed the case that $\mu = 2\mu_1$ where μ_1 refers to the standard representation of C_{2n+m} which restricts to the standard representation of $\mathrm{Sp}(n, n+m)$. This splits on the maximal compact subgroup $\tilde{\mathbf{K}} = \mathrm{Sp}(n) \times \mathrm{Sp}(n+m)$ as the sum of the representation of $\mathrm{Sp}(n)$ with highest weight μ' and the representation of $\mathrm{Sp}(n+m)$ with highest weight μ''. When we pass to the adjoint representation of $\mathfrak{g}$ we get a decomposition into the representations with highest weights $2\mu'$ and $2\mu''$ and the representation with highest weight $\mu' + \mu''$. The first pair corresponds to the adjoint representation of **K** which is reducible, and the last is the faithful irreducible representation of **K** on $\mathfrak{g}^{-\sigma}$.

5. **Examples.** The boundaries $\Gamma \cong \mathbf{G}/\mathbf{P}(\mathfrak{z})$ which occur are flag manifolds which for the real simple algebras with classical Dynkin diagrams can be given suggestive descriptions. Where the Dynkin diagram is of type C_n or BC_n, $n \geq 1$, Γ is the manifold of isotropic lines. For the Dynkin diagrams of type A_n, $n \geq 2$ the flag has the form $V_1 \subset V_n$ where V_1 is a 1-dimensional subspace and V_n is an n-dimensional subspace of the basic $(n+1)$-dimensional vector space. Finally, for the orthogonal Dynkin diagrams B_n, $n \geq 3$, and D_n, $n \geq 4$, Γ is a manifold of isotropic planes. These descriptions are, of course, only heuristic in general.

Recall some notation. Σ is a system of positive (restricted) roots for $\mathfrak{a}$. Write $\Sigma_1 = \{\xi \in \Sigma : \frac{1}{2}\xi \notin \Sigma\}$. For $\xi \in \Sigma$ we put H_ξ for the corresponding co-root. Let ζ be the highest root. We put

$$\Phi = \{\xi \in \Sigma : \xi H_\zeta > 0\}, \quad \Phi_1 = \Phi \setminus \{\zeta\}.$$

For the nil-radical $\mathfrak{n}(\mathfrak{z})$ of $\mathfrak{p}(\mathfrak{z})$ we have

$$\mathfrak{n}(\mathfrak{z}) = \sum_{\xi \in \Phi} \mathfrak{g}(\xi).$$

We write $m(\xi)$ for the multiplicity of a root $\xi \in \Sigma$, we single out $d = m(\zeta)$

It is banal that Φ must contain at least one primitive root, *i e* a minimal positive root for the given ordering We say that a simple root is "extreme" if it has at most one neighbor in the Dynkin diagram The first observation is

PROPOSITION 5 01 *All real simple Lie algebras* $\mathfrak{g}$ *fall into one of the five following classes*

(O) *there is a unique simple root* $\delta \in \Phi$ *and* $\zeta H_\delta = 1$, δ *is not extreme, but it has an extreme neighbor* γ *such that the the expansion of* ζ *in terms of primitive roots has* $\zeta = \gamma + 2\delta +$, *and* $m(\delta) = d$ *(Here the rank of* $\mathfrak{g}$ *must be at least 3 and* $d = 1$ *or* 2 *)*

(EFG) *there is a unique simple root* $\delta \in \Phi$ *and* $\zeta H_\delta = 1$, δ *is extreme, and, if* γ *is its neighbor, the expansion of* ζ *in terms of primitive roots has* $\zeta = 2\delta + 3\gamma +$, *and* $m(\delta) = d$ *(Here the rank of* $\mathfrak{g}$ *must be at least 2 and* $d = 1$ *or* 2 *)*

(C) *there is a unique simple root* $\delta \in \Phi$, $\zeta H_\delta = 2$, *and* $\zeta/2$ *is not a root,* δ *is extreme, and the expansion in terms of primitive roots is* $\zeta = \delta$ *in rank 1 and* $\zeta = 2\delta +$ *otherwise (Here* d *is arbitrary in rank 1 and* $d = 1, 2,$ *or* 3 *otherwise)*

(BC) *there is a unique simple root* $\delta \in \Phi$ *and* $\zeta = 2\lambda$ *for some* $\lambda \in \Phi_1$, δ *is extreme, the expansion is* $\zeta = 2(\delta +$), *and* $m(\xi)$ *is divisible by* $d + 1$ *for all* $\xi \in \Phi_1$ *(Here* d *must be odd)*

(A) *there are two simple roots* $\delta, \delta' \in \Phi$, $\zeta H_\delta = 1 = \zeta H_{\delta'}$, *both are extreme, the expansion is* $\zeta = \delta + \delta' +$, *and* $m(\xi) = d$ *for all* $\xi \in \Phi$ *(Here the rank is at least 2 and* $d = 1, 2, 4,$ *or* 8 *)*

In all cases $\Phi_1 \subset \Sigma_1$

PROOF Since ζ is the highest root, $\zeta H_\xi \geq 0$ for all $\xi \in \Sigma$, and $\Phi = \{\xi \in \Sigma \; \zeta H_\xi > 0\}$ Let $\zeta = \sum a_\alpha \alpha$ be the expansion in terms of primitive roots α of Σ Each a_α is a positive integer Evaluation $2 = \zeta H_\zeta$ gives immediately that either there is a unique δ with $\delta H_\zeta > 0$ and $\alpha H_\zeta = 0$ for $\alpha \neq \delta$, or we are in case (**A**) Observe that $\xi H_\zeta \geq 2$ implies that $\xi - 2\zeta$ is a root which is impossible unless $\xi = \zeta$ This proves the final sentence and shows that, excluding case (**A**) and the situation in which $\zeta = \delta$ is the unique positive root, we have $a_\delta = 2$ The possibilities with $a_\delta = 2$ are partitioned Note that if d is odd and $d > 1$ then ζ must be twice a root This can only occur in case (**BC**) Consider the possibilities where $\zeta = 2\delta +$ and $\zeta H_\delta = 1$ If δ is extreme and γ is its only neighbor then we have $\zeta = 2\delta + c\gamma +$ where $c\gamma H_\delta = -3$ and $c \in \mathbb{Z}_+$ The possibility $\gamma H_\delta = -3$ can be eliminated with the result that $c = 3$ and $\gamma H_\delta = -1$ (One also sees from $\zeta H_\gamma = 0$ that either the rank is 2 and $\delta H_\gamma = -3$ or the rank is at least 4 and $\delta H_\gamma = -1$ Chasing further one has that the exceptional Dynkin diagrams are the only possibilities here) If δ is not extreme it has at least two neighbors, say γ and β, with $\zeta = 2\delta + c\gamma + b\beta$ Since both $c > 0$ and $b > 0$ and $-c\gamma H_\delta - b\beta H_\delta \leq 3$, for at least one of these, say γ, we must have $c = 1$, $\gamma H_\delta = -1$ The fact that $\zeta H_\gamma = 0$ forces γ to be extreme with $\delta H_\gamma = -1$ Similar arguments are used in the other cases The statements about multiplicities are not needed

When $d = 1$, case (**A**) occurs exactly for $\mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{R})$, $n \geq 2$, and, when n is even, the adjoint group is $\mathrm{SL}(n+1, \mathbb{R})$. View $\mathfrak{g}$ as the $(n+1) \times (n+1)$ matrices of trace 0, and take Z to be the element with 1 in the upper right-hand corner, 0 elsewhere. Here $\mathfrak{n}(\mathfrak{z})$ consists of the elements with entry 0 everywhere except after the first position in row 1 and before the last position in column $n+1$. Put ϵ for the diagonal matrix with -1 in the first and last positions and +1 elsewhere. Thus $\epsilon = \exp(\pi i H_\zeta)$. Note that H_δ is the diagonal matrix $\{1, -1, 0, \ldots, 0\}$ and $H_{\delta'}$ is the diagonal matrix $\{0, \ldots, 0, 1, -1\}$. We see that

$$\mathbf{Q}(z) = \{I, \epsilon\} \times \mathrm{SL}(n-1, \mathbb{R}) \times \mathbf{A}(z) \times \exp\bigl(\mathbb{R}\mathfrak{n}(\mathfrak{z})\bigr)$$

where $\mathrm{SL}(n-1, \mathbb{R})$ is viewed as the subgroup of $\mathrm{SL}(n+1, \mathbb{R})$ situated by having rows 1 and $n+1$ as well as these columns having 0 everywhere except the diagonal positions which are 1. Clearly ϵ is not in the identity component. Note that $\mathbf{P}(\mathfrak{z})$ modulo its identity component is a 4-group. The case (**A**) with odd rank comes under Theorem 3.08. The rank n is the height of H_ζ; the altimeter is given by $2A = \operatorname{diag}\{n, n-2, \ldots, 2-n, -n\}$. That $\mathbf{Q}(z)$ is connected in this situation is the same as saying that ϵ is equivalent to the diagonal matrix with +1 in the first and last positions and -1 elsewhere which is in the identity component of $\mathbf{Q}(z)$. In all cases the maximal compact Lie subalgebra is $\mathfrak{so}(n+1)$ acting on $\mathfrak{g}^{-\sigma}$ according to the representation with highest weight $2\varpi_1$ where ϖ_1 corresponds to the standard representation inside $\mathrm{SL}(n+1, \mathbb{R})$ and $2\varpi_1$ the action on symmetric matrices of trace 0. The weight ϖ_1 is a fundamental weight except for the case $n = 2$ where $\mathfrak{so}(3) \cong \mathfrak{su}(2)$ and $\varpi_1 = 2\varpi$, ϖ being the unique fundamental weight; it corresponds to the standard representation of $\mathbf{SU}(2)$. More details will be given below.

One can generalize the above to treat everything which comes under case (**A**). We may consider Lie algebras $\mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{F})$, $n \geq 2$ where $\mathbb{F}$ is a "field". The construction is modelled on the idea of taking V, a right vector space of dimension $n+1$ over $\mathbb{F}$. The boundary Γ is the one associated with the two extreme roots of the Dynkin diagram. Let V_k denote a k-dimensional subspace of V. Then

$$\Gamma = \{(V_1, V_n) : V_1 \subset V_n\}.$$

Put V' for the left vector space of $\mathbb{F}$-linear functionals on V. Let Ω_1 be a connected component of the set of pairs $(u, f) \in (V \setminus \{0\}) \times (V' \setminus \{0\})$ such that $fu = 0$. The tangent space may be given the identification

$$T_{(u,f)}(\Omega_1) = \{(v, g) \in V \oplus V' : fv + gu = 0\}.$$

We define an $\mathbb{F}$-valued 2-form ω on Ω_1 by

$$\omega(u, f; v, g; w, h) = gw - hv.$$

Let $\mathbf{G}_+$ be the component of the identity in the group of $\mathbb{F}$-automorphisms of V. The natural action of $\mathbf{G}_+$ on Ω_1 is given by $S(u, f) = (Su, fS^{-1})$. This preserves the form ω. We should like $\Re\omega$ to be a symplectic form, but this is not true: let $c \in \mathbb{F}$ be a non-zero scalar and consider the tangent vector $(uc, -cf)$. We have

$$\Re\omega(u, f; v, g; uc, -cf) = \Re(guc + cfv) = \Re\bigl(c(gu + fv)\bigr) = 0.$$

In order to eliminate this we put $\mathbf{\Omega} = \mathbf{\Omega}_1/\mathbb{F}_+$ where $\mathbb{F}_+$ is the component of 1 in the invertible elements of $\mathbb{F}$, and the identification is $(uc, fc^{-1}) \sim (u, v)$ when $c \in \mathbb{F}_+$. Now let $\mathbf{C}$ be the centre of $\mathbb{F}_*$, the invertible elements of $\mathbb{F}$, and put $\mathbf{C}_+ = \mathbf{C} \cap \mathbb{F}_+$ viewed as diagonal elements of $\mathbf{G}_1$. In this case $\mathbf{G} = \mathbf{G}_+/\mathbf{C}_+$ acts faithfully as symplectic transformations of $(\mathbf{\Omega}, \Re\omega)$. The Lie algebra of $\mathbf{G}_+$ may be viewed as $V \otimes_{\mathbb{F}} V'$ where $u \otimes f$ acts as the endomorphism $(u \otimes f)v = u(fv)$. The Lie algebra $\mathfrak{g}$ of $\mathbf{G}$ is this modulo the centre. The map $\mathbf{\Omega} \longrightarrow \mathfrak{g}$ arising from $(f, u) \longmapsto u \otimes_{\mathbb{F}} f$ is k-to-1 where k is the index of $\mathbb{F}_+$ in the subgroup of $\mathbb{F}_*$ which preserves components of $\mathbf{\Omega}_1$. In this way we get a k-fold covering $\tilde{\Delta}$ of the minimal contact manifold Δ arising from the adjoint orbit of uf. For all $n > 1$ we may take $\mathbb{F}$ to be one of the three standard fields $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$. In the case of the complex numbers we have $\mathbb{F}_+ = \mathbb{F}_*$, and $\mathbf{G} = \mathbf{PL}(n, \mathbb{C})$ which is the same as $\mathrm{Int}(\mathfrak{g})$. In the case of the quaternions $\mathbb{H}$ we again have $\mathbb{F}_+ = \mathbb{F}_*$. Here one has $\mathbf{C}_+ = \mathbb{R}_* I$; so, once again, $\mathbf{G} = \mathrm{Int}(\mathfrak{g})$. Since $\mathbf{G}_1$ is simply connected when $\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{H}$ we get that Δ is simply connected. Note that when $\mathbb{F} = \mathbb{R}$ the index k is 2 when n is odd and 1 otherwise. The only remaining case is $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{O})$ where $\mathbb{O}$ is the Cayley numbers. This is best examined by noting that, in general, the Lie algebras corresponding to $\mathbf{P}(\mathfrak{z})$ and $\mathbf{Q}(z)$ are given by

$$\mathfrak{m}(\mathfrak{z}) \cong \mathfrak{so}(d) \oplus \mathfrak{sl}(n-1, \mathbb{F}), \quad \mathfrak{q}(z) \cong \mathfrak{so}(d-1) \oplus \mathfrak{sl}(n-1, \mathbb{F}) \oplus \mathbb{R}^2 \oplus \mathfrak{n}(\mathfrak{z}),$$

where $d = \dim_{\mathbb{R}}(\mathbb{F})$. This arises from the identifications $\mathfrak{sl}(2, \mathbb{F}) = \mathfrak{so}(1, 1+d)$ and $\mathfrak{sl}(1, \mathbb{F}) = 0$. One always has $\dim \mathfrak{n}(\mathfrak{z}) = \dim \Gamma = (2n-1)d$ and $\dim \Delta = 2nd - 1$. Thus, the exceptional $\mathfrak{sl}(3, \mathbb{O})$, a real form of E_6, has $\dim \Delta = 31$. Since the maximal compact subgroup is compact F_4, one sees with a little more effort that $\Gamma \cong \mathbf{F}_4/\,\mathrm{Spin}(8)$ and $\Delta \cong \mathbf{F}_4/\,\mathrm{Spin}(7)$. Each of these manifolds is simply connected because $\mathbf{F}_4$ is. For the genuine fields the passage to a maximal compact subgroup arises from imposing a Hilbert space structure on V and putting $f = v^*$ where $\langle u, v \rangle = 0$. Thus we get

$$\Gamma \cong \mathbf{U}(n+1, \mathbb{F})/\big(\mathbb{F}_1 \times \mathbb{F}_1 \times \mathbf{U}(n-1, \mathbb{F})\big), \quad \Delta \cong \mathbf{U}(n+1, \mathbb{F})/\big(\mathbb{F}_1 \times \mathbf{U}(n-1, \mathbb{F})\big),$$

where $\mathbb{F}_1 = \{c \in \mathbb{F} : |c| = 1\}$. When $\mathbb{F} = \mathbb{R}$ we always have a double covering $\Delta_2 \cong \mathbf{SO}(n+1)/\mathbf{SO}(n-1)$ of Δ which is the Stiefel manifold of 2-frames in $(n+1)$-space; it is simply connected [Steenrod 1951, p. 132] when $n > 2$. In the exceptional case, $n = 2$ the universal covering space of Δ is S^3 which we identify with $\mathbb{H}_1$, the unit quaternions viewed as a maximal compact subgroup of $\tilde{\mathbf{G}}$, the universal covering group of $\mathrm{SL}(3, \mathbb{R})$. Δ_2 corresponds to $\mathbb{H}_1/\{\pm 1\} \cong \mathrm{Proj}(3, \mathbb{R})$ while Δ itself may be viewed as $\mathbb{H}/\{i^m\}$. The boundary is $\Gamma \cong \mathbb{H}_1/Q$ where Q is the quaternion group viewed as $Q = \{\pm 1, \pm i, \pm j, \pm k\}$. The tangent space at a point $q \in \mathbb{H}_1$ may be viewed as the set of elements $\xi \in \mathbb{H}$ which are imaginary multiples of q. One gets a contact form by putting

$$\theta(q; \xi) = \Re(\xi i \bar{q}).$$

This is clearly invariant under the action of left-multiplication by unit quaternions. It is also invariant under right-multiplication by powers of i. It changes sign under right-multiplication by j or k which shows why Γ does not carry an invariant contact structure.

We have proved that $\mathrm{Int}(\mathfrak{g})$ is uniquely faithfully representable as a transitive group of contact transformations by the minimal contact manifold Δ except in the case $\mathfrak{g} = \mathfrak{sl}(n+1,\mathbb{R})$, n even, where it has a second representation by a double covering $\tilde{\Delta}$ with the stability groups being the identity components of the $\mathbf{Q}(z)$. It remains to consider what happens when $\mathbf{G}$ is a finite-covering group of $\mathrm{Int}(\mathfrak{g})$. We have just seen that, when $n > 1$ is odd, $\mathrm{SL}(n+1,\mathbb{R})$, the double covering of its adjoint group, acts faithfully on a double covering $\tilde{\Delta}$ of Δ. For $\mathfrak{sl}(3,\mathbb{R})$ we get a special case which has been considered. The rank 1 Lie algebras need to be dealt with separately. Here the distinct Lie algebras which arise are $\mathfrak{so}(1,1+d), d \geq 1$; $\mathfrak{su}(1,1+m), m \geq 1$; $\mathfrak{sp}(1,1+m), m \geq 1$; and the rank 1 version of F_4. In the first three cases the boundary Γ may be viewed as the manifold of isotropic lines relative to the appropriate indefinite hermitean form. In all cases the boundary Γ is a sphere. The $\mathfrak{s}u$ cases correspond to holomorphic Lie algebras in which $\Delta = \Gamma$. The $\mathfrak{s}p$ cases have $d = 3$; so Δ is an S^2 bundle over S^{4m-1} which is simply connected. The F_4 case has $d = 7$ and Δ is an S^6-bundle over S^{15}; it is simply connected. For $\mathfrak{so}(1,1+d)$, Corollary 2.07 gives Δ as an S^{d-1}-bundle over S^d when $d \geq 2$ which is simply connected for $d > 2$. The exceptions $\mathfrak{so}(1,2) \cong \mathfrak{sp}(1,\mathbb{R})$ and $\mathfrak{so}(1,3) \cong \mathfrak{sp}(1,\mathbb{C})$ will be dealt with in the next paragraph.

Let (V,ω) be a symplectic space over $\mathbb{R}$ with $\dim(V) = 2n$. Put $\tilde{\Omega} = V \setminus \{0\}$. The group $\mathbf{G} = \mathrm{Sp}(n,\mathbb{R})$ is a transitive group of automorphisms of of $(\tilde{\Omega},\omega)$ which commutes with the action of scalar multiplication by $\mathbb{R}_+$. Here ω is homogeneous of degree 1 if we take the action to be $(v,c) \longmapsto vc^{1/2}$. The contact form is κ defined by

$$\kappa(v,\xi) = \tfrac{1}{2}\omega(v,\xi), \quad v \in \tilde{\Omega},\ \xi \in T(\tilde{\Omega}) = V.$$

We obtain $\tilde{\Delta} \cong S^{2n-1}$. Put $\mathcal{H} = \mathrm{Proj}(n-1,\mathbb{C})$. One has that $\tilde{\Omega}$ is a holomorphic $\mathbb{C}_*$-bundle over $\mathcal{H}$ corresponding to a Kähler form ω_1 which gives a generator of $H^2(\mathcal{H},\mathbb{Z})$ identified as a subgroup of $H^2(\mathcal{H},\mathbb{R})$. The holomorphic $\mathbb{C}_*$-bundle corresponding to $2\omega_1$ may be identified with $\Omega = \tilde{\Omega}/\{\pm I\}$. The map $\tilde{\Omega} \longrightarrow \Omega$ is given by $v \longmapsto Z(v)$ where $Z(v)$ can be identified with the linear transformation of V given by $Z(v)u = v\omega(v,u)$. It can also be viewed as the image of $\tilde{\Omega}$ in $V \otimes V$ under the map $v \longmapsto v \otimes v$. Here we have $\Delta \cong \mathrm{Proj}(2n-1,\mathbb{R})$. For $n > 1$ this gives the complete picture. In the case $n = 1$, if $\mathbf{G}$ is any connected Lie group with finite centre having Lie algebra $\mathfrak{sp}(1,\mathbb{R})$ and $\mathbf{G} = \mathbf{KAN}$ is its Iwasawa decomposition, then $\mathbf{G}/\mathbf{AN}$ is a contact manifold on which $\mathbf{G}$ acts faithfully by contact transformations.

One can repeat the above for V a vector space of dimension $2n$ over $\mathbb{C}$ and ω a complex symplectic form. In this case $\Re\omega$ gives the gives the symplectic structure. Here $\mathrm{Sp}(n,\mathbb{C})$ acts faithfully as a transitive group of contact transformations of $\tilde{\Delta} \cong S^{4n-1}$ while its adjoint group acts similarly on $\Delta \cong \mathrm{Proj}(4n-1,\mathbb{R})$. In this case the boundary Γ is identified with $\tilde{\Omega}/C_* \cong \mathrm{Proj}(2n-1,\mathbb{C})$.

For our purposes it is convenient to list the complex simple Lie algebras as $C_n, n \geq 1$, $A_n, n \geq 2$, $B_n, n \geq 3$, $D_n, n \geq 4$, and the five exceptional algebras. Only the C_n have representations on contact manifolds other than the minimal one.

Case **O** of Proposition 5.01 when $d = 1$ corresponds to the Lie algebras $\mathfrak{so}(n, n+m)$ with $n \geq 3$ and $n+m \geq 4$. We may view $\mathfrak{g}$ as the Lie algebra of endomorphisms of (V, ϕ) where V is a vector space of dimension $2n + m$ over $\mathbb{R}$ and ϕ is an indefinite quadratic form with n plus signs and $n + m$ minus signs. In this case Δ is the manifold of *oriented* 2-dimensional isotropic subspaces while Γ is the manifold of 2-dimensional isotropic subspaces. This is indicated by the relevant primitive root δ being in the second position. Put **G** for the connected component of the group of automorphisms of (V, ϕ). A maximal compact subgroup **K** has the form $\mathbf{K}_1 \times \mathbf{K}_2$ where $\mathbf{K}_1 \cong \mathbf{SO}(n)$ and $\mathbf{K}_2 \cong \mathbf{SO}(n+m)$. The subgroup **L** leaving an element $z \in \Delta$ fixed can be written as $\mathbf{L}_0 \times \mathbf{L}_1 \times \mathbf{L}_2$ where $\mathbf{L}_0 \cong \mathbf{SO}(2)$, $\mathbf{L}_1 \cong \mathbf{SO}(n-2)$, and $\mathbf{L}_2 \cong \mathbf{SO}(n+m-2)$. Here $\mathbf{L}_0 \times \mathbf{L}_j$ is imbedded as a subgroup of $\mathbf{K}_j$. One way to view Δ is as an $\mathcal{S}_{n,n-2}$ bundle over Q_{n+m} where Q_{n+m} is the manifold of oriented 2-planes in $n+m$-dimensional real space (a compact complex quadric) and $\mathcal{S}_{n,n-2}$ is the Stiefel manifold of $(n-2)$-frames in n dimensional space. Both these manifolds are simply connected. Therefore Δ is simply connected. It is irrelevant whether **G** is isomorphic to $\mathrm{Int}(\mathfrak{g})$ or is a double covering, for in the latter case the element $-I$ acts trivially on Δ.

The Lie algebras $\mathfrak{so}(2, 2+m)$, $m \geq 2$ come under case (**C**). The only difference from what was just described is that $\Gamma = \Delta$ for the isotropic 2-planes split into two components according to the orientation. Of course, here we have $\Delta \cong \mathcal{S}_{2+m,m}$ which is simply connected for the m under consideration. For the Langlands decomposition of the parabolic subgroup $\mathbf{P}(\mathfrak{z}) = \mathbf{Q}(z)$ one gets $\mathfrak{m}(\mathfrak{z}) \cong \mathfrak{so}(m) + \mathfrak{sl}(2, \mathbb{R})$. This case comes under the next if one takes enough generality.

Let (V, ω) be a sesqui-symplectic space over a "field" $\mathbb{F}$ containing $\mathbb{R}$. This is to say that V is a right vector space over $\mathbb{F}$ and we have a form $\omega \colon \bar{V} \otimes_{\mathbb{F}} V \longrightarrow \mathbb{F}$, where $\bar{V}$ is the opposite vector space, which we write in the form $\omega(u, v)$ with $(u, v) \in V$ and $\omega(u\alpha, v\beta) = \bar{\alpha}\omega(u, v)\beta$. We also suppose that $\omega(v, u) = -\overline{\omega(u, v)}$. The final hypothesis is that for all there exists a linear transformation J of V with $J^2 = -I$ such that J preserves ω and $\omega(u, Ju) > 0$ for all $u \in V \setminus \{0\}$. Put $\Omega_1 = \{u \in V : \omega(u, u) = 0\}$. We may identify the tangent space

$$T_u(\Omega_1) = \{v \in V : \Im\omega(u, v) = 0\}.$$

We should like to have $\Re\omega$ a symplectic form, but note that ua is a non-trivial element of $T_u(\Omega_1)$ whenever $a \in \Im\mathbb{F} \setminus \{0\}$ while $\Re\omega(ua, v) = \Re\big(\bar{a}\omega(u, v)\big) = 0$ for all v with $\Im\omega(u, v) = 0$. Therefore we have to replace Ω_1 by $\Omega = \Omega_1 / \mathbb{F}_1$ where $\mathbb{F}_1$ is the multiplicative group generated by the exponentials of purely imaginary elements of $\mathbb{F}$. We obtain a Lie algebra $\mathfrak{g}_1$ generated by the endomorphisms of V of the form

$$X(u, v) = \tfrac{1}{2}\big(u\omega(v, \cdot) + v\omega(u, \cdot)\big).$$

Put $\mathfrak{c}$ for the elements of $\mathfrak{g}_1$ with the property that $Xv = vc$ for some fixed $c \in \Im\mathbb{F}$ and all $v \in V$. Then $\mathfrak{g} = \mathfrak{g}_1 / \mathfrak{c}$ acts faithfully as infinitesimal transformations of Ω, and its exponentials generate a group **G** of automorphisms of (V, ω). Note that the map $\Omega \longrightarrow \mathfrak{g}$, $u \longmapsto X(u, u)$ is k-to-1 where k is the index of $\mathbb{F}_1$ in the group of elements of unit norm in $\mathbb{F}$.

In any case, the image of Ω in $\mathfrak{g}$ modulo $\mathbb{R}_+$ is exactly the manifold Γ of isotropic lines in (V,ω). The Lie algebra $\mathfrak{g}$ is always of holomorphic type, and we obtain all holomorphic Lie algebras this way with sufficient imagination (V might not really exist). We shall denote the Lie algebra in question by $\mathfrak{sq}(n,m,\mathbb{F})$ where n is the dimension of a maximal isotropic subspace of (V,ω) and $2n+m$ is the dimension of V over $\mathbb{F}$. The Lie algebra is said to be of *tube* type if $m=0$ and *non-tube* type if $m>0$. It is easy to see that the stability group of a point $z\in\Omega/\mathbb{R}_+$ is

$$\mathbf{SQ}(n-1,m,\mathbb{F})\times(\mathbb{F}_*/F_1)\times\exp\big(\mathfrak{n}(z)\big)$$

where $\mathfrak{n}(z)$ is genereated by the elements of the form $X(u,v)$ where $u\in z$ and $\omega(u,v)=0$. If the dimension of V over $\mathbb{R}$ is $2nf+mf'$ then

$$\dim\Delta=2(n-1)f+mf'+1;$$

and mf' is necessarily even. In case $n=1$ one can take $\mathbb{F}$ to be anything and $\mathfrak{sq}(1,m,\mathbb{F})\cong\mathfrak{su}(1,1+mf'/2)$. When $n=2$ the tube type Lie algebras may be subsumed under $\mathfrak{so}(2,2+f)$. For $n>2$ the tube type algebras all correspond to classical fields $\mathbb{F}=\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$ with the single exception of $\mathfrak{sq}(3,0,\mathbb{O})$ which is a form of E_7. In this case $f=8$ and $\dim\Delta=33$; here $\mathbf{K}=\mathbb{T}\times\mathbf{E}_6$ where $\mathbf{E}_6$ is the compact simply-connected version of the Lie algebra E_6. The Lie algebra $\mathfrak{m}(\mathfrak{z})$ is isomorphic to $\mathfrak{so}(2,10)$ according to our general calculation, and the maximal compact subgroup in $\mathbf{Q}(z)$ is $\mathbb{T}\times\mathrm{Spin}(10)$. Here Δ is minimal and simply connected. Note that $\mathbf{SQ}(n,0,\mathbb{F})$ has a minimal boundary B_n of dimension $\frac{1}{2}n(n-1)f+n)$ given by the manifold of Lagrangian (n-dimensional isotropic) subspaces of V; when $n=2, f>1$ or $n=3, f>2$ it has lower dimension than Δ. The non-tube cases arise only for $\mathfrak{sq}(n,m,\mathbb{C})$ with $n>0$, $m>0$ arbitrary, $\mathfrak{sq}(n,1,\mathbb{H})$, and an exceptional $\mathfrak{sq}(2,1,\mathbb{F})$ where $\mathbb{F}$ is a strange object with $f=6$ and $f'=8$ corresponding to a form of E_6. For this the maximal compact subgroup is $\mathbf{K}\cong\mathbb{T}\times\mathrm{Spin}(10)$ while $\mathfrak{m}(\mathfrak{z})\cong\mathfrak{su}(1,5)$ according to our general formulas. One obtains $\Delta\cong\mathrm{Spin}(10)/\mathbf{SU}(5)$ which is a 21-dimensional simply-connected manifold.

NOTE. $\mathbf{SU}(5)$ is a subgroup of $\mathbf{SO}(10)$ in an obvious way. The pullback to the double covering Spin(10) has two components, and we take the component of the identity. Having eliminated the exceptional cases we can consider the standard cases. When $\mathbb{F}=\mathbb{R}$ we have obtained a double covering since the index of $\mathbb{F}_1$ in the elements of norm 1 is 2. In case $\mathbb{F}=\mathbb{C},\mathbb{H}$ the index is 1.

When $d=1$ and the rank is 4 one obtains case (**F**) under what was described at the beginning as case (**EFG**). The boundary Γ which occurs here corresponds to the points of the projective F_4-geometries described in [Freudenthal & deVries 1969, §73]. Four "fields" occur corresponding to $F_4, f=1$; $E_6, f=2$, $E_7, f=4$; and $E_8, f=8$. We get

$$\dim(\Delta)=6f+9.$$

The normal real forms of G_2, E_6, E_7, and E_8 require special geometric considerations which we do not go into. In the case of G_2 we get $\dim(\Delta)=5$. All the other dimensions are as in F_4 versions. Boundaries of minimal dimension are obtained for G_2, F_4, and E_8. For E_6 and E_7 the root δ is not in the right position on the Dynkin diagram to give a boundary of minimal dimension.

All that remains are case (**C**) and case (**BC**) with $d = 3$. These are the Lie algebras $\mathfrak{sp}(n, n+m)$. The boundary Γ corresponds to the manifold of isotropic lines for the appropriate indefinite quaternionic hermitean form.

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