Probability in the Engineering and Informational Sciences, **16**, 2002, 85–100. Printed in the U.S.A.

BOUNDS AND PERFORMANCE LIMITS OF CHANNEL ASSIGNMENT POLICIES IN CELLULAR NETWORKS

NICOLE BÄUERLE AND ANJA HOUDEK
Department of Mathematics VII
University of Ulm
89069 Ulm, Germany
E-mail: baeuerle@mathematik.uni-ulm.de

We investigate the performance of channel assignment policies for cellular networks. The networks are given by an interference graph which describes the reuse constraints for the channels. In the first part, we derive lower bounds on the expected (weighted) number of blocked calls under any channel assignment policy over finite time intervals as well as in the average case. The lower bounds are solutions of deterministic control problems. As far as the average case is concerned, the control problem can be replaced by a linear program. In the second part, we consider the cellular network in the limit, when the number of available channels as well as the arrival intensities are linearly increased. We show that the network obeys a functional law of large numbers and that a fixed channel assignment policy which can be computed from a linear program is asymptotically optimal. Special networks like fully connected and star networks are considered.

1. INTRODUCTION

Cellular telephony has become a standard in many countries and is still a rapidly growing part of the telecommunication industry. The coverage area is divided into mutually disjoint cells, each with its own base station in the center. A mobile customer in a cell can be connected via its base station to the international wire-line telephone network. The available bandwidth is divided into channels, which we assume to be frequency slots. Due to interference, neighboring cells are not permit-

 0269-9648/02 $12.50

https://doi.org/10.1017/S0269964802161067 Published online by Cambridge University Press

ted to use the same channel. However, the possibility of channel reuse exists for cells with a certain distance.

When a call request appears in cell i, then either a free channel which does not violate the interference constraints is assigned to this call or else it is blocked. This may also involve rearrangements of the channels assigned to calls already in progress. A common objective of channel assignment policies is to minimize the (weighted) number of blocked calls. Outstanding channel assignment policies are as follows.

1.1. Fixed Channel Assignment

The channels are permanently assigned to cells in such a way that the interference constraints are not violated. A call request in cell i is then accepted if there still exists a free channel in cell i.

1.2. Maximum Packing

A call is accepted, whenever this is possible. Channel reassignments may be necessary. Robinson [12] was one of the first to consider the problem as a Markov decision process. In principle, the optimal policy can be found by implementing the standard Markov decision algorithms (policy iteration algorithm, value iteration algorithm; see, e.g., Chapter 3 in Tijms [13] or Chapter 8 in Puterman [11]). Due to the large state space of the problem (the state space can grow exponentially in the number of cells), a numerical computation is often intractable. In several special cases, conditions can be given under which certain channel assignment policies are optimal. Robinson [12], for example, gives necessary and sufficient conditions for the optimality of a fixed channel assignment in the star network. Houdek [4] and Kind, Niessen, and Mathar [8] have shown that under sufficiently light traffic, maximum packing is optimal in any type of cellular network. In order to deal with the general problem, our focus in this article is restricted to asymptotically optimal policies. In McEliece and Sivarajan [10] a lower bound on the average number of blocked calls per channel in a cellular system has been derived which holds for any channel assignment policy. Moreover, it has been shown that this bound is asymptotically achieved by a fixed channel assignment policy when the number of channels as well as the traffic intensity are linearly increased. We will strengthen these results by deriving lower bounds on the number of blocked calls over finite time intervals, as well as in the average case. The limit behavior of the number of blocked calls under any fixed channel assignment policy will be given explicitly as a function of time. The analysis involves the proof of functional laws of large numbers and is in the spirit of Hunt and Laws [6], Hunt and Kurtz [5], and Alanyali and Hajek [1]. The asymptotic performance of the maximum packing policy has been investigated in Kulshreshtha and Sivarajan [9] (cf. also Kelly [7]).

Our article is organized as follows. In the next section, we present the model in the framework of Markov decision processes. In Section 3, we derive a lower bound on the expected number of lost calls over a finite time interval, as well as in the average case. The lower bounds are obtained as solutions of deterministic control problems. Section 4 contains some auxiliary results about the convergence of the state and action processes if the number of available channels as well as the arrival intensities are linearly increased. Next, we show that the value of the lower bound in the average case coincides with the value of a linear program. In Section 6, we characterize the limit behavior of the number of blocked calls under an arbitrary fixed channel assignment policy as the unique solution of an initial value problem. Further, we show that the fixed channel assignment policy which can be constructed from the solution of the linear program is asymptotically optimal in the sense that the lower bound will be achieved in the limit. Some numerical examples are given in Section 7.

2. FORMULATION AS A MARKOV DECISION PROCESS

Our cellular network consists of n cells and a set C of k channels, $C = \{1, \dots, k\}$. Call requests arrive in cell i according to a Poisson process with parameter $\lambda_i > 0$. The arrival processes for the cells are supposed to be independent of each other. A call request in cell i can be accepted if there is a free channel which can be assigned to this call. This may also involve rearrangements of the channels assigned to calls already in progress. Blocked calls are lost. Due to interference, a channel which is in use in cell i cannot be used simultaneously in a neighboring cell. We suppose that these restrictions are given by an interference graph $G = (V, E)$, where the cells form the set of vertices $V = \{1, \dots, n\}$ and an edge $(i, j) \in E$ indicates that cells i and j are neighbors and have to use different channels. Thus, the *state space* of our network is given by the set S of all admissible channel assignments

$$S = \{x \in \mathbb{N}_0^n \mid \text{there exist } M_1, \dots, M_n \subset C, \text{ s.t. } |M_i| = x_i, M_i \cap M_j = \emptyset, \ \forall (i, j) \in E\}.$$

For $x = (x_1, \dots, x_n) \in S$, x_i gives the number of channels which are in use in cell i. It has been shown in Kulshreshtha and Sivarajan [9] that S can also be written as

$$S = \left\{x \in \mathbb{N}_0^n \mid \text{there exists a } z \in \mathbb{N}_0^m, \text{ s.t. } Az \geq x, \sum_{j=1}^{m} z_j \leq k\right\}$$

with $A \in \{0,1\}^{(n,m)}$. The state process itself is denoted by $(X_t) = (X_1(t), \dots, X_n(t))$, where $X_i(t)$ gives the number of connected calls in cell i at time $t > 0$. All holding times of the calls are independent of each other and exponentially distributed with parameter $\mu_i > 0$ in cell i. Upon arrival of a new call request, we have to decide whether to accept (if possible) or reject the call. A randomized decision will also be allowed. The *action space* is therefore given by $A = [0,1]^n$, where a_i gives the probability with which the next call request in cell i will be accepted. Of course, the *set of admissible actions* in state x is given by $D(x) = \{a \in A | a_i > 0 \Rightarrow x + e_i \in S, i = 1, \dots, n\}$, where e_i denotes the ith unit vector. A (stationary) channel assignment policy for the Markov decision process is given by a decision rule $f: S \to A$ with $f(x) \in D(x)$. f selects (depending on the current state x) the acceptance probabilities of new calls for any cell. For a given channel assignment policy, the state process is obviously a continuous-time Markov chain. Now, suppose that the system is in state x and action a is chosen. The off-diagonal elements of the intensity matrix $Q = (q(x, a, x'))$ of the controlled state process are given by

https://doi.org/10.1017/S0269964802161067 Published online by Cambridge University Press

$$q(x,a,x') = \begin{cases} \lambda_i a_i, & x' = x + e_i \\ \mu_i x_i, & x' = x - e_i \\ 0 & \text{else.} \end{cases}$$

Our aim is to minimize the long-run average cost due to blocked calls. As cost rate function $c : S \times A \to \mathbb{R}_+$ we choose $c(x,a) = \sum_{i=1}^n (1-a_i)\lambda_i c_i$, with $c_i \in \mathbb{R}_+$. The terms $(1-a_i)\lambda_i$ are the rates of blocked calls in cell i, so $c(x,a)$ is a weighted sum of the blocking rates per cell. For an arbitrary channel assignment policy f, we define the associated long-run average cost, starting the system in state x by

$$G_f(x) = \limsup_{T\to\infty} \frac{1}{T} E_x\left[\int_0^T c(X_t, f(X_t))\, dt\right].$$

Since the corresponding Markov chain has only one positive recurrent class for all decision rules f, the long-run average cost does not depend on the starting state x (i.e., $G_f(x) = G_f$ for all $x \in S$). Hence, the optimization problem is given by

$$G = \inf_f G_f$$

and G is the minimal average cost. Let us first look at two special types of networks.

2.1. Star Network

A star network consists of a sun (cell 1) and planets (cells 2 through n) such that interference occurs only between the sun and each of the planets. A seven-cell star network is depicted in Figure 1. The state space S is here given by $S = \{x \in \mathbb{N}_0^n \mid x_1 + x_i \le k,\ i = 2,\dots,n\}$. In the case $\mu_1 = \cdots = \mu_n = 1$ and $c_1 = \cdots = c_n = 1$, Robinson [12] has shown that the fixed channel assignment which gives no channels to the sun and all channels to the planets is optimal if and only if

$$\sum_{i=2}^n \mathrm{Erl}_k(\lambda_i) \ge 1,$$

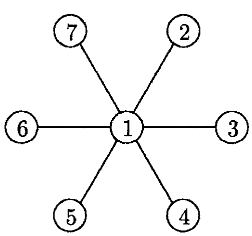

FIGURE 1. A star of seven vertices.

https://doi.org/10.1017/S0269964802161067 Published online by Cambridge University Press

where

$$\mathrm{Erl}_k(\rho) = \frac{\rho^k/k!}{\sum_{j=0}^{k} \rho^j/j!}$$

is the usual Erlang B-formula.

2.2. Fully Connected Network

A cellular network is called fully connected if any two cells are neighbors. A six-cell fully connected network is depicted in Figure 2. The state space S is given by $S = \{x \in \mathbb{N}_0^n | x_1 + \cdots + x_n \leq k\}$. Houdek [4] has given a condition for general networks under which maximum packing is optimal. For fully connected networks with $\mu_1 = \cdots = \mu_n = \mu$, this condition reduces to

$$\frac{\sum_{i=1}^{n} \lambda_i c_i}{\sum_{i=1}^{n} \lambda_i} \frac{\mathrm{Erl}_k(\rho)}{\mathrm{Erl}_{k-1}(\rho)} \leq \min_{i=1,\ldots,n} c_i$$

with $\rho = \sum_{i=1}^{n}(\lambda_i/\mu)$.

The following definitions will be used in the sequel. For a measurable function $v : \mathbb{R}_+^n \to \mathbb{R}$, the generator $\mathcal{A}_f$ of the state process is given by

$$\mathcal{A}_f v(x) = \sum_{x' \in S} (v(x') - v(x)) q(x, f(x), x').$$

Thus, if we plug in $v_i(x) = x_i$, $i = 1, \ldots, n$, we obtain

$$\mathcal{A}_f v_i(x) = \lambda_i f_i(x) - \mu_i x_i.$$

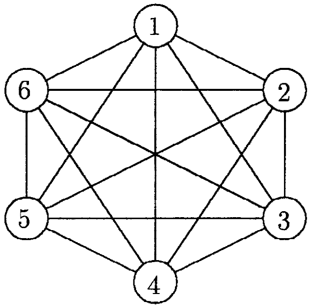

FIGURE 2. A complete graph of six vertices.

https://doi.org/10.1017/S0269964802161067 Published online by Cambridge University Press

In what follows, we denote $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $\mu = (\mu_1, \ldots, \mu_n)$. To simplify notation, we will write $x \circ y = (x_1 y_1, \ldots, x_n y_n)$ to denote the coordinatewise product of two vectors of equal dimension, and $x'y = \sum_{i=1}^n x_i y_i$ is the usual scalar product.

3. LOWER BOUNDS

Let us first define the state space

$$S^\infty = \left\{ x \in \mathbb{R}_+^n \mid \text{there exists a } z \in \mathbb{R}_+^m, \text{ s.t. } Az \geq x, \sum_{j=1}^m z_j \leq k \right\}.$$

Since S^∞ is the projection of a polyhedron, S^∞ is a polyhedron itself and thus can be written as $S^\infty = \{x \in \mathbb{R}_+^n \,|\, \bar{A}x \leq \bar{b}\}$ with a matrix $\bar{A}$ and a vector $\bar{b}$. Note that S^∞ is bounded. The following deterministic control problem will play a crucial role:

$$(C) \begin{cases} \limsup_{T\to\infty} \dfrac{1}{T} \displaystyle\int_0^T \left(\sum_{i=1}^n (1 - a_i(t)) \lambda_i c_i \right) dt \to \min, \\ x_t = x_0 + \displaystyle\int_0^t (\lambda \circ a_s - \mu \circ x_s)\, ds, \\ \bar{A}x_t \leq \bar{b}, \\ x_t \geq 0, \\ a_t = (a_1(t), \ldots, a_n(t)) \in [0,1]^n. \end{cases}$$

It is easy to see that we have the same problem when we replace the target function by

$$\lambda' c - \liminf_{T\to\infty} \frac{1}{T} \int_0^T \left(\sum_{i=1}^n c_i \mu_i x_i(t) \right) dt \to \min.$$

The finite horizon control problem with target function

$$\int_0^T \left(\sum_{i=1}^n (1 - a_i(t)) \lambda_i c_i \right) dt \to \min$$

and the same constraints are denoted by (C_T). If we denote by $V^C(x)$ and $V_T^C(x)$ the optimal values of the control problems (C) and (C_T), respectively, for starting state x, then $V^C(x)$ provides a nontrivial lower bound for the minimal average cost G of the stochastic problem and $V_T^C(x)$ for the finite horizon problem.

THEOREM 1: *For all initial states $x \in S$ and time points $T \geq 0$, it holds that*

(a) $V_T^C(x) \leq \inf_f E_x \left[\int_0^T c(X_t, f(X_t))\, dt \right].$

(b) $V^C(x) \leq G.$

https://doi.org/10.1017/S0269964802161067 Published online by Cambridge University Press

PROOF: Suppose that $x \in S$ is the initial state. Let f be an arbitrary channel assignment policy. The induced state process (X_t) is a Markov process with intensity matrix $Q = (q(x, f(x), x'))$. Hence, it holds that

$$X_t - x - \int_0^t (\lambda \circ f(X_s) - \mu \circ X_s)\, ds = M_t,$$

where (M_t) is a martingale and $M_0 = 0$. Taking the expectation on both sides and denoting $x_t := E_x[X_t]$ and $a_t := E_x[\, f(X_t)]$ for $t \geq 0$, we obtain

$$x_t = x + \int_0^t (\lambda \circ a_s - \mu \circ x_s)\, ds.$$

Moreover, since f is admissible, we get for all $t \geq 0$ a.s. $Az(X_t) \geq X_t$, $\sum_{j=1}^n z_j(X_t) \leq k$, $X_t \geq 0$, $z(X_t) \geq 0$, and $f(X_t) \in [0,1]^n$. Thus, it holds for all $t \geq 0$ that $x_t \in S^\infty$ (i.e., $\bar{A}x_t \leq \bar{b}$, $x_t \geq 0$, and $a_t \in [0,1]^n$). This means that the pair $\{(x_t, a_t),\ t \geq 0\}$ is admissible for the deterministic control problems (C) and (C_T) for every channel assignment policy f. Moreover,

$$E_x\left[\int_0^T \left(\sum_{i=1}^n (1 - f_i(X_t))\lambda_i c_i\right) dt\right] = \int_0^T \left(\sum_{i=1}^n (1 - a_i(t))\lambda_i c_i\right) dt,$$

which implies the statement. ■

4. CONVERGENCE

We will now study the performance of the system under an arbitrary channel assignment policy, when the number of available channels as well as the arrival intensities of the calls get large. The increase is linear in both the number of available channels as well as the arrival intensities. Let $f: S \to A$ be an arbitrary channel assignment policy for a problem with γk channels, $\gamma \in \mathbb{N}$. The process $(\hat{X}_t^\gamma)$ under f is given by the intensity matrix (only the off-diagonal elements are indicated)

$$q^\gamma(x, f(x), x') = \begin{cases} \gamma\lambda_i f_i(x), & x' = x + e_i \\ \mu_i x_i, & x' = x - e_i \\ 0, & \text{else} \end{cases}$$

and initial state γx. The scaled process is defined by $X_t^\gamma := (1/\gamma)\hat{X}_t^\gamma$, $t \geq 0$. (X_t^γ) is a Markov process on the state space

$$S^\gamma := \frac{1}{\gamma}\left\{x \in \mathbb{N}_0^n \mid \exists z \in \mathbb{N}_0^m \text{ s.t. } Az \geq x,\ \sum_{j=1}^m z_j \leq \gamma k\right\}.$$

Further, we define the integrated scaled action process $A_t^\gamma = (A_1^\gamma(t), \ldots, A_n^\gamma(t))$ by

$$A_i^\gamma(t) := \int_0^t f_i(\gamma X_s^\gamma)\, ds, \qquad i = 1, \ldots, n.$$

We understand the processes (X_t^γ, A_t^γ) as random elements with values in $D^n[0,\infty)$, which is the space of $\mathbb{R}^n$-valued functions on $[0,\infty)$ that are right continuous and have left-hand limits. The space is endowed with the Skorokhod topology. By $\Rightarrow$ we denote the weak convergence of the processes as $\gamma \to \infty$. The next theorem states that every sequence of scaled state and action processes has a further subsequence which converges weakly and the limit satisfies almost surely the constraints of the deterministic control problem (C). Thus, every convergent subsequence satisfies the following functional law of large numbers.

THEOREM 2: *Every sequence (X_t^γ, A_t^γ) has a further subsequence $(X_t^{\gamma_n}, A_t^{\gamma_n})$ such that $(X_t^{\gamma_n}, A_t^{\gamma_n}) \Rightarrow (X_t, A_t = \int_0^t a_s\, ds)$ and the limit satisfies a.s. for all $t \geq 0$*

(i) $X_t = x_0 + \int_0^t (\lambda \circ a_s - \mu \circ X_s)\, ds$
(ii) $\bar{A} X_t \leq \bar{b}$
(iii) $X_t \geq 0$
(iv) $a_t \in [0,1]^n$.

The proof follows by showing the tightness of the sequence (X_t^γ, A_t^γ) (cf. also Hunt and Laws [6], Hunt and Kurtz [5], and Alanyali and Hajek [1]). For an arbitrary channel assignment policy f, the scaled average cost are defined by

$$G_f^\gamma = \limsup_{T \to \infty} \frac{1}{T} E_x \left[\int_0^T c(X_t^\gamma, f(\gamma X_t^\gamma))\, dt \right].$$

Remark: Note that the lower bound given in Theorem 1 holds for every scaled system; that is, we have

$$G^\gamma := \inf_f G_f^\gamma \geq V^C(x)$$

for all $\gamma > 0$ and all x.

5. AN LP APPROACH

Let us next consider the following linear program, where we denote $\eta_i := \lambda_i / \mu_i$, $i = 1, \ldots, n$, and $\eta = (\eta_1, \ldots, \eta_n)$.

$$(LP) \begin{cases} \lambda' c - \sum_{i=1}^n c_i \mu_i x_i \to \min, \\ \bar{A} x \leq \bar{b}, \\ 0 \leq x \leq \eta. \end{cases}$$

https://doi.org/10.1017/S0269964802161067 Published online by Cambridge University Press

It is easy to see that the linear program has an optimal solution and we denote by V^{LP} the optimal value of the problem. It is now possible to show that the optimal value of the linear program coincides with the optimal value of the deterministic control problem (C), independent of the initial state. The proof of Theorem 3 also shows how to construct an optimal solution for (C) using the optimal solution of (LP).

THEOREM 3: *For all $x \in \mathbb{R}_+^n$, we have $V^C(x) = V^{LP}$.*

PROOF: Suppose that x_t is admissible for (C). In particular, x_t is a solution of the initial value problem

$$\dot{x}_i(t) = \lambda_i a_i(t) - \mu_i x_i(t)$$

and $x_i(0) = x_i$ is given. Hence,

$$x_i(t) = e^{-t\mu_i} x_i(0) + \int_0^t \lambda_i a_i(s) e^{-\mu_i(t-s)}\, ds$$

and

$$\int_0^t x_i(s)\, ds = \frac{x_i(0)}{\mu_i}(1 - e^{-t\mu_i}) + \int_0^t \eta_i a_i(s)[1 - e^{-\mu_i(t-s)}]\, ds.$$

As a result, the average integrated state of any admissible trajectory of (C) is bounded above by

$$\frac{1}{t}\int_0^t x_i(s)\, ds \le \frac{1}{t}\left[\frac{x_i(0)}{\mu_i}(1 - e^{-t\mu_i})\right] + \eta_i\left[1 + \frac{1}{\mu_i}\left(\frac{e^{-t\mu_i}}{t} - \frac{1}{t}\right)\right],$$

which converges to η_i for $t \to \infty$. In particular, every limit point $\xi_i = \lim_{n\to\infty}(1/t_n)\int_0^{t_n} x_i(s)\, ds$ of a sequence (t_n), $t_n \to \infty$ for $n \to \infty$, satisfies $0 \le \xi_i \le \eta_i$, $i = 1, \ldots, n$, and $\bar{A}\xi \le \bar{b}$. Thus, $V^{LP} \le V^C(x)$.

Now, suppose that $\bar{x}$ is admissible for (LP). Define $f\colon \mathbb{R}_+^n \to A$ by

$$f_i(x) = f_i(x_i) = \begin{cases} 1 & \text{if } x_i < \bar{x}_i \\ \dfrac{\mu_i x_i}{\lambda_i} & \text{if } x_i = \bar{x}_i \\ 0 & \text{if } x_i > \bar{x}_i. \end{cases}$$

The initial value problem

$$\dot{x}_i(t) = \lambda_i f_i(x_i(t)) - \mu_i x_i(t),$$

https://doi.org/10.1017/S0269964802161067 Published online by Cambridge University Press

$x_i(0) = x_i$ given, has exactly one solution (cf. Hartman [3, Thm. 6.2]). For $\bar{x}_i \geq x_i(0)$ the solution is given by

$$x_i(t) = \begin{cases} e^{-\mu_i t} x_i(0) + \eta_i (1 - e^{-\mu_i t}), & t \leq -\frac{1}{\mu_i} \log\left(\frac{\eta_i - \bar{x}_i}{\eta_i - x_i(0)}\right) \\ \bar{x}_i, & t > -\frac{1}{\mu_i} \log\left(\frac{\eta_i - \bar{x}_i}{\eta_i - x_i(0)}\right) \end{cases} \qquad \textbf{(1)}$$

and for $\bar{x}_i \leq x_i(0)$ by

$$x_i(t) = \begin{cases} e^{-\mu_i t} x_i(0), & t \leq -\frac{1}{\mu_i} \log\left(\frac{\bar{x}_i}{x_i(0)}\right) \\ \bar{x}_i, & t > -\frac{1}{\mu_i} \log\left(\frac{\bar{x}_i}{x_i(0)}\right). \end{cases} \qquad \textbf{(2)}$$

Moreover, $x_t \geq 0$. $\bar{A} x_t \leq \bar{b}$ can be obtained by first driving those cells with $x_i(0) > \bar{x}_i$ to $\bar{x}_i$ and then filling up those with $x_i(0) \leq \bar{x}_i$. Since $\bar{A} x_0 \leq \bar{b}$, the inequality holds for all $t > 0$. Last but not least $a_t = f(x_t) \in [0,1]^n$. Obviously, this control for problem (C) yields the value $\lambda' c - \sum_{i=1}^n c_i \mu_i \bar{x}_i$ and it follows that $V^{LP} = V^C(x)$. ■

Remark: Instead of solving the deterministic control problem (C), we have shown that it is sufficient to solve the linear program (LP). However, due to the complicated state space, the worst-case complexity of (LP) is exponential in the number of cells (see McEliece and Sivarajan [10]).

6. ASYMPTOTIC OPTIMALITY OF FIXED CHANNEL ASSIGNMENT

In this section, we investigate the limit behavior of fixed channel assignment policies. The limit of the state process is defined here by the unique solution of an initial value problem. Moreover, suppose that x^* is the optimal solution of (LP) with value V^{LP}. From Theorem 1, we know that V^{LP} is a lower bound for $G^\gamma, \gamma > 0$. In this section, we show that this lower bound can be achieved in the limit by implementing a fixed channel assignment policy which is given by the solution x^* of (LP). More precisely, in the scaled model with $\gamma > 0$, we use the fixed channel assignment which assigns $\lfloor x_i^* \gamma \rfloor$ channels to cell i. We will denote this policy by FCA; that is, for $x \in S$

$$\mathrm{FCA}_i(x) = \begin{cases} 1 & \text{if } x_i + 1 \leq \lfloor x_i^* \gamma \rfloor \\ 0 & \text{else.} \end{cases}$$

We obtain the following theorem.

https://doi.org/10.1017/S0269964802161067 Published online by Cambridge University Press

THEOREM 4:

(a) *Under a fixed channel assignment policy given by* $\bar{x} \in S^\infty$*, it holds that* $(X_t^\gamma) \Rightarrow (x_t)$ *and the limit* (x_t) *is given by (1) and (2).*
(b) *The fixed channel assignment policy which is given by* x^* *is asymptotically optimal; that is,*

$$\lim_{\gamma \to \infty} G_{\text{FCA}}^\gamma = V^{LP}.$$

PROOF: Let $\gamma > 0$ be fixed. Under a fixed channel assignment policy given by $\bar{x}$, the following additional equations are almost surely fulfilled for the stochastic processes (X_t^γ, A_t^γ):

$$\int_0^t \left[\frac{\lfloor \bar{x}_i \gamma \rfloor}{\gamma} - \min\left(X_i^\gamma(s), \frac{\lfloor \bar{x}_i \gamma \rfloor}{\gamma} \right) \right]^+ d(s - A_i^\gamma(s)) = 0,$$

$$\int_0^t \left[\max\left(X_i(s), \frac{\lfloor \bar{x}_i \gamma \rfloor}{\gamma} \right) - \frac{\lfloor \bar{x}_i \gamma \rfloor}{\gamma} \right]^+ dA_i^\gamma(s) = 0.$$

Since $\lfloor \bar{x}_i \gamma \rfloor / \gamma \to \bar{x}_i$ for $\gamma \to \infty$, it follows with Lemma 2.4 of Dai and Williams [2] that for any convergent subsequence $(X_t^{\gamma_n}, A_t^{\gamma_n})$ (which exists due to Theorem 2) with limit (X_t, A_t) the preceding expressions converge against

$$\int_0^t [\bar{x}_i - \min(X_i(s), \bar{x}_i)]^+ \, d(s - A_i(s)) = 0,$$

$$\int_0^t [\max(X_i(s), \bar{x}_i) - \bar{x}_i]^+ dA_i(s) = 0.$$

Thus, it follows that under the fixed channel assignment policy, the limit control a_t at time $t \geq 0$ satisfies almost everywhere

$$a_i(t) = f_i(X_t) = f_i(X_i(t)) = \begin{cases} 1 & \text{if } X_i(t) < \bar{x}_i \\ \dfrac{\mu_i X_i(t)}{\lambda_i} & \text{if } X_i(t) = \bar{x}_i \\ 0 & \text{if } X_i(t) > \bar{x}_i. \end{cases}$$

In addition, Theorem 2 tells us that every limit (X_t, A_t) satisfies

$$X_t = x_0 + \int_0^t (\lambda \circ a_s - \mu \circ X_s) \, ds.$$

However, from the proof of Theorem 3, we then know that the limit (X_t, A_t) is uniquely defined (and the same for every convergent subsequence) and given as stated. If $\bar{x} = x^*$, we obtain

https://doi.org/10.1017/S0269964802161067 Published online by Cambridge University Press

$$\lambda' c - \liminf_{T\to\infty} \frac{1}{T}\int_0^T \sum_{i=1}^n X_i(s)\mu_i c_i\, ds = \lambda' c - \sum_{i=1}^n x_i^*(s)\mu_i c_i = V^{LP}.$$

Now, for fixed $\gamma > 0$, the state process (X_t^γ) has a unique stationary distribution π^γ and

$$G_{\mathrm{FCA}}^\gamma = \lambda' c - \liminf_{T\to\infty} \frac{1}{T} E_x\left[\int_0^T \sum_{i=1}^n c_i\,\mu_i\, X_i^\gamma(t)\, dt\right] = \lambda' c - E_{\pi^\gamma}\left[\sum_{i=1}^n c_i\,\mu_i\, X_i^\gamma(t)\right].$$

Since (X_t^γ) is stochastically dominated by a Poisson process with intensity $(\bar\lambda\gamma + \bar\mu k\gamma)$ and jumps of height $1/\gamma$, we obtain, with dominated convergence for $t > 0$ large enough,

$$\lim_{\gamma\to\infty} G_{\mathrm{FCA}}^\gamma = \lambda' c - \sum_{i=1}^n c_i\,\mu_i\, x_i^*$$

and the proof is complete. ■

7. NUMERICAL RESULTS

In this section, we illustrate our results by some numerical examples. We investigate the performance of the asymptotically optimal fixed channel assignment policy FCA for different scaling parameters for the star network and the fully connected network. It turns out that FCA is, in general, different from the optimal assignment policy. Indeed, there are cases where FCA is suboptimal for any scaling parameter γ. On the positive side, there are examples where FCA is optimal for all γ.

7.1. Star Network

The limiting state space of the star network is $S^\infty = \{x \in \mathbb{R}_+^n \,|\, x_1 + x_i \le k,\ i = 2,\ldots,n\}$. The linear program of Section 5 (without constant) is given by

$$(LP)\begin{cases} \sum_{i=1}^n c_i\,\mu_i\, x_i \to \max, \\ x_1 + x_i \le k, \qquad i = 2,\ldots,n, \\ 0 \le x \le \eta \end{cases}$$

and reduces to the one-dimensional optimization problem

$$\begin{cases} c_1\mu_1 x_1 + \sum_{i=1}^n c_i\,\mu_i \min(k - x_1, \eta_i) \to \max, \\ 0 \le x_1 \le \eta_1. \end{cases}$$

https://doi.org/10.1017/S0269964802161067 Published online by Cambridge University Press

We have chosen the following data for our numerical example: $n = 7$ cells, $k = 4$ channels, service rates $\mu_1 = \cdots = \mu_7 = 1$, and cost rates $c_1 = \cdots = c_7 = 1$. The arrival rates are given by $\lambda_1 = 7$, $\lambda_2 = 6$, $\lambda_3 = 4.8$, $\lambda_4 = 5.3$, $\lambda_5 = 4.7$, $\lambda_6 = 5.1$, and $\lambda_7 = 4.8$. We will multiply the arrival rates λ_i by a parameter λ and vary λ from 1 to 1.5. It is easy to see that the solution of the linear program is the fixed channel assignment policy which gives no channels to the sun and all channels to the planets (independent of $\lambda \in [1, 1.5]$). If we introduce the scaling parameter γ, it holds that (cf. McEliece and Sivarajan [10, Sect. 6])

$$\lim_{\gamma\to\infty} \mathrm{Erl}_{k\cdot\gamma}(\rho\gamma) = \begin{cases} \dfrac{\rho - k}{\rho} & \text{if } \rho > k \\ 0 & \text{if } \rho \leq k \end{cases}$$

and the mapping $\gamma \mapsto \mathrm{Erl}_{k\cdot\gamma}(\rho\gamma)$ is decreasing in γ. Thus, for our data, it holds that $\sum_{i=2}^{7} \mathrm{Erl}_{4\cdot\gamma}(\lambda_i\gamma) \geq 1.277 > 1$, which implies that the same fixed channel assignment policy is, indeed, optimal for all values of γ (see Section 2). This is an example where the asymptotically optimal fixed channel assignment policy coincides with the optimal channel assignment policy for all γ. In Figure 3, the expected minimal average costs are plotted for different arrival rates (multiplied by λ) and different scaling parameters $\gamma = 1$, 5, and 50. The lower black line is the lower bound of the linear program.

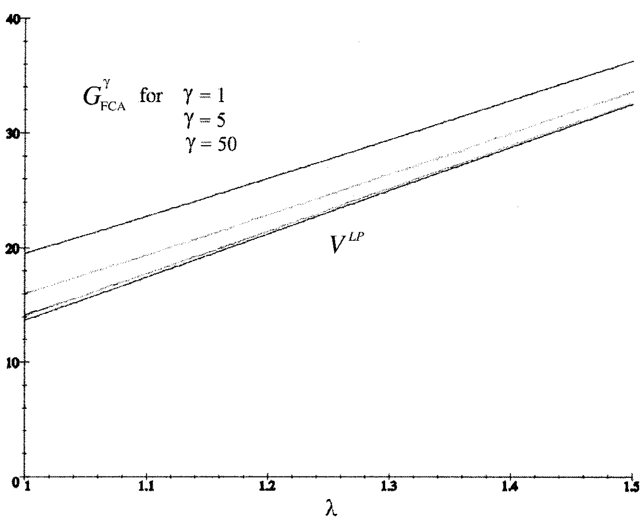

FIGURE 3. V^{LP} and $G^{\gamma}_{\mathrm{FCA}} = G^{\gamma}$ for $\gamma = 1$, 5, and 50 and $1 \leq \lambda \leq 1.5$.

https://doi.org/10.1017/S0269964802161067 Published online by Cambridge University Press

7.2. Fully Connected Network

The limiting state space of the fully connected network is $S^\infty = \{x \in \mathbb{R}_+^n \mid x_1 + \cdots + x_n \le k\}$. Consequently, the linear program of Section 5 (without constant) reduces to the following knapsack problem:

$$(LP)\begin{cases} \sum_{i=1}^{n} c_i\,\mu_i\,x_i \to \max, \\ \sum_{i=1}^{n} x_i \le k, \\ 0 \le x \le \eta. \end{cases}$$

Suppose that $c_1\mu_1 \ge c_2\mu_2 \ge \cdots \ge c_n\mu_n$; then, the solution of the linear program is given by $x_1^* = \eta_1, \ldots, x_\nu^* = \eta_\nu$, $x_{\nu+1}^* = k - \sum_{i=1}^{\nu}\eta_i$, $x_{\nu+2}^* = 0, \ldots, x_n^* = 0$, where $\nu = \max\{m \mid \sum_{i=1}^{m}\eta_i \le k\}$. Hence, the asymptotically optimal fixed channel assignment policy assigns $\lfloor \eta_i \rfloor$ channels to cell i, $i = 1, \ldots, \nu$, $\lfloor k - \sum_{i=1}^{\nu-1}\eta_i \rfloor$ channels to cell $\nu + 1$ and no channels to the remaining cells. We have chosen the following data for our numerical example: $n = 6$ cells, $k = 4$ channels, service rates $\mu_1 = \cdots = \mu_7 = 1$, and cost rates $c_1 = \cdots = c_7 = 1$. The arrival rates are given by $\lambda_1 = \cdots = \lambda_6 = \lambda$. The parameter λ will be varied from 0 to 4. In this case, it follows easily (see Section 2) that maximum packing is optimal for any γ.

In Figure 4, the performance of maximum packing is compared to the performance of the asymptotically optimal fixed channel assignment policy with $\gamma = 1$. This is an example in which the asymptotically optimal fixed channel assignment

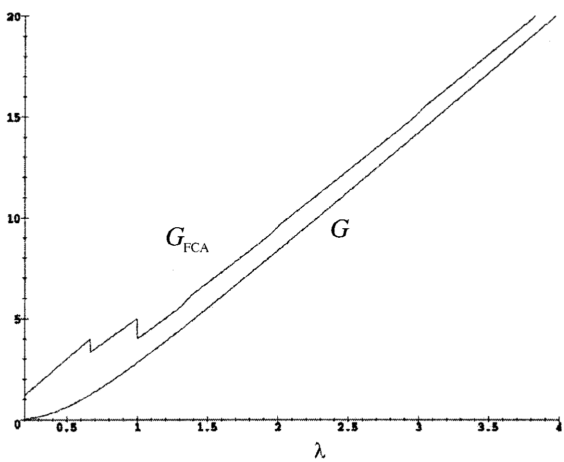

Figure 4. $G = G_{\text{max pack}}$ and G_{FCA} for $0 \le \lambda \le 4$.

https://doi.org/10.1017/S0269964802161067 Published online by Cambridge University Press

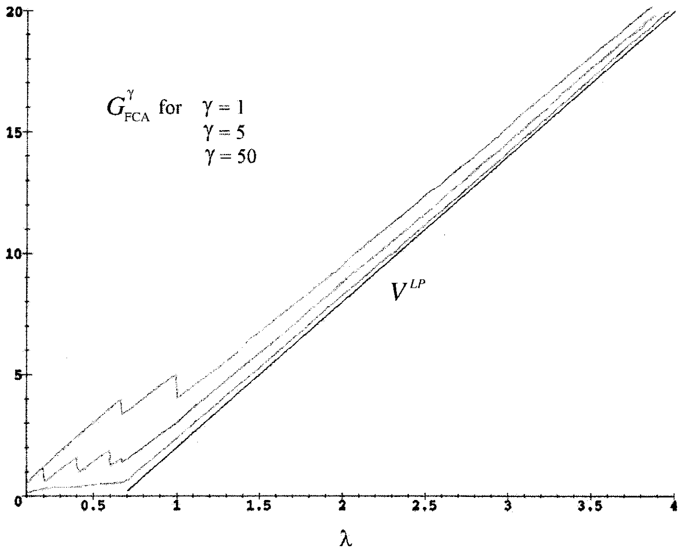

FIGURE 5. V^{LP} and G^{γ}_{FCA} for $\gamma = 1$, 5, and 50 and $0 \leq \lambda \leq 4$.

policy is indeed suboptimal for all γ and differs completely from the optimal assignment policy. For γ tending to infinity, both assignment rules achieve the lower bound. Figure 5 shows the expected average cost under FCA for different arrival rates λ and different scaling parameters $\gamma = 1$, 5, and 50. The lower black line is the lower bound of the linear program.

References

1. Alanyali, M. & Hajek, B. (1998). Analysis of simple algorithms for dynamic load balancing. *Mathematics of Operations Research* 22: 840–871.
2. Dai, J. & Williams, R. (1995). Existence and uniqueness of semimartingale reflecting Brownian motions in convex polyhedrons. *Theory of Probability and Its Applications* 40: 1–40.
3. Hartman, P. (1982). *Ordinary differential equations*. Boston: Birkhäuser Verlag.
4. Houdek, A. (2000). Call admission in cellular radio networks. [Unpublished] dissertation, University of Ulm.
5. Hunt, P.J. & Kurtz, T.G. (1994). Large loss networks. *Stochastic Processes and Their Applications* 53: 363–378.
6. Hunt, P.J. & Laws, C.N. (1993). Asymptotically optimal loss network control. *Mathematics of Operations Research* 18: 880–900.
7. Kelly, F.P. (1986). Blocking probabilities in large circuit-switched networks. *Advances in Applied Probability* 18: 473–505.
8. Kind, J., Niessen, T., & Mathar, R. (1998). Theory of maximum packing and related channel assignment strategies for cellular radio networks. *Mathematics and Methods of Operations Research* 48: 1–16.

9. Kulshreshtha, A. & Sivarajan, K.N. (1999). Maximum packing channel assignment in cellular networks. *IEEE Transactions on Vehicular Technology* 48: 858–872.
10. McEliece, R.J. & Sivarajan, K.N. (1994). Performance limits for channelized cellular telephone systems. *IEEE Transactions on Information Theory* 40: 21–34.
11. Puterman, M.L. (1994). *Markov decision processes, discrete stochastic dynamic programming*. Wiley Series in Probability and Mathematical Statistics. Chichester: Wiley.
12. Robinson, D. (1992). The optimality of fixed channel assignment policies for cellular radio systems. *Advances in Applied Probability* 24: 474–495.
13. Tijms, H.C. (1994). *Stochastic models: An algorithmic approach*. Chichester: Wiley.