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Divergence of solutions of polynomial finite-difference equations

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A theorem is proven for kth-order polynomial finite-difference equations that guarantees the divergence of solutions. A 'basin of divergence' is characterized and an order of divergence is provided. The basin of divergence is shown to depend on k independent parameters. An unconventional compactification method is used. Applications include the multi-step method in the numerical integration of ordinary differential equations, quadratic equations and the Henon map.

1. Introduction

Let u_n, $n = 0, 1, 2, \ldots,$ denote scalars. Let y, y_n, x, x_n and p be column vectors in $\mathbb{R}^k$. Let

$$\begin{aligned} y^\dagger &= (y^1, y^2, \ldots, y^k), & y_n^\dagger &= (y_n^1, y_n^2, \ldots, y_n^k), \\ x^\dagger &= (x^1, x^2, \ldots, x^k), & x_n^\dagger &= (x_n^1, x_n^2, \ldots, x_n^k), \\ p^\dagger &= (p^1, p^2, \ldots, p^k) \end{aligned}$$

denote row vectors that are the transposes of y, y_n, x, x_n, p, respectively. Let $h(y)$ be a scalar polynomial of degree $L > 1$. The primary result of this paper is the following theorem.

Theorem 1.1. *Given the kth-order polynomial finite-difference equation, we have*

$$u_{n+k} = h(u_n, u_{n+1}, \ldots, u_{n+k-1}) = h_0 + h_1 + \cdots + h_L, \tag{1.1}$$

where $h_j(u_n, u_{n+1}, \ldots, u_{n+k-1})$ are homogeneous polynomials of degree j.

Let $L > 1$, L being the degree of $h_L(u_n, u_{n+1}, \ldots, u_{n+k-1})$ and let the scalar p^k be such that

$$[p^k]^2 = 1, \qquad [p^k]^{L+1} = \frac{h_L(0, 0, \ldots, 0, p^k)}{|h_L(0, 0, \ldots, 0, p^k)|}. \tag{1.2}$$

Then, (1.1) possesses a k-parameter family of solutions such that

$$\lim_{n\to\infty} \frac{u_{n+j-1}}{u_{n+k-1}} = 0, \quad j = 1, 2, \ldots, k-1, \qquad \lim_{n\to\infty} u_n = \pm\infty. \tag{1.3}$$

The + or − sign that is used in (1.3) depends on whether p^k *in (1.2) is* $+1$ *or* -1*, respectively.*

REMARK 1.2. If L is odd and $h(0,0,\dots,0,1)<0$, then the above conclusion need not hold. The reader will find it easy to substantiate this remark. For example, one can show that the solution sequence to $u_{n+1}=-u_n^L$ possesses one subsequence that converges to ∞ and another subsequence that converges to $-\infty$.

An immediate conclusion of theorem 1.1 is the following.

COROLLARY 1.3. *No finite fixed point of (1.1) that is subject to (1.3) can be globally asymptotically stable.*

kth-order polynomial difference equations occur in numerous instances in both theory and applications. They are a means for modelling natural phenomena. They also occur in conjunction with polynomial differential equations when their discretization is called for in numerical approximations. This is substantiated by a voluminous literature. For modelling and various applications, see, for example, [1, 4, 5, 7, 13, 21, 24, 31]. For a partial list of the immense literature on dynamical systems, see, for example, [4, 18, 21, 26, 28–30]. For the numerical treatment of ordinary differential equations, see, for example, [4,6,19,20,24]. For textbooks that treat finite-difference equations and systems, see, for example, [4, 11, 22–24, 30, 31]. Theorem 1.1 could shed some light on the stability and instability of numerical methods associated with polynomial autonomous differential equations. It is noteworthy that multi-step numerical methods applied to polynomial autonomous differential equations lead in a natural way to kth-order polynomial autonomous difference equations. Moreover, the location of a bounded invariant set, of a bounded attractor or of a strange attractor is of considerable interest in studies of polynomial autonomous difference equations. Therefore, theorem 1.1 could help in searching for these sets, as it guarantees the existence of a basin of divergence. It goes without saying that the location of a basin of divergence is mutually exclusive to, say, the location of a bounded attractor set. In a chaotic system, slight errors in initial conditions could give rise to results that differ wildly from the correct result. In this sense, chaos is also intimately related to stability. The fact that a kth-order polynomial finite-difference equation could possess a k-parameter family of solutions that diverge reflects its affinity to becoming chaotic for a large set of large initial data. For an intimate relation between stability and chaos see [10]. For related work on asymptotic behaviour see [8, 15, 16].

Theorem 1.1 will be proved in several steps. The proof uses an unconventional mapping as a compactification tool. Unlike [2, 3, 17, 26, 28, 29], here the compactification proposed in [14] that was applied in [12] to dynamical systems is used. The content of the remaining sections of this paper is as follows. Sections 2 and 3 bring to the fore a few lemmas that are needed in order to prove theorem 1.1. In § 2 the original difference equation (1.1) that engages the scalar variables u_n is converted into an equivalent vector difference equation that engages a new vector variable y_n. The latter vector equation is then transformed by the mapping $y_n=(1-x_n^{\dagger}x_n)^{-1}x_n$ into a compactified difference vector system that engages the bounded variable x_n. Section 3 elaborates on a perturbation lemma. Section 4 concludes the proof of theorem 1.1. Section 5 discusses some applications.

2. Preliminary lemmas

LEMMA 2.1. *Define*

$$y_n^1 := u_n, \qquad y_n^j := u_{n+j-1}, \dots, y_n^k := u_{n+k-1}, \qquad y_n^\dagger = [y_n^1, y_n^2, \dots, y_n^k]. \tag{2.1}$$

The scalar equation (1.1) can be cast in the form

$$\left.\begin{aligned} u_{n+k} = h(y_n) = h_0 + h_1(y_n) + \cdots + h_{L-1}(y_n) + h_L(y_n), \\ h_j(y_n) = h_j(y_n^1, y_n^2, \dots, y_n^k), \end{aligned}\right\} \tag{2.2}$$

where $h_j(y_n)$, $j = 0, 1, 2, \dots, L$, $L > 1$, *are homogeneous scalar polynomials, in the variables* $y_n^1, y_n^2, \dots, y_n^k$, *of degree* j.

Moreover, the scalar equation (1.1) is equivalent to the first-order system of difference equations

$$y_{n+1} = f(y_n) = \begin{bmatrix} y_n^2 \\ y_n^3 \\ \vdots \\ y_n^k \\ h(y_n) \end{bmatrix} = f_0 + f_1(y_n) + \cdots + f_{L-1}(y_n) + f_L(y_n), \tag{2.3}$$

where

$$f_0 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ h_0 \end{bmatrix}, \qquad f_1(y_n) = \begin{bmatrix} y_n^2 \\ y_n^3 \\ \vdots \\ y_n^k \\ h_1(y_n) \end{bmatrix}, \qquad f_j(y_n) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ h_j(y_n) \end{bmatrix}, \quad j = 2, \dots, L,$$

and $f_j(y_n)$, $j = 0, 1, 2, \dots, L$, *are homogeneous vector polynomials, in the variables* y_n^1, $y_n^2, \dots, y_n^k$, *of degree* j.

Proof. It is easily verified that the relations (2.1) and (2.2) are equivalent to (1.1). This is a standard procedure of converting a kth-order finite-difference equation into a 'companion' vector system (see, for example, [11, 22]). □

The following lemma is based on [14] and elaborates on properties of an unconventional compactification mapping.

LEMMA 2.2. *Denote by* U *the unit ball and by* ∂U *its boundary:*

$$U := \{x_n \in \mathbb{R}^k \mid x_n^\dagger x_n \leqslant 1\}, \qquad \partial U := \{x_n \in \mathbb{R}^k \mid x_n^\dagger x_n = 1\}. \tag{2.4}$$

Define

$$r_n := \sqrt{y_n^\dagger y_n} = \|y_n\|, \qquad \|x_n\| := \sqrt{x_n^\dagger x_n} = R_n. \tag{2.5}$$

Consider the transformation

$$y_n = (1 - R_n^2)^{-1} x_n, \tag{2.6}$$

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and define its inverse by the branch

$$x_n = \frac{2y_n}{1+\sqrt{1+4y_n^\dagger y_n}}. \tag{2.7}$$

Then, the transformation (2.6) is a bijection from $\mathbb{R}^k$ *onto the interior of* U *and is also a bijection from the ideal set* $\mathrm{ID} := \{\infty p \mid p^\dagger p = 1\}$ *onto* ∂U*.*

Moreover,

$$r_n = \frac{R_n}{1-R_n^2}, \qquad R_n = \frac{2r_n}{1+\sqrt{1+4r_n^2}}.$$

Proof. This is proven in [14]. □

In the following, the difference system (2.3) is converted into a difference system that engages the vector variable x_n.

Lemma 2.3. *Define*

$$\begin{aligned}\tilde{f} &= \tilde{f}(x_n,(1-R_n^2))\\ &:= (1-R_n^2)^L f((1-R_n^2)^{-1}x_n)\\ &= f_L(x_n) + (1-R_n^2)^L f_0(x_n) + (1-R_n^2)^{L-1} f_1(x_n)\\ &\quad + \cdots + (1-R_n^2)^2 f_{L-2}(x_n) + (1-R_n^2)^1 f_{L-1}(x_n).\end{aligned} \tag{2.8}$$

Then, the transformation (2.6) takes the difference system (2.3) into

$$d_{n+1}x_{n+1} = \tilde{f}(x_n,(1-R_n^2)) = \tilde{f}, \quad x_{n+1} = d_{n+1}^{-1}\tilde{f}, \tag{2.9}$$

that are called the compactified equations, where

$$d_{n+1}^{-1} = \frac{2}{(1-R_n^2)^L + \sqrt{(1-R_n^2)^{2L} + 4\tilde{f}^\dagger\tilde{f}}}. \tag{2.10}$$

Proof. Upon substitution of (2.6) into (2.3) we obtain

$$\begin{aligned}y_{n+1} &= (1-R_{n+1}^2)^{-1}x_{n+1}\\ &= f(y_n)\\ &= f((1-R_n^2)^{-1}x_n)\\ &= (1-R_n^2)^{-L}\tilde{f}(x_n,(1-R_n^2)).\end{aligned} \tag{2.11}$$

Equation (2.11) is an implicit system of difference equations for x_{n+1}. In order to convert it into an explicit equation in x_{n+1}, we use the quantity

$$d_{n+1} := (1-R_{n+1}^2)^{-1}(1-R_n^2)^L, \quad (1-R_{n+1}^2) \neq 0. \tag{2.12}$$

d_{n+1} is ill behaved as

$$\lim_{n\to\infty}\|y_n\| = \infty.$$

This is the case because then

$$\lim_{n\to\infty}(1-R_{n+1}^2) = \lim_{n\to\infty}(1-R_n^2) = 0.$$

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However, a straightforward calculation yields a pleasant surprise. Using the solution of (2.13),

$$(1 - R_{n+1}^2)^2 + a_n^2(1 - R_{n+1}^2) - a_n^2 = 0, \tag{2.13}$$

we obtain

$$(1 - R_{n+1}^2)^{-1} = \frac{(1 - R_n^2)^{2L} + \sqrt{(1 - R_n^2)^{4L} + 4(1 - R_n^2)^{2L}\tilde{f}^\dagger\tilde{f}}}{2(1 - R_n^2)^{2L}}. \tag{2.14}$$

Consequently,

$$d_{n+1} := \frac{(1 - R_n^2)^L}{(1 - R_{n+1}^2)} = \frac{(1 - R_n^2)^L + \sqrt{(1 - R_n^2)^{2L} + 4\tilde{f}^\dagger\tilde{f}}}{2}, \tag{2.15}$$

and the result follows. □

It will now be shown that certain vectors p, $p^\dagger = (0, 0, \dots, 0, p^k)$, $[p^k]^2 = 1$ solve a certain nonlinear eigenvalue problem. These vectors p will be instrumental in the determination of certain 'fixed points at infinity' for

$$u_{n+k} = h(u_n, u_{n+1}, \dots, u_{n+k-1})$$

or, equivalently, for the companion system (2.3). In order to deal soundly with vector functions that diverge to infinity, which is a central theme in this paper, we consider the ideal set ID $:= \{\infty p \mid p^\dagger p = 1\}$ and add the following.

Definition 2.4. Define $\lim_{n\to\infty} y_n = \infty p$ if and only if

$$\lim_{n\to\infty} \|y_n\| = \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{\|y_n\|} y_n = p. \tag{2.16}$$

The following lemma holds.

Lemma 2.5.

(i) $\lim_{n\to\infty} y_n = \infty p$ *if and only if* $\lim_{n\to\infty} x_n = p$.

(ii) *If* $\lim_{n\to\infty} y_n = \infty p$, *then* p *must solve the nonlinear eigenvalue problem*

$$sp = f_L(p) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ h_L(p) \end{bmatrix}, \qquad s = |h_L(p)|. \tag{2.17}$$

(iii) *The relation (2.17) possesses a solution* p *if there exists* p^k *defined by the implicit relation*

$$p^k := \lim_{x_n \to p} |h_L(x_n)|^{-1} h_L(x_n) \tag{2.18}$$

such that

$$p^\dagger p = 1, \quad p = \begin{bmatrix} p^1 \\ p^2 \\ \vdots \\ p^k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ |h_L(p)|^{-1} h_L(p) \end{bmatrix} \implies p^j = 0, \quad j = 1, 2, \dots, k-1,$$

$$[p^k]^2 = 1. \tag{2.19}$$

The relation $|h_L(p)|^{-1} h_L(p)$ *is then defined as* p^k *even when* $h_L(p) = 0$.

Proof. Part (i) follows from the relation (2.7). Part (ii) follows by taking the limit $x_n \to p$ in the compactified equation $d_{n+1} x_{n+1} = \tilde{f}(x_n, (1 - R_n^2))$ in (2.9). Note that, then, $\lim_{x_n \to p}(1 - R_n^2) = 0$ and, therefore,

$$\begin{aligned} \lim_{x_n \to p} \frac{(1 - R_n^2)^L + \sqrt{(1 - R_n^2)^{2L} + 4\tilde{f}^\dagger \tilde{f}}}{2} x_{n+1} &= sp = f_L(p) \\ &= \lim_{x_n \to p} \tilde{f}(x_n, (1 - R_n^2)). \end{aligned} \tag{2.20}$$

If, in addition, we assume that $h_L(0, 0, \dots, 0, 1) \neq 0$, then

$$h_L(0, 0, \dots, 0, p^k) = h_L(0, 0, \dots, 0, 1)[p^k]^L, \tag{2.21}$$

and

$$\begin{aligned} p^k &= \frac{h_L(0, 0, \dots, 0, 1)[p^k]^L}{|h_L(0, 0, \dots, 0, 1)[p^k]^L|} \\ &= \begin{cases} +1 & \text{if } L \text{ is even and } h_L(0, 0, \dots, 0, 1) > 0, \\ -1 & \text{if } L \text{ is even and } h_L(0, 0, \dots, 0, 1) < 0, \\ \pm 1 & \text{if } L \text{ is odd and } \ h_L(0, 0, \dots, 0, 1) > 0, \\ \exp\left[\dfrac{(2m+1)\pi \mathrm{i}}{L-1}\right], \ m = 0, 1, 2, \dots & \text{if } L \text{ is odd and } \ h_L(0, 0, \dots, 0, 1) < 0. \end{cases} \end{aligned} \tag{2.22}$$

Thus, it has been demonstrated that if $h_L(0, 0, \dots, 0, 1) \neq 0$, then some vector p, $p^\dagger = (0, 0, \dots, 0, p^k)$ with $[p^k]^2 = 1$ is a solution of (2.17) unless L is odd and $h_L(0, 0, \dots, 0, 1) < 0$. □

3. A perturbation lemma

The next aim is to transform the compactified equations (2.9) into an equivalent finite-difference system. This is done in the following perturbation lemma.

LEMMA 3.1.

(i) *Let* $y_{n+1} = f(y_n)$ *possess a fixed point* ∞p.

(ii) *Assume that* $f_L(p) \neq \mathbf{0}$. *Then, the compactified equation* $x_{n+1} = d_{n+1}^{-1}\tilde{f}$ *is equivalent to*

$$x_{n+1} - p = A(x_n - p) + g, \tag{3.1}$$

where

$$A := [f_L^\dagger(p) f_L(p)]^{-1/2}[I - pp^\dagger][Jf_L(p) - 2f_{L-1}(p)p^\dagger], \qquad g = \mathcal{O}(\|x_n - p\|^2), \tag{3.2}$$

as $x_n \to p$ *and* $g = g((x_n - p))$ *has a Taylor series expansion that is absolutely convergent in a disc with centre at* $x_n = p$.

Proof. Let a, b, c be three k-dimensional column vectors. Then one can easily verify that the following (non-associative and non-commutative) relations hold:

$$(a^\dagger b)c = (b^\dagger a)c = (ca^\dagger)b = (cb^\dagger)a. \tag{3.3}$$

This is so because

$$(a^\dagger b)c = \begin{bmatrix} c_1b_1 & c_1b_2 & \cdots & c_1b_k \\ c_2b_1 & c_2b_2 & \cdots & c_2b_k \\ \vdots & \vdots & \ddots & \vdots \\ c_kb_1 & c_kb_2 & \cdots & c_kb_k \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \end{bmatrix} = (cb^\dagger)a = (ca^\dagger)b. \tag{3.4}$$

Next we expand $d_{n+1}^{-1}\tilde{f}$ in a Taylor series expansion up to second-order terms in $(x_n - p)$. Evidently,

$$\left.\begin{aligned} 1 - R_n^2 &= -[2p^\dagger + (x_n - p)^\dagger](x_n - p), \\ (1 - R_{n+1}^2) &= -[2p^\dagger + (x_{n+1} - p)^\dagger](x_{n+1} - p). \end{aligned}\right\} \tag{3.5}$$

Hence, as $x_n \to p$, we have

$$1 - R_n^2 = \mathcal{O}(\|x_n - p\|), \qquad (1 - R_n^2)^L = \mathcal{O}(\|x_n - p\|^L). \tag{3.6}$$

We focus now on the expansion of $\tilde{f}$ into a polynomial that depends on the variable $(x_n - p)$. Note that

$$f_L(x_n) = \{f_L(p) + [Jf_L(p)](x_n - p) + \Delta_2\}, \tag{3.7}$$

where

$$\Delta_2 := f_L(x_n) - \{f_L(p) + [Jf_L(p)](x_n - p)\} = \mathcal{O}(\|x_n - p\|^2) \tag{3.8}$$

and Δ_2 is a polynomial in the variable $(x_n - p)$. We focus on the contribution of $(1 - R_n^2)^1 f_{L-1}(x_n)$ in $\tilde{f}$ to the Jacobian A. With the help of (3.5), we have

$$\begin{aligned} (1 - R_n^2)^1 f_{L-1}(x_n) &= -[2p^\dagger + (x_n - p)^\dagger](x_n - p)[f_{L-1}(p) + (f_{L-1}(x_n) - f_{L-1}(p))] \\ &= -2p^\dagger(x_n - p)f_{L-1}(p) + \Delta_3, \end{aligned}$$

where

$$\Delta_3 := (1 - R_n^2)^1 f_{L-1}(x_n) + 2p^\dagger(x_n - p)f_{L-1}(p) = O(\|x_n - p\|^2) \quad \text{as } x_n \to p,$$

and Δ_3 is a polynomial in the variable $(x_n - p)$. By (3.3) we obtain

$$(1 - R_n^2)^1 f_{L-1}(x_n) = -2[f_{L-1}(p)p^\dagger](x_n - p) + \Delta_3.$$

By (3.5) we have

$$\begin{aligned}\Delta_4 &:= (1 - R_n^2)^L f_0(x_n) + (1 - R_n^2)^{L-1} f_1(x_n) + \cdots + (1 - R_n^2)^2 f_{L-2}(x_n) \\ &= \mathcal{O}(\|x_n - p\|^2) \end{aligned} \tag{3.9}$$

as $x_n \to p$. Δ_4 is a polynomial in the variable $(x_n - p)$. A summary of the above implies that

$$\tilde{f}(x_n, (1 - R_n^2)) = f_L(p) + [Jf_L(p) - 2f_{L-1}(p)p^\dagger](x_n - p) + \Delta_5, \tag{3.10}$$

where

$$\Delta_5 := \Delta_2 + \Delta_3 + \Delta_4 = \mathcal{O}(\|x_n - p\|^2) \tag{3.11}$$

as $x_n \to p$ and Δ_5 is a polynomial in the variable $(x_n - p)$. Next we expand the function d_{n+1}^{-1} as a Taylor series about the point p. We have

$$\begin{aligned} d_{n+1}^{-1} &= \frac{2}{(1 - R_n^2)^L + \sqrt{(1 - R_n^2)^{2L} + 4\tilde{f}^\dagger \tilde{f}}} \\ &= \frac{1}{\sqrt{\tilde{f}^\dagger \tilde{f}}} \sqrt{1 + \frac{(1 - R_n^2)^{2L}}{4\tilde{f}^\dagger \tilde{f}}} = [\tilde{f}^\dagger \tilde{f}]^{-1/2} \left[1 + \frac{(1 - R_n^2)^{2L}}{4\tilde{f}^\dagger \tilde{f}}\right]^{1/2}. \end{aligned} \tag{3.12}$$

Note that $\tilde{f}(p) = f_L(p)$ for $p^\dagger p = 1$. Due to (3.10), we have

$$W := f_L(p) + [Jf_L(p) - 2f_{L-1}(p)p^\dagger](x_n - p) + \Delta_5, \tag{3.13}$$

$$\begin{aligned}[\tilde{f}^\dagger \tilde{f}]^{-1/2} &= [W^\dagger W]^{-1/2} \\ &= \{f_L^\dagger(p) f_L(p) + 2f_L^\dagger(p)[Jf_L(p) - 2f_{L-1}(p)p^\dagger](x_n - p) + \Delta_6\}^{-1/2}, \end{aligned} \tag{3.14}$$

where

$$\Delta_6 := W^\dagger W - \{f_L^\dagger(p) f_L(p) + 2f_L^\dagger(p)[Jf_L(p) - 2f_{L-1}(p)p^\dagger](x_n - p)\}, \tag{3.15}$$

and Δ_6 is a polynomial in the variable $(x_n - p)$. It is now possible to obtain

$$\begin{aligned} &[f_L^\dagger(p) f_L(p)]^{-1/2} \left[1 + \frac{2f_L^\dagger(p)[Jf_L(p) - 2f_{L-1}(p)p^\dagger](x_n - p) + \Delta_6}{f_L^\dagger(p) f_L(p)}\right]^{-1/2} \\ &\quad = [f_L^\dagger(p) f_L(p)]^{-1/2} \left[1 - \frac{f_L^\dagger(p)[Jf_L(p) - 2f_{L-1}(p)p^\dagger](x_n - p) + \frac{1}{2}\Delta_6}{f_L^\dagger(p) f_L(p)} + \Delta_7\right], \end{aligned} \tag{3.16}$$

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where $\triangle_7$ is defined by

$$\Delta_7 := \left[1 + \frac{2f_L^\dagger(p)[Jf_L(p) - 2f_{L-1}(p)p^\dagger](x_n - p) + \Delta_6}{f_L^\dagger(p)f_L(p)}\right]^{-1/2} - \left[1 - \frac{f_L^\dagger(p)[Jf_L(p) - 2f_{L-1}(p)p^\dagger](x_n - p) + \frac{1}{2}\Delta_6}{f_L^\dagger(p)f_L(p)}\right]. \tag{3.17}$$

By virtue of the Taylor series expansions

$$\Delta_6 := \mathcal{O}(\|x_n - p\|^2), \qquad \Delta_7 := \mathcal{O}(\|x_n - p\|^2) \quad \text{as } x_n \to p, \tag{3.18}$$

Δ_7 is an absolutely convergent Taylor series in the vector variable $(x_n - p)$ in a disc $\{x_n | \|x_n - p\| \leqslant \gamma_1\}$, for some positive γ_1. Now combine (3.16) with (3.12) to obtain

$$\begin{aligned} d_{n+1}^{-1} &= [\tilde{f}^\dagger \tilde{f}]^{-1/2}\left[1 + \frac{(1 - R_n^2)^{2L}}{4\tilde{f}^\dagger \tilde{f}}\right]^{1/2} \\ &= [f_L^\dagger(p)f_L(p)]^{-1/2} \\ &\quad \times \left[1 - \frac{f_L^\dagger(p)[Jf_L(p) - 2f_{L-1}(p)p^\dagger](x_n - p)}{f_L^\dagger(p)f_L(p)} + \Delta_7\right]\left[1 + \frac{(1 - R_n^2)^{2L}}{4\tilde{f}^\dagger \tilde{f}}\right]^{1/2}. \end{aligned} \tag{3.19}$$

Thus,

$$d_{n+1}^{-1} = [f_L^\dagger(p)f_L(p)]^{-1/2} - [f_L^\dagger(p)f_L(p)]^{-3/2} f_L^\dagger(p)[Jf_L(p) - 2f_{L-1}(p)p^\dagger](x_n - p) + \Delta_8, \tag{3.20}$$

where

$$\begin{aligned} \Delta_8 &:= d_{n+1}^{-1} - [f_L^\dagger(p)f_L(p)]^{-1/2} \\ &\qquad - [f_L^\dagger(p)f_L(p)]^{-3/2} f_L^\dagger(p)[Jf_L(p) - 2f_{L-1}(p)p^\dagger](x_n - p) \\ &= \mathcal{O}(\|x_n - p\|^2) \end{aligned} \tag{3.21}$$

as $x_n \to p$. Δ_8 is an absolutely convergent Taylor series in the vector variable $(x_n - p)$ in a disc $\{x_n | \|x_n - p\| \leqslant \gamma_2\}$, for some positive γ_2.

With the aid of (3.10), we have

$$d_{n+1}^{-1}\tilde{f} = d_{n+1}^{-1}f_L(p) + Q + \Delta_9, \tag{3.22}$$

where

$$Q := d_{n+1}^{-1}[Jf_L(p) - 2f_{L-1}(p)p^\dagger](x_n - p), \qquad \Delta_9 := d_{n+1}^{-1}\Delta_5, \tag{3.23}$$

and Δ_9 is an absolutely convergent Taylor series in the vector variable $x_n - p$ in a disc $\{x_n \mid \|x_n - p\| \leqslant \gamma_3\}$ for some positive γ_3. Recall (3.20). An elaboration on $d_{n+1}^{-1}f_L(p)$ reveals that with

$$S := [f_L^\dagger(p)f_L(p)]^{-1/2} - [f_L^\dagger(p)f_L(p)]^{-3/2} f_L^\dagger(p)[Jf_L(p) - 2f_{L-1}(p)p^\dagger](x_n - p) + \Delta_8,$$

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we have

$$\begin{aligned} d_{n+1}^{-1} f_L(p) &= S f_L(p) \\ &= [f_L^\dagger(p) f_L(p)]^{-1/2} f_L(p) \\ &\quad - \{[f_L^\dagger(p) f_L(p)]^{-3/2} f_L^\dagger(p)[J f_L(p) - 2 f_{L-1}(p) p^\dagger](x_n - p)\} f_L(p) + \Delta_{10}. \end{aligned}$$

After modifying the middle term in the formula above for $d_{n+1}^{-1} f_L(p)$ according to (3.3), we obtain

$$\begin{aligned} d_{n+1}^{-1} f_L(p) = [f_L^\dagger(p) f_L(p)]^{-1/2} f_L(p) & \\ - [f_L^\dagger(p) f_L(p)]^{-3/2} [f_L(p) f_L^\dagger(p)][J f_L(p) - 2 f_{L-1}(p) p^\dagger](x_n - p) & \\ + \Delta_{10}. & \end{aligned} \tag{3.24}$$

Δ_{10} is an absolutely convergent Taylor series in the vector variable $(x_n - p)$ in a disc $\{x_n | \|x_n - p\| \leqslant \gamma_4\}$, for some positive γ_4. Note that

$$[f_L^\dagger(p) f_L(p)]^{-1/2} f_L(p) = p$$

because of (2.17). Also, take into consideration that

$$\Delta_{10} := \Delta_8 f_L(p) = \mathcal{O}(\|x_n - p\|^2) \quad \text{as } x_n \to p.$$

All of these imply that

$$\begin{aligned} d_{n+1}^{-1} f_L(p) = p - [f_L^\dagger(p) f_L(p)]^{-3/2} [f_L(p) f_L^\dagger(p)][J f_L(p) - 2 f_{L-1}(p) p^\dagger](x_n - p) & \\ + \mathcal{O}(\|x_n - p\|^2). & \end{aligned} \tag{3.25}$$

Turn now to the middle term Q in (3.22). By definition, we have

$$\begin{aligned} Q = \{[f_L^\dagger(p) f_L(p)]^{-1/2} - [f_L^\dagger(p) f_L(p)]^{-3/2} f_L^\dagger(p)[J f_L(p) - 2 f_{L-1}(p) p^\dagger](x_n - p) + \Delta_8\} & \\ \times [J f_L(p) - 2 f_{L-1}(p) p^\dagger](x_n - p). & \end{aligned}$$

Consequently,

$$Q = [f_L^\dagger(p) f_L(p)]^{-1/2} [J f_L(p) - 2 f_{L-1}(p) p^\dagger](x_n - p) + \Delta_{11}. \tag{3.26}$$

Δ_{11} is an absolutely convergent Taylor series in the vector variable $x_n - p$ in a disc $\{x_n | \|x_n - p\| \leqslant \gamma_5\}$, for some positive γ_5. By its definition,

$$\Delta_{11} := Q - [f_L^\dagger(p) f_L(p)]^{-1/2} [J f_L(p) - 2 f_{L-1}(p) p^\dagger](x_n - p) = \mathcal{O}(\|x_n - p\|^2), \tag{3.27}$$

as $x_n \to p$. By the discussion above we conclude that

$$\begin{aligned} x_{n+1} &= d_{n+1}^{-1} \tilde{f} \\ &= p - [f_L^\dagger(p) f_L(p)]^{-3/2} [f_L(p) f_L^\dagger(p)][J f_L(p) - 2 f_{L-1}(p) p^\dagger](x_n - p) \\ &\quad + [f_L^\dagger(p) f_L(p)]^{-1/2} [J f_L(p) - 2 f_{L-1}(p) p^\dagger](x_n - p) + \Delta_{12}, \end{aligned} \tag{3.28}$$

where

$$\Delta_{12} := \Delta_{10} + \Delta_{11} = \mathcal{O}(\|x_n - p\|^2) \quad \text{as } x_n \to p.$$

Δ_{12} is an absolutely convergent Taylor series in the vector variable $(x_n - p)$ in a disc $\{x_n \| \|x_n - p\| \leqslant \gamma_6\}$, for some positive γ_6. Note that $f_L(p) f_L^\dagger(p) = [f_L^\dagger(p) f_L(p)]^1 pp^\dagger$. Hence, the expression

$$[f_L^\dagger(p) f_L(p)]^{-3/2} [f_L(p) f_L^\dagger(p)]$$

in (3.28) simplifies to $[f_L^\dagger(p) f_L(p)]^{-1/2} pp^\dagger$. Keeping this simplification in mind and transferring the term p from the right-hand side of (3.28) to the left yields

$$\begin{aligned} x_{n+1} - p = \Delta_{12} + [f_L^\dagger(p) f_L(p)]^{-1/2} \\ \times \{[J f_L(p) - 2 f_{L-1}(p) p^\dagger] - pp^\dagger [J f_L(p) - 2 f_{L-1}(p) p^\dagger]\}(x_n - p). \end{aligned} \tag{3.29}$$

Finally, we have $x_{n+1} - p = A(x_n - p) + \Delta_{12}$ with A and Δ_{12} of the desired form. □

4. Concluding the proof of theorem 1.1

The conclusion of the proof of theorem 1.1 involves two steps. The first step elaborates on the Jacobian A for the special companion system (2.3). The second step then consists of a fixed-point argument.

Proof. We shall now determine the form of the Jacobian A in (3.1) for the special case of (2.3) and p, $p^\dagger = (0, 0, \ldots, 0, p^k)$, $[p^k]^2 = 1$. Observe that

$$\left.\begin{aligned} [\tilde{f}_L^\dagger(p) \tilde{f}_L(p)]^{-1/2} &= |a_L|^{-1}, \\ [I - pp^\dagger] = I - \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ p^k \end{bmatrix} [0, 0, \ldots, 0, p^k] &= \begin{bmatrix} I_{k-1} & O_{12} \\ O_{21} & 0 \end{bmatrix}. \end{aligned}\right\} \tag{4.1}$$

I_{k-1} is the $(k-1) \times (k-1)$ identity matrix, O_{12} is the $(k-1) \times 1$ matrix with entries that are zero and O_{21} is the $1 \times (k-1)$ matrix with entries that are zero. We must now distinguish between two cases: the case where $L = 2$ is to be treated after the treatment of the simpler case, where $L > 2$ is analysed.

We have with $L > 2$ that

$$J f_L(p) = \begin{bmatrix} O_1 \\ \nabla h_L(p) \end{bmatrix}, \tag{4.2}$$

$$-2 f_{L-1}(p) p^\dagger = -2 \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ h_{L-1}(p) \end{bmatrix} [0, 0, \ldots, 0, p^k] = -2 \begin{bmatrix} O_{11} & O_{12} \\ O_{21} & p^k h_{L-1}(p) \end{bmatrix}, \tag{4.3}$$

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where O_{11} is the $(k-1)\times(k-1)$ matrix with entries that are zero. Next we have

$$\left.\begin{aligned} Jf_L(p) - 2f_{L-1}(p)p^\dagger &= \begin{bmatrix} O_{11} & O_{12} \\ \varphi & \psi \end{bmatrix}, \\ \varphi &:= \left[\frac{\partial h_L}{\partial y^1}(p), \frac{\partial h_L}{\partial y^2}(p), \dots, \frac{\partial h_L}{\partial y^{k-1}}(p)\right], \\ \psi &:= \frac{\partial h_L}{\partial y^k}(p) - 2p^k h_{L-1}(p). \end{aligned}\right\} \tag{4.4}$$

φ is the $1\times(k-1)$ matrix with entries that are defined by the first $(k-1)$ components of $\nabla h_L(p)$ evaluated at the point p, $p^\dagger = [0,0,\dots,0,p^k]$. Consequently, we have

$$\begin{aligned} A &= -2|a_L|^{-1} \begin{bmatrix} I_{k-1} & O_{12} \\ O_{21} & 0 \end{bmatrix} \begin{bmatrix} O_{11} & O_{12} \\ \varphi & \psi \end{bmatrix} \\ &= -2|a_L|^{-1} \begin{bmatrix} I_{k-1}O_{11} + O_{12}\varphi & I_{k-1}O_{12} + O_{12}\psi \\ O_{21}O_{11} + 0\varphi & O_{21}O_{12} + 0\psi \end{bmatrix} \\ &= O. \end{aligned} \tag{4.5}$$

The matrix O has all of its entries zero. Recall that $g = \mathcal{O}(\|x_n - p\|^2)$ as $x_n \to p$ and $g = g((x_n - p))$ has a Taylor series expansion that is absolutely convergent in a disc with centre at $x_n = p$. This implies, with a suitable norm on matrices, that

$$x_{n+1} - p = g \equiv M((x_n - p))[x_n - p] \implies \|x_{n+1} - p\| \leqslant \|M((x_n - p))\| \|x_n - p\|. \tag{4.6}$$

$M((x_n - p))$ is a certain $k \times k$ matrix function that has a Taylor series expansion that is absolutely convergent in a disc $\{x_n \| \|x_n - p\| \leqslant \gamma\}$, for some positive γ. Moreover, $M(\mathbf{0}) = O$ and, without loss of generality, we may assume that, in the disc above, $\|M((x_n - p))\| \leqslant \sigma < 1$ for some constant σ. Hence, the fixed point p is asymptotically stable in the x_n space and its basin of attraction contains as a subset an entire disc. This in turn implies that there exists a non-trivial basin of divergence in the y space that depends on a k-parameter family of variables. Each point y_0 in this basin of divergence gives rise to a sequence y_n, $n = 0,1,2,\dots$, such that $\lim_{n\to\infty} y_n^\dagger = \infty[0,\dots,p^k]$.

The case that $L = 2$ requires modifications since the term $f_1(p)$ changes form in (4.3). Then we have

$$-2f_1(p)p^\dagger = -2 \begin{bmatrix} 0 \\ \vdots \\ 0 \\ p^k \\ h_1(p) \end{bmatrix} [0,0,\dots,0,p^k] = -2 \begin{bmatrix} \hat{O}_{11} & \hat{O}_{12} \\ \hat{O}_{21} & \theta \\ \hat{O}_{21} & \eta \end{bmatrix}, \tag{4.7}$$

with

$$\theta := [0,1], \qquad \eta := [0,\ p^k h_1(p)]. \tag{4.8}$$

The blocks in the above matrix are of the following sizes. $\hat{O}_{11}$ is a $(k-2)\times(k-2)$ matrix with all of its entries zero, $\hat{O}_{12}$ is a $(k-2)\times 2$ matrix with all of its entries

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zero, $\hat{O}_{21}$ is a $1\times(k-2)$ matrix with all of its entries zero. We now rewrite $Jf_L(p)$ as a 3×2 matrix with entries being blocks of the same size as the blocks of $-2f_1(p)p^\dagger$:

$$Jf_2(p) = \begin{bmatrix} O_1 \\ \nabla h_2(p) \end{bmatrix} = \begin{bmatrix} \hat{O}_{11} & \hat{O}_{12} & \\ \hat{O}_{21} & 0 & 0 \\ \phi & \tau & \omega \end{bmatrix}. \tag{4.9}$$

O_1 is a $(k-1)\times k$ matrix with all of its entries zero. The symbols ϕ, τ, ω stand for the following:

$$\phi := \left[\frac{\partial h_2}{\partial y^1}(p), \frac{\partial h_2}{\partial y^2}(p), \dots, \frac{\partial h_2}{\partial y^{k-2}}(p)\right], \qquad \tau := \frac{\partial h_2}{\partial y^{k-1}}(p), \qquad \omega := \frac{\partial h_2}{\partial y^k}(p). \tag{4.10}$$

Hence,

$$Jf_2(p) - 2f_1(p)p^\dagger = \begin{bmatrix} \hat{O}_{11} & \hat{O}_{12} & \\ \hat{O}_{21} & 0 & 0 \\ \phi & \tau & \omega \end{bmatrix} - 2\begin{bmatrix} \hat{O}_{11} & \hat{O}_{12} \\ \hat{O}_{21} & \theta \\ \hat{O}_{31} & \eta \end{bmatrix} = \begin{bmatrix} \hat{O}_{11} & \hat{O}_{12} \\ \hat{O}_{21} & [0\ -2] \\ \phi & [\tau\mu] \end{bmatrix}, \tag{4.11}$$

with

$$\mu := \frac{\partial h_2}{\partial y^k}(p) - p^k h_1(p). \tag{4.12}$$

We now need to express the matrix $[I - pp^\dagger]$ as a 3×3 matrix with entries that are blocks. This is in preparation for the multiplication of $[I - pp^\dagger]$ by $[Jf_2(p) - 2f_1(p)p^\dagger]$ in order to obtain the final form of A. Let $\tilde{O}_{k-2}$ be a $(k-2)\times 1$ column vector, with all of its entries zero. We set

$$[I - pp^\dagger] = \begin{bmatrix} I_{k-1} & O_{12} \\ O_{21} & 0 \end{bmatrix} = \begin{bmatrix} I_{k-2} & \tilde{O}_{k-2} & \tilde{O}_{k-2} \\ \tilde{O}^\dagger_{k-2} & 1 & 0 \\ \tilde{O}^\dagger_{k-2} & 0 & 0 \end{bmatrix}. \tag{4.13}$$

We are now ready to compute

$$\begin{aligned} A &= -2|a_L|^{-1}\begin{bmatrix} I_{k-2} & \tilde{O}_{k-2} & \tilde{O}_{k-2} \\ \tilde{O}^\dagger_{k-2} & 1 & 0 \\ \tilde{O}^\dagger_{k-2} & 0 & 0 \end{bmatrix}\begin{bmatrix} \hat{O}_{11} & \hat{O}_{12} \\ \hat{O}_{21} & [0\ -2] \\ \phi & [\tau\mu] \end{bmatrix} \\ &= -2|a_L|^{-1}\begin{bmatrix} I_{k-2}\hat{O}_{11} + \tilde{O}_{k-2}\hat{O}_{21} + \tilde{O}_{k-2}\phi & I_{k-2}\hat{O}_{12} + \tilde{O}_{k-2}[0\ -2] + \tilde{O}_{k-2}[\tau\mu] \\ \tilde{O}^\dagger_{k-2}\hat{O}_{11} + 1\hat{O}_{21} + 0\phi & \tilde{O}^\dagger_{k-2}\hat{O}_{12} + 1[0\ -2] + 0[\tau\mu] \\ \tilde{O}^\dagger_{k-2}\hat{O}_{11} + 0\hat{O}_{21} + 0\phi & \tilde{O}^\dagger_{k-2}\hat{O}_{12} + 0[0\ -2] + 0[\tau\mu] \end{bmatrix} \\ &= 4|a_L|^{-1}\begin{bmatrix} \hat{O}_{11} & \tilde{O}_{k-2} & \tilde{O}_{k-2} \\ \tilde{O}^\dagger_{k-2} & 0 & 1 \\ \tilde{O}^\dagger_{k-2} & 0 & 0 \end{bmatrix}. \end{aligned} \tag{4.14}$$

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Thus, A is a $k \times k$ matrix with one non-zero entry in its $((k-1),k)$ location. It is then easily verified that $A^2 = O$. This has the following implications:

$$x_{n+1} - p = A(x_n - p) + g((x_n - p)) \implies x_{n+2} - p = A(x_{n+1} - p) + g((x_{n+1} - p)), \tag{4.15}$$

$$\begin{aligned} x_{n+2} - p &= A[A(x_n - p) + g((x_n - p))] + g([A(x_n - p) + g((x_n - p))]) \\ &= G((x_n - p)), \end{aligned} \tag{4.16}$$

where

$$G((x_n - p)) := \mathbf{0} + Ag((x_n - p)) + g([A(x_n - p) + g((x_n - p))]). \tag{4.17}$$

Evidently, $G((x_n - p)) = \mathcal{O}(\|x_n - p\|^2)$ as $x_n \to p$ and $G((x_n - p))$ has a Taylor series expansion that is absolutely convergent in a disc $\{x_n \| \|x_n - p\| \leqslant \gamma\}$ for some positive γ. Without loss of generality, we may assume that, in this disc, we have

$$\begin{aligned} x_{n+2} - p = G((x_n - p)) &\equiv M((x_n - p))[x_n - p] \\ &\implies \|x_{n+2} - p\| \leqslant \|M((x_n - p))\| \|x_n - p\|. \end{aligned} \tag{4.18}$$

$M((x_n - p))$ is a certain $k \times k$ matrix function. Moreover, $M(\mathbf{0}) = O$ and, without loss of generality, we may assume that, in the disc above, $\|M((x_n - p))\| \leqslant \sigma < 1$ for some constant σ. We consider the sequence x_n that satisfies

$$x_{n+1} - p = A(x_n - p) + g((x_n - p)) \quad \text{with } \|x_0\| < 1$$

as a union of two subsequences. The subsequence SE $:= \{x_0, x_{2,}, \dots\}$ (a sequence with even indexes) and the subsequence SO $:= \{x_1, x_{2,}, \dots\}$ (a sequence with odd indexes). We claim that if x_n, $n = 0, 1, 2, \dots$, is a sequence solution satisfying the compactified equation $x_{n+1} = d_{n+1}^{-1} \tilde{f}$, then

$$\|x_0\| < 1 \implies \|x_n\| < 1 \quad \text{for } n = 1, 2, \dots.$$

This follows by observing that $|d_{n+1}^{-1} \tilde{f}| < 1$. This, of course, also implies that none of the elements x_n, $n = 0, 1, 2, \dots$, is equal to p. Each subsequence satisfies the relation (4.18). Hence, the fixed point p is asymptotically stable in the x_n space and its basin of attraction contains an entire disc as a subset. This in turn implies that there exists a non-trivial basin of divergence in the y space. Each point y_0 in this basin of divergence gives rise to a sequence y_n, $n = 0, 1, 2, \dots$, such that

$$\lim_{n \to \infty} y_n^\dagger = \infty[0, \dots, p^k].$$

This implies that

$$\lim_{n \to \infty} y_n^k = \lim_{n \to \infty} \frac{x_n^k}{1 - x_n^\dagger x_n} = \lim_{n \to \infty} \frac{-[p^k + (x_n^k - p^k)]}{[2p^\dagger + (x_n - p)^\dagger](x_n - p)} = \infty p^k$$

and

$$\lim_{n \to \infty} \frac{u_{n+j-1}}{u_{n+k-1}} = \lim_{n \to \infty} \frac{y_n^j}{y_n^k} = \lim_{n \to \infty} \frac{x_n^j}{x_n^k} = \frac{0}{p^k} = 0, \quad j = 1, 2, \dots, k-1.$$

□

REMARK 4.1. It follows from the contraction relations (4.6) and (4.18) that the order of divergence is $\mathcal{O}(\mu^n)$ for some $\mu > 1$.

5. Some applications

A few applications of theorem 1.1 to specific equations are provided below.

EXAMPLE 5.1. The Henon map

$$u_{n+2} = h(u_n, u_{n+1}) := 1 + bu_n - au_{n+1}^2, \quad a \neq 0, \tag{5.1}$$

has attracted a considerable amount of attention. So far, studies of the Henon map in the literature exclude the analysis of fixed points at infinity (see, for example, [22, 31]). Theorem 1.1 makes it possible to broaden our understanding of this equation, of its direction field for large values of u_n and, of course, of its divergent solutions. First, note that, consistent with the notation in (1.1), the decomposition of h into homogeneous parts is as follows:

$$L = 2, \qquad h_0 = 1, \qquad h_1 = bu_n, \qquad h_2 = -au_{n+1}^2, \qquad h_2(0,1) = -a.$$

Consider the case where $a > 0$. We enquire first about the existence of fixed points of the compactified system (2.9) that relate to fixed points at infinity of the Henon map. In this case, we have

$$[p^k]^2 = 1, \qquad [p^k]^{2+1} = \frac{h_2(0,p^k)}{|h_2(0,p^k)|} = -1 \implies p^k = -1.$$

The initial conditions $y_0^1 = u_0$, $y_0^2 = u_1$ for (5.1) need to be chosen in such a manner that x_0^1 is close to zero and x_0^2 be is close to $p^k = -1$. This is done in order to satisfy the conditions of theorem 1.1 as given in the fixed-point arguments of § 4. We use the scalar version of the compactification mapping

$$u_n = y_n^1 = (1 - R_n^2)^{-1} x_n^1, \qquad u_{n+1} = y_n^2 = (1 - R_n^2)^{-1} x_n^2. \tag{5.2}$$

Note that if x_0^1 is close to zero and if x_0^2 is close to $p^k = -1$, then $(1 - R_0^2)^{-1}$ is large and positive. From the ratio of the two equations in (5.2) we obtain

$$\frac{u_0}{u_1} = \frac{y_0^1}{y_0^2} = \frac{x_0^1}{x_0^2},$$

and we can conclude that the ratio x_0^1/x_0^2 must be small. Therefore, there exists a positive number ε, $1 > \varepsilon > 0$ and a positive number $K > 0$ such that if $u_1 < -K$ and $|u_0/u_1| < \varepsilon$, then

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} u_{n+1} = -\infty \quad \text{and} \quad \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = 0.$$

Consider the case where $a < 0$. Then the following relations hold:

$$[p^k]^2 = 1, \qquad [p^k]^{2+1} = \frac{h_2(0,p^k)}{|h_2(0,p^k)|} = 1 \implies p^k = 1.$$

Theorem 1.1 then guarantees that there exists a positive number $K > 0$ and a positive number ε, $1 > \varepsilon > 0$, such that if $u_1 > K$ and $|u_0/u_1| < \varepsilon$, then

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} u_{n+1} = \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = 0.$$

A more general quadratic difference equation is given in the example below.

Example 5.2. The references (for example, [4, 9, 18, 21, 28]) bring to the fore a wealth of quadratic dynamical systems and equations. Numerous variants of their discretization lead to the second-order quadratic difference equations with real coefficients that are of the form

$$u_{n+2} = a_0 + a_{1,0}u_n + a_{0,1}u_{n+1} + a_{2,0}u_n^2 + a_{1,1}u_n u_{n+1} + a_2 u_{n+1}^2. \tag{5.3}$$

Modelling in various disciplines, with (5.3), is also described in, for example, [7, 11, 22]. The difference equation (5.3) is amenable to theorem 1.1 with $a_2 \neq 0$. The analysis of it is similar to the analysis of the Henon map. Theorem 1.1 guarantees that there exists a two-parameter family of solutions that diverge for each of the two cases, $a_2 < 0$ and $a_2 > 0$.

We now consider an example that is motivated by numerical analysis.

Example 5.3. A rich field for applications of theorem 1.1 is also provided by the multi-step method (see, for example, [6, 19]). Consider a polynomial scalar differential equation

$$u' = f(u) = a_0 + a_1 u + \cdots a_{L-1}u^{L-1} + a_L u^L, \quad a_L \neq 0,\ u, a_j \in \mathbb{R},\ j = 0, 1, 2, \dots, L,$$

with an initial condition $u(t_0) = u_0$.

A linear multi-step method will convert the above initial-value problem into the discrete form

$$\begin{aligned} u_{n+k} &= h(u_n, u_{n+1}, \dots, u_{n+k-1}) \\ &:= c_{k-1}u_{n+k-1} + c_{k-2}u_{n+k-2} + \cdots + c_0 u_n \\ &\qquad + \theta[b_k f(u_{n+k}) + b_{k-1} f(u_{n+k-1}) + b_{k-2} f(u_{n+k-2}) + \cdots + b_0 f(u_n)], \end{aligned} \tag{5.4}$$

where $\theta > 0$ denotes the time-step size and

$$c_{k-1}, c_{k-2}, \dots, c_0, b_k, b_{k-1}, b_{k-2}, \dots, b_0$$

are certain constants that determine the method. Consider the case where $b_k = 0$, namely, that the method is explicit, with $b_{k-1}a_L > 0$. We need to determine the highest degree term h_L on the right-hand side of (5.4). An examination of (5.4) reveals that

$$h_L = \theta a_L[b_{k-1}u_{n+k-1}^L + b_{k-2}u_{n+k-2}^L + \cdots + b_0 u_n^L], \tag{5.5}$$

which implies that

$$h_L(0, 0, \dots, 0, p^k) = \theta a_L b_{k-1} u_{n+k-1}^L. \tag{5.6}$$

The nonlinear eigenvalue relation (2.17) requires

$$[p^k]^2 = 1, \quad [p^k]^{L+1} = \frac{h_L(0,0,\dots,0,p^k)}{|h_L(0,0,\dots,0,p^k)|} = \frac{\theta a_L b_{k-1}[p^k]^L}{|\theta a_L b_{k-1}[p^k]^L|} = [p^k]^L \implies p^k = 1.$$

Fix $\theta > 0$. Theorem 1.1 then predicts that there are $K > 0$ and $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{k-2}$, $(k-1)$ positive numbers such that if $u_{k-1} > K$ and $|u_j/u_{k-1}| < \varepsilon_j$, $j = 0, 1, \dots,$ $(k-2)$, then (no matter how small the step size θ is) divergence will occur such that

$$\lim_{n\to\infty} \frac{u_{n+j-1}}{u_{n+k-1}} = 0, \ j = 1, 2, \dots, k-1, \qquad \lim_{n\to\infty} u_n = \infty.$$

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