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BOUNDED CONJUGACY CLASSES AND CONJUGACY CLASSES SUPPORTING INVARIANT MEASURES AND AUTOMORPHISMS

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Abstract

We consider conjugacy classes in a locally compact group G that support finite G-invariant measures. If G is a property (M) extension of an abelian group, in particular, if G is a metabelian group, then any such conjugacy class is relatively compact. As an application, centralisers of lattices in such groups have bounded conjugacy classes. We use the same techniques to obtain results in the case of totally disconnected, locally compact groups.

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1. Introduction

Let G be a locally compact group. For $x \in G$, let $C_x = \{gxg^{-1} \mid g \in G\}$ be the conjugacy class containing x and let $B(G) = \{x \in G \mid \overline{C_x} \text{ is compact}\}$. It is easily seen that $B(G)$ is a characteristic subgroup containing the centre $Z(G) = \{x \in G \mid gx = xg$ for all $g \in G\}$. In general, $B(G)$ is not closed. There are locally compact groups G with $B(G)$ as a proper dense subgroup (see [9, Proposition 3]), but if G is a totally disconnected, locally compact (tdlc) group that is generated by a compact set, then [7, Theorem 2] shows that $B(G)$ is closed. The subgroup $B(G)$ plays a crucial role in the location of finite central (positive) measures. A measure is called central if it is invariant under the conjugate action of the group G (see [4, Theorem 1.5]).

A conjugacy class supporting a central measure is in $B(G)$ for connected Lie groups [5, Theorem 1′]. We prove this result for a certain extension of abelian groups (see Theorem 1.1) and obtain interesting applications (see Corollary 3.1) as in [5, Theorem 3]. We also obtain some general results in the case of tdlc groups (see Section 4).

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There is a close connection with finite covolume subgroups (see [4, 5]). In this context, we say that a locally compact group has property (M) if, for any closed subgroup H of G with G/H admitting a finite G-invariant measure, G/H is compact (see [1, 8] for further details on property (M)). It can be easily seen that abelian groups and compact groups have property (M). We obtain the following result.

THEOREM 1.1. *Let G be a locally compact group and let $x \in G$ be such that $\mu(C_x) = 1$ for some conjugate-invariant finite measure μ. If G has a closed abelian normal subgroup A such that G/A has property (M), then C_x has compact closure: that is, $x \in B(G)$.*

In particular, if G is a metabelian locally compact group, then C_x has compact closure: that is, $x \in B(G)$.

REMARK 1.2. An example of a metabelian group G having a finite covolume subgroup that is not cocompact is provided in [1]. Hence, groups under consideration in Theorem 1.1 need not have property (M).

2. Automorphisms of bounded displacement

Following [9], a (bicontinuous) automorphism α of a locally compact group G is called an automorphism of bounded displacement if $\{\alpha(x)x^{-1} \mid x \in G\}$ has compact closure. If α is an inner automorphism defined by $x \in G$ (that is, $\alpha(g) = xgx^{-1}$), then α is an automorphism of bounded displacement if and only if $x \in B(G)$. Thus, in order to identify elements in $B(G)$, we study automorphisms of bounded displacement. We obtain the following result.

PROPOSITION 2.1. *Let G be a locally compact group containing a closed abelian normal subgroup A such that G/A has property (M). Suppose that α is an automorphism of G such that G/H has a finite G-invariant measure, where $H = \{x \in G \mid \alpha(x) = x\}$ and $\alpha(A) = A$. Then α is an automorphism of bounded displacement.*

Our proof relies on certain shift-invariant properties in convolutions of probability measures on locally compact groups. For a locally compact group X, let $\mathcal{M}(X)$ be the space of all regular Borel probability measures (that is, positive measures with total measure one) on X equipped with the weak* topology: that is, $\rho_n \to \rho \in \mathcal{M}(X)$ if $\int f(x)\, d\rho_n(x) \to \int f(x)\, d\rho(x)$ for all continuous bounded functions f on X. For $x \in X$ and ρ in $\mathcal{M}(X)$, $x\rho$ and $\rho x \in \mathcal{M}(X)$ are defined by $x\rho(E) = \rho(x^{-1}E)$ and $\rho x(E) = \rho(Ex^{-1})$ for any Borel set E in X. The convolution of two measures $\mu, \lambda \in \mathcal{M}(X)$ is denoted by $\mu * \lambda$ and is defined by $\mu * \lambda(E) = \int \mu(Ex^{-1})\, d\lambda(x)$ for any Borel set E in X. For any automorphism α of X and $\mu \in \mathcal{M}(X)$, define $\alpha(\mu)$ by $\alpha(\mu)(E) = \mu(\alpha^{-1}(E))$ for any Borel set E in X (see [6] for more details on probability measures on groups).

LEMMA 2.2. *Let G be a locally compact group and let A be a closed abelian normal subgroup of G. Suppose that $\beta\colon G \to A$ is a continuous map such that $\beta(xy) = \beta(x)x\beta(y)x^{-1}$ for all $x, y \in G$. Then $H = \{x \in G \mid \beta(x) = e\}$ is a closed subgroup. If G/H has a finite G-invariant measure, then $\beta(A)$ is contained in a compact subgroup of A.*

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Proof. Since β is continuous, H is a closed subgroup. Assume that G/H carries a finite G-invariant measure μ, say. Normalising μ, we may assume that μ is a probability measure. For $x \in G$ and $h \in H$, we have $\beta(xh) = \beta(x)$. Hence, β induces a continuous map $\tilde{\beta}\colon G/H \to A$ defined by $\tilde{\beta}(xH) = \beta(x)$ for all $x \in G$. Take $\lambda = \tilde{\beta}(\mu)$. Then λ is a Borel probability measure on A such that $\beta(x)x\lambda x^{-1} = \lambda$ for all $x \in G$. Since A is an abelian group, $\lambda = \beta(a)a\lambda a^{-1} = \beta(a)\lambda$. Let $M = \{a \in A \mid a\lambda = \lambda\}$. Then, by [6, Theorems 1.2.4 and 1.2.7], M is a compact subgroup of A and $\beta(A) \subset M$. □

Proof of Proposition 2.1. Let $N = \overline{AH}$. Then N is a closed α-invariant subgroup of G. Since $H \subset N$, G/N has a finite G-invariant measure. Since G/N is a quotient of G/A, the assumption that G/A has property (M) implies that G/N is compact.

Define β on N by $\beta(x) = \alpha(x)x^{-1}$ for all $x \in N$. Since $\alpha(x) = x$ for all $x \in H$, we have $\beta(x) \in A$. Thus, $\beta\colon N \to A$ is a well-defined continuous map. Also, $\beta(xy) = \beta(x)x\beta(y)x^{-1}$ for all $x, y \in N$ and $H = \{x \in N \mid \beta(x) = e\}$. Since G/H has a finite G-invariant measure and $H \subset N$, it follows that N/H has a finite N-invariant measure. By Lemma 2.2, there is a compact subgroup M of A such that $\beta(A) \subset M$. Since $\beta(ah) = \beta(a)$, we have $\beta(AH) \subset M$ and, since AH is dense in N, $\beta(N) \subset M$. Thus, $\alpha(x) \subset Mx$ for all $x \in N$ and M is a compact subgroup of A.

Since G/N is compact, there is a compact subset C of G such that $G = CN$. For $g \in G$, there are $x \in N$ and $y \in C$ such that $g = yx$. This implies that $\alpha(g)g^{-1} = \alpha(y)\alpha(x)x^{-1}y^{-1} \subset \alpha(C)MC^{-1}$. Thus, α is an automorphism of bounded displacement. □

3. Theorem 1.1 and applications

Proof of Theorem 1.1. Let α be the inner automorphism defined by x^{-1} and define $H = \{g \in G \mid gx = xg\}$. The group G has a canonical action on G/H on the left and G acts on C_x by conjugation. The map $\theta\colon C_x \to G/H$ defined by $\theta(gxg^{-1}) = gH$ is a well-defined G-equivariant Borel isomorphism and $\theta(\mu)$ is a finite G-invariant measure on G/H. It follows from Proposition 2.1 that α is an automorphism of bounded displacement. Thus, there is a compact set C such that $\alpha(g) \subset Cg$ for all $g \in G$. This implies that $x^{-1}gxg^{-1} \subset C$ for all $g \in G$ and hence $gxg^{-1} \subset xC$ for all $g \in G$. Thus, the conjugacy class C_x containing x has compact closure. □

The following result on the centraliser of finite covolume subgroups is a kind of density theorem in the sense that any element commuting with a finite covolume subgroup commutes with all elements up to a compact set.

Corollary 3.1. *Let G be a locally compact group containing a closed abelian normal subgroup A such that G/A has property (M) and H is a finite covolume subgroup of G. Then $Z(H) \subset B(G)$.*

Proof. Let $x \in Z(H)$. Then $Z(x)$ contains H, and hence $G/Z(x)$ has a finite G-invariant measure. Let $\eta\colon G/Z(x) \to C_x$ be $\eta(g) = gxg^{-1}$. Then η is a well-defined continuous map preserving the G-action. Hence, the conjugacy class of x supports a finite

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G-invariant measure. Therefore, by Theorem 1.1, the conjugacy class of x has compact closure. Thus, $Z(H) \subset B(G)$. □

We now look at the case when A, as above, has no compact subgroups. As examples, $\mathbb{R}^n$ and $\mathbb{Z}^n$ have no compact subgroups.

COROLLARY 3.2. *Let G, H, A and α be as in Proposition 2.1. Suppose that A has no compact subgroups. Then G/H is compact. In particular, if $x \in G$ is such that $\mu(C_x) > 0$ for some conjugate-invariant finite measure μ, then the corresponding conjugacy class C_x is compact.*

PROOF. Let $\beta\colon A \to A$ be $\beta(a) = \alpha(a)a^{-1}$ for all $a \in A$. Then β is a continuous map and $\beta(xy) = \beta(x)x\beta(y)x^{-1} = \beta(x)\beta(y)$ as A is abelian. By Lemma 2.2, $\beta(A)$ is contained in a compact subgroup of A. Since A has no compact subgroup, β is trivial. Thus, $A \subset H$ and G/H is a quotient of G/A. By the assumption that G/A has property (M), G/H is compact. □

4. Tdlc groups

We now consider tdlc groups. For an automorphism α of a tdlc group G, define the α-invariant subgroups $U_\alpha = \{x \in G \mid \lim_{n\to\infty} \alpha^n(x) = e\}$ and $M_\alpha = \{x \in G \mid \overline{\{\alpha^n(x) \mid n \in \mathbb{Z}\}}$ is compact$\}$, where U_α is called the contraction group of the automorphism α (see [2] for various results on U_α and M_α). In the tdlc case, we obtain the following result.

PROPOSITION 4.1. *Let G be a tdlc group and let α be an automorphism of G. Suppose that G/H has a finite G-invariant measure, where $H = \{x \in G \mid \alpha(x) = x\}$. Then α fixes a compact open subgroup of G and M_α is a subgroup of finite index in G.*

PROOF. Let $N = \overline{HU_\alpha}$. Since $\alpha(x) = x$ for $x \in H$, H normalises U_α. Then N is a tdlc group invariant under α. Let $\beta\colon N/H \to N$ be $\beta(xH) = \alpha(x)x^{-1}$ for all $x \in N$. Since $\alpha(x) = x$ for all $x \in H$, β is a well-defined continuous map on N/H. Since $N = \overline{HU_\alpha}$ and $\alpha(x) = x$ for all $x \in H$, we have $\beta(N) \subset \overline{U_\alpha}$. By [2, Corollaries 3.27 and 3.30], $\overline{U_\alpha} = U_0U_\alpha$, where U_0 is an α-invariant compact subgroup of G. In fact, we have $U_0 = \overline{U_\alpha} \cap \overline{U_{\alpha^{-1}}}$. This implies that $\alpha^n(x)U_0 \to U_0$ for all $x \in \overline{U_\alpha}$.

Let $\bar{\alpha}\colon N/H \to N/H$ be $\bar{\alpha}(xH) = \alpha(x)H$ for all $x \in N$. Since $\alpha(H) = H$, $\bar{\alpha}$ is a continuous map (in fact, $\bar{\alpha}$ is an N-equivariant homeomorphism).

For $x \in N$, $\beta(\bar{\alpha}(xH)) = \alpha^2(x)\alpha(x^{-1}) = \alpha(\beta(x))$. Thus, $\beta\bar{\alpha} = \alpha\beta$.

Since $H \subset N$, N/H has an N-invariant probability measure μ, say. Let $\lambda = \beta(\mu)$. Since β is a continuous map, λ is a probability measure on N. Since μ is an N-invariant probability measure and $\bar{\alpha}(gxH) = \alpha(g)\bar{\alpha}(xH)$ for all $g, x \in G$, it follows that μ is $\bar{\alpha}$-invariant. Since $\beta\bar{\alpha} = \alpha\beta$, λ is α-invariant, and hence $\lambda = \alpha^n(\lambda)$ for all n.

Let ρ be the normalised Haar measure on the compact subgroup U_0. Since $\alpha(U_0) = U_0$, ρ is α-invariant. Since $\alpha^n(x)U_0 \to U_0$ for all $x \in \overline{U_\alpha}$, we have $\alpha^n(\lambda * \rho) \to \rho$. But $\alpha^n(\lambda * \rho) = \alpha^n(\lambda) * \alpha^n(\rho) = \lambda * \rho$, so $\lambda * \rho = \rho$. By considering the supports of the measures, we see that λ is supported on U_0.

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Since μ is an N-invariant measure on N/H, the support of μ is N. Since $\beta(\mu) = \lambda$, $\overline{\beta(N)}$ is the support of λ, and hence $\beta(N) \subset U_0$. This implies that $\alpha(x) \subset U_0 x$ for all $x \in U_\alpha$. Let $x \in U_\alpha$. Then, for each $n \geq 1$, we have $\alpha^n(x) = g_n x$ for some $g_n \in U_0$. Since U_0 is compact and $\alpha^n(x) \to e$ as $n \to \infty$, we have $x \in U_0$. Thus, $U_\alpha \subset U_0 \subset \overline{U_\alpha}$, and hence $\overline{U_\alpha} = U_0$. Similarly, we may show that $\overline{U_{\alpha^{-1}}} = U_0$. By [2, Proposition 3.24], α fixes a compact open subgroup K, say.

Since $M_\alpha = \{x \in G \mid \overline{\{\alpha^n(x) \mid n \in \mathbb{Z}\}} \text{ is compact}\}$, $K \subset M_\alpha$, and so M_α is open. Since $\alpha(x) = x$ for all $x \in H$, $H \subset M_\alpha$. This implies that G/M_α is a discrete G-space with a finite G-invariant measure and hence G/M_α is finite. □

As a consequence, we have the following result on property (M) for tdlc groups.

Corollary 4.2. *Let G be a tdlc group and let H be a closed subgroup of G such that G/H has a finite G-invariant measure. If H is a compactly generated abelian group, then G/H is compact.*

Remark 4.3. For the metabelian counter-example provided in [1], the finite covolume subgroup H, although abelian, is not compactly generated. It may be noted that any compactly generated abelian tdlc group is a direct product of $\mathbb{Z}^n$ and a compact group.

Proof of Corollary 4.2. Let H be a compactly generated abelian subgroup of G with finite covolume. For $x \in H$, let α_x be the inner automorphism defined by x on G: that is, $\alpha_x(g) = xgx^{-1}$ for all $g \in G$. Since H is abelian, $H \subset Z(x)$. Thus, $G/Z(x)$ has a finite G-invariant measure. By Proposition 4.1, each $x \in H$ fixes a compact open subgroup in G. Since H is a compactly generated abelian group, [10, Theorem 5.9] implies that H normalises a compact open subgroup K of G. Then HK is a open subgroup G and hence G/HK is a discrete G-space with a finite G-invariant measure. Therefore, G/HK is finite. Thus, G/H is compact. □

We also have the following result for expansive automorphisms. An automorphism α of a tdlc group G is called expansive if $\bigcap \alpha^n(U) = \{e\}$ for some compact open subgroup U of G (see [3] for more details on expansive automorphisms on tdlc groups).

Corollary 4.4. *Let G be a tdlc group and let α be an expansive automorphism of G. Suppose that G/H has a finite G-invariant measure, where $H = \{x \in G \mid \alpha(x) = x\}$. Then α fixes a compact open subgroup of G that is normalised by H and G/H is compact.*

Proof. By Proposition 4.1, $\overline{U_\alpha} = \overline{U_{\alpha^{-1}}} = U_0 = \overline{U_\alpha} \cap \overline{U_{\alpha^{-1}}}$. Since α is expansive on G, Lemma 1.1 of [3] implies that $U_\alpha U_{\alpha^{-1}}$ is open, and hence U_0 is open. Since $\alpha(x) = x$ for all $x \in H$ and $U_0 = \overline{U_\alpha}$, it follows that H normalises U_0. Since G/H has a finite G-invariant measure, G/HU_0 is a discrete G-space with a finite G-invariant measure and hence G/HU_0 is finite. Thus, G/H is compact. □

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