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THE NONARITHMETICITY OF THE PREDICATE LOGIC OF STRICTLY PRIMITIVE RECURSIVE REALIZABILITY

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Abstract. A notion of strictly primitive recursive realizability is introduced by Damnjanovic in 1994. It is a kind of constructive semantics of the arithmetical sentences using primitive recursive functions. It is of interest to study the corresponding predicate logic. It was argued by Park in 2003 that the predicate logic of strictly primitive recursive realizability is not arithmetical. Park's argument is essentially based on a claim of Damnjanovic that intuitionistic logic is sound with respect to strictly primitive recursive realizability, but that claim was disproved by the author of this article in 2006. The aim of this paper is to present a correct proof of the result of Park.

§1. Introduction. Recursive realizability introduced by Kleene [4] can be considered as an interpretation of the informal intuitionistic meaning of arithmetical sentences on the basis of the theory of recursive functions. The main idea of recursive realizability is to replace the vague intuitionistic concept of an effective operation by the exact notion of a partial recursive function. On the other hand, many other more restrictive classes of computable functions are studied in mathematics. It is of interest to consider variants of realizability based on subrecursive classes of functions. One of them is the class of primitive recursive functions.

Damnjanovic [2] introduced the notion of strictly primitive recursive realizability for the language of formal arithmetic. This kind of realizability combines the ideas of recursive realizability and Kripke models. Strictly primitive recursive realizability can be considered as a kind of constructive semantics of arithmetical sentences. For any semantics, it is of interest to study the corresponding logic. The predicate logic of Kleene's recursive realizability was considered by the author. It was proved that the set of recursively realizable predicate formulas is not arithmetical [11].

A similar result for the strictly primitive recursive realizability was obtained by Park [7]. His proof of the nonarithmeticity of the predicate logic of strictly primitive recursive realizability is essentially based on the claim [2] that every predicate formula deducible in the intuitionistic predicate calculus is strictly primitive recursively realizable (Lemma 4.3 and Theorem 5.1). Later this claim was disproved by the author [8, 10]. Nevertheless the result of Park remains valid. The aim of this paper is to present a correct proof of this result.

Another primitive recursive realizability was introduced by Salehi [12], which was compared with the strictly primitive recursive realizability of Damnjanovic in [8–10]. It was proved that there exists an arithmetical formula that is primitive recursively

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realizable, in the sense of [12], but not strictly primitive recursively realizable, in the sense of [2]; while its negation is strictly primitive recursively realizable but not primitive recursively realizable.

§2. Indexing of primitive recursive functions.

Definition 2.1. *Primitive recursive functions are obtained by substitution and recursion from the basic functions* $O(x) = 0$, $S(x) = x + 1$, $I_i^m(x_1, \ldots, x_m) = x_i$ $(m = 1, 2, \ldots;$ $1 \leq i \leq m)$.

Definition 2.2. *The class of elementary functions is the least class containing the functions* $x \mapsto 1$, I_i^m, $+$, $\div$, *where* $x \div y = \begin{cases} 0 & \text{if } x < y, \\ x - y & \text{if } x \geq y, \end{cases}$ *and closed under substitution, summation* $(\mathbf{x}, y) \mapsto \sum_{i=0}^{y} \psi(\mathbf{x}, i)$, *and multiplication* $(\mathbf{x}, y) \mapsto \prod_{i=0}^{y} \psi(\mathbf{x}, i)$, $\mathbf{x}$ *being the list* $x_1, \ldots, x_m$.

If $a_0, a_1, \ldots, a_n$ are natural numbers, then $\langle a_0, a_1, \ldots, a_n \rangle$ denotes the number $p_0^{a_0} \cdot p_1^{a_1} \cdot \ldots \cdot p_n^{a_n}$, where $p_0, p_1, \ldots, p_n$ are sequential prime numbers ($p_0 = 2$, $p_1 = 3$, $p_2 = 5, \ldots$). The functions $\pi(i) = p_i$ and $(x, y) \mapsto \langle x, y \rangle$ are elementary. In what follows, for $a \geq 1$ and $i \geq 0$, let $[a]_i$ denote the exponent of p_i under decomposition of a into prime factors. Therefore, $[a]_i = a_i$ if $a = \langle a_0, \ldots, a_n \rangle$. For definiteness, let $[0]_i = 0$ for every i. Note that the function $(x, i) \mapsto [x]_i$ is elementary.

Definition 2.3. *An* $(m + 1)$*-ary function* f *is obtained by bounded recursion from an* m*-ary function* g, *an* $(m + 2)$*-ary function* h, *and an* $(m + 1)$*-ary function* j *if the following conditions are fulfilled for any* $x_1, \ldots, x_m, y$:

$$f(0, x_1, \ldots, x_m) = g(x_1, \ldots, x_m),$$

$$f(y + 1, x_1, \ldots, x_m) = h(y, f(y, x_1, \ldots, x_m), x_1, \ldots, x_m),$$

$$f(y, x_1, \ldots, x_m) \leq j(y, x_1, \ldots, x_m).$$

Thus f is obtained by bounded recursion from g, h, j if f is obtained by primitive recursion from g, h and is bounded by j.

For given functions $\theta_1, \ldots, \theta_k$, let $\mathbf{E}[\theta_1, \ldots, \theta_k]$ be the least class containing $\theta_1, \ldots, \theta_k$, S, I_i^m, the constant functions and closed under substitution and bounded recursion. Consider the following sequence of functions:

$$f_0(x, y) = y + 1,$$

$$f_1(x, y) = x + y,$$

$$f_2(x, y) = (x + 1) \cdot (y + 1),$$

$$f_{n+1}(0, y) = f_n(y + 1, y + 1),$$

$$f_{n+1}(x + 1, y) = f_{n+1}(x, f_{n+1}(x, y)),$$

for $n \geq 2$. Grzegorczyk [3] introduced a hierarchy of classes $\mathcal{E}^n$, where $\mathcal{E}^n = \mathbf{E}[f_n]$. The class $\mathcal{E}^3$ contains all elementary functions. It was shown by Grzegorczyk [3] that $\bigcup_{n=0}^{\infty} \mathcal{E}^n$ is exactly the class of primitive recursive functions.

Axt [1] has shown that in the definition of the classes $\mathcal{E}^n$ for $n \geq 4$, the usual bounded recursion can be replaced by the following scheme applicable to every triple of functions g, h, j of appropriate arities:

$$\begin{cases} f(0, \mathbf{x}) = g(\mathbf{x}), \\ f(y+1, \mathbf{x}) = h(y, f(y, \mathbf{x}), \mathbf{x}) \cdot \mathsf{sg}(j(y, \mathbf{x}) \dot{-} f(y, \mathbf{x})) \cdot \mathsf{sg}(f(y, \mathbf{x})), \end{cases}$$

where $\mathsf{sg}(x) = \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$

For any collection of functions $\Theta = \{\theta_1, \dots, \theta_m\}$, let $\mathbf{E}^4[\Theta]$ be the least class including Θ, containing S, the constant functions, I_i^m, sg, $\dot{-}$, f_4 and closed under substitution and Axt's bounded recursion. A way of indexing the functions which are primitive recursive relative to $\theta_1, \dots, \theta_m$ is proposed in [6]. It can be adapted to an indexing of the class $\mathbf{E}^4[\Theta]$. The functions in $\mathbf{E}^4[\Theta]$ get indexes according to their definition from the initial functions by substitution and Axt's bounded recursion. We list below the possible defining schemes for a function φ and indicate on the right its index.

( )	$\varphi(x_1, \dots, x_{k_i}) = \theta_i(x_1, \dots, x_{k_i})$	$\langle 0, k_i, i \rangle$,
(I)	$\varphi(x) = x + 1$	$\langle 1, 1 \rangle$,
(II)	$\varphi(x_1, \dots, x_n) = q$	$\langle 2, n, q \rangle$,
(III)	$\varphi(x_1, \dots, x_n) = x_i$ (where $1 \leq i \leq n$)	$\langle 3, n, i \rangle$,
(IV)	$\varphi(x) = \mathsf{sg}(x)$	$\langle 4, 1 \rangle$,
(V)	$\varphi(x, y) = x \dot{-} y$	$\langle 5, 2 \rangle$,
(VI)	$\varphi(x, y) = f_4(x, y)$	$\langle 6, 2 \rangle$,
(VII)	$\varphi(\mathbf{x}) = \psi(\chi_1(\mathbf{x}), \dots, \chi_k(\mathbf{x}))$	$\langle 7, m, g, h_1, \dots, h_k \rangle$,
(VIII)	$\begin{cases} \varphi(0, \mathbf{x}) = \psi(\mathbf{x}) \\ \varphi(y+1, \mathbf{x}) = \chi(y, f(y, \mathbf{x}), \mathbf{x}) \cdot \\ \cdot \mathsf{sg}(\zeta(y, x) \dot{-} \varphi(y, \mathbf{x})) \cdot \mathsf{sg}(\varphi(y, \mathbf{x})) \end{cases}$	$\langle 8, m+1, g, h, j \rangle$.

Here $\mathbf{x}$ is the list $x_1, \dots, x_m$ and $g, h_1, \dots, h_k, h, j$ are indexes of the functions $\psi, \chi_1, \dots, \chi_k, \chi, \zeta$ respectively.

Let $In^{\Theta}(b)$ mean that b is an index of a function in the class $\mathbf{E}^4[\Theta]$. It is shown in [1] that $In^{\Theta}(b)$ is an elementary predicate. If $In^{\Theta}(b)$ holds, then ef_b^{Θ} denotes a $[b]_1$-ary function in $\mathbf{E}^4[\Theta]$ indexed by b. Following [1], we set

$$\mathsf{ef}^{\Theta}(b, a) = \begin{cases} \mathsf{ef}_b^{\Theta}([a]_0, \dots, [a]_{[b]_1 \dot{-} 1}) & \text{if } In^{\Theta}(b), \\ 0 & \text{else.} \end{cases}$$

The function ef^{Θ} is universal for the class $\mathbf{E}^4[\Theta]$ and is not in this class. Following [1], a binary function e_n is defined inductively as follows:

$$\mathsf{e}_0(b, a) = 0, \ \mathsf{e}_{n+1}(b, a) = \mathsf{ef}^{[\mathsf{e}_0, \dots, \mathsf{e}_n]}(b, a).$$

Finally, the class $\mathbf{E}_n$ is defined as $\mathbf{E}^4[\mathsf{e}_0, \dots, \mathsf{e}_n]$. It is proved (see [1]) that for any $n \geq 0$, $\mathbf{E}_n = \mathcal{E}^{n+4}$.

Let $In(n, b)$ mean that b is an n-index, i.e., an index of a function in $\mathbf{E}_n$. It is shown in [1] that the predicate $In(n, b)$ is elementary. It follows immediately from the definition of the indexing that $In(n, b)$ and $m > n$ imply $In(m, b)$. Note that if an n-index b of a function $\varphi(\mathbf{x})$ is given, then for any $\mathbf{m}$, we can compute the value $\varphi(\mathbf{m})$. Indeed, computing the value $\varphi(\mathbf{x})$ is reduced to computing the values of a finite number of functions whose indexes are less than b. Thus the process of computing should be terminated.

Notation. If n is a natural number, then $\Lambda x.n$ will denote the number $\langle 2, 1, n\rangle$, a_k will denote the number $\Lambda j.\Lambda x.k$, and a will denote the number a_0. Thus $\Lambda x.n$ is a 0-index of the constant function $x \mapsto n$, a_k is a 0-index of the function $x \mapsto \Lambda x.k$, and a is a 0-index of the function $x \mapsto \Lambda x.0$.

§3. Strictly primitive recursively realizable arithmetical formulas. We consider arithmetical sentences in a purely predicate language. Namely, the language of formal arithmetic Ar is a first-order language without any functional symbols and individual constants consisting of a unary predicate symbol Z, binary predicate symbols E and S, and ternary predicate symbols A and M. In the standard model of arithmetic $\mathfrak{N}$ these predicate symbols have the following meaning: $Z(x)$ means $x = 0$, $E(x, y)$ means $x = y$, $S(x, y)$ means $x + 1 = y$, $A(x, y, z)$ means $x + y = z$, and $M(x, y, z)$ means $x \cdot y = z$. Besides, we consider an extended arithmetical language Ar^* obtained by adding to Ar individual constants $0, 1, 2, \ldots$ for all natural numbers.

Ar-formulas and Ar^*-formulas are built from atomic ones by means of the connectives $\&$, $\vee$, $\rightarrow$, $\neg$ and the quantifiers $\forall$, $\exists$. The expression $\Phi \equiv \Psi$ is an abbreviation for the formula $(\Phi \rightarrow \Psi) \,\&\, (\Psi \rightarrow \Phi)$. If a formula A does not contain any free variables except $x_1, \ldots, x_n$, we denote this formula $A(x_1, \ldots, x_n)$. In this case, $A(k_1, \ldots, k_n)$ denotes the Ar^*-formula obtained from A by substituting the constants $k_1, \ldots, k_n$ for $x_1, \ldots, x_n$ respectively. For brevity, we sometimes write $\forall \mathbf{x}$ instead of $\forall x_1 \ldots \forall x_n$ and $\exists \mathbf{x}$ instead of $\exists x_1 \ldots \exists x_n$, where $\mathbf{x}$ is the list of variables $x_1, \ldots, x_n$.

We say that a closed Ar^*-formula Φ is *true* iff $\mathfrak{N} \models \Phi$ in the usual classical sense.

The notion of strictly primitive recursive realizability for arithmetical formulas is introduced by Damnjanovic [2]. The relation "*a natural number e strictly primitive recursively realizes a closed Ar^*-formula A at level n*" ($e \Vdash_n A$) is defined by induction on the number of logical symbols in A.

1) If A is an atomic formula, then $e \Vdash_n A$ means that $e = 0$ and A is true.
2) If A is $B \,\&\, C$, then $e \Vdash_n A$ means that $[e]_0 \Vdash_n B$ and $[e]_1 \Vdash_n C$.
3) If A is $B \vee C$, then $e \Vdash_n A$ means that either $[e]_0 = 0$ and $[e]_1 \Vdash_n B$ or $[e]_0 \neq 0$ and $[e]_1 \Vdash_n C$.
4) If A is $B \rightarrow C$, then $e \Vdash_n A$ means that $In(n, e)$ holds and for any $j \geq n$, $In(j, \mathsf{e}_{n+1}(e, \langle j\rangle))$ holds, and for any y, if $y \Vdash_j B$, then $\mathsf{e}_{j+1}(\mathsf{e}_{n+1}(e, \langle j\rangle), \langle y\rangle) \Vdash_j C$.
5) If A is $\neg B$, then $e \Vdash_n A$ means that $e \Vdash_n (B \rightarrow E(0, 1))$.
6) If A is $\exists x\, B(x)$, then $e \Vdash_n A$ means that $[e]_1 \Vdash B([e]_0)$.
7) If A is $\forall x\, B(x)$, then $e \Vdash_n A$ means that $In(n, e)$ holds and for any m, $\mathsf{e}_{n+1}(e, \langle m\rangle) \Vdash_n B(m)$.

A closed Ar^*-formula A is called *strictly primitive recursively realizable* (*spr-realizable*) iff there are e and n such that $e \Vdash_n A$. It is proved in [2] that $e \Vdash_m A$ if

$e \Vdash_n A$ and $m \geq n$. Since any closed Ar-formula is a closed Ar^*-formula, it follows that the notion of strictly primitive recursive realizability is defined for closed Ar-formulas too.

The following properties of spr-realizability immediately follow from the definition.

Proposition 3.4. *For any closed Ar^*-formulas A and B,*

1) *$A \,\&\, B$ is spr-realizable iff A and B are both spr-realizable;*
2) *$A \vee B$ is spr-realizable iff at least one of A, B is spr-realizable;*
3) *if A is spr-realizable and B is not spr-realizable, then $A \to B$ is not spr-realizable;*
4) *if $k \Vdash_n B$, then $\mathsf{a}_k \Vdash_n (A \to B)$ (see Notation);*
5) *if A is not spr-realizable, then for any k, $\mathsf{a}_k \Vdash_0 (A \to B)$;*
6) *if $\neg A$ is spr-realizable, then $\mathsf{a} \Vdash_0 \neg A$.*

Proof. The statements 1)–3) are obvious.

4) Suppose $k \Vdash_n B$. Then $k \Vdash_j B$ for any $j \geq n$. We prove that $\mathsf{a}_k \Vdash_n (A \to B)$, i.e., that (a) $In(n, \mathsf{a}_k)$ holds, (b) $In(j, \mathsf{e}_{n+1}(\mathsf{a}_k, \langle j \rangle))$ holds for any $j \geq n$, and (c) for any $j \geq n$ and any y, if $y \Vdash_j A$, then $\mathsf{e}_{j+1}(\mathsf{e}_{n+1}(\mathsf{a}_k, \langle j \rangle), \langle y \rangle) \Vdash_j B$. Note, that $\Lambda x.k$ is a 0-index of the constant function with the only value k and a_k is a 0-index of the constant function whose only value is $\langle 2, 1, k \rangle$. Thus the conditions (a) and (b) are fulfilled. The condition (c) is also obvious, because for any j and y, we have $\mathsf{e}_{j+1}(\mathsf{e}_{n+1}(\mathsf{a}_k, \langle j \rangle), \langle y \rangle) = \mathsf{e}_{j+1}(\Lambda x.k, \langle y \rangle) = k$, and $k \Vdash_j B$.

5) Assume that A is not *spr*-realizable. This means that for any y, j, the condition $y \Vdash_j A$ does not hold. We have to prove that $\mathsf{a}_k \Vdash_0 (A \to E(0,1))$. It was shown above that $In(0, \mathsf{a}_k)$ holds and for any j, $In(j, \mathsf{e}_1(\mathsf{a}_k, \langle j \rangle))$ holds. Thus it is enough to prove that for any y, $y \Vdash_j A \Rightarrow \mathsf{e}_{j+1}(\mathsf{e}_1(\mathsf{a}_k, \langle j \rangle), \langle y \rangle) \Vdash_j B$, but this is obvious because the premise is false.

The statement 6) follows from 5) because A is not *spr*-realizable if $\neg A$ is spr-realizable. □

Proposition 3.5. *Suppose that an Ar^*-formula $A(x, y)$ is such that for any m and n, the Ar^*-formula $\neg A(m, n)$ is spr-realizable. Then the formula $\forall x \forall y \neg A(x, y)$ is spr-realizable at level* 0.

Proof. For any m and n, if $\neg A(m, n)$ is *spr*-realizable, then by Proposition 3.4, $\mathsf{a} \Vdash_0 \neg A(m, n)$. We show that $\Lambda x.\Lambda y.\mathsf{a} \Vdash_0 \forall x \forall y \neg A(x, y)$, i.e., $In(0, \Lambda x.\Lambda y.\mathsf{e})$ holds (this is obvious) and for any m, $\mathsf{e}_1(\Lambda x.\Lambda y.\mathsf{a}, \langle m \rangle) \Vdash_0 \forall y \neg A(m, y)$. Note that $\mathsf{e}_1(\Lambda x.\Lambda y.\mathsf{a}, \langle\ m \rangle) = \Lambda y.\mathsf{a}$. Thus we have to prove that $\Lambda y.\mathsf{a} \Vdash_0 \forall y \neg A(m, y)$, i.e., $In(0, \Lambda y.\mathsf{a})$ holds (this is evident) and for any n, $\mathsf{e}_1(\Lambda y.\mathsf{a}, \langle n \rangle \Vdash_0 \neg A(m, n)$. This is also evident because $\mathsf{e}_1(\Lambda y.\mathsf{a}, \langle n \rangle) = \mathsf{a}$. □

Definition 3.6. *An Ar^*-formula Φ will be called completely negative iff it does not contain logical symbols $\vee$ and $\exists$ and every atom Ψ occurs in Φ only in subformulas of the form $\neg\Psi$.*

An inductive definition of a completely negative Ar^*-formula is the following:

- if Φ is an atomic Ar^*-formula, then $\neg\Phi$ is completely negative;
- if Φ and Ψ are completely negative Ar^*-formulas, then $\neg\Phi$, $(\Phi \,\&\, \Psi)$, and $(\Phi \to \Psi)$ are completely negative;
- if Φ is a completely negative Ar^*-formula and x is an individual variable, then $\forall x\, \Phi$ is completely negative.

PROPOSITION 3.7. *If $A(x_1, \dots, x_n)$ is a completely negative Ar^*-formula, then there exists a natural number n_A such that for any natural numbers $k_1, \dots, k_n$, the following conditions are equivalent*:

1) *the formula $A(k_1, \dots, k_n)$ is true*;
2) $n_A \Vdash_0 A(k_1, \dots, k_n)$;
3) *the formula $A(k_1, \dots, k_n)$ is spr-realizable.*

Proof. We construct the number n_A inductively. Obviously, it is enough to verify two conditions:

(i) $n_A \Vdash_0 A(k_1, \dots, k_n)$ iff the formula $A(k_1, \dots, k_n)$ is true;
(ii) if the formula $A(k_1, \dots, k_n)$ is *spr*-realizable, then $n_A \Vdash_0 A(k_1, \dots, k_n)$.

If $A(x_1, \dots, x_n)$ is of the form $\neg\Phi(x_1, \dots, x_n)$ for an atomic formula Φ, let $n_A = \mathsf{a}$. We now verify the item (i). Assume that $\mathsf{a} \Vdash_0 \neg\Phi(k_1, \dots, k_n)$. Then the formula $\Phi(k_1, \dots, k_n)$ is not *spr*-realizable. It follows from the definition of spr-realizability for atomic formulas that $\Phi(k_1, \dots, k_n)$ is not true. Then the formula $\neg\Phi(k_1, \dots, k_n)$ is true. Conversely, assume that the formula $\neg\Phi(k_1, \dots, k_n)$ is true. Then $\Phi(k_1, \dots, k_n)$ is not true. It follows from the definition of spr-realizability for atomic formulas that $\Phi(k_1, \dots, k_n)$ is not *spr*-realizable. Then by Proposition 3.4, $\mathsf{a} \Vdash_0 \neg\Phi(k_1, \dots, k_n)$. The item (ii) follows from Proposition 3.4

Suppose that for formulas $B(x_1, \dots, x_n)$ and $C(x_1, \dots, x_n)$ we have found the numbers n_B and n_C such that for any $k_1, \dots, k_n$, the following conditions hold:

(ib) $n_B \Vdash_0 B(k_1, \dots, k_n)$ iff the formula $B(k_1, \dots, k_n)$ is true;
(ic) $n_C \Vdash_0 C(k_1, \dots, k_n)$ iff the formula $C(k_1, \dots, k_n)$ is true;
(iib) if the formula $B(k_1, \dots, k_n)$ is *spr*-realizable, then $n_B \Vdash_0 B(k_1, \dots, k_n)$;
(iic) if the formula $C(k_1, \dots, k_n)$ is *spr*-realizable, then $n_C \Vdash_0 C(k_1, \dots, k_n)$.

If $A(x_1, \dots, x_n)$ is $B(x_1, \dots, x_n) \,\&\, C(x_1, \dots, x_n)$, let $n_A = \langle n_B, n_C \rangle$. We prove that n_A is a required number. Let us show the item (i). Assume that $A(k_1, \dots, k_n)$ is true. Then $B(k_1, \dots, k_n)$ and $C(k_1, \dots, k_n)$ are both true. By (ib) and (ic), we have $n_B \Vdash_0 B(k_1, \dots, k_n)$ and $n_C \Vdash_0 C(k_1, \dots, k_n)$. Then $n_A \Vdash_0 A(k_1, \dots, k_n)$ by the definition of spr-realizability because $[n_A]_0 = n_B$, $[n_A]_1 = n_C$. Conversely, if $n_A \Vdash_0 A(k_1, \dots, k_n)$, then $n_B \Vdash_0 B(k_1, \dots, k_n)$ and $n_C \Vdash_0 C(k_1, \dots, k_n)$ by the definition of spr-realizability for the conjunction. In this case, by the items (ib) and (ic), the formulas $B(k_1, \dots, k_n)$ and $C(k_1, \dots, k_n)$ are both true; therefore, $A(k_1, \dots, k_n)$ is also true.

We now prove the item (ii). Assume that $A(k_1, \dots, k_n)$ is *spr*-realizable. Then $B(k_1, \dots, k_n)$ and $C(k_1, \dots, k_n)$ are *spr*-realizable. It follows from (iib) and (iic) that $n_B \Vdash_0 B(k_1, \dots, k_n)$ and $n_C \Vdash_0 C(k_1, \dots, k_n)$. This gives $n_A \Vdash_0 A(k_1, \dots, k_n)$ by the definition of spr-realizability.

If $A(x_1, \dots, x_n)$ is $B(x_1, \dots, x_n) \to C(x_1, \dots, x_n)$, then we set $n_A = \mathsf{a}_{n_C}$ and prove that n_A is a required number. Let us verify the item (i). Assume that the formula $A(k_1, \dots, k_n)$ is true. Then either the formula $B(k_1, \dots, k_n)$ is false or the formula $C(k_1, \dots, k_n)$ is true. In the first case, $B(k_1, \dots, k_n)$ is not *spr*-realizable. Indeed, suppose the opposite; then it should follow from (iib) that $n_B \Vdash_0 B(k_1, \dots, k_n)$ and by (ib), $B(k_1, \dots, k_n)$ should be true. By Proposition 3.4, $n_A \Vdash_0 A(k_1, \dots, k_n)$. If $C(k_1, \dots, k_n)$ is true, then by (ic), $n_C \Vdash_0 C(k_1, \dots, k_n)$. By Proposition 3.4, $n_A \Vdash_0 A(k_1, \dots, k_n)$. Now assume that $n_A \Vdash_0 A(k_1, \dots, k_n)$. We prove that $A(k_1, \dots, k_n)$ is true. Suppose the opposite. Then the formula $B(k_1, \dots, k_n)$ is true and the formula $C(k_1, \dots, k_n)$ is false. By (ib), the formula $B(k_1,\dots,k_n)$ is *spr*-realizable. Therefore, the formula $C(k_1,\dots,k_n)$ should be *spr*-realizable. But in this case, by the item (iic), $n_C \Vdash_0 C(k_1,\dots,k_n)$ and by (ic), the formula $C(k_1,\dots,k_n)$ is true. This contradiction shows that the formula $A(k_1,\dots,k_n)$ is true.

We now show the item (ii). Assume that $A(k_1,\dots,k_n)$ is *spr*-realizable. In this case, if the formula $B(k_1,\dots,k_n)$ is not *spr*-realizable, then by Proposition 3.4, $n_A \Vdash_0 A(k_1,\dots,k_n)$. If $B(k_1,\dots,k_n)$ is *spr*-realizable, then $C(k_1,\dots,k_n)$ is also *spr*-realizable. It follows from (iic) that $n_C \Vdash_0 C(k_1,\dots,k_n)$. Then by Proposition 3.4, $n_A \Vdash_0 A(k_1,\dots,k_n)$.

The case when the formula $A(x_1,\dots,x_n)$ is of the form $\neg B(x_1,\dots,x_n)$ is reduced to the case of the implication $B(x_1,\dots,x_n) \to E(0,1)$. In this case we can set $n_A = \mathsf{a}$.

Finally, assume that $A(x_1,\dots,x_n)$ is of the form $\forall y\, B(y,x_1,\dots,x_n)$ and there is a number n_B such that for any natural numbers $m,k_1,\dots,k_n$, the following conditions are fulfilled:

(iB) $n_B \Vdash_0 B(m,k_1,\dots,k_n)$ iff the formula $B(m,k_1,\dots,k_n)$ is true;
(iiB) if $B(m,k_1,\dots,k_n)$ is *spr*-realizable, then $n_B \Vdash_0 B(m,k_1,\dots,k_n)$.

Let $n_A = \Lambda x.n_B$. Obviously, $In(0,n_A)$ holds. We prove the item (i). Assume that the formula $A(k_1,\dots,k_n)$ is true. Then for any m, the formula $B(m,k_1,\dots,k_n)$ is true and by (iB), $n_B \Vdash_0 B(m,k_1,\dots,k_n)$. Since $\mathsf{e}_1(n_A,\langle m\rangle) = n_B$, it follows that for any m, $\mathsf{e}_1(n_A,\langle m\rangle) \Vdash_0 B(m,k_1,\dots,k_n)$. This means that $n_A \Vdash_0 A(k_1,\dots,k_n)$.

Conversely, suppose $n_A \Vdash_0 A(k_1,\dots,k_n)$. Thus $\mathsf{e}_1(n_A,\langle m\rangle) \Vdash_0 B(m,k_1,\dots,k_n)$, i.e., $n_B \Vdash_0 B(m,k_1,\dots,k_n)$, for any m. By (iB), the formula $B(m,k_1,\dots,k_n)$ is true for any m; therefore, the formula $A(k_1,\dots,k_n)$ is true.

We now show the item (ii). Assume that $A(k_1,\dots,k_n)$ is *spr*-realizable. Then for any m, the formula $B(m,k_1,\dots,k_n)$ is *spr*-realizable and by the condition (iiB), $n_B \Vdash_0 B(m,k_1,\dots,k_n)$. It follows that for any m, $\mathsf{e}_1(n_A,\langle m\rangle) \Vdash_0 B(m,k_1,\dots,k_n)$. This means that $n_A \Vdash_0 A(y,k_1,\dots,k_n)$ what we wanted to prove. □

PROPOSITION 3.8. *Suppose that $A(\mathbf{x})$ is an $Ar*$-formula of the form $\exists \mathbf{y}\, \Psi(\mathbf{x},\mathbf{y})$, where $\mathbf{x}$ is the list of variables $x_1,\dots,x_n$, $\mathbf{y}$ is the list of variables $y_1,\dots,y_m$, and $\Psi(\mathbf{x},\mathbf{y})$ is a completely negative quantifier-free Ar^*-formula. Then for any list of natural numbers $\mathbf{k} = k_1,\dots,k_n$, the formula $A(\mathbf{k})$ is spr-realizable iff it is true.*

Proof. Assume that the formula $A(\mathbf{k})$ is *spr*-realizable. This means that there is the list of natural numbers $\mathbf{l} = l_1,\dots,l_m$ such that the formula $\Psi(\mathbf{k},\mathbf{l})$ is *spr*-realizable. By Proposition 3.7, this formula is true. It follows that the formula $A(\mathbf{k})$ is also true. Conversely, if the formula $A(\mathbf{k})$ is true, then there exists the list of natural numbers $\mathbf{l}$ such that the formula $\Psi(\mathbf{k},\mathbf{l})$ is true. By Proposition 3.7, this formula is *spr*-realizable. It follows that the formula $A(\mathbf{k})$ is also realizable. □

§4. An arithmetical theory sound with respect to spr-realizability. Let β be Gödel's Beta Function (see [5]). This ternary function has the following property: for any list of natural numbers $k_0,k_1,\dots,k_n$, there exist natural numbers a and b such that $\beta(a,b,i) = k_i$ for all $i \le n$. The predicate $\beta([x]_0,[x]_1,y) = 0$ is recursively enumerable. By Matiyasevich Theorem, it can be defined in $\mathfrak{N}$ by an arithmetical formula $B(x,y)$ of the form $\exists \mathbf{z}\, \Phi(x,y,\mathbf{z})$, where $\Phi(x,y,\mathbf{z})$ is an atomic arithmetical formula. It follows easily that the predicate $\beta([x]_0,[x]_1,y) = 0$ can be defined in $\mathfrak{N}$ by an Ar-formula of the form $\exists \mathbf{z}\, \Phi(x, y, \mathbf{z})$, where $\Phi(x, y, \mathbf{z})$ is a completely negative quantifier-free Ar-formula. Thus for any natural numbers a and i,

$$\beta([a]_0, [a]_1, i) = 0 \Rightarrow \mathfrak{N} \models B(a, i);$$

$$\beta([a]_0, [a]_1, i) \neq 0 \Rightarrow \mathfrak{N} \models \neg B(a, i).$$

Let a be a natural number. Then $\{a\}$ will denote a unary primitive recursive function g defined as follows: $g(n) = \mathsf{e}_{[a]_0+1}([a]_1, \langle n \rangle)$. Since the predicate $\{x_1\}(x_2) = x_3$ is recursively enumerable, it follows that it can be defined in $\mathfrak{N}$ by an Ar-formula $G(x_1, x_2, x_3)$ of the form $\exists \mathbf{y}\, \Psi(\mathbf{y}, x_1, x_2, x_3)$, where $\Psi(\mathbf{y}, x_1, x_2, x_3)$ is a completely negative quantifier-free Ar-formula.

Definition 4.9. *Let $H(x_1, x_2)$ be the formula $\exists z\, (\neg\neg Z(z) \,\&\, G(x_1, x_2, z))$.*

Obviously, this formula $H(x_1, x_2)$ defines in $\mathfrak{N}$ the predicate $\{x_1\}(x_2) = 0$.

Definition 4.10. *Let Q be the conjunction of the following formulas*:

A_1. $\forall x\, \neg\neg E(x, x)$;
A_2. $\forall x \forall y\, (\neg\neg E(x, y) \to \neg\neg E(y, x))$;
A_3. $\forall x \forall y \forall z\, (\neg\neg E(x, y) \,\&\, \neg\neg E(y, z) \to \neg\neg E(x, z))$;
A_4. $\exists x\, \neg\neg Z(x)$;
A_5. $\forall x \forall y\, (\neg\neg Z(x) \,\&\, \neg\neg Z(y) \to \neg\neg E(x, y))$;
A_6. $\forall x \forall y\, (\neg\neg E(x, y) \,\&\, \neg\neg Z(x) \to \neg\neg Z(y))$;
A_7. $\forall x \exists y\, \neg\neg S(x, y)$;
A_8. $\forall x \forall y \forall z\, (\neg\neg S(x, y) \,\&\, \neg\neg S(x, z) \to \neg\neg E(y, z))$;
A_9. $\forall x_1 \forall x_2 \forall y_1 \forall y_2\, (\neg\neg E(x_1, x_2) \,\&\, \neg\neg E(y_1, y_2) \,\&\, \neg\neg S(x_1, y_1) \to \neg\neg S(x_2, y_2))$;
A_{10}. $\forall x \forall y \forall z\, (\neg\neg S(x, z) \,\&\, \neg\neg S(y, z) \to \neg\neg E(x, y))$;
A_{11}. $\forall x \forall y\, (\neg\neg S(x, y) \to \neg Z(y))$;
A_{12}. $\forall x\, (\neg\neg Z(x) \vee \exists y\, \neg\neg S(y, x))$;
A_{13}. $\forall x \forall y \exists z\, \neg\neg A(x, y, z)$;
A_{14}. $\forall x \forall y \forall z_1 \forall z_2\, (\neg\neg A(x, y, z_1) \,\&\, \neg\neg A(x, y, z_2) \to \neg\neg E(z_1, z_2))$;
A_{15}. $\forall x_1 \forall x_2 \forall y_1 \forall y_2 \forall z_1 \forall z_2\, (\neg\neg E(x_1, x_2) \,\&\, \neg\neg E(y_1, y_2) \,\&\, \neg\neg E(z_1, z_2) \,\&\, \neg\neg A(x_1, y_1, z_1) \to \neg\neg A(x_2, y_2, z_2))$;
A_{16}. $\forall x \forall y\, (\neg\neg Z(y) \to \neg\neg A(x, y, x))$;
A_{17}. $\forall x \forall y \forall z \forall u \forall v\, (\neg\neg S(y, z) \,\&\, \neg\neg A(x, y, u) \,\&\, \neg\neg S(u, v) \to \neg\neg A(x, z, v))$;
A_{18}. $\forall x \forall y \exists z\, \neg\neg M(x, y, z)$;
A_{19}. $\forall x \forall y \forall z_1 \forall z_2\, (\neg\neg M(x, y, z_1) \,\&\, \neg\neg M(x, y, z_2) \to \neg\neg E(z_1, z_2))$;
A_{20}. $\forall x_1 \forall x_2 \forall y_1 \forall y_2 \forall z_1 \forall z_2\, (\neg\neg E(x_1, x_2) \,\&\, \neg\neg E(y_1, y_2) \,\&\, \neg\neg E(z_1, z_2) \,\&\, \neg\neg M(x_1, y_1, z_1) \to \neg\neg M(x_2, y_2, z_2))$;
A_{21}. $\forall x \forall y\, (\neg\neg Z(y) \to \neg\neg M(x, y, y))$;
A_{22}. $\forall x \forall y \forall z \forall u \forall v (\neg\neg S(y, z) \& \neg\neg M(x, y, u) \& \neg\neg A(u, x, v) \to \neg\neg M(x, z, v))$;
A_{23}. $\forall x \forall y \forall z_1 \forall z_2\, (G(x, y, z_1) \,\&\, G(x, y, z_2) \to \neg\neg E(z_1, z_2))$;
A_{24}. $\forall y \forall z\, \neg\neg \exists v \forall x\, (x \leq z \to (\neg\neg B(v, x) \equiv \neg\neg H(y, x)))$;
A_{25}. $\forall x \forall y\, (\neg B(x, y) \vee \neg\neg B(x, y))$.

Theorem 4.11. *The Ar-formula Q is spr-realizable.*

Proof. We prove that everyone of the formulas A_1–A_{25} is *spr*-realizable. All these formulas are evidently true. All of them except A_4, A_7, A_{12}, A_{13}, A_{18}, A_{23}, A_{24}, and A_{25} are completely negative. Therefore they are *spr*-realizable at level 0 by Proposition 3.7

Recall that a is a 0-index of the function $x \mapsto \Lambda x.0$ (see Notation in Section 2).

Consider the formula A_4. Let $a = 3^{\mathsf{a}}$. Obviously, $a \Vdash_0 \exists x \neg\neg Z(x)$.

Consider the formula A_7. The function $f(x) = 2^{x+1} \cdot 3^{\mathsf{a}}$ is in the class $\mathbf{E}_0$. Let a be its 0-index. We prove that $a \Vdash_0 \forall x \exists y \neg\neg S(x, y)$, i.e., $In(0, a)$ holds (this condition is obviously fulfilled) and for any k, $[f(k)]_1 \Vdash_0 \neg\neg S(k, [f(k)]_0)$, i.e., $\mathsf{a} \Vdash_0 \neg\neg S(k, k+1)$, but this easily follows from Proposition 3.4 Thus the formula A_7 is *spr*-realizable.

Consider the formula A_{12}. The function $f(x) = 2^{\mathsf{sg}(x)} \cdot 3^{\overline{\mathsf{sg}}(x)\cdot \mathsf{a} + \mathsf{sg}(x)\cdot 2^{x \dot{-} 1}\cdot 3^{\mathsf{a}}}$ is in the class $\mathbf{E}_0$. Let a be its 0-index. We prove that $a \Vdash_0 \forall x (\neg\neg Z(x) \vee \exists y \neg\neg S(y, x))$, i.e., $In(0, a)$ holds (this condition is fulfilled) and for any k, either $[f(k)]_0 = 0$ and $[f(k)]_1 \Vdash_0 \neg\neg Z(k)$ or $[f(k)]_0 \neq 0$ and $[f(k)]_1 \Vdash_0 \exists y \neg\neg S(y, k)$. Note, that

$$f(k) = \begin{cases} 3^{\mathsf{a}} & \text{if } k = 0, \\ 2 \cdot 3^{2^{k-1}\cdot 3^{\mathsf{a}}} & \text{if } k > 0. \end{cases}$$

If $k = 0$, then $[f(k)]_0 = 0$, $[f(k)]_1 = \mathsf{a}$. In this case, $[f(0)]_1 \Vdash_0 \neg\neg Z(0)$ by Proposition 3.4 because $Z(0)$ is true and *spr*-realizable. If $k > 0$, then $[f(k)]_0 = 1$, $[f(k)]_1 = 2^{k-1} \cdot 3^{\mathsf{a}}$. Since the formula $S(k-1, k)$ is true and *spr*-realizable, it follows by Proposition 3.4, that $\mathsf{a} \Vdash_0 \neg\neg S(k-1, k)$. Therefore $[f(k)]_1 \Vdash_0 \exists y \neg\neg S(y, k)$ what we wanted to prove. Thus the formula A_{12} is *spr*-realizable.

Consider the formula A_{13}. The function $f(x, y) = 2^{x+y} \cdot 3^{\mathsf{a}}$ is in the class $\mathbf{E}_0$. Let a be its 0-index and $g(x) = \langle 7, 1, a, \langle 2, 1, x\rangle, \langle 3, 1, 1\rangle\rangle$. Obviously, $g \in \mathbf{E}_0$. Let b be a 0-index of g. We prove that $b \Vdash_0 \forall x \forall y \exists z \neg\neg A(x, y, z)$, i.e., $In(0, b)$ holds (this condition is fulfilled) and for any k,

$$\mathsf{e}_1(b, \langle k\rangle) \Vdash_0 \forall y \exists z \neg\neg A(k, y, z). \tag{1}$$

Note that $\mathsf{e}_1(b, \langle k\rangle) = g(k) = \langle 7, 1, a, \langle 2, 1, k\rangle, \langle 3, 1, 1\rangle\rangle$. Thus $g(k)$ is a 0-index of the function h obtained by substituting the unary constant function with the only value k and the identity function to the function f, i.e., for any ℓ, $h(\ell) = f(k, \ell)$. Thus (1) means that $f(k, \ell) \Vdash_0 \exists z \neg\neg A(k, \ell, z)$, i.e., $[f(k, \ell)]_1 \Vdash_0 \neg\neg A(k, \ell, [f(k, \ell)]_0)$. Note that $[f(k, \ell)]_0 = k + \ell$, $[f(k, \ell)]_1 = \mathsf{a}$. We have to prove that $\mathsf{a} \Vdash_0 \neg\neg A(k, \ell, k + \ell)$, but this easily follows from Proposition 3.4 Thus the formula A_{13} is *spr*-realizable.

The case of the formula A_{18} is considered in the same way with the function $f(x, y) = 2^{x\cdot y} \cdot 3^{\mathsf{a}}$.

Consider the formula A_{23}. Let natural numbers k, ℓ, m_1, m_2 be fixed. Suppose that the formula $G(k, \ell, m_1) \,\&\, G(k, \ell, m_2)$ is realizable. It follows that $G(k, \ell, m_1)$ and $G(k, \ell, m_2)$ are both *spr*-realizable. By Proposition 3.8, they are true. This means that $\{k\}(\ell) = m_1$ and $\{k\}(\ell) = m_2$. Then $m_1 = m_2$ and by Proposition 3.4, $\mathsf{a} \Vdash_0 \neg\neg E(m_1, m_2)$, $\Lambda w.\mathsf{a} \Vdash_0 G(k, \ell, m_1) \,\&\, G(k, \ell, m_2) \to \neg\neg E(m_1, m_2)$. Then the number $\Lambda x.\Lambda y.\Lambda z_1.\Lambda z_2.\Lambda w.\mathsf{a}$ spr-realizes the formula A_{23} at level 0.

Consider the formula A_{24}. By Proposition 3.5, it is enough to prove that for any fixed natural numbers m and n, the formula

$$\neg\exists v \forall x (x \leq n \to (\neg\neg B(v, x) \equiv \neg\neg H(m, x)))$$

is not *spr*-realizable. We prove that the formula

$$\exists v \forall x (x \leq n \to (\neg\neg B(v, x) \equiv \neg\neg H(m, x))) \tag{2}$$

is *spr*-realizable. Consider the list of natural numbers $k_0, k_1, \ldots, k_n$, where $k_i = 0$ iff $\neg\neg H(m, i)$ is *spr*-realizable, i.e., $\{m\}(i) = 0$. By the property of the function β, there

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exists a natural number a such that $\beta([a]_0,[a]_1,i)=k_i$ for all $i\le n$. Assume that $i\le n$. If $k_i=0$, then both formulas $\neg\neg B(a,i)$ and $\neg\neg H(m,i)$ are *spr*-realizable by a at level 0. By Proposition 3.4, we have

$$\mathsf{a}_{\mathsf{a}} \Vdash_0 (\neg\neg B(a,i)\to\neg\neg H(m,i)) \tag{3}$$

and

$$\mathsf{a}_{\mathsf{a}} \Vdash_0 (\neg\neg H(m,i)\to\neg\neg B(a,i)). \tag{4}$$

If $k_i\ne 0$, then $\neg\neg B(a,i)$ and $\neg\neg H(m,i)$ are not *spr*-realizable. In this case, (3) and (4) hold by Proposition 3.4. It follows that $\mathsf{b}\Vdash_0(\neg\neg B(a,i)\equiv\neg\neg H(m,i))$, where $\mathsf{b}=\langle \mathsf{a}_{\mathsf{a}},\mathsf{a}_{\mathsf{a}}\rangle$. Then by Proposition 3.4, $\mathsf{a}_{\mathsf{b}}\Vdash_0(i\le n\to(\neg\neg B(a,i)\equiv\neg\neg H(m,i)))$ for any i. It follows that $\Lambda x.\mathsf{a}_{\mathsf{b}}\Vdash_0\forall x\,(x\le n\to((\neg\neg B(a,x)\equiv\neg\neg H(m,x)))$. This means that the formula (2) is *spr*-realizable, what we wanted to prove.

Consider the formula A_{25}. Let

$$\phi(x,y)=\mathsf{sg}(\beta([x]_0,[x]_1,y))\cdot 3^{\mathsf{a}}+\overline{\mathsf{sg}}(\beta([x]_0,[x]_1,y))\cdot 2\cdot 3^{\mathsf{a}}.$$

The function ϕ is primitive recursive. Thus there is n such that $\phi\in\mathbf{E}_n$. Let a be an n-index of ϕ and $g(x)=\langle 7,1,a,\langle 2,1,x\rangle,\langle 3,1,1\rangle\rangle$. Obviously, $g\in\mathbf{E}_0$. Let b be a 0-index of g. We prove that $b\Vdash_n\forall x\forall y\,(\neg B(x,y)\vee\neg\neg B(x,y))$, i.e., $In(n,b)$ holds (this condition is fulfilled) and for any k,

$$\mathsf{e}_{n+1}(b,\langle k\rangle)\Vdash_n\forall y\,(\neg B(k,y)\vee\neg\neg B(k,y)). \tag{5}$$

Note that $\mathsf{e}_{n+1}(b,\langle k\rangle)=g(k)=\langle 7,1,a,\langle 2,1,k\rangle,\langle 3,1,1\rangle\rangle$. Thus $g(k)$ is an n-index of the unary function h obtained by substituting the unary constant function with the only value k and the identity function to the function ϕ, i.e., for any ℓ, $h(\ell)=\phi(k,\ell)$. Thus (5) means that $\phi(k,\ell)\Vdash_n(\neg B(k,\ell)\vee\neg\neg B(k,\ell))$, i.e., if $[\phi(k,\ell)]_0=0$, then $[\phi(k,\ell)]_1\Vdash_n\neg B(k,\ell)$, and $[\phi(k,\ell)]_1\Vdash_n\neg\neg B(k,\ell)$ if $[\phi(k,\ell)]_0\ne 0$. Note that $[\phi(k,\ell)]_0=\begin{cases}0 & \text{if } \beta([k]_0,[k]_1,\ell)\ne 0,\\ 1 & \text{if } \beta([k]_0,[k]_1,\ell)=0;\end{cases}$ $[\phi(k,\ell)]_1=\mathsf{a}$. Therefore, if $[\phi(k,\ell)]_0=0$, then $\beta([k]_0,[k]_1,\ell)\ne 0$. In this case, $\neg B(k,\ell)$ is *spr*-realizable. By Proposition 3.4, $\mathsf{a}\Vdash_0\neg B(k,\ell)$, i.e., $[\phi(k,\ell)]_1\Vdash_n\neg B(k,\ell)$. If $[\phi(k,\ell)]_0\ne 0$, then $\beta([k]_0,[k]_1,\ell)=0$. In this case, $\neg\neg B(k,\ell)$ is *spr*-realizable. By Proposition 3.4, $\mathsf{a}\Vdash_0\neg\neg B(k,\ell)$, i.e., $[\phi(k,\ell)]_1\Vdash_n\neg\neg B(k,\ell)$. Thus the formula A_{25} is realizable. □

§5. Strictly primitive recursively realizable predicate formulas. The language of predicate logic consists of individual variables $x_0,x_1,x_2,\ldots$, predicate symbols $P_i^{n_i}$ $(i,n_i\in\mathbb{N})$ ($P_i^{n_i}$ is called an n_i-ary predicate symbol), logical symbols $\neg,\&,\vee,\to,\forall,\exists$, and auxiliary symbols ,, (,). Predicate symbols are also called *predicate variables* and individual variables are called *terms*.

Atomic formulas have the form $P(t_1,\ldots,t_n)$, where P is an n-ary predicate symbol, and $t_1,\ldots,t_n$ are terms.

Sometimes we use a notation $\mathbf{x}$ for the list of individual variables $x_1,\ldots,x_n$.

DEFINITION 5.12. *Predicate formulas are defined inductively as follows*:

1) *atomic formulas are formulas*;
2) *if A is a formula, then $\neg A$ is a formula*;

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3) *if A and B are formulas, then $(A \,\&\, B)$, $(A \vee B)$, and $(A \to B)$ are formulas*;
4) *if A is a formula, x is an individual variable, then $\forall x\, A$ and $\exists x\, A$ are formulas.*

Free and bound occurrences of a variable in a formula are defined as usual. A formula A is *closed* if no variable is free in A. If $A(x_1, \dots, x_n)$ is a formula, then $A(t_1, \dots, t_n)$ denotes *the result of substituting* terms $t_1, \dots, t_n$ for free occurrences of the variables $x_1, \dots, x_n$ in A. An expression $(A \equiv B)$ is an abbreviation for $(A \to B) \,\&\, (B \to A)$.

A predicate formula A will be referred as $A(P_1, \dots, P_n, y_1, \dots, y_m)$ if it does not contain any predicate variables except $P_1, \dots, P_n$ and any free individual variables except $y_1, \dots, y_m$.

Let $A(P_1, \dots, P_n)$ be a predicate formula. We say that the list of Ar^*-formulas $\Phi_1, \dots, \Phi_n$ is *admissible for substituting in A*, if for any $i = 1, \dots, n$, the formula Φ_i does not contain free variables except $x_1, \dots, x_m$, where m is arity of the predicate variable P_i. In this case, $A(\Phi_1, \dots, \Phi_n)$ will denote the result of substituting the formulas $\Phi_1, \dots, \Phi_n$ for the predicate variables $P_1, \dots, P_n$ in A (bound individual variables should be renamed in order to avoid any collision). The formula $A(\Phi_1, \dots, \Phi_n)$ will be referred as an arithmetical instance of a predicate formula $A(P_1, \dots, P_n)$.

A closed predicate formula $A(P_1, \dots, P_n)$ is called strictly primitive recursively realizable (*spr*-realizable) if for any list of Ar^*-formulas $\Phi_1, \dots, \Phi_k$ admissible for substituting in A, the closed Ar^*-formula $A(\Phi_1 \dots \Phi_k)$ is *spr*-realizable.

A predicate formula $A(P_1, \dots, P_n)$ will be called *completely negative* if it does not contain logical symbols $\vee$ and $\exists$ and each predicate variable P_i occurs in A only in subformulas of the form $\neg P_i(y_1, \dots, y_k)$, i.e., an inductive definition of a completely negative predicate formula is the following:

- if A is an atomic formula, then $\neg A$ is a completely negative predicate formula;
- if A and B are completely negative predicate formulas, then $\neg A$, $(A \,\&\, B)$, and $(A \to B)$ are completely negative predicate formulas;
- if A is a completely negative predicate formula, x is an individual variable, then $\forall x\, A$ is a completely negative predicate formula.

PROPOSITION 5.13. *If $F(P_1, \dots, P_n, y_1, \dots, y_m)$ is a completely negative predicate formula, then there exists a natural number a_F such that for any list of Ar^*-formulas $\Phi_1, \dots, \Phi_n$ admissible for substituting in F and any natural $k_1, \dots, k_m$, the following conditions are equivalent*:

1) *the formula $F(\Phi_1, \dots, \Phi_n, k_1, \dots, k_m)$ is spr-realizable*;
2) $a_F \Vdash_0 F(\Phi_1, \dots, \Phi_n, k_1, \dots, k_m)$.

Proof. Induction on the complexity of a completely negative predicate formula F. Let F be of the form $\neg P(y_1, \dots, y_m)$, where P is an m-ary predicate variable, $\Phi(x_1, \dots, x_m)$ is an Ar^*-formula admissible for substituting in F, and $k_1, \dots, k_m$ are natural numbers. Then $F(\Phi_1, \dots, \Phi_n, k_1, \dots, k_m)$ is $\neg\Phi(k_1, \dots, k_m)$. By Proposition 3.4, in this case we can define a_F as a.

Let $F(P_1, \dots, P_n, y_1, \dots, y_m)$ be of the form

$$G(P_1, \dots, P_n, y_1, \dots, y_m) \,\&\, H(P_1, \dots, P_n, y_1, \dots, y_m),$$

and for the formulas G are H the corresponding numbers a_G and a_H are defined. Then one can define a_F as $\langle a_G, a_H \rangle$.

If $F(P_1, \dots, P_n, y_1, \dots, y_m)$ is of the form

$$G(P_1, \dots, P_n, y_1, \dots, y_m) \to H(P_1, \dots, P_n, y_1, \dots, y_m),$$

we can define a_F as a_{a_H}. Let $\Phi_1, \dots, \Phi_n$ be the list of Ar^*-formulas admissible for substituting in F and $k_1, \dots, k_m$ be natural numbers. Assume that the formula $F(\Phi_1, \dots, \Phi_n, k_1, \dots, k_m)$ is *spr*-realizable. We prove that

$$a_F \Vdash_0 F(\Phi_1, \dots, \Phi_n, k_1, \dots, k_m).$$

If $G(\Phi_1, \dots, \Phi_n, k_1, \dots, k_m)$ is not *spr*-realizable, then by Proposition 3.4,

$$a_F \Vdash_0 G(\Phi_1, \dots, \Phi_n, k_1, \dots, k_m) \to H(\Phi_1, \dots, \Phi_n, k_1, \dots, k_m). \tag{6}$$

If $G(\Phi_1, \dots, \Phi_n, k_1, \dots, k_m)$ is *spr*-realizable, then $H(\Phi_1, \dots, \Phi_n, k_1, \dots, k_m)$ is *spr*-realizable. By the inductive hypothesis, $a_H \Vdash_0 H(\Phi_1, \dots, \Phi_n, k_1, \dots, k_m)$. Then, by Proposition 3.4, (6) holds.

If F is $\neg G(P_1, \dots, P_n, y_1, \dots, y_m)$, then for any list of Ar^*-formulas $\Phi_1, \dots, \Phi_n$ admissible for substituting in F and any $k_1, \dots, k_m$, $\neg G(\Phi_1, \dots, \Phi_n, k_1, \dots, k_m)$ can be replaced by $G(\Phi_1, \dots, \Phi_n, k_1, \dots, k_m) \to E(0, 1)$. Thus this case is reduced to the case of implication and we can define a_F as a.

If F is $\forall y\, G(P_1, \dots, P_n, y, y_1, \dots, y_m)$ and the number a_G is defined for the formula $G(P_1, \dots, P_n, y, y_1, \dots, y_m)$, then we define a_F as $\Lambda y.a_G$. The fact that this number satisfies the conclusion of the theorem is proved by the same argument as in the proof of Proposition 3.7 □

§6. Simulating arithmetic in the predicate language. Let A be an Ar-formula. Then A can be considered as a predicate formula if we treat the predicate symbols as predicate variables. In this case, we refer to A as a predicate Ar-formula.

The predicate Ar-formula Q defined in Section 4 does not contain any predicate variables except Z, S, A, M, E; thus it will be denoted $Q(Z, S, A, M, E)$. The list of Ar-formulas $\mathbf{\Phi} = \mathcal{Z}(x_1), \mathcal{S}(x_1, x_2), \mathcal{A}(x_1, x_2, x_3), \mathcal{M}(x_1, x_2, x_3), \mathcal{E}(x_1, x_2)$ admissible for substituting in $Q(Z, S, A, M, E)$ will be called *an interpretation* of Q iff the Ar-formula $Q(\mathcal{Z}, \mathcal{S}, \mathcal{A}, \mathcal{M}, \mathcal{E})$ is *spr*-realizable.

Assume that an interpretation $\mathbf{\Phi}$ of the formula Q is fixed. If $F(Z, S, A, M, E)$ is a predicate formula, then the Ar-formula $F(\mathcal{Z}, \mathcal{S}, \mathcal{A}, \mathcal{M}, \mathcal{E})$ will be denoted $\widetilde{F}$. This is the meaning we attach to the expression $\widetilde{Q}$.

Proposition 6.14. *If $\mathbf{\Phi}$ is an interpretation of Q, then for any completely negative Ar-formula $\Psi(x_1, \dots, x_n)$, the Ar-formula*

$$\forall \mathbf{x}, \mathbf{y}\, (\neg\neg\mathcal{E}(x_1, y_1) \,\&\, \cdots \,\&\, \neg\neg\mathcal{E}(x_n, y_n) \,\&\, \widetilde{\Psi}(\mathbf{x}) \to \widetilde{\Psi}(\mathbf{y})), \tag{7}$$

where $\mathbf{x}$ is the list $x_1, \dots, x_n$ and $\mathbf{y}$ is $y_1, \dots, y_n$ is spr-realizable.

Proof. See Appendix A. □

For a natural number n, a predicate Ar-formula $[n](x)$ is defined inductively:

- $[0](x)$ is $\neg\neg Z(x)$;
- $[n+1](x)$ is $\neg Z(x) \,\&\, \forall y\, (\neg\neg S(y, x) \to [n](y))$.

Note that for any n, the formula $[n](x)$ is completely negative.

PROPOSITION 6.15. *If* $\mathbf{\Phi}$ *is an interpretation of* Q, *then for any* n, *the* Ar*-formula*

$$\forall x \forall y\, (\widetilde{[n]}(x)\,\&\,\widetilde{[n]}(y) \to \neg\neg\mathcal{E}(x,y)) \tag{8}$$

is spr-realizable.

Proof. We prove spr-realizability of (8) by induction on n. If $n = 0$, then $[n](x)$ is $\neg\neg Z(x)$. Thus we have to prove that $\forall x \forall y\, (\neg\neg\mathcal{Z}(x)\,\&\,\neg\neg\mathcal{Z}(y) \to \neg\neg\mathcal{E}(x,y))$ is *spr*-realizable. But this is the formula $\widetilde{A_5}$, which is *spr*-realizable because $\mathbf{\Phi}$ is an interpretation of Q. Now assume that the formula (8) is *spr*-realizable for a given n. We verify that the formula

$$\forall x \forall y\, (\widetilde{[n+1]}(x)\,\&\,\widetilde{[n+1]}(y) \to \neg\neg\mathcal{E}(x,y)) \tag{9}$$

is *spr*-realizable. First we prove that for any k, ℓ, the formula

$$\widetilde{[n+1]}(k)\,\&\,\widetilde{[n+1]}(\ell) \to \neg\neg\mathcal{E}(k,\ell) \tag{10}$$

is *spr*-realizable. Assume that the premise of the formula (10) is *spr*-realizable. Then $\neg\mathcal{Z}(k)\,\&\,\forall y\,(\neg\neg\mathcal{S}(y,k) \to \widetilde{[n]}(y))$ and $\neg\mathcal{Z}(\ell)\,\&\,\forall y\,(\neg\neg\mathcal{S}(y,\ell) \to \widetilde{[n]}(y))$ are *spr*-realizable. We prove that the formula $\neg\neg\mathcal{E}(k,\ell)$ is *spr*-realizable. Since the formula $\widetilde{A_{12}}$ is *spr*-realizable, it follows that the formula $\exists y\, \neg\neg\mathcal{S}(y,k)$ is *spr*-realizable. This means that there exists a natural number p such that $\neg\neg\mathcal{S}(p,k)$ is realizable. It follows that the formula $\widetilde{[n]}(p)$ is *spr*-realizable. By the same reason, there exists a natural number q such that the formulas $\neg\neg\mathcal{S}(q,\ell)$ and $\widetilde{[n]}(q)$ are *spr*-realizable. By the inductive hypothesis, the formula $\neg\neg\mathcal{E}(p,q)$ is *spr*-realizable. Since the formulas $\neg\neg\mathcal{E}(k,k)$ and $\widetilde{A_9}$ are *spr*-realizable, it follows that the formula $\neg\neg\mathcal{S}(q,k)$ is realizable. Since the formula $\widetilde{A_8}$ is *spr*-realizable, it follows that the formula $\neg\neg\mathcal{E}(k,\ell)$ is *spr*-realizable. By Proposition 3.4, $\mathsf{a} \Vdash_0 \neg\neg\mathcal{E}(k,\ell)$ and $\mathsf{a_a}$ spr-realizes the formula (10) at level 0. If the premise of the formula (10) is not *spr*-realizable, then by Proposition 3.4, $\mathsf{a_a}$ spr-realizes the formula (10). Thus we have proved that for any k, ℓ the number $\mathsf{a_a}$ spr-realizes the formula (10). Now it is evident that the number $\Lambda x.\Lambda y.\mathsf{a_a}$ spr-realizes the formula (9). □

PROPOSITION 6.16. *If* $\mathbf{\Phi}$ *is an interpretation of* Q *and* $e \Vdash_m \widetilde{Q}$, *then for any natural numbers* k *and* n *we can effectively determine whether the* Ar^**-formula* $\widetilde{[n]}(k)$ *is spr-realizable.*

Proof. The proof is by induction on n. If $n = 0$, then $\widetilde{[n]}(k)$ is the formula $\neg\neg\widetilde{\mathcal{Z}(k)}$. Let a natural number k be given. We can extract an spr-realization of the formula $\widetilde{A_{12}}$ from e. Thus we can determine which of the formulas $\neg\neg\widetilde{\mathcal{Z}(k)}$ and $\exists y\, \neg\neg\mathcal{S}(y,k)$ is *spr*-realizable. If the first one is *spr*-realizable, then the formula $\widetilde{[0]}(k)$ is *spr*-realizable. If the formula $\exists y\, \neg\neg\mathcal{S}(y,k)$ is *spr*-realizable, then there exists a natural number ℓ such that the formula $\neg\neg\mathcal{S}(\ell,k)$ is *spr*-realizable. Since the formula $\widetilde{A_{11}}$ is *spr*-realizable, it follows that $\neg\mathcal{Z}(k)$ is *spr*-realizable. This means that the formula $\widetilde{[0]}(k)$ is not *spr*-realizable. Now suppose that for a given n and any ℓ we can effectively determine whether the formula $\widetilde{[n]}(\ell)$ is *spr*-realizable. Let a natural number k be given and we want to determine whether the formula $\widetilde{[n+1]}(k)$, i.e., $\neg\widetilde{\mathcal{Z}(k)}\,\&\,\forall y\,(\neg\neg\mathcal{S}(y,k) \to \widetilde{[n]}(y))$, is realizable. Using an spr-realization of the formula $\widetilde{A_{12}}$ we determine which of the formulas $\neg\neg\mathcal{Z}(k)$ and $\exists y\, \neg\neg\widetilde{\mathcal{S}(y,k)}$ is *spr*-realizable. If the formula $\neg\neg\mathcal{Z}(k)$ is *spr*-realizable, then the formula $\widetilde{[n+1]}(k)$ is not *spr*-realizable. Assume that the formula $\exists y\, \neg\neg\mathcal{S}(y,k)$ is *spr*-realizable. Then we have a natural number ℓ such that the formula $\neg\neg\mathcal{S}(\ell,k)$ is *spr*-realizable. We determine whether the formula $\widetilde{[n]}(\ell)$ is *spr*-realizable. If

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this formula is not realizable, then obviously the formula $\widetilde{[n+1]}(k)$ is not *spr*-realizable. Assume that the formula $\widetilde{[n]}(\ell)$ is *spr*-realizable. We show that the formula $\widetilde{[n+1]}(k)$ is *spr*-realizable. It is enough to prove that the formula $\forall y\,(\neg\neg\mathcal{S}(y,k) \to \widetilde{[n]}(y))$ is *spr*-realizable. First we prove that for any p the formula $\neg\neg\mathcal{S}(p,k) \to \widetilde{[n]}(p)$ is *spr*-realizable. First consider the case when the formula $\neg\neg\mathcal{S}(p,k)$ is *spr*-realizable. Recall that the formula $\neg\neg\mathcal{S}(\ell,k)$ is *spr*-realizable as well. Since the formula $\widetilde{A_{10}}$ is *spr*-realizable, it follows that the formula $\neg\neg\mathcal{E}(\ell,p)$ is *spr*-realizable. Recall that the formula $\widetilde{[n]}(\ell)$ is *spr*-realizable. Since the formula $[n](x)$ is completely negative, it follows from Proposition 6.14 that the formula $\widetilde{[n]}(p)$ is *spr*-realizable. By Proposition 5.13, $a_{[n](x)} \Vdash_0 \widetilde{[n]}(p)$. By Proposition 3.4,

$$\mathsf{a}_{a_{[n](x)}} \Vdash_0 \neg\neg\mathcal{S}(p,k) \to \widetilde{[n]}(p). \tag{11}$$

If the formula $\neg\neg\mathcal{S}(p,k)$ is not *spr*-realizable, then (11) holds by Proposition 3.4 Thus (11) holds for any p. Then $\Lambda y.\mathsf{a}_{a_{[n](x)}} \Vdash_0 \forall y\,(\neg\neg\mathcal{S}(y,k) \to \widetilde{[n]}(y))$. This means that the formula $\widetilde{[n+1]}(k)$ is *spr*-realizable. □

Proposition 6.17. *If* $\mathbf{\Phi}$ *is an interpretation of* Q, *then there exists a primitive recursive function* f *such that for any natural number* n, *the* Ar^**-formula* $\widetilde{[n]}(\mathsf{f}(n))$ *is spr-realizable.*

Proof. Let $\mathbf{\Phi}$ be an interpretation of the formula Q. So, there exist natural numbers e and m such that $e \Vdash_m \widetilde{Q}$. It follows from the definition of spr-realizability, that the formula $\widetilde{A_4}$ is realizable at level m. Thus, there exists a natural number a_4 such that $a_4 \Vdash_m \exists x\,\neg\neg\mathcal{Z}(x)$. Then

$$[a_4]_1 \Vdash_m \neg\neg\mathcal{Z}([a_4]_0). \tag{12}$$

Let $\mathsf{f}(0) = [a_4]_0$. Note that by Proposition 3.4, we have $\mathsf{a} \Vdash_0 \widetilde{[0]}(\mathsf{f}(0))$.

Suppose that the value $\mathsf{f}(n)$ is defined for a given n. Since $\widetilde{A_7}$ is *spr*-realizable, it follows that there exists a natural number a_7 such that $a_7 \Vdash_m \forall x \exists y\,\neg\neg\mathcal{S}(x,y)$. Then $\mathsf{e}_{m+1}(a_7, \langle \mathsf{f}(n)\rangle) \Vdash_m \exists y\,\neg\neg\mathcal{S}(\mathsf{f}(n), y)$. Therefore,

$$[\mathsf{e}_{m+1}(a_7, \langle \mathsf{f}(n)\rangle)]_1 \Vdash_m \neg\neg\mathcal{S}(\mathsf{f}(n), [\mathsf{e}_{m+1}(a_7, \langle \mathsf{f}(n)\rangle)]_0).$$

Let $\mathsf{f}(n+1) = [\mathsf{e}_{m+1}(a_7, \langle \mathsf{f}(n)\rangle)]_0$. Note that the formula

$$\neg\neg\mathcal{S}(\mathsf{f}(n), \mathsf{f}(n+1)) \tag{13}$$

is *spr*-realizable. Thus we have the following definition of the function f:

$$\begin{cases} \mathsf{f}(0) = [a_4]_0; \\ \mathsf{f}(n+1) = [\mathsf{e}_{m+1}(a_7, \langle \mathsf{f}(n)\rangle)]_0, \end{cases}$$

where the numbers a_4, a_7, and m depend only on the formula $\widetilde{Q}$. The function e_{m+1} is primitive recursive. Obviously, the function f is primitive recursive.

Let us prove that for any n, the formula $\widetilde{[n]}(\mathsf{f}(n))$ is *spr*-realizable. The proof is by induction on n. If $n = 0$, this condition follows from (12) because $\neg\neg\mathcal{Z}(x)$ is $\widetilde{[0]}(x)$ and $\mathsf{f}(0) = [a_4]_0$. Now suppose that for a given n, the formula $\widetilde{[n]}(\mathsf{f}(n))$ is *spr*-realizable. Then by Proposition 5.13, $a_F \Vdash_0 \widetilde{[n]}(\mathsf{f}(n))$, where F is $[n](x)$. We prove that the formula $\widetilde{[n+1]}(\mathsf{f}(n+1))$ is *spr*-realizable. This means that the formulas

$$\neg\mathcal{Z}(\mathsf{f}(n+1)) \tag{14}$$

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and

$$\forall y\,(\neg\neg\mathcal{S}(y, \mathsf{f}(n+1)) \to \widetilde{[n]}(y)) \tag{15}$$

are *spr*-realizable. Since the formulas (13) and $\widetilde{A_{11}}$ are *spr*-realizable, it follows that the formula (14) is *spr*-realizable. Now we prove that the formula (15) is *spr*-realizable. First we prove that for any k, the formula

$$\neg\neg\mathcal{S}(k, \mathsf{f}(n+1)) \to \widetilde{[n]}(k) \tag{16}$$

is *spr*-realizable. Assume that the premise of the formula (16), i.e., $\neg\neg\mathcal{S}(k, \mathsf{f}(n+1))$, is *spr*-realizable. Since the formulas (13) and $\widetilde{A_{10}}$ are *spr*-realizable, it follows that the formula $\neg\neg\mathcal{E}(\mathsf{f}(n), k)$ is *spr*-realizable. By the inductive hypothesis, $\widetilde{[n]}(\mathsf{f}(n))$ is *spr*-realizable. By Proposition 6.14, $\widetilde{[n]}(k)$ is *spr*-realizable and by Proposition 5.13, $a_F \Vdash_0 \widetilde{[n]}(k)$. By Proposition 3.4, the number a_{a_F} spr-realizes the formula (16) at level 0. If the premise of the formula (16) is not *spr*-realizable, then by Proposition 3.4, the number a_{a_F} spr-realizes the formula (16) at level 0. Thus we have proved that for any k, the number a_{a_F} spr-realizes the formula (16) at level 0. Now it is evident that $\Lambda x.\mathsf{a}_{a_F}$ spr-realizes the formula (15) at level 0. □

PROPOSITION 6.18. *If $\mathbf{\Phi}$ is an interpretation of Q, then for any natural numbers n, m, and a, if the Ar^*-formulas $\widetilde{[n]}(a)$ and $\widetilde{[m]}(a)$ are spr-realizable, then $m = n$.*

Proof. Assume that $\widetilde{[n]}(a)$ and $\widetilde{[m]}(a)$ are *spr*-realizable. We prove that $n < m$ is impossible by induction on n. Let $n = 0$, $m = k+1$. Thus we have that the formulas $\widetilde{[0]}(a)$, i.e., $\neg\neg\mathcal{Z}(a)$, and $\widetilde{[k+1]}(a)$, i.e., $\neg\mathcal{Z}(a)\,\&\,\forall y\,(\neg\neg\mathcal{S}(y,a) \to \widetilde{[k]}(y))$, are *spr*-realizable. This is obviously impossible. Now let $m = k+1 > n+1$, and assume that the formulas $\widetilde{[n+1]}(a)$, i.e., $\neg\mathcal{Z}(a)\,\&\,\forall y\,(\neg\neg\mathcal{S}(y,a) \to \widetilde{[n]}(y))$, and $\widetilde{[k+1]}(a)$, i.e., $\neg\mathcal{Z}(a)\,\&\,\forall y\,(\neg\neg\mathcal{S}(y,a) \to \widetilde{[k]}(y))$, are *spr*-realizable. Since the formulas $\neg\mathcal{Z}(a)$ and $\widetilde{A_{12}}$ are *spr*-realizable, it follows that there exists a number b such that the formula $\neg\neg\mathcal{S}(b,a)$ is *spr*-realizable. Then the formulas $\widetilde{[n]}(b)$ and $\widetilde{[k]}(b)$ are *spr*-realizable. This is impossible by the inductive hypothesis. □

For any n, let $\tilde{n}$ denote the number $\mathsf{f}(n)$.

PROPOSITION 6.19. *Let $\mathbf{\Phi}$ be an interpretation of Q. Then for any quantifier-free completely negative Ar-formula $\Psi(y_1,\dots,y_m)$ without any free variables except $y_1,\dots,y_m$ and any natural numbers $k_1,\dots,k_m$, the Ar^*-formula $\Psi(k_1,\dots,k_m)$ is true iff the Ar^*-formula $\widetilde{\Psi}(\tilde{k}_1,\dots,\tilde{k}_m)$ is spr-realizable.*

Proof. See Appendix B. □

Recall that $\{a\}$ denotes the function $g(n) = \mathsf{e}_{[a]_0+1}([a]_1, \langle n\rangle)$ and the Ar-formula $G(x_1,x_2,x_3)$ of the form $\exists\mathbf{y}\,\Psi(\mathbf{y},x_1,x_2,x_3)$, where $\Psi(\mathbf{y},x_1,x_2,x_3)$ is a completely negative quantifier-free Ar-formula, defines in $\mathfrak{N}$ the predicate $\{x_1\}(x_2) = x_3$.

PROPOSITION 6.20. *If $\mathbf{\Phi}$ is an interpretation of Q, then for any natural numbers e, n, and k, the Ar^*-formula $\widetilde{G}(\tilde{e},\tilde{n},\tilde{k})$ is spr-realizable iff $\{e\}(n) = k$.*

Proof. Assume that $\{e\}(n) = k$. Then the formula $G(e,n,k)$ is true. This means that there exists the list of natural numbers $\mathbf{b} = b_1,\dots,b_m$ such that the quantifier-free Ar^*-formula $\Psi(\mathbf{b},e,n,k)$ is true. By Proposition 6.19, the formula $\widetilde{\Psi}(\widetilde{\mathbf{b}},\tilde{e},\tilde{n},\tilde{k})$, where $\widetilde{\mathbf{b}} = \widetilde{b}_1,\dots,\widetilde{b}_m$, is *spr*-realizable. Then the formula $\exists\mathbf{y}\,\widetilde{\Psi}(y,\tilde{e},\tilde{n},\tilde{k})$, i.e., $\widetilde{G}(\tilde{e},\tilde{n},\tilde{k})$, is *spr*-realizable.

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Conversely, assume that the formula $\widetilde{G}(\tilde{e},\tilde{n},\tilde{k})$ is *spr*-realizable. Obviously, there exists a natural number ℓ such that $\{e\}(n)=\ell$. Then the formula $G(e,n,\ell)$ is true. Arguing as above, we conclude that the formula $\widetilde{G}(\tilde{e},\tilde{n},\tilde{\ell})$ is realizable. Since the formula $\widetilde{A_{23}}$ is *spr*-realizable, it follows that the formula $\neg\neg\mathcal{E}(\tilde{\ell},\tilde{k})$ is realizable. By Proposition 6.17, the formula $\widetilde{[k]}(\tilde{k})$ is *spr*-realizable. It follows from Proposition 6.14 that the formula $\widetilde{[k]}(\tilde{\ell})$ is *spr*-realizable. On the other hand, the formula $\widetilde{[\ell]}(\tilde{\ell})$ is *spr*-realizable. Then, by Proposition 6.18, $k=\ell$. Thus we have proved that $\{e\}(n)=k$. □

Recall that $H(x_1,x_2)$ is the formula $\exists z\,(\neg\neg Z(z)\,\&\,G(x_1,x_2,z))$.

Proposition 6.21. *If* $\mathbf{\Phi}$ *is an interpretation of* Q, *then for any natural numbers* e *and* n, *the* Ar^**-formula* $\widetilde{H}(\tilde{e},\tilde{n})$ *is spr-realizable iff* $\{e\}(n)=0$.

Proof. Assume that $\{e\}(n)=0$. Then by Proposition 6.21, $\widetilde{G}(\tilde{e},\tilde{n},\tilde{0})$ is *spr*-realizable. On the other hand, $\neg\neg\mathcal{Z}(\tilde{0})$ is also *spr*-realizable. Then the formula $\neg\neg\mathcal{Z}(\tilde{0})\,\&\,\widetilde{G}(\tilde{e},\tilde{n},\tilde{0})$ is *spr*-realizable. It follows that $\exists z\,(\neg\neg\mathcal{Z}(z)\,\&\,\widetilde{G}(\tilde{e},\tilde{n},z))$, i.e., $\widetilde{H}(\tilde{e},\tilde{n})$, is realizable. Conversely, assume that the formula $\widetilde{H}(\tilde{e},\tilde{n})$ is *spr*-realizable. Then there is a natural number k such that the formulas $\neg\neg\mathcal{Z}(k)$, i.e., $\widetilde{[0]}(k)$, and $\widetilde{G}(\tilde{e},\tilde{n},k)$ are *spr*-realizable. Since the formula $\widetilde{[0]}(\tilde{0})$ is also *spr*-realizable, it follows from Proposition 6.15 that the formula $\neg\neg\mathcal{E}(k,\tilde{0})$ is *spr*-realizable. Then it follows from Proposition 6.14 that the formula $\widetilde{G}(\tilde{e},\tilde{n},\tilde{0})$ is *spr*-realizable. By Proposition 6.21, $\{e\}(n)=0$. □

Definition 6.22. *Assume that an interpretation* $\mathbf{\Phi}$ *of the formula* Q *is fixed. A natural number* a *will be called* $\mathbf{\Phi}$*-standard if there exists a natural number* n *such that the formula* $\widetilde{[n]}(a)$ *is spr-realizable.*

Note that the numbers $\tilde{0},\tilde{1},\tilde{2},\ldots$ are $\mathbf{\Phi}$-standard.

Proposition 6.23. *If a natural number* a *is* $\mathbf{\Phi}$*-standard and* b *is a natural number such that the* Ar^**-formula* $\neg\neg\mathcal{S}(a,b)$ *is spr-realizable, then the number* b *is* $\mathbf{\Phi}$*-standard.*

Proof. Assume that the formula $\widetilde{[n]}(a)$ is *spr*-realizable. Since the predicate *Ar*-formula $[n](x)$ is completely negative, it follows from Proposition 5.13 that there exists a number d, namely $a_{[n](x)}$, such that $d\Vdash_0\widetilde{[n]}(a)$. Assume that b is a natural number such that the formula $\neg\neg\mathcal{S}(a,b)$ is *spr*-realizable. We prove that the formula $\widetilde{[n+1]}(b)$, i.e., $\neg\mathcal{Z}(b)\,\&\,\forall y\,(\neg\neg\mathcal{S}(y,b)\to\widetilde{[n]}(y))$, is *spr*-realizable. Since the formula $\widetilde{A_{11}}$ is *spr*-realizable, it follows that the formula $\neg\mathcal{Z}(b)$ is *spr*-realizable. We prove that the formula $\forall y\,(\neg\neg\mathcal{S}(y,b)\to\widetilde{[n]}(y))$ is *spr*-realizable at level 0. Let c be an arbitrary natural number. We prove that

$$\mathsf{a}_d\Vdash_0\neg\neg\mathcal{S}(c,b)\to\widetilde{[n]}(c).\tag{17}$$

If the formula $\neg\neg\mathcal{S}(c,b)$ is not *spr*-realizable, then (17) holds by Proposition 3.4 Assume that the formula $\neg\neg\mathcal{S}(c,b)$ is *spr*-realizable. Since the formulas $\neg\neg\mathcal{S}(a,b)$, $\neg\neg\mathcal{S}(c,b)$, and $\widetilde{A_{10}}$ are *spr*-realizable, it follows that the formula $\neg\neg\mathcal{E}(a,c)$ is *spr*-realizable. Since the formula $\widetilde{[n]}(a)$ is realizable, it follows from Proposition 6.14 that the formula $\widetilde{[n]}(c)$ is *spr*-realizable. By Proposition 5.13, $d\Vdash_0\widetilde{[n]}(c)$. Now (17) follows from Proposition 3.4 Then $\Lambda y.\mathsf{a}_d\Vdash_0\forall y\,(\neg\neg\mathcal{S}(y,b)\to\widetilde{[n]}(y))$. □

Definition 6.24. *Let* $x\leq y$ *be the formula* $\exists z\,\neg\neg A(z,x,y)$.

Proposition 6.25. *Suppose that* $\mathbf{\Phi}$ *is an interpretation of the formula* Q. *For any natural numbers* n *and* b, *if* b *is not* $\mathbf{\Phi}$*-standard, then the* Ar^**-formula* $\tilde{n}\widetilde{\leq}b$ *is spr-realizable.*

Proof. Induction on n. Let $n = 0$. Since the formula $\widetilde{A_{16}}$ is *spr*-realizable, it follows that the formula $\neg\neg\mathcal{Z}(\tilde{0}) \to \neg\neg\mathcal{A}(b, \tilde{0}, b)$ is *spr*-realizable. Since the formula $\neg\neg\mathcal{Z}(\tilde{0})$ is realizable, it follows that the formula $\neg\neg\mathcal{A}(b, \tilde{0}, b)$ is *spr*-realizable. Therefore the formula $\exists z\, \neg\neg\mathcal{A}(z, \tilde{0}, b)$, i.e., $\tilde{0}\widetilde{\leq}b$, is *spr*-realizable.

Now suppose that for any natural number b which is not $\mathbf{\Phi}$-standard, the formula $\tilde{n}\widetilde{\leq}b$ is *spr*-realizable. We have to prove that for any such number b, the formula $\widetilde{n+1}\widetilde{\leq}b$ is *spr*-realizable. Since the number b is not $\mathbf{\Phi}$-standard, it follows that the formula $\neg\neg\mathcal{Z}(b)$ is not *spr*-realizable. Since the formula $\widetilde{A_{12}}$ is realizable, it follows that the formula $\exists y\, \neg\neg\mathcal{S}(y, b)$ is *spr*-realizable. This means that there exists a natural number a such that the formula $\neg\neg\mathcal{S}(a, b)$ is *spr*-realizable. By Proposition 6.23, the number a cannot be $\mathbf{\Phi}$-standard. By the inductive hypothesis, the formula $\exists z\, \neg\neg\mathcal{A}(z, \tilde{n}, a)$ is *spr*-realizable. This means that there exists a natural number c such that the formula $\neg\neg\mathcal{A}(c, \tilde{n}, a)$ is *spr*-realizable. Since the formula $\neg\neg S(n, n+1)$ is true, it follows from Proposition 6.19 that the formula $\neg\neg\mathcal{S}(\tilde{n}, \widetilde{n+1})$ is realizable. Since the formula $\widetilde{A_{17}}$ is *spr*-realizable, it follows that the formula $\neg\neg\mathcal{A}(c, \widetilde{n+1}, b)$ is *spr*-realizable. Therefore the formula $\exists z\, \neg\neg\mathcal{A}(z, \widetilde{n+1}, b)$, i.e., $\widetilde{n+1}\widetilde{\leq}b$, is *spr*-realizable. □

§7. Non-arithmeticity of the *spr*-realizability.

THEOREM 7.26. *If* $\mathbf{\Phi}$ *is an interpretation of the formula* Q, *then every natural number is* $\mathbf{\Phi}$-*standard.*

Proof. Let an interpretation $\mathbf{\Phi}$ of Q be given. Then $\widetilde{A_{25}}$ is *spr*-realizable. This means that there are natural numbers q and p such that $q \Vdash_p \forall x \forall y\, (\neg\tilde{B}(x, y) \vee \neg\neg\tilde{B}(x, y))$. Then for any k, ℓ, we have $\mathsf{e}_{p+1}(\mathsf{e}_{p+1}(q, \langle k\rangle)), (\langle \ell\rangle) \Vdash_p \neg\tilde{B}(k, \ell) \vee \neg\neg\tilde{B}(k, \ell))$, i.e., if $[\mathsf{e}_{p+1}(\mathsf{e}_{p+1}(q, \langle k\rangle)), (\langle \ell\rangle)]_0 = 0$, then the formula $\neg\tilde{B}(k, \ell)$ is *spr*-realizable, else the formula $\neg\neg\tilde{B}(k, \ell)$ is *spr*-realizable. Let a unary function g be defined as follows:

$$g(n) = \begin{cases} 0 & \text{if the formula } \neg\tilde{B}(n, \tilde{n}) \text{ is } spr\text{-realizable,} \\ 1 & \text{otherwise.} \end{cases}$$

By Proposition 6.17, the number $\tilde{n}$ can be found from n by means of a primitive recursive function f. Thus we can represent the function g in the following form:

$$g(n) = \mathsf{sg}((\mathsf{e}_{p+1}([\mathsf{e}_{p+1}(q, \langle n\rangle), \langle \mathsf{f}(n)\rangle]_0).$$

It is obvious that the function g is primitive recursive. Therefore, there exists a natural number e such that for all n, $g(n) = \mathsf{e}_{[e]_0+1}(\langle [e]_1, n\rangle)$.

Since $\mathbf{\Phi}$ is an interpretation of the formula Q, it follows that the formula $\widetilde{A_{24}}$, i.e.,

$$\forall y \forall z\, \neg\neg\exists v \forall x\, (x\widetilde{\leq}z \to (\neg\neg\widetilde{B}(v, x) \equiv \neg\neg\widetilde{H}(y, x))), \tag{18}$$

is *spr*-realizable. Suppose that there exists a natural number b which is not $\mathbf{\Phi}$-standard. Since the formula (18) is *spr*-realizable, it follows that the formula

$$\neg\neg\exists v \forall x\, (x\widetilde{\leq}b \to (\neg\neg\widetilde{B}(v, x) \equiv \neg\neg\widetilde{H}(\tilde{e}, x))) \tag{19}$$

is *spr*-realizable. Now suppose that the formula

$$\exists v \forall x\, (x\widetilde{\leq}b \to (\neg\neg\widetilde{B}(v, x) \equiv \neg\neg\widetilde{H}(\tilde{e}, x))) \tag{20}$$

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is *spr*-realizable. Then there exists a natural number a such that the formula

$$\forall x\,(x\widetilde{\leq}b \rightarrow (\neg\neg\widetilde{B}(a,x) \equiv \neg\neg\widetilde{H}(\tilde{e},x)))$$

is *spr*-realizable. It follows that for any n, the formula

$$\tilde{n}\widetilde{\leq}b \rightarrow (\neg\neg\widetilde{B}(a,\tilde{n}) \equiv \neg\neg\widetilde{H}(\tilde{e},\tilde{n}))$$

is *spr*-realizable. By Proposition 6.25, the formula $\tilde{n}\widetilde{\leq}b$ is *spr*-realizable. Then for any n, the formula $\neg\neg\widetilde{B}(a,\tilde{n}) \equiv \neg\neg\widetilde{H}(\tilde{e},\tilde{n}))$ is *spr*-realizable. Thus we have that for any n,

(a) the formula $\neg\neg\widetilde{B}(a,\tilde{n})$ is *spr*-realizable iff the formula $\neg\neg\widetilde{H}(\tilde{e},\tilde{n})$ is realizable.

On the other hand, by Proposition 6.21, for any n,

(b) the formula $\neg\neg\widetilde{H}(\tilde{e},\tilde{n})$ is *spr*-realizable iff $g(n) = 0$.

Further, by the definition of the function g, we have

(c) $g(n) = 0$ iff the formula $\neg\widetilde{B}(n,\tilde{n})$ is *spr*-realizable.

It follows from the statements (a), (b), and (c) that for a given natural number a and any natural number n, the following equivalence holds: the formula $\neg\neg\widetilde{B}(a,\tilde{n})$ is *spr*-realizable iff the formula $\neg\widetilde{B}(n,\tilde{n})$ is *spr*-realizable, and we have a contradiction if $n = a$. This contradiction means that the formula (20) is not *spr*-realizable. Then its negation is *spr*-realizable contrary to the fact that the formula (19) is *spr*-realizable. Thus we have proved that there exists no natural number which is not $\mathbf{\Phi}$-standard. □

PROPOSITION 7.27. *If $\mathbf{\Phi}$ is an interpretation of Q and $e \Vdash_m \widetilde{Q}$, then for any natural number k, we can effectively find a natural number n such that the formula $\widetilde{[n]}(k)$ is spr-realizable.*

Proof. Assume that $\mathbf{\Phi}$ is an interpretation of Q and numbers e and m such that $e \Vdash_m \widetilde{Q}$ are given. Let a natural number k be given. By Proposition 6.16, for any natural number n we can effectively determine whether the formula $\widetilde{[n]}(k)$ is *spr*-realizable. Sequentially iterating over natural numbers starting from 0, we find n such that the formula $\widetilde{[k]}(n)$ is *spr*-realizable because otherwise the number k should be not $\mathbf{\Phi}$-standard in contradiction with Theorem 7.26. □

PROPOSITION 7.28. *Suppose that $\mathbf{\Phi}$ is an interpretation of the formula Q. Then for any completely negative Ar-formula $\Psi(y_1,\dots,y_n)$ without any free variables except $y_1,\dots,y_n$ and any natural numbers $k_1,\dots,k_n$, the Ar^*-formula $\Psi(k_1,\dots,k_n)$ is true iff the Ar^*-formula $\widetilde{\Psi}(\widetilde{k_1},\dots,\widetilde{k_n})$ is spr-realizable.*

Proof. Induction on the logical complexity of a completely negative Ar-formula $\Psi(y_1,\dots,y_n)$.

The case of an atomic formula $\Psi(y_1,\dots,y_n)$ is considered in Proposition 6.19 The proof of Proposition 6.19 contains also a consideration of the cases when the formula Ψ is of the form $\Psi_1 \,\&\, \Psi_2$, $\Psi_1 \rightarrow \Psi_2$ or $\neg\Psi_1$ and the statement is true for the formulas Ψ_1 and Ψ_2.

Assume that $\Psi(y_1,\dots,y_n)$ is of the form $\forall y\,\Psi_0(y,y_1,\dots,y_n)$. Then $\widetilde{\Psi}(y_1,\dots,y_n)$ is the formula $\forall y\,\widetilde{\Psi_0}(y,y_1,\dots,y_n)$. The inductive hypothesis states that for any natural numbers $k,k_1,\dots,k_n$, the Ar^*-formula $\Psi_0(k,k_1,\dots,k_n)$ is true iff the Ar^*-formula $\widetilde{\Psi_0}(\tilde{k},\widetilde{k_1},\dots,\widetilde{k_n})$ is realizable. Assume that $\Psi(k_1,\dots,k_n)$ is true. It follows that for any natural number k, the formula $\Psi_0(k,k_1,\dots,k_n)$ is true and by the inductive hypothesis, the formula $\widetilde{\Psi_0}(\tilde{k},\widetilde{k_1},\dots,\widetilde{k_n})$ is *spr*-realizable. We prove that $\forall y\,\widetilde{\Psi_0}(y,\widetilde{k_1},\dots,\widetilde{k_n})$ is *spr*-realizable. Let ℓ be an arbitrary natural number. By Theorem 7.26, the number ℓ is $\mathbf{\Phi}$-standard. This means that there exists a natural number k such that $\widetilde{[k]}(\ell)$ is *spr*-realizable. (Moreover, by Proposition 7.27, we can find k effectively.) It

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follows from Proposition 6.15 that $\mathcal{E}(\tilde{k}, \ell)$ is *spr*-realizable. By Proposition 6.14, $\mathcal{E}(\tilde{k}, \ell) \,\&\, \widetilde{\Psi}_0(\tilde{k}, \tilde{k}_1, \dots, \tilde{k}_n) \to \widetilde{\Psi}_0(\ell, \tilde{k}_1, \dots, \tilde{k}_n)$ is *spr*-realizable. Since the premise of this formula is *spr*-realizable, it follows that its conclusion $\widetilde{\Psi}_0(\ell, \tilde{k}_1, \dots, \tilde{k}_n)$ is *spr*-realizable. Note that Ψ_0 is completely negative. By Proposition 5.13, there exists a number a_{Ψ_0} such that $a_{\Psi_0} \Vdash_0 \widetilde{\Psi}_0(\ell, \tilde{k}_1, \dots, \tilde{k}_n)$ for any number ℓ. Then $\Lambda y.a_{\Psi_0} \Vdash_0 \forall y \widetilde{\Psi}_0(y, \tilde{k}_1, \dots, \tilde{k}_n)$, i.e., the formula $\widetilde{\Psi}(\tilde{k}_1, \dots, \tilde{k}_n)$ is *spr*-realizable, what we wanted to prove.

Conversely, assume that $\widetilde{\Psi}(\tilde{k}_1, \dots, \tilde{k}_n)$ is *spr*-realizable. This implies that for any natural number k, the formula $\widetilde{\Psi}_0(\tilde{k}, \tilde{k}_1, \dots, \tilde{k}_n)$ is *spr*-realizable. By the inductive hypothesis, for any k, the formula $\Psi_0(k, k_1, \dots, k_n)$ is true. This means that the formula $\Psi(k_1, \dots, k_n)$ is true. □

THEOREM 7.29. *For any closed completely negative Ar-formula Ψ one can effectively construct a closed predicate formula Ψ^* such that Ψ^* is spr-realizable iff Ψ is true.*

Proof. If Ψ is a closed completely negative Ar-formula, let Ψ^* be the predicate Ar-formula $Q \to \Psi$. We show that Ψ^* is the required predicate formula.

CLAIM I. *If the predicate formula Ψ^* is spr-realizable, then the Ar-formula Ψ is true.*

Proof. Assume that Ψ^* is *spr*-realizable. This means that for any list of Ar-formulas $\mathbf{\Phi}$ admissible for substituting in Ψ^*, the formula $\widetilde{\Psi^*}$ is *spr*-realizable. In particular, this holds if $\mathbf{\Phi}$ is the list of the formulas $Z(x)$, $S(x, y)$, $A(x, y, z)$, $M(x, y, z)$, and $E(x, y)$. In this case, $\widetilde{\Psi^*}$ is just the formula $Q \to \Psi$. By Theorem 4.11, the formula Q is *spr*-realizable. Thus the formula Ψ is also *spr*-realizable. Since Ψ is a completely negative formula, it follows from Proposition 3.7 that Ψ is true, what we wanted to prove. $\square_{\text{I}}$

Claim II. *If a closed completely negative Ar-formula Ψ is true, then the predicate formula Ψ^* is spr-realizable.*

Proof. Assume that a closed completely negative Ar-formula Ψ is true. We prove that the predicate Ar-formula Ψ^* is *spr*-realizable. Assume that the list of Ar-formulas $\mathbf{\Phi}$ is admissible for substituting in Ψ^*. Since Ψ is completely negative, it follows from Proposition 5.13 that there exists a number a_Ψ such that the formula $\widetilde{\Psi}$ is *spr*-realizable iff $a_\Psi \Vdash_0 \widetilde{\Psi}$. We prove that $\Lambda n.\Lambda y.a_\Psi \Vdash_0 \widetilde{\Psi^*}$. By the definition of spr-realizability, it is enough to prove that for any n and any b such that $b \Vdash_n \widetilde{Q}$, we have $a_\Psi \Vdash_n \widetilde{\Psi}$. Assume that $b \Vdash_n \widetilde{Q}$. This means that $\mathbf{\Phi}$ is an interpretation of the formula Q. By Proposition 7.28, the formula $\widetilde{\Psi}$ is *spr*-realizable because the formula Ψ is true. As it was remarked above, in this case $a_\Psi \Vdash_0 \widetilde{\Psi}$. It follows that $a_\Psi \Vdash_n \widetilde{\Psi}$, what we wanted to prove. $\square_{\text{II}}$

Claims I and II yield the statement of Theorem 7.29.

THEOREM 7.30. *The set of spr-realizable predicate formulas is not arithmetical.*

Proof. It is obvious that the set of true completely negative closed Ar-formulas is recursively isomorphic to the set of true closed Ar-formulas which is not arithmetical by Tarski's Undefinability Theorem. It follows that the set of true completely negative closed Ar-formulas is not arithmetical too. By Theorem 7.29, the set of true completely negative closed Ar-formulas is 1-1-reducible to the set of *spr*-realizable predicate formulas. It follows that the set of *spr*-realizable predicate formulas is not arithmetical. □

§A. Appendix: Proof of Proposition 6.14. Let

$$\mathbf{\Phi} = \mathcal{Z}(x_1), \mathcal{S}(x_1, x_2), \mathcal{A}(x_1, x_2, x_3), \mathcal{M}(x_1, x_2, x_3), \mathcal{E}(x_1, x_2)$$

be an interpretation of the formula Q. Note that the formula $\widetilde{\Psi}(x_1, \dots, x_n)$ is built from the formulas $\neg\mathcal{E}(x_i, x_j)$, $\neg\mathcal{Z}(x_i)$, $\neg\mathcal{S}(x_i, x_j)$, $\neg\mathcal{A}(x_i, x_j, x_k)$, and $\neg\mathcal{M}(x_i, x_j, x_k)$, where $1 \le i, j, k \le n$, using the logical symbols $\neg$, &, $\to$, and $\forall$. Note that (7) is an arithmetical instance of the completely negative predicate Ar-formula

$$\forall \mathbf{x}, \mathbf{y}\, (\neg\neg E(x_1, y_1) \,\&\, \cdots \,\&\, \neg\neg E(x_n, y_n) \,\&\, \Psi(\mathbf{x}) \to \Psi(\mathbf{y})). \tag{A.1}$$

By Proposition 5.13, there exists a number that spr-realizes at level 0 every *spr*-realizable arithmetical instance of (A.1). Thus it is enough to prove that the formula (7) is *spr*-realizable at level 0 without paying any attention to the construction of a concrete spr-realization.

We prove that the formula (7) is *spr*-realizable by induction on the complexity of $\Psi(x_1, \dots, x_n)$.

1) If $\Phi(x_1, \dots, x_n)$ is $\neg E(x_i, x_j)$, where $1 \le i, j \le n$, then $\widetilde{\Psi}(x_1, \dots, x_n)$ is $\neg\mathcal{E}(x_i, x_j)$ and in this case the formula (7) is

$$\forall \mathbf{x}, \mathbf{y}\, (\neg\neg\mathcal{E}(x_1, y_1) \,\&\, \cdots \,\&\, \neg\neg\mathcal{E}(x_n, y_n) \,\&\, \neg\mathcal{E}(x_i, x_j) \to \neg\mathcal{E}(y_i, y_j)). \tag{A.2}$$

First we prove that for any $k_1, \dots, k_n, \ell_1, \dots, \ell_n$, the Ar^*-formula

$$\neg\neg\mathcal{E}(k_1, \ell_1) \,\&\, \cdots \,\&\, \neg\neg\mathcal{E}(k_n, \ell_n) \,\&\, \neg\mathcal{E}(k_i, k_j) \to \neg\mathcal{E}(\ell_i, \ell_j) \tag{A.3}$$

is *spr*-realizable.

Suppose $i = j$. Since the formula $\widetilde{A_1}$ is *spr*-realizable, it follows that $\neg\neg\mathcal{E}(k_i, k_j)$ is *spr*-realizable. In this case, the premise of the formula (A.3) is not *spr*-realizable, and by Proposition 3.4, the number $\mathsf{a_a}$ spr-realizes the formula (A.3) at level 0.

Now suppose $i \ne j$. If the premise of the formula (A.3) is not *spr*-realizable, then by Proposition 3.4, the number $\mathsf{a_a}$ spr-realizes the formula (A.3) at level 0. Suppose that the premise of the formula (A.3) is *spr*-realizable. Then the formulas $\neg\neg\mathcal{E}(k_i, \ell_i)$, $\neg\neg\mathcal{E}(k_j, \ell_j)$, and $\neg\mathcal{E}(k_i, k_j)$ are *spr*-realizable. We prove that the conclusion of the formula (A.3), i.e., $\neg\mathcal{E}(\ell_i, \ell_j)$, is *spr*-realizable as well. By Proposition 3.4, it is enough to prove that the formula $\neg\neg\mathcal{E}(\ell_i, \ell_j)$ is not *spr*-realizable. Suppose that this formula is *spr*-realizable. Thus we have that the formulas $\neg\neg\mathcal{E}(k_i, \ell_i)$, $\neg\neg\mathcal{E}(k_j, \ell_j)$, and $\neg\neg\mathcal{E}(\ell_i, \ell_j)$ are *spr*-realizable. Since the formula $\widetilde{A_3}$ is *spr*-realizable, it follows that the formula $\neg\neg\mathcal{E}(k_i, \ell_j)$ is also *spr*-realizable. Since the formula $\widetilde{A_2}$ is *spr*-realizable, it follows that the formula $\neg\neg\mathcal{E}(\ell_j, k_j)$ is *spr*-realizable. Since the formula $\widetilde{A_3}$ is *spr*-realizable, it follows that the formula $\neg\neg\mathcal{E}(k_i, kj)$ is also *spr*-realizable in contradiction with the assumption that the formula $\neg\mathcal{E}(k_i, k_j)$ is *spr*-realizable. Thus $\neg\mathcal{E}(\ell_i, \ell_j)$ is *spr*-realizable. By Proposition 3.4, $\mathsf{a} \Vdash_0 \neg\mathcal{E}(\ell_i, \ell_j)$ and the number $\mathsf{a_a}$ spr-realizes the formula (A.3) at level 0. Thus we have proved that for any $k_1, \dots, k_n, \ell_1, \dots, \ell_n$, the number $\mathsf{a_a}$ spr-realizes (A.3) at level 0. Evidently, the number $\Lambda x_1.\dots\Lambda x_n.\Lambda y_1.\dots\Lambda y_n.\mathsf{a_a}$ spr-realizes the formula (A.2) at level 0.

2) If $\Psi(x_1, \dots, x_n)$ is $\neg Z(x_i)$, where $1 \le i \le n$, then $\widetilde{\Psi}(x_1, \dots, x_n)$ is $\neg\mathcal{Z}(x_i)$ and in this case the formula (7) is

$$\forall \mathbf{x}, \mathbf{y}\, (\neg\neg\mathcal{E}(x_1, y_1) \,\&\, \cdots \,\&\, \neg\neg\mathcal{E}(x_n, y_n) \,\&\, \neg\mathcal{Z}(x_i) \to \neg\mathcal{Z}(y_i)). \tag{A.4}$$

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First we prove that for any $k_1, \dots, k_n, \ell_1, \dots, \ell_n$, the Ar^*-formula

$$\neg\neg\mathcal{E}(k_1, \ell_1) \,\&\, \cdots \,\&\, \neg\neg\mathcal{E}(k_n, \ell_n) \,\&\, \neg\mathcal{Z}(k_i) \to \neg\mathcal{Z}(\ell_i) \tag{A.5}$$

is *spr*-realizable.

If the premise of the formula (A.5) is not *spr*-realizable, then by Proposition 3.4, the number a_a spr-realizes the formula (A.5) at level 0. Suppose that the premise of the formula (A.5) is *spr*-realizable. Then the formulas $\neg\neg\mathcal{E}(k_i, \ell_i)$ and $\neg\mathcal{Z}(k_i)$ are *spr*-realizable. We prove that the conclusion of the formula (A.5), i.e., $\neg\mathcal{Z}(\ell_i)$, is *spr*-realizable as well. By Proposition 3.4, it is enough to prove that the formula $\neg\neg\mathcal{Z}(\ell_i)$ is not *spr*-realizable. Suppose that this formula is *spr*-realizable. Thus we have that the formulas $\neg\neg\mathcal{E}(k_i, \ell_i)$ and $\neg\neg\mathcal{Z}(\ell_i)$ are *spr*-realizable. Since the formula $\widetilde{A_2}$ is *spr*-realizable, it follows that the formula $\neg\neg\mathcal{E}(\ell_i, k_i)$ is also *spr*-realizable. Note that the formula $\widetilde{A_6}$ is *spr*-realizable. It follows that $\neg\neg\mathcal{E}(\ell_i, k_i) \,\&\, \neg\neg\mathcal{Z}(\ell_i) \to \neg\neg\mathcal{Z}(k_i)$ is realizable. Since the premise of this implication is *spr*-realizable, it follows that its conclusion $\neg\neg\mathcal{Z}(k_i)$ is realizable in contradiction with the assumption that the formula $\neg\mathcal{Z}(k_i)$ is *spr*-realizable. Thus $\neg\mathcal{Z}(\ell_i)$ is *spr*-realizable. By Proposition 3.4, $\mathsf{a} \Vdash_0 \neg\mathcal{Z}(\ell_i)$ and the number a_a spr-realizes (A.5) at level 0. We have proved that for any $k_1, \dots, k_n, \ell_1, \dots, \ell_n$, the number a_a spr-realizes (A.5) at level 0. Evidently, the number $\Lambda x_1. \dots \Lambda x_n. \Lambda y_1. \dots \Lambda y_n. \mathsf{a}_\mathsf{a}$ spr-realizes the formula (A.4) at level 0.

3) If $\Psi(x_1, \dots, x_n x)$ is $\neg S(x_i, x_j)$, where $1 \le i, j \le n$, then $\widetilde{\Psi}(x_1, \dots, x_n)$ is $\neg\mathcal{S}(x_i, x_j)$ and in this case the formula (7) is

$$\forall \mathbf{x}, \mathbf{y}\, (\neg\neg\mathcal{E}(x_1, y_1) \,\&\, \cdots \,\&\, \neg\neg\mathcal{E}(x_n, y_n) \,\&\, \neg\mathcal{S}(x_i, x_j) \to \neg\mathcal{S}(y_i, y_j)). \tag{A.6}$$

First we prove that for any $k_1, \dots, k_n, \ell_1, \dots, \ell_n$, the formula

$$\neg\neg\mathcal{E}(k_1, \ell_1) \,\&\, \cdots \,\&\, \neg\neg\mathcal{E}(k_n, \ell_n) \,\&\, \neg\mathcal{S}(k_i, k_j) \to \neg\mathcal{S}(\ell_i, \ell_j) \tag{A.7}$$

is *spr*-realizable.

If the premise of the formula (A.7) is not *spr*-realizable, then by Proposition 3.4, the number a_a spr-realizes the formula (A.7) at level 0. Suppose that the premise of the formula (A.7) is *spr*-realizable. Then the formulas $\neg\neg\mathcal{E}(k_i, \ell_i)$, $\neg\neg\mathcal{E}(k_j, \ell_j)$, and $\neg\mathcal{S}(k_i, k_j)$ are *spr*-realizable. We prove that the conclusion of the formula (A.7), i.e., $\neg\mathcal{S}(\ell_i, \ell_j)$, is *spr*-realizable as well. By Proposition 3.4, it is enough to prove that the formula $\neg\neg\mathcal{S}(\ell_i, \ell_j)$ is not *spr*-realizable. Suppose that this formula is *spr*-realizable. Thus we have that the formulas $\neg\neg\mathcal{E}(k_i, \ell_i)$, $\neg\neg\mathcal{E}(k_j, \ell_j)$, and $\neg\neg\mathcal{S}(\ell_i, \ell_j)$ are *spr*-realizable. Since the formula $\widetilde{A_2}$ is *spr*-realizable, it follows that the formulas $\neg\neg\mathcal{E}(\ell_i, k_i)$ and $\neg\neg\mathcal{E}(\ell_j, k_j)$ are also *spr*-realizable. Note that the formula $\widetilde{A_9}$ is *spr*-realizable. It follows that $\neg\neg\mathcal{E}(\ell_i, k_i) \,\&\, \neg\neg\mathcal{E}(\ell_j, k_j) \,\&\, \neg\neg\mathcal{S}(\ell_i, \ell_j) \to \neg\neg\mathcal{S}(k_i, k_j)$ is spr- realizable. Since the premise of this implication is *spr*-realizable, it follows that its conclusion $\neg\neg\mathcal{S}(k_i, k_j)$ is *spr*-realizable in contradiction with the assumption that $\neg\mathcal{S}(k_i, k_j)$ is *spr*-realizable. Thus $\neg\mathcal{S}(\ell_i, \ell_j)$ is *spr*-realizable. By Proposition 3.4, $\mathsf{a} \Vdash_0 \neg\mathcal{S}(\ell_i, \ell_j)$ and the number a_a spr-realizes (A.7) at level 0. We have proved that for any $k_1, \dots, k_n, \ell_1, \dots, \ell_n$, the number a_a spr-realizes (A.7) at level 0. Evidently, the number $\Lambda x_1. \dots \Lambda x_n. \Lambda y_1. \dots \Lambda y_n. \mathsf{a}_\mathsf{a}$ spr-realizes the formula (A.6) at level 0.

4) If $\Psi(x_1, \dots, x_n)$ is the formula $\neg A(x_i, x_j, x_m)$, where $1 \le i, j, m \le n$, then $\widetilde{\Psi}(x_1, \dots, x_n)$ is $\neg\mathcal{A}(x_i, x_j, x_m)$ and in this case the formula (7) is

$$\forall \mathbf{x}, \mathbf{y}\, (\neg\neg\mathcal{E}(x_1, y_1) \,\&\, \cdots \,\&\, \neg\neg\mathcal{E}(x_n, y_n) \,\&\, \neg\mathcal{A}(x_i, x_j, x_m) \to \neg\mathcal{A}(y_i, y_j, y_m)). \tag{A.8}$$

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First we prove that for any $k_1, \dots, k_n, \ell_1, \dots, \ell_n$ the formula

$$\neg\neg\mathcal{E}(k_1,\ell_1)\,\&\cdots\&\,\neg\neg\mathcal{E}(k_n,\ell_n)\,\&\,\neg\mathcal{A}(k_i,k_j,k_m) \to \neg\mathcal{A}(\ell_i,\ell_j,\ell_m) \tag{A.9}$$

is *spr*-realizable.

If the premise of the formula (A.9) is not *spr*-realizable, then by Proposition 3.4, the number $\mathsf{a_a}$ spr-realizes the formula (A.9) at level 0. Suppose that the premise of the formula (A.9) is *spr*-realizable. Then the formulas $\neg\neg\mathcal{E}(k_i,\ell_i)$, $\neg\neg\mathcal{E}(k_j,\ell_j)$, $\neg\neg\mathcal{E}(k_m,\ell_m)$, and $\neg\mathcal{A}(k_i,k_j,k_m)$ are realizable. We prove that the conclusion of the formula (A.9), i.e., $\neg\mathcal{A}(\ell_i,\ell_j,\ell_m)$, is realizable as well. By Proposition 3.4, it is enough to prove that the formula $\neg\neg\mathcal{A}(\ell_i,\ell_j,\ell_m)$ is not realizable. Suppose that this formula is *spr*-realizable. Thus we have that the formulas $\neg\neg\mathcal{E}(k_i,\ell_i)$, $\neg\neg\mathcal{E}(k_j,\ell_j)$, $\neg\neg\mathcal{E}(k_m,\ell_m)$, and $\neg\neg\mathcal{A}(\ell_i,\ell_j,\ell_m)$ are *spr*-realizable. Since the formula $\widetilde{A_2}$ is *spr*-realizable, it follows that the formulas $\neg\neg\mathcal{E}(\ell_i,k_i)$, $\neg\neg\mathcal{E}(\ell_j,k_j)$, and $\neg\neg\mathcal{E}(\ell_m,k_m)$ are also *spr*-realizable. Note that the formula $\widetilde{A_{15}}$ is *spr*-realizable. It follows that the formula

$$\neg\neg\mathcal{E}(\ell_i,k_i)\,\&\,\neg\neg\mathcal{E}(\ell_j,k_j)\,\&\,\neg\neg\mathcal{E}(\ell_m,k_m)\,\&\,\neg\neg\mathcal{A}(\ell_i,\ell_j,\ell_m) \to \neg\neg\mathcal{A}(k_i,k_j,k_m)$$

is *spr*-realizable. Since the premise of this implication is *spr*-realizable, it follows that its conclusion $\neg\neg\mathcal{A}(k_i,k_j,k_m)$ is *spr*-realizable in contradiction with the assumption that $\neg\mathcal{A}(k_i,k_j,k_m)$ is *spr*-realizable. Thus $\neg\mathcal{A}(\ell_i,\ell_j,\ell_m)$ is *spr*-realizable. By Proposition 3.4, we have $\mathsf{a} \Vdash_0 \neg\mathcal{A}(\ell_i,\ell_j,\ell_m)$ and the number $\mathsf{a_a}$ spr-realizes (A.9) at level 0. We have proved that for any $k_1, \dots, k_n, \ell_1, \dots, \ell_n$, the number $\mathsf{a_a}$ spr-realizes (A.9) at level 0. Evidently, the number $\Lambda x_1.\dots\Lambda x_n.\Lambda y_1.\dots\Lambda y_n.\mathsf{a_a}$ spr-realizes the formula (A.8) at level 0.

5) If $\Psi(x_1,\dots,x_n)$ is the formula $\neg M(x_i,x_j,x_m)$, where $1 \le i,j,m \le n$, then $\widetilde{\Psi}(x_1,\dots,x_n)$ is $\neg\mathcal{M}(x_i,x_j,x_m)$ and in this case the formula (7) is

$$\forall \mathbf{x},\mathbf{y}(\neg\neg\mathcal{E}(x_1,y_1)\&\cdots\&\neg\neg\mathcal{E}(x_n,y_n)\&\neg\mathcal{M}(x_i,x_j,x_m) \to \neg\mathcal{M}(y_i,y_j,y_m)). \tag{A.10}$$

spr-realizability of the formula (A.10) is proved by the same argument as in the case of the formula (A.8); only one should replace $\mathcal{A}$ by $\mathcal{M}$ and use the formula $\widetilde{A_{20}}$ instead of $\widetilde{A_{15}}$.

6) If $\Psi(x_1,\dots,x_n)$ is $\neg\Psi_0(x_1,\dots,x_n)$, then $\widetilde{\Psi}(x_1,\dots,x_n)$ is $\neg\widetilde{\Psi_0}(x_1,\dots,x_n)$. By the inductive hypothesis, the formula

$$\forall \mathbf{x},\mathbf{y}\,(\neg\neg\mathcal{E}(x_1,y_1)\,\&\cdots\&\,\neg\neg\mathcal{E}(x_n,y_n)\,\&\,\widetilde{\Psi_0}(\mathbf{x}) \to \widetilde{\Psi_0}(\mathbf{y})) \tag{A.11}$$

is *spr*-realizable. We have to prove that the formula

$$\forall \mathbf{x},\mathbf{y}\,(\neg\neg\mathcal{E}(x_1,y_1)\,\&\cdots\&\,\neg\neg\mathcal{E}(x_n,y_n)\,\&\,\neg\widetilde{\Psi_0}(\mathbf{x}) \to \neg\widetilde{\Psi_0}(\mathbf{y})) \tag{A.12}$$

is *spr*-realizable.

First we prove that for any $k_1, \dots, k_n, \ell_1, \dots, \ell_n$ the formula

$$\neg\neg\mathcal{E}(k_1,\ell_1)\,\&\cdots\&\,\neg\neg\mathcal{E}(k_n,\ell_n)\,\&\,\neg\widetilde{\Psi_0}(\mathbf{k}) \to \neg\widetilde{\Psi_0}(\mathbf{l})), \tag{A.13}$$

where $\mathbf{k}$ is the list $k_1,\dots,k_n$ and $\mathbf{l}$ is the list $\ell_1,\dots,\ell_n$, is *spr*-realizable.

If the premise of (A.13) is not *spr*-realizable, then by Proposition 3.4, the number $\mathsf{a_a}$ spr-realizes the formula (A.13) at level 0. Suppose that the premise of the formula (A.13) is *spr*-realizable. Then the formulas $\neg\neg\mathcal{E}(k_i,\ell_i)(i = 1,\dots,n)$ and $\neg\widetilde{\Psi_0}(\mathbf{k})$ are *spr*-realizable. We prove that the conclusion of the formula (A.13), i.e., $\neg\widetilde{\Psi_0}(\mathbf{l})$, is *spr*-realizable as well. By Proposition 3.4, it is enough to prove that the formula $\widetilde{\Psi_0}(\mathbf{l})$ is not *spr*-realizable. Suppose the opposite. Thus we have that the formulas $\neg\neg\mathcal{E}(k_i,\ell_i)(i =$

$1, \ldots, n)$ and $\widetilde{\Psi}_0(\mathbf{l})$ are *spr*-realizable. Since $\widetilde{A_2}$ is *spr*-realizable, it follows that the formulas $\neg\neg\mathcal{E}(\ell_i, k_i)(i = 1, \ldots, n)$ are also *spr*-realizable. Since the formula (A.11) is *spr*-realizable, then it follows that the formula $\neg\neg\mathcal{E}(\ell_1, k_1) \mathbin{\&} \cdots \mathbin{\&} \neg\neg\mathcal{E}(\ell_n, k_n) \mathbin{\&} \widetilde{\Psi}_0(\mathbf{l}) \to \widetilde{\Psi}_0(\mathbf{k})$ is *spr*-realizable. Since the premise of this implication is *spr*-realizable, it follows that its conclusion $\widetilde{\Psi}_0(\mathbf{k})$ is *spr*-realizable in contradiction with the assumption that the formula $\neg\widetilde{\Psi}_0(\mathbf{k})$ is *spr*-realizable. Thus $\neg\widetilde{\Psi}_0(\mathbf{l})$ is *spr*-realizable. By Proposition 3.4, $\mathsf{a} \Vdash_0 \neg\widetilde{\Psi}_0(\mathbf{l})$ and the number a_a spr-realizes (A.13) at level 0. We have proved that for any $k_1, \ldots, k_n, \ell_1, \ldots, \ell_n$, the number a_a spr-realizes (A.13) at level 0. Evidently, the number $\Lambda x_1. \ldots \Lambda x_n.\Lambda y_1. \ldots \Lambda y_n.\mathsf{a}$ spr-realizes the formula (A.12) at level 0.

7) If $\Psi(x_1, \ldots, x_n)$ is $\Psi_1(x_1, \ldots, x_n) \mathbin{\&} \Psi_2(x_1, \ldots, x_n)$, then $\widetilde{\Psi}(x_1, \ldots, x_n)$ is the formula $\widetilde{\Psi}_1(x_1, \ldots, x_n) \mathbin{\&} \widetilde{\Psi}_2(x_1, \ldots, x_n)$. By the inductive hypothesis, the formulas

$$\forall \mathbf{x}, \mathbf{y}\, (\neg\neg\mathcal{E}(x_1, y_1) \mathbin{\&} \cdots \mathbin{\&} \neg\neg\mathcal{E}(x_n, y_n) \mathbin{\&} \widetilde{\Psi}_1(\mathbf{x}) \to \widetilde{\Psi}_1(\mathbf{y})) \tag{A.14}$$

and

$$\forall \mathbf{x}, \mathbf{y}\, (\neg\neg\mathcal{E}(x_1, y_1) \mathbin{\&} \cdots \mathbin{\&} \neg\neg\mathcal{E}(x_n, y_n) \mathbin{\&} \widetilde{\Psi}_2(\mathbf{x}) \to \widetilde{\Psi}_2(\mathbf{y})) \tag{A.15}$$

are *spr*-realizable. We have to prove that the formula

$$\forall \mathbf{x}, \mathbf{y}\, (\neg\neg\mathcal{E}(x_1, y_1) \mathbin{\&} \cdots \mathbin{\&} \neg\neg\mathcal{E}(x_n, y_n) \mathbin{\&} \widetilde{\Psi}_1(\mathbf{x}) \mathbin{\&} \widetilde{\Psi}_2(\mathbf{x}) \to \widetilde{\Psi}_1(\mathbf{y}) \mathbin{\&} \widetilde{\Psi}_2(\mathbf{y})) \tag{A.16}$$

is *spr*-realizable.

First we prove that for any $k_1, \ldots, k_n, \ell_1, \ldots, \ell_n$ the formula

$$\neg\neg\mathcal{E}(k_1, \ell_1) \mathbin{\&} \cdots \mathbin{\&} \neg\neg\mathcal{E}(k_n, \ell_n) \mathbin{\&} \widetilde{\Psi}_1(\mathbf{k}) \mathbin{\&} \widetilde{\Psi}_2(\mathbf{k}) \to \widetilde{\Psi}_1(\mathbf{l}) \mathbin{\&} \widetilde{\Psi}_2(\mathbf{l}) \tag{A.17}$$

is *spr*-realizable. Suppose that the premise of (A.17) is *spr*-realizable. Then the formulas $\neg\neg\mathcal{E}(k_i, \ell_i)(i = 1, \ldots, n)$ and $\widetilde{\Psi}_1(\mathbf{k}) \mathbin{\&} \widetilde{\Psi}_2(\mathbf{k})$ are *spr*-realizable. By Proposition 3.4, the formulas $\widetilde{\Psi}_1(\mathbf{k})$ and $\widetilde{\Psi}_2(\mathbf{k})$ are *spr*-realizable. We prove that the conclusion of (A.17), i.e., $\widetilde{\Psi}_1(\mathbf{l}) \mathbin{\&} \widetilde{\Psi}_2(\mathbf{l})$, is *spr*-realizable. By Proposition 3.4, it is enough to prove that $\widetilde{\Psi}_1(\mathbf{l})$ and $\widetilde{\Psi}_2(\mathbf{l})$ are realizable. But this follows from spr-realizability of (A.14) and (A.15). By Proposition 5.13, $a_{\Psi_i} \Vdash_0 \Psi_i(\ell_1, \ldots, \ell_n)$ for $i = 1, 2$. It follows that $\langle a_{\Psi_1}, a_{\Psi_2}\rangle \Vdash_0 \Psi_1(\ell_1, \ldots, \ell_n) \mathbin{\&} \Psi_2(\ell_1, \ldots, \ell_n)$ and $\mathsf{a}_{\langle a_{\Psi_1}, a_{\Psi_2}\rangle}$ spr-realizes (A.17) at level 0. If the premise of (A.17) is not *spr*-realizable, then by Proposition 3.4, the number $\mathsf{a}_{\langle a_{\Psi_1}, a_{\Psi_2}\rangle}$ spr-realizes (A.17) at level 0. We see that for any $k_1, \ldots, k_n, \ell_1, \ldots, \ell_n$, the number $\mathsf{a}_{\langle a_{\Psi_1}, a_{\Psi_2}\rangle}$ spr-realizes the formula (A.17) at level 0. Evidently, the number $\Lambda x_1. \ldots \Lambda x_n.\Lambda y_1. \ldots \Lambda y_n.\mathsf{a}_{\langle a_{\Psi_1}, a_{\Psi_2}\rangle}$ spr-realizes the formula (A.16) at level 0.

8) Assume that $\Psi(x_1, \ldots, x_n)$ is the formula $\Psi_1(x_1, \ldots, x_n) \to \Psi_2(x_1, \ldots, x_n)$ and the formulas (A.14) and (A.15) are *spr*-realizable. We have to prove that the formula

$$\forall \mathbf{x}, \mathbf{y}(\neg\neg\mathcal{E}(x_1, y_1) \mathbin{\&} \cdots \mathbin{\&} \neg\neg\mathcal{E}(x_n, y_n) \mathbin{\&} (\widetilde{\Psi}_1(\mathbf{x}) \to \widetilde{\Psi}_2(\mathbf{x})) \to (\widetilde{\Psi}_1(\mathbf{y}) \to \widetilde{\Psi}_2(\mathbf{y})) \tag{A.18}$$

is *spr*-realizable.

First we prove that for any $k_1, \ldots, k_n, \ell_1, \ldots, \ell_n$, the formula

$$\neg\neg\mathcal{E}(k_1, \ell_1) \mathbin{\&} \cdots \mathbin{\&} \neg\neg\mathcal{E}(k_n, l_n) \mathbin{\&} (\widetilde{\Psi}_1(\mathbf{k}) \to \widetilde{\Psi}_2(\mathbf{k})) \to (\widetilde{\Psi}_1(\mathbf{l}) \to \widetilde{\Psi}_2(\mathbf{l})) \tag{A.19}$$

is *spr*-realizable. Suppose that the premise of the formula (A.19) is *spr*-realizable. This means that the formulas $\neg\neg\mathcal{E}(k_i, \ell_i)$ $(i = 1, \ldots, n)$ and $\widetilde{\Psi}_1(\mathbf{k}) \to \widetilde{\Psi}_2(\mathbf{k})$ are *spr*-realizable. Since $\widetilde{A_2}$ is *spr*-realizable, it follows that the formulas $\neg\neg\mathcal{E}(\ell_i, k_i)(i = 1, \ldots, n)$ are also *spr*-realizable. We prove that the conclusion of (A.19), i.e., $\widetilde{\Psi}_1(\mathbf{l}) \to \widetilde{\Psi}_2(\mathbf{l})$, is *spr*-realizable. If the formula $\widetilde{\Psi}_1(\mathbf{l})$ is *spr*-realizable, then it follows from spr-realizability of the formula (A.14) that the formula $\widetilde{\Psi}_1(\mathbf{k})$ is *spr*-realizable.

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Therefore, the formula $\widetilde{\Psi_2}(\mathbf{k})$ is also *spr*-realizable. Now spr-realizability of $\widetilde{\Psi_2}(\mathbf{l})$ follows from spr-realizability of the formula (A.15). Since the predicate Ar-formula Ψ_2 is completely negative, it follows from Proposition 5.13 that $a_{\Psi_2} \Vdash_0 \Psi_2(\mathbf{l})$. Therefore, by Proposition 3.4,

$$\mathsf{a}_{a_{\Psi_2}} \Vdash_0 \Psi_1(\mathbf{l}) \to \Psi_2(\mathbf{l}). \tag{A.20}$$

If the formula $\widetilde{\Psi_1}(\mathbf{l})$ is not *spr*-realizable, then (A.20) holds by Proposition 3.4. Let $b = \mathsf{a}_{a_{\Psi_2}}$. By Proposition 3.4, the number a_b spr-realizes the formula (A.19) at level 0. If the premise of the formula (A.19) is not realizable, then the number a_b spr-realizes the formula (A.19) at level 0. Thus we have proved that for any $k_1, \dots, k_n, \ell_1, \dots, \ell_n$, the number a_b spr-realizes the formula (A.19) at level 0. Now it is evident that the number $\Lambda x_1 . \dots \Lambda x_n . \Lambda y_1 . \dots \Lambda y_n . \mathsf{a}_b$ spr-realizes the formula (A.18) at level 0.

9) Assume that $\Psi(x_1, \dots, x_n)$ is the formula $\forall x\, \Psi_0(x, x_1, \dots, x_n)$. The inductive hypothesis states that the formula

$$\forall x \forall \mathbf{x} \forall y \forall \mathbf{y}\, (\neg\neg\mathcal{E}(x, y) \,\&\, \neg\neg\mathcal{E}(x_1, y_1) \,\&\, \neg\neg\mathcal{E}(x_n, y_n) \,\&\, \widetilde{\Psi_0}(x, \mathbf{x}) \to \widetilde{\Psi_0}(y, \mathbf{y})) \tag{A.21}$$

is *spr*-realizable. We have to prove that the formula

$$\forall \mathbf{x}, \mathbf{y}\, (\neg\neg\mathcal{E}(x_1, y_1) \,\&\, \cdots \,\&\, \neg\neg\mathcal{E}(x_n, y_n) \,\&\, \forall x\, \widetilde{\Psi_0}(x, \mathbf{x}) \to \forall x\, \widetilde{\Psi_0}(x, \mathbf{y})) \tag{A.22}$$

is *spr*-realizable. First we prove that for any $k_1, \dots, k_n, \ell_1, \dots, \ell_n$, the formula

$$\neg\neg\mathcal{E}(k_1, \ell_1) \,\&\, \cdots \,\&\, \neg\neg\mathcal{E}(k_n, \ell_n) \,\&\, \forall x\, \widetilde{\Psi_0}(x, \mathbf{k}) \to \forall x\, \widetilde{\Psi_0}(x, \mathbf{l}) \tag{A.23}$$

is *spr*-realizable. Assume that the premise of the formula (A.23) is *spr*-realizable. This means that the formulas $\neg\neg\mathcal{E}(k_i, \ell_i)$ $(i = 1, \dots, n)$ and $\forall x\, \widetilde{\Psi}(x, \mathbf{k})$ are *spr*-realizable. We prove that the conclusion of the formula (A.23), i.e., $\forall x\, \widetilde{\Psi_0}(x, \mathbf{l})$ is *spr*-realizable. First we prove that for any ℓ, the formula $\widetilde{\Psi_0}(\ell, \mathbf{l})$ is *spr*-realizable. Note that the formula $\widetilde{\Psi_0}(\ell, \mathbf{k})$ is *spr*-realizable. Since the formula $\widetilde{A_1}$ is *spr*-realizable, it follows that the formula $\neg\neg\mathcal{E}(\ell, \ell)$ is *spr*-realizable. Now it follows from spr-realizability of the formula (A.21) that $\widetilde{\Psi_0}(\ell, \mathbf{l})$ is *spr*-realizable. Since Ψ_0 is a completely negative formula, it follows from Proposition 5.13 that $a_{\Psi_0} \Vdash_0 \widetilde{\Psi_0}(\ell, \mathbf{l})$. It is evident that $\Lambda x . a_{\Psi_0} \Vdash_0 \forall x\, \widetilde{\Psi_0}(x, \mathbf{l})$. By Proposition 3.4, the number $\mathsf{a}_{\Lambda x . a_{\Psi_0}}$ spr-realizes the formula (A.23) at level 0. If the premise of the formula (A.23) is not *spr*-realizable, then by Proposition 3.4, the number $\mathsf{a}_{\Lambda x . a_{\Psi_0}}$ spr-realizes the formula (A.23) at level 0. Thus we have proved that for any $k_1, \dots, k_n, \ell_1, \dots, \ell_n$, the number $\mathsf{a}_{\Lambda x . a_{\Psi_0}}$ spr-realizes the formula (A.23) at level 0. Now it is evident that the number $\Lambda x_1 . \dots \Lambda x_n . \Lambda y_1 . \dots \Lambda y_n . \mathsf{a}_{\Lambda x . a_\Psi}$ spr-realizes the formula (A.22) at level 0.

§B. Appendix: Proof of Proposition 6.19.

Assume that

$$\mathbf{\Phi} = \mathcal{Z}(x_1), \mathcal{S}(x_1, x_2), \mathcal{A}(x_1, x_2, x_3), \mathcal{M}(x_1, x_2, x_3), \mathcal{E}(x_1, x_2)$$

is an interpretation of the formula Q. The formula $\widetilde{\Psi}(x_1, \dots, x_n)$ is built from the formulas $\neg\mathcal{Z}(x_i)$, $\neg\mathcal{S}(x_i, x_j)$, $\neg\mathcal{A}(x_i, x_j, x_k)$, $\neg\mathcal{M}(x_i, x_j, x_k)$, and $\neg\mathcal{E}(x_i, x_j)$, where $1 \le i, j, k \le n$, using the logical symbols $\neg$, &, and $\to$. We prove the proposition by induction on the complexity of $\Psi(x_1, \dots, x_n)$.

1) If $\Psi(x_1, \dots, x_n)$ is $\neg E(x_i, x_j)$, where $1 \le i, j \le n$, then $\widetilde{\Psi}(x_1, \dots, x_n)$ is $\neg\mathcal{E}(x_i, x_j)$. If $i = j$, then the formula $\neg E(k_i, k_j)$ is not true. On the other hand, since the formula

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$\widetilde{A_1}$ is *spr*-realizable, it follows that the formula $\neg\neg\mathcal{E}(\widetilde{k_i},\widetilde{k_j})$ is *spr*-realizable and the formula $\neg\mathcal{E}(\widetilde{k_i},\widetilde{k_j})$ is not *spr*-realizable. Thus $\neg E(k_i,k_j)$ is true iff $\neg\mathcal{E}(\widetilde{k_i},\widetilde{k_j})$ is *spr*-realizable. Now assume that $i \neq j$. Suppose that the formula $\neg E(k_i,k_j)$ is true. This means that $k_i \neq k_j$. We prove that the formula $\neg\mathcal{E}(\widetilde{k_i},\widetilde{k_j})$ is *spr*-realizable. By Proposition 3.4, it is enough to prove that the formula $\neg\neg\mathcal{E}(\widetilde{k_i},\widetilde{k_j})$ is not *spr*-realizable. Assume the opposite. By Proposition 6.17, we have that the formula $\widetilde{[k_i]}(\widetilde{k_i})$ is *spr*-realizable. Then by Proposition 6.14, the formula $\widetilde{[k_i]}(\widetilde{k_j})$ is *spr*-realizable as well. On the other hand, the formula $\widetilde{[k_j]}(\widetilde{k_j})$ is *spr*-realizable. By Proposition 6.18, $k_i = k_j$ in contradiction with the assumption that the formula $\neg E(k_i,k_j)$ is true. Thus the formula $\neg\mathcal{E}(\widetilde{k_i},\widetilde{k_j})$ is realizable. Conversely, assume that the formula $\neg\mathcal{E}(\widetilde{k_i},\widetilde{k_j})$ is *spr*-realizable. We prove that the formula $\neg E(k_i,k_j)$ is true, i.e., $k_i \neq k_j$. Suppose the opposite. Then $\widetilde{k_i} = \widetilde{k_j}$. Since the formula $\widetilde{A_1}$ is *spr*-realizable, it follows that the formula $\neg\neg\mathcal{E}(\widetilde{k_i},\widetilde{k_j})$ is *spr*-realizable in contradiction with the assumption that the formula $\neg\mathcal{E}(\widetilde{k_i},\widetilde{k_j})$ is *spr*-realizable. Thus $k_i \neq k_j$ and the formula $\neg E(k_i,k_j)$ is true.

2) If $\Psi(x_1,\dots,x_n)$ is $\neg Z(x_i)$, where $1 \leq i \leq n$, then $\widetilde{\Psi}(x_1,\dots,x_n)$ is $\neg\mathcal{Z}(x_i)$. Assume that the formula $\neg Z(k_i)$ is true. This means that $k_i \neq 0$. We prove that the formula $\neg\mathcal{Z}(\widetilde{k_i})$ is *spr*-realizable. Suppose the opposite. Then the formula $\neg\neg\mathcal{Z}(\widetilde{k_i})$, i.e., $\widetilde{[0]}(\widetilde{k_i})$ is *spr*-realizable. On the other hand, the formula $\widetilde{[k_i]}(\widetilde{k_i})$ is *spr*-realizable as well. By Proposition 6.18, $k_i = 0$ in contradiction with the assumption that the formula $\neg Z(k_i)$ is true. Conversely, assume that the formula $\neg\mathcal{Z}(\widetilde{k_i})$ is *spr*-realizable. Then the formula $\neg\neg\mathcal{Z}(\widetilde{k_i})$, i.e., $\widetilde{[0]}(\widetilde{k_i})$, is not *spr*-realizable. We prove that the formula $\neg Z(k_i)$ is true, i.e., $k_i \neq 0$. Suppose the opposite. Then the formula $\widetilde{[0]}(\widetilde{k_i})$ is *spr*-realizable. This contradiction proves that the formula $\neg Z(k_i)$ is true.

3) If $\Psi(x_1,\dots,x_n)$ is $\neg S(x_i,x_j)$, where $1 \leq i,j \leq n$, then $\widetilde{\Psi}(x_1,\dots,x_n)$ is the formula $\neg\mathcal{S}(x_i,x_j)$. Assume that the formula $\neg S(k_i,k_j)$ is true. This means that $k_j \neq k_i + 1$. We prove that the formula $\neg\mathcal{S}(\widetilde{k_i},\widetilde{k_j})$ is *spr*-realizable. Suppose the opposite. Then the formula $\neg\neg\mathcal{S}(\widetilde{k_i},\widetilde{k_j})$ is *spr*-realizable. Let us prove that the formula $\widetilde{[k_i+1]}(\widetilde{k_j})$, i.e.,

$$\neg\mathcal{Z}(\widetilde{k_j})\,\&\,\forall y\,(\neg\neg\mathcal{S}(y,\widetilde{k_j}) \to \widetilde{[k_i]}(y)), \tag{B.1}$$

is *spr*-realizable. Since the formulas $\neg\neg\mathcal{S}(\widetilde{k_i},\widetilde{k_j})$ and $\widetilde{A_{11}}$ are *spr*-realizable, it follows that the formula $\neg\mathcal{Z}(\widetilde{k_j})$ is *spr*-realizable. Let ℓ be an arbitrary natural number. We prove that the formula

$$\neg\neg\mathcal{S}(\ell,\widetilde{k_j}) \to \widetilde{[k_i]}(\ell) \tag{B.2}$$

is *spr*-realizable. Assume that the premise of the formula (B.2), i.e., $\neg\neg\mathcal{S}(\ell,\widetilde{k_j})$, is *spr*-realizable. Thus we have that the formulas $\neg\neg\mathcal{S}(\widetilde{k_i},\widetilde{k_j})$ and $\neg\neg\mathcal{S}(\ell,\widetilde{k_j})$ are *spr*-realizable. Since the formula $\widetilde{A_{10}}$ is *spr*-realizable, it follows that the formula $\neg\neg\mathcal{E}(\widetilde{k_i},\ell)$ is *spr*-realizable. It follows from Proposition 6.14 that the formula $\widetilde{[k_i]}(\ell)$ is *spr*-realizable. By Proposition 5.13, $a_{[k_i](x)} \Vdash_0 \widetilde{[k_i]}(\ell)$ and by Proposition 3.4, the number $\mathsf{a}_{a_{[k_i](x)}}$ spr-realizes the formula (B.2) at level 0. If the premise of the formula (B.2) is not *spr*-realizable, then by Proposition 3.4, the number $\mathsf{a}_{a_{[k_i](x)}}$ spr-realizes the formula (B.2) at level 0. Thus, for any ℓ, the number $\mathsf{a}_{a_{[k_i](x)}}$ spr-realizes (B.2) at level 0. Now it is evident that $\Lambda y.\mathsf{a}_{a_{[k_i](x)}} \Vdash_0 \forall y\,(\neg\neg\mathcal{S}(y,\widetilde{k_j}) \to \widetilde{[k_i]}(y))$. Thus it is proved that the formula $\widetilde{[k_i+1]}(\widetilde{k_j})$ is *spr*-realizable. Since the formula $\widetilde{[k_j]}(\widetilde{k_j})$ is also *spr*-realizable, it follows

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from Proposition 6.18 that $k_i + 1 = k_j$ in contradiction with the assumption that the formula $\neg S(k_i, k_j)$ is true. Thus the formula $\neg\mathcal{S}(\widetilde{k_i}, \widetilde{k_j})$ is *spr*-realizable. Conversely, assume that the formula $\neg\mathcal{S}(\widetilde{k_i}, \widetilde{k_j})$ is realizable. We prove that the formula $\neg S(k_i, k_j)$ is true, i.e., $k_j \neq k_i + 1$. Suppose the opposite. Then the formula $[\widetilde{k_i + 1}](\widetilde{k_j})$, i.e., (B.1), is *spr*-realizable. It follows that the formula $\neg\mathcal{Z}(k_j)$ is *spr*-realizable. Since the formula $\widetilde{A_{12}}$ is *spr*-realizable, it follows that there exists a natural number ℓ such that the formula $\neg\neg\mathcal{S}(\ell, \widetilde{k_j})$ is *spr*-realizable. On the other hand, since the formula (B.1) is *spr*-realizable, it follows that $\neg\neg\mathcal{S}(\ell, \widetilde{k_j}) \to [\widetilde{k_i}](\ell)$ is *spr*-realizable, thus the formula $[\widetilde{k_i}](\ell)$ is realizable. Since the formula $[\widetilde{k_i}](\widetilde{k_i})$ is also *spr*-realizable, it follows from Proposition 6.15 that the formula $\neg\neg\mathcal{E}(\ell, \widetilde{k_i})$ is *spr*-realizable. By Proposition 6.14, the formula $\neg\neg\mathcal{S}(\widetilde{k_i}, \widetilde{k_j})$ is *spr*-realizable in contradiction with the assumption that the formula $\neg\mathcal{S}(\widetilde{k_i}, \widetilde{k_j})$ is *spr*-realizable. Thus it is proved that the formula $\neg S(k_i, k_j)$ is true.

4) If $\Psi(x_1, \dots, x_n)$ is the formula $\neg A(x_i, x_j, x_m)$, where $1 \leq i, j, m \leq n$, then $\widetilde{\Psi}(x_1, \dots, x_n)$ is $\neg\mathcal{A}(x_i, x_j, x_m)$. We prove the proposition by induction on k_j. Let $k_j = 0$. Then $\widetilde{k_j} = \tilde{0}$ and the formula $[\tilde{0}](\widetilde{k_j})$, i.e., $\neg\neg\mathcal{Z}(\widetilde{k_j})$, is *spr*-realizable. Assume that the formula $\neg A(k_i, k_j, k_m)$ is true. This means that $k_m \neq k_i$. We prove that the formula $\neg\mathcal{A}(\widetilde{k_i}, \widetilde{k_j}, \widetilde{k_m})$ is *spr*-realizable. Suppose the opposite. Then the formula $\neg\neg\mathcal{A}(\widetilde{k_i}, \widetilde{k_j}, \widetilde{k_m})$ is *spr*-realizable. Since the formula $\widetilde{A_{16}}$ is realizable, it follows that the formula $\neg\neg\mathcal{A}(\widetilde{k_i}, \widetilde{k_j}, \widetilde{k_i})$ is *spr*-realizable. Since the formula $\widetilde{A_{14}}$ is *spr*-realizable, it follows that the formula $\neg\neg\mathcal{E}(\widetilde{k_m}, \widetilde{k_i})$ is *spr*-realizable. As it was proved above in the case 1), the formula $E(k_m, k_i)$ is true, i.e., $k_m = k_i$ in contradiction with the assumption that the formula $\neg A(k_i, k_j, k_m)$ is true. Thus we have proved that the formula $\neg\mathcal{A}(\widetilde{k_i}, \widetilde{k_j}, \widetilde{k_m})$ is *spr*-realizable. Conversely, assume that the formula $\neg\mathcal{A}(\widetilde{k_i}, \widetilde{k_j}, \widetilde{k_m})$ is *spr*-realizable. We prove that the formula $\neg A(k_i, k_j, k_m)$ is true, i.e., $k_m \neq k_i$. Suppose the opposite. Then $\widetilde{k_m} = \widetilde{k_i}$. Since the formula $\widetilde{A_{16}}$ is *spr*-realizable, it follows that the formula $\neg\neg\mathcal{A}(\widetilde{k_i}, \widetilde{k_j}, \widetilde{k_m})$ is *spr*-realizable in contradiction with the assumption that the formula $\neg\mathcal{A}(\widetilde{k_i}, \widetilde{k_j}, \widetilde{k_m})$ is *spr*-realizable. Thus we have proved that the formula $\neg A(k_i, k_j, k_m)$ is true. Now suppose that $k_j = \ell_j + 1$ and for any k_i, ℓ_m, the formula $\neg A(k_i, \ell_j, \ell_m)$ is true iff the formula $\neg\mathcal{A}(\widetilde{k_i}, \widetilde{\ell_j}, \widetilde{\ell_m})$ is *spr*-realizable. Note that the formula $\neg S(\ell_j, k_j)$ is not true. As it was proved in the case 3) above, the formula $\neg\mathcal{S}(\widetilde{\ell_j}, \widetilde{k_j})$ is not *spr*-realizable. Then the formula $\neg\neg\mathcal{S}(\widetilde{\ell_j}, \widetilde{k_j})$ is *spr*-realizable. Assume that the formula $\neg A(k_i, k_j, k_m)$ is true. This means that $k_m \neq k_i + k_j$. We prove that the formula $\neg\mathcal{A}(\widetilde{k_i}, \widetilde{k_j}, \widetilde{k_m})$ is *spr*-realizable. Suppose the opposite. Then the formula $\neg\neg\mathcal{A}(\widetilde{k_i}, \widetilde{k_j}, \widetilde{k_m})$ is *spr*-realizable. On the other hand, since the formula $\neg A(k_i, \ell_j, k_i + \ell_j)$ is not true, then by the inductive hypothesis, the formula $\neg\mathcal{A}(\widetilde{k_i}, \widetilde{\ell_j}, \widetilde{k_i + \ell_j})$ is not *spr*-realizable. Then the formula $\neg\neg\mathcal{A}(\widetilde{k_i}, \widetilde{\ell_j}, \widetilde{k_i + \ell_j})$ is *spr*-realizable. Let $\ell_m = k_i + \ell_j + 1$. Then the formula $S(k_i + \ell_j, \ell_m)$ is true and the formula $\neg S(k_i + \ell_j, \ell_m)$ is not true. As it was proved above in the case 3), the formula $\neg\mathcal{S}(\widetilde{k_i + \ell_j}, \widetilde{\ell_m})$ is not *spr*-realizable. Then the formula $\neg\neg\mathcal{S}(\widetilde{k_i + \ell_j}, \widetilde{\ell_m})$ is *spr*-realizable. Since the formula $\widetilde{A_{17}}$ is *spr*-realizable, it follows that the formula $\neg\neg\mathcal{A}(\widetilde{k_i}, \widetilde{k_j}, \widetilde{\ell_m})$ is *spr*-realizable. Since the formula $\widetilde{A_{14}}$ is *spr*-realizable, it follows that the formula $\neg\neg\mathcal{E}(k_m, \ell_m)$ is *spr*-realizable. As it was proved above, the formula $E(k_m, \ell_m)$ is true, i.e., $k_m = \ell_m = k_i + \ell_j + 1 = k_i + k_j$. This means that the formula $A(k_i, k_j, k_m)$ is true in contradiction with the assumption that the formula $\neg A(k_i, k_j, k_m)$ is true. Thus we have proved that the formula $\neg\mathcal{A}(\widetilde{k_i}, \widetilde{k_j}, \widetilde{k_m})$

is *spr*-realizable. Conversely, assume that the formula $\neg\mathcal{A}(\widetilde{k_i},\widetilde{k_j},\widetilde{k_m})$ is *spr*-realizable. We prove that the formula $\neg A(k_i,k_j,k_m)$ is true. Suppose the opposite. Then the formula $A(k_i,k_j,k_m)$ is true. This means that $k_m = k_i + k_j$. Let $\ell_m = k_i + \ell_j$. Then $k_m = \ell_m + 1$ and the formula $\neg A(k_i,\ell_j,\ell_m)$ is not true. By the inductive hypothesis, the formula $\neg\mathcal{A}(\widetilde{k_i},\widetilde{\ell_j},\widetilde{\ell_m})$ is not *spr*-realizable. Then the formula $\neg\neg\mathcal{A}(\widetilde{k_i},\widetilde{\ell_j},\widetilde{\ell_m})$ is *spr*-realizable. On the other hand, since the formulas $S(\ell_j,k_j)$ and $S(\ell_m,k_m)$ are true, it follows that the formulas $\neg S(\ell_j,k_j)$ and $\neg S(\ell_m,k_m)$ are not true. It follows from the case 3) above that the formulas $\neg\mathcal{S}(\widetilde{\ell_j},\widetilde{k_j})$ and $\neg\mathcal{S}(\widetilde{\ell_m},\widetilde{k_m})$ are not *spr*-realizable. Then the formulas $\neg\neg\mathcal{S}(\widetilde{\ell_j},\widetilde{k_j})$ and $\neg\neg\mathcal{S}(\widetilde{\ell_m},\widetilde{k_m})$ are *spr*-realizable. Since the formula $\widetilde{A_{17}}$ is *spr*-realizable, it follows that the formula $\neg\neg\mathcal{A}(\widetilde{k_i},\widetilde{k_j},\widetilde{k_m})$ is *spr*-realizable in contradiction with the assumption that the formula $\neg\mathcal{A}(\widetilde{k_i},\widetilde{k_j},\widetilde{k_m})$ is *spr*-realizable. Thus we have proved that the formula $\neg A(k_i,k_j,k_m)$ is true.

5) If $\Psi(x_1,\dots,x_n)$ is the formula $\neg M(x_i,x_j,x_m)$, where $1 \le i,j,m \le n$, then $\widetilde{\Psi}(x_1,\dots,x_n)$ is $\neg\mathcal{M}(x_i,x_j,x_m)$. We prove the proposition by induction on k_j. Let $k_j = 0$. Then the formula $\widetilde{[0]}(\widetilde{k_j})$, i.e., $\neg\neg\mathcal{Z}(\widetilde{k_j})$, is *spr*-realizable. Assume that the formula $\neg M(k_i,k_j,k_m)$ is true. We prove that the formula $\neg\mathcal{M}(\widetilde{k_i},\widetilde{k_j},\widetilde{k_m})$ is *spr*-realizable. Suppose the opposite. Then the formula $\neg\neg\mathcal{M}(\widetilde{k_i},\widetilde{k_j},\widetilde{k_m})$ is *spr*-realizable. Since the formula $\widetilde{A_{21}}$ is *spr*-realizable, it follows that $\neg\neg\mathcal{M}(\widetilde{k_i},\widetilde{k_j},\widetilde{k_j})$ is *spr*-realizable. Since the formula $\widetilde{A_{19}}$ is realizable, it follows that the formula $\neg\neg\mathcal{E}(\widetilde{k_m},\widetilde{k_j})$ is *spr*-realizable. As it was proved above in the case 1), the formula $E(k_m,k_j)$ is true, i.e., $k_m = k_j$. Then the formula $M(k_i,k_j,k_m)$ is true in contradiction with the assumption that the formula $\neg M(k_i,k_j,k_m)$ is true. Thus we have proved that the formula $\neg\mathcal{M}(\widetilde{k_i},\widetilde{k_j},\widetilde{k_m})$ is *spr*-realizable. Conversely, assume that the formula $\neg\mathcal{M}(\widetilde{k_i},\widetilde{k_j},\widetilde{k_m})$ is *spr*-realizable. We prove that the formula $\neg M(k_i,k_j,k_m)$ is true. Suppose the opposite. Then $k_m = k_j$ and $\widetilde{k_m} = \widetilde{k_j}$. Since the formula $\widetilde{A_{21}}$ is realizable, it follows that the formula $\neg\neg\mathcal{M}(\widetilde{k_i},\widetilde{k_j},\widetilde{k_m})$ is *spr*-realizable in contradiction with the assumption that the formula $\neg\mathcal{M}(\widetilde{k_i},\widetilde{k_j},\widetilde{k_m})$ is *spr*-realizable. Thus we have proved that the formula $\neg M(k_i,k_j,k_m)$ is true. Now suppose that $k_j = \ell_j + 1$ and for any k_i,ℓ_m, the formula $\neg M(k_i,\ell_j,\ell_m)$ is true iff the formula $\neg\mathcal{M}(\widetilde{k_i},\widetilde{\ell_j},\widetilde{\ell_m})$ is *spr*-realizable. As it was shown in the case 3) above, the formula $\neg\neg\mathcal{S}(\widetilde{\ell_j},\widetilde{k_j})$ is *spr*-realizable. Assume that the formula $\neg M(k_i,k_j,k_m)$ is true. We prove that the formula $\neg\mathcal{M}(\widetilde{k_i},\widetilde{k_j},\widetilde{k_m})$ is *spr*-realizable. Suppose the opposite. Then the formula $\neg\neg\mathcal{M}(\widetilde{k_i},\widetilde{k_j},\widetilde{k_m})$ is *spr*-realizable. Since the formula $M(k_i,\ell_j,k_i\cdot\ell_j)$ is true, then the formula $\neg M(k_i,\ell_j,k_i\cdot\ell_j)$ is not true. By the inductive hypothesis, the formula $\neg\mathcal{M}(\widetilde{k_i},\widetilde{\ell_j},\widetilde{k_i\cdot\ell_j})$ is not *spr*-realizable. Then the formula $\neg\neg\mathcal{M}(\widetilde{k_i},\widetilde{\ell_j},\widetilde{k_i\cdot\ell_j})$ is *spr*-realizable. Let $\ell_m = k_i\cdot\ell_j + k_i$. Then the formula $A(k_i\cdot\ell_j,k_i,\ell_m)$ is true and the formula $\neg A(k_i\cdot\ell_j,k_i,\ell_m)$ is not true. As it was proved in the case 5) above, the formula $\neg\mathcal{A}(\widetilde{k_i\cdot\ell_j},\widetilde{k_i},\widetilde{\ell_m})$ is not *spr*-realizable. Then the formula $\neg\neg\mathcal{A}(\widetilde{k_i\cdot\ell_j},\widetilde{k_i},\widetilde{\ell_m})$ is *spr*-realizable. Since the formula $\widetilde{A_{22}}$ is *spr*-realizable, it follows that the formula $\neg\neg\mathcal{M}(\widetilde{k_i},\widetilde{k_j},\widetilde{\ell_m})$ is realizable. Since the formula $\widetilde{A_{19}}$ is *spr*-realizable, it follows that $\neg\neg\mathcal{E}(k_m,\ell_m)$ is *spr*-realizable. As it was proved in the case 1) above, $E(k_m,\ell_m)$ is true, i.e., $k_m = \ell_m = k_i + k_i\cdot\ell_j = k_i\cdot k_j$. This means that the formula $M(k_i,k_j,k_m)$ is true in contradiction with the assumption that the formula $\neg M(k_i,k_j,k_m)$ is true. Thus we have proved that the formula $\neg\mathcal{M}(\widetilde{k_i},\widetilde{k_j},\widetilde{k_m})$ is *spr*-realizable. Conversely, assume that the formula $\neg\mathcal{M}(\widetilde{k_i},\widetilde{k_j},\widetilde{k_m})$ is *spr*-realizable. We prove that the formula $\neg M(k_i,k_j,k_m)$ is true. Suppose the opposite. Then $k_m = k_i\cdot k_j$.

Let $\ell_m = k_i \cdot \ell_j$. Then the formula $M(k_i, \ell_j, \ell_m)$ is true and $k_m = \ell_m + k_i$. This means that the formula $A(\ell_m, k_i, k_m)$ is true. Then the formulas $\neg M(k_i, \ell_j, \ell_m)$ and $\neg A(\ell_m, k_i, k_m)$ are not true. By the inductive hypothesis, the formula $\neg\mathcal{M}(\widetilde{k_i}, \widetilde{\ell_j}, \widetilde{\ell_m})$ is not *spr*-realizable. Then the formula $\neg\neg\mathcal{M}(\widetilde{k_i}, \widetilde{\ell_j}, \widetilde{\ell_m})$ is *spr*-realizable. Since the formula $\neg A(\ell_m, k_i, k_m)$ is not true, it follows from the case 4) above that the formula $\neg\mathcal{A}(\widetilde{\ell_m}, \widetilde{k_i}, \widetilde{k_m})$ is not *spr*-realizable. Then the formula $\neg\neg\mathcal{A}(\widetilde{\ell_m}, \widetilde{k_i}, \widetilde{k_m})$ is *spr*-realizable. Since the formula $\widetilde{A_{22}}$ is realizable, it follows that the formula $\neg\neg\mathcal{M}(\widetilde{k_i}, \widetilde{k_j}, \widetilde{k_m})$ is *spr*-realizable in contradiction with the assumption that the formula $\neg\mathcal{M}(\widetilde{k_i}, \widetilde{k_j}, \widetilde{k_m})$ is *spr*-realizable. Thus we have proved that the formula $\neg M(k_i, k_j, k_m)$ is true.

6) If $\Psi(x_1, \dots, x_n)$ is $\neg\Psi_0(x_1, \dots, x_n)$, then $\widetilde{\Psi}(x_1, \dots, x_n)$ is $\neg\widetilde{\Psi_0}(x_1, \dots, x_n)$. By the inductive hypothesis, for any natural $k_1, \dots, k_m$, the formula $\Psi_0(k_1, \dots, k_m)$ is true iff the formula $\widetilde{\Psi_0}(\tilde{k}_1, \dots, \tilde{k}_m)$ is *spr*-realizable. Assume that the formula $\neg\Psi_0(k_1, \dots, k_m)$ is true. Then the formula $\Psi_0(k_1, \dots, k_m)$ is not true. By the inductive hypothesis, the formula $\widetilde{\Psi_0}(\tilde{k}_1, \dots, \tilde{k}_m)$ it is not *spr*-realizable. By Proposition 3.4, $\neg\widetilde{\Psi_0}(\tilde{k}_1, \dots, \tilde{k}_m)$ is *spr*-realizable. Conversely, if $\neg\widetilde{\Psi_0}(\tilde{k}_1, \dots, \tilde{k}_m)$ is *spr*-realizable, then the formula $\widetilde{\Psi_0}(\tilde{k}_1, \dots, \tilde{k}_m)$ is not *spr*-realizable. By the inductive hypothesis, the formula $\Psi_0(k_1, \dots, k_m)$ it is not true and $\neg\Psi_0(k_1, \dots, k_m)$ is true.

7) If $\Psi(x_1, \dots, x_n)$ is $\Psi_1(x_1, \dots, x_n) \,\&\, \Psi_2(x_1, \dots, x_n)$, then $\widetilde{\Psi}(x_1, \dots, x_n)$ is the formula $\widetilde{\Psi_1}(x_1, \dots, x_n) \,\&\, \widetilde{\Psi_2}(x_1, \dots, x_n)$. By the inductive hypothesis, for $i = 1, 2$ and any natural numbers $k_1, \dots, k_m$, the formula $\Psi_i(k_1, \dots, k_m)$ is true iff the formula $\widetilde{\Psi_i}(\tilde{k}_1, \dots, \tilde{k}_m)$ is realizable. The formula $\Psi(k_1, \dots, k_m)$ is true iff the formulas $\Psi_1(k_1, \dots, k_m)$ and $\Psi_2(k_1, \dots, k_m)$ are both true. By the inductive hypothesis, this is possible iff the formulas $\widetilde{\Psi_1}(\tilde{k}_1, \dots, \tilde{k}_m)$ and $\widetilde{\Psi_2}(\tilde{k}_1, \dots, \tilde{k}_m)$ are both *spr*-realizable and, by Proposition 3.4, the formula $\widetilde{\Psi}(\tilde{k}_1, \dots, \tilde{k}_m)$ is *spr*-realizable.

8) If $\Psi(x_1, \dots, x_n)$ is $\Psi_1(x_1, \dots, x_n) \to \Psi_2(x_1, \dots, x_n)$, then $\widetilde{\Psi}(x_1, \dots, x_n)$ is the formula $\widetilde{\Psi_1}(x_1, \dots, x_n) \to \widetilde{\Psi_2}(x_1, \dots, x_n)$. The inductive hypothesis is the same as in the previous case. Assume that the formula $\Psi(k_1, \dots, k_m)$ is true. This means that the formula $\Psi_1(k_1, \dots, k_m)$ is false or the formula $\Psi_2(k_1, \dots, k_m)$ is true. In the first case, by the inductive hypothesis, the formula $\widetilde{\Psi_1}(\tilde{k}_1, \dots, \tilde{k}_m)$ is not *spr*-realizable and, by Proposition 3.4, the formula $\widetilde{\Psi}(\tilde{k}_1, \dots, \tilde{k}_m)$ is *spr*-realizable. In the second case, by the inductive hypothesis, the formula $\widetilde{\Psi_2}(\tilde{k}_1, \dots, \tilde{k}_m)$ is *spr*-realizable and, by Proposition 3.4, the formula $\widetilde{\Psi}(\tilde{k}_1, \dots, \tilde{k}_m)$ is *spr*-realizable. Conversely, assume that the formula $\widetilde{\Psi}(\tilde{k}_1, \dots, \tilde{k}_m)$ is *spr*-realizable. Then the formula $\Psi(k_1, \dots, k_m)$ should be true because otherwise the formula $\Psi_1(k_1, \dots, k_m)$ is true and the formula $\Psi_2(k_1, \dots, k_m)$ is false. By the inductive hypothesis, the formula $\widetilde{\Psi_1}(\tilde{k}_1, \dots, \tilde{k}_m)$ is *spr*-realizable and the formula $\widetilde{\Psi_2}(\tilde{k}_1, \dots, \tilde{k}_m)$ is not realizable. But this is impossible if the formula $\widetilde{\Psi}(\tilde{k}_1, \dots, \tilde{k}_m)$ is *spr*-realizable.

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