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WEAKENING THE INDEPENDENCE ASSUMPTION ON POLAR COMPONENTS: LIMIT THEOREMS FOR GENERALIZED ELLIPTICAL DISTRIBUTIONS

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Abstract

By considering the extreme behavior of bivariate random vectors with a polar representation $R(u(T), v(T))$, it is commonly assumed that the radial component R and the angular component T are stochastically independent. We investigate how to relax this rigid independence assumption such that conditional limit theorems can still be deduced. For this purpose, we introduce a novel measure for the dependence structure and present convenient criteria for validity of limit theorems possessing a geometrical meaning. Thus, our results verify a stability of the available limit results, which is essential in applications where the independence of the polar components is not necessarily present or exactly fulfilled.

Keywords: Conditional extreme value model; polar representation; elliptical distribution; Gumbel max-domain of attraction; random norming

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Secondary 60G70; 62E20; 62G32

1. Introduction

Analyzing and predicting extreme events for random vectors is of particular interest in numerous applications. In contrast to the univariate case, there are different ways to define the term *extreme event* by a requirement that one or several or even all of the vector components have to be large simultaneously. An effective approach for the analysis of multivariate extreme values was introduced by Heffernan and Tawn (2004). They examined the distribution tail of a random vector in terms of the conditional distribution, given that one of the components of the random vector becomes large. This approach was further developed and extended to the *conditional extreme value model* by Heffernan and Resnick (2007) and Das and Resnick (2011).

Conditional limit statements for elliptical and more general random vectors possessing a polar representation $(X, Y) = R(u(T), v(T))$ with radial component R and angular component T were intensively investigated among others by Berman (1983), Abdous *et al.* (2005), Fougères and Soulier (2010), Hashorva (2012), and Seifert (2014). The latter three articles show that rather weak and local assumptions on the coordinate functions u and v are sufficient to derive limit statements. But one assumption made is very rigid, namely that R and T are stochastically independent. This requirement is global and unstable such that its validity is assumed to hold even in regions which are not important for the limit behavior.

A possibility to weaken this independence assumption would underscore a certain stability of the polar extreme value model with respect to the above mentioned limit results, which is of particular interest for statistical inferences in applications. Hence, a natural question remains

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open: how much can we deviate from the stochastic independence of the polar components R and T such that we still obtain a conditional limit result for $(X, Y) = R(u(T), v(T))$? Up to now, to the best of the author's knowledge, there is only one result in this direction, presented in the book of Balkema and Embrechts (2007), which does not explicitly use polar representations.

In this paper we introduce a novel approach for weakening the independence assumption. After definitions in Section 2, we present in Section 3 the extreme value model with independent polar components where the radial component R belongs to the Gumbel max-domain of attraction with some auxiliary function ψ. We unite two important results from Fougères and Soulier (2010) and Seifert (2014) in Theorem 1, and deduce a new Theorem 2 for *dependent* polar components which states: the limit results still hold if the conditional distributions of R, given $T = t$, have a similar tail behavior with asymptotically equivalent auxiliary functions ψ_t.

In order to verify such a condition in empirical applications, we develop convenient and model-independent geometric criteria. We describe the dependence between R and T by comparing $R(u(T), v(T))$ with some *reference model* $\widetilde{R}(u(T), v(T))$, where $\widetilde{R}$ and T are independent. The difference between the conditional distribution functions $H_t(r) = \mathbb{P}(R \le r \mid T = t)$ and the reference distribution function $\widetilde{H}(r) = \mathbb{P}(\widetilde{R} \le r)$ is measured by *shifts* $\delta_t(r)$. In Section 4 we show in Theorem 3 that the limit results still hold for *relative shifts* $\delta_t(r)/r$ vanishing asymptotically for $r \to \infty$. Furthermore, we deduce in Theorem 4 limit results for relative shifts which tend to a t-dependent limit so that the auxiliary functions ψ_t are no more asymptotically equivalent. In Section 5 we compare our approach for weakening the independence assumption with Balkema and Embrechts' approach. In Theorem 5 we analyze cases where Theorem 4 extends their results.

2. Preliminaries

First we give the definitions and important properties of the regular and of Γ-variation; see Resnick (1987), Geluk and de Haan (1987), and de Haan and Ferreira (2006). All considered functions are assumed to be Lebesgue measurable. Two functions f and g are said to be asymptotically equivalent if $f(x)/g(x) \to 1$ for $x \to \infty$ (written $f \sim g$).

Definition 1. An eventually positive function $f\colon (0, \infty) \to \mathbb{R}$ is said to be regularly varying at ∞ with index $\alpha \in \mathbb{R}$ (written $f \in \mathrm{RV}_\alpha(\infty)$), if for all $\lambda > 0$ it holds that

$$\lim_{x\to\infty} \frac{f(\lambda x)}{f(x)} = \lambda^\alpha. \tag{1}$$

A function $g\colon \mathbb{R} \to \mathbb{R}$ is said to be regularly varying at t_0 with index $\alpha \in \mathbb{R}$ (written $g \in \mathrm{RV}_\alpha(t_0)$) if for all $\lambda > 0$ it holds that

$$\lim_{s\to 0} \frac{g(t_0 + \lambda s)}{g(t_0 + s)} = \lambda^\alpha. \tag{2}$$

If $g \in \mathrm{RV}_\alpha(t_0)$ fulfills $\lim_{s\downarrow 0} |g(t_0 + s)/g(t_0 - s)| = 1$, we call it *infinitesimally symmetric* and write $g \in \mathrm{RV}^s_\alpha(t_0)$.

Remark 1. (i) The convergence in (1) and (2) is locally uniform in λ.

(ii) For $g \in \mathrm{RV}_\alpha(t_0)$ it follows that $g(t) \neq 0$ for $t \in (t_0 - \varepsilon, t_0 + \varepsilon) \setminus \{t_0\}$ for some $\varepsilon > 0$.

(iii) For $f \in \mathrm{RV}_\alpha(\infty)$ it holds that $\lim_{x\to\infty} f(x) = 0$ for $\alpha < 0$, $\lim_{x\to\infty} f(x) = \infty$ for $\alpha > 0$.

(iv) If f is eventually positive and $\lim_{x\to\infty} f(\lambda x)/f(x)$ exists, is finite and positive for all λ in a set of positive Lebesgue measure, then $f \in \mathrm{RV}_\alpha(\infty)$ for some $\alpha \in \mathbb{R}$ (De Haan and Ferreira (2006, Theorem B.1.3)).

Definition 2. A nondecreasing function f is said to be $\Gamma(\psi)$-varying (with a positive *auxiliary function* ψ) if for all $z \in \mathbb{R}$ it holds that $\lim_{x\to\infty} f(x + z\psi(x))/f(x) = \mathrm{e}^z$.

We say that a random variable R on $[0, \infty)$ and its distribution function H (respectively survival function $\overline{H} = 1 - H$) are of type $\Gamma(\psi)$ if $1/\overline{H}$ is $\Gamma(\psi)$-varying, i.e. if for all $z \in \mathbb{R}$ it holds that

$$\lim_{r\to\infty} \frac{\mathbb{P}\{R > r + z\psi(r)\}}{\mathbb{P}\{R > r\}} = \lim_{r\to\infty} \frac{\overline{H}(r + z\psi(r))}{\overline{H}(r)} = \mathrm{e}^{-z}. \tag{3}$$

Remark 2. (i) The auxiliary function ψ is unique up to asymptotic equivalence, i.e. a positive function ψ_2 is an auxiliary function for R of type $\Gamma(\psi_1)$ if and only if $\psi_1 \sim \psi_2$. It can be chosen to be differentiable satisfying $\lim_{r\to\infty} \psi'(r) = 0$ (Geluk and de Haan (1987, Theorem 1.28(ii), Corollary 1.29)).

(ii) A random variable R is of type $\Gamma(\psi)$ if and only if R is in the Gumbel max-domain of attraction with infinite right endpoint $\sup\{r : H(r) < 1\} = \infty$.

(iii) A distribution function H is of type $\Gamma(\psi)$ if and only if it possesses a von Mises representation:

$$\overline{H}(r) = 1 - H(r) = a(r) \exp\left(-\int_0^r 1/\psi(u)\,\mathrm{d}u\right) \tag{4}$$

with $\lim_{r\to\infty} a(r) = a \in (0, \infty)$, Resnick (1987, Proposition 1.4).

3. From independent to dependent polar components

We consider bivariate random vectors $(X, Y) \in [0, \infty) \times (-\infty, \infty)$ on the right half-plane, since we are interested in the asymptotic behavior for X becoming large. (X, Y) can be represented in *Euclidean polar coordinates* $(X, Y) \overset{\mathrm{D}}{=} A(\cos\Theta, \sin\Theta)$ with Euclidean distance $A \geq 0$ and Euclidean angle $\Theta \in [-\pi/2, \pi/2]$.

The popular class of *elliptical distributions* is described conveniently by

$$(X, Y) \overset{\mathrm{D}}{=} R(\cos T, \rho\cos T + \sqrt{1-\rho^2}\sin T), \qquad \rho \in (-1, 1) \tag{5}$$

with stochastically independent R and T, where T is uniformly distributed.

More generally, we investigate random vectors which possess a *polar representation*

$$(X, Y) \overset{\mathrm{D}}{=} R(u(T), v(T)) \tag{6}$$

with *polar components* R and T and quite arbitrary *coordinate functions* u and v.

Such elliptical and generalized distributions were intensively investigated with respect to their conditional limit behavior. A detailed overview of this research field is given in Fougères and Soulier (2010) and Seifert (2014). In this paper we start with random vectors with polar representation (6) fulfilling the following three assumptions.

Assumption 1. *The following hold:*

(i) *R takes values in $[0, \infty)$ and T in $[-\pi/2, \pi/2]$;*

(ii) *(R, T) possesses a positive, continuous joint density f_{RT};*

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(iii) *there exists a diffeomorphism* τ *of* $[-\pi/2, \pi/2]$ *with derivative* $\tau' > 0$ *such that* $\Theta = \tau(T)$ *for the Euclidean angle* Θ.

Assumption 2. *The following hold:*

(i) *R is of type* $\Gamma(\psi)$*;*

(ii) *R and T are stochastically independent.*

Assumption 3. *It holds that*

$$u(t) = u_{\max} - l(t) \quad \text{for } t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

with some $u_{\max} \in (0, \infty)$*,*

$$v(t) = \tan(t)u(t) \quad \text{for } t \in (t_0 - \varepsilon_0, t_0 + \varepsilon_0)$$

with some $\varepsilon_0 > 0$*, where* $l\colon [-\pi/2, \pi/2] \to [0, u_{\max}]$ *has a unique zero at* $t_0 \in (-\pi/2, \pi/2)$ *and its derivative* l' *is* $\mathrm{RV}^s_{\kappa-1}(t_0)$ *for some* $\kappa > 0$. *We denote* $\rho := (v/u)(t_0) = \tan(t_0)$.

According to Remark 1(ii), in some ε-neighborhood $(t_0 - \varepsilon, t_0 + \varepsilon) \setminus \{t_0\}$ it holds that $l' \neq 0$ and, hence, $u' \neq 0$. Thus, u increases strictly on $(t_0 - \varepsilon, t_0)$ and decreases strictly on $(t_0, t_0 + \varepsilon)$. As a consequence, l possesses two branches of inverses $l_\pm^{-1} \in \mathrm{RV}_{1/\kappa}(0)$ on $(-\varepsilon, 0)$ respectively $(0, \varepsilon)$. Assumption 3 implies that l is $\mathrm{RV}^s_\kappa(t_0)$ and that u has the unique global maximum $u_{\max}$ at $t = t_0$.

Remark 3. There is much freedom to select a polar representation (6). The angular component T simply labels the rays $y = \gamma x$, $\gamma \in \mathbb{R}$ in the (x, y)-plane. We specify T (in Assumption 3) to coincide locally with the Euclidean angle Θ and assume (in Assumption 1) a diffeomorphism τ between T and Θ, which enables us to obtain the density f_{XY} of (X, Y) by using $f_{XY}(a\cos(\vartheta), a\sin(\vartheta)) = f_{A\Theta}(a, \vartheta)/a$ such that

$$f_{XY}(ru(t), rv(t)) = \frac{f_{RT}(r, t)}{r(u^2(t) + v^2(t))\tau'(t)} \tag{7}$$

as $a^2 = r^2(u^2(t) + v^2(t))$, $\vartheta = \tau(t)$. For the radial component R only a linear scaling is allowed: $(X, Y) = R(u(T), v(T))$ also possesses the polar representation $(X, Y) = R^*(u^*(T), v^*(T))$ with $R^* = cR$ and $u^* = u/c$, $v^* = v/c$. If $c \in (0, \infty)$ is a *constant* then this is a global rescaling of R, often done to obtain $\max u^* = 1$ as in (5). If c is a *nonconstant* function of T then this changes the dependence structure between the polar components; in Theorem 4 we take advantage of this possibility.

With $k = \kappa^{-1/\kappa} l_+^{-1}$ for $\zeta \geq 0$, $k = -\kappa^{-1/\kappa} l_-^{-1}$ for $\zeta < 0$, as well as

$$G(\zeta) = \frac{1}{2\kappa^{1/\kappa - 1}\Gamma(1/\kappa)} \int_{-\infty}^{\zeta} \exp\left(-\frac{|s|^\kappa}{\kappa}\right) \mathrm{d}s,$$

we state results of Fougères and Soulier (2010) for variation indices $\kappa > 1$, and of Seifert (2014) for $\kappa > 0$ using *random norming* (cf. Heffernan and Resnick (2007)) with the bound on Y evaluated not at the threshold x but at the actual value X.

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Theorem 1. (Independent polar components.) *Let* $(X, Y) = R(u(T), v(T))$ *satisfy Assumptions 1, 2, and 3. Then for all* $\xi > 0$, $\zeta \in \mathbb{R}$, *and* $\kappa > 1$ *it holds that*

$$\lim_{x\to\infty} \mathbb{P}\bigg(X \le x + \psi(x)\xi,\, Y \le \rho x + xk\bigg(\frac{\psi(x)}{x}\bigg)\zeta \;\bigg|\; X > x\bigg) = (1 - \mathrm{e}^{-\xi})G(\zeta), \tag{8}$$

and for arbitrary $\kappa > 0$ *it holds that*

$$\lim_{x\to\infty} \mathbb{P}\bigg(X \le x + \psi(x)\xi,\, Y \le \rho X + Xk\bigg(\frac{\psi(X)}{X}\bigg)\zeta \;\bigg|\; X > x\bigg) = (1 - \mathrm{e}^{-\xi})G(\zeta). \tag{9}$$

Note that the radial component R influences the limit statements of Theorem 1 *only* by its tail behavior characterized by the auxiliary function ψ.

Now we deduce a generalization of Theorem 1, where R and T do not have to be stochastically independent anymore. We assume the conditional distribution functions

$$H_t(r) = \mathbb{P}(R \le r \mid T = t), \qquad r \in (0, \infty),\ t \in \bigg[-\frac{\pi}{2}, \frac{\pi}{2}\bigg] \tag{10}$$

to be of type $\Gamma(\psi_t)$ with asymptotically equivalent auxiliary functions ψ_t, i.e. there exists some ψ with $\psi_t \sim \psi$ for all t. Then the distinction among the H_t is captured by the a_t in the von Mises representation, cf. Remarks 2(i) and 2(iii).

Theorem 2. (Dependent polar components with similar conditional tails.) *Let* $(X, Y) = R(u(T), v(T))$ *satisfy Assumptions 1 and 3. Instead of Assumption 2, let* T *have a positive, continuous marginal density and* $\overline{H}_t(r) = a_t(r)\exp(-\int_0^r 1/\psi(u)\,\mathrm{d}u)$ *with* $a_t(r) \to a_t > 0$ *uniformly in* t *for* $r \to \infty$. *Then the limit statements (8) and (9) hold.*

The proof of Theorem 2 is provided in Section 6.

Remark 4. The distribution of (X, Y) according to Theorem 2 might differ substantially from those with independent polar components, even in the asymptotic region; see Example 2 with Figure 2.

4. Geometric dependence measure and criteria for limit theorems

Here we present criteria formulated in terms of the distributions (not using the auxiliary functions) which allow us to apply Theorem 2. We describe the dependence between R and T by comparing $(X, Y) = R(u(T), v(T))$ with a *reference model* $(\widetilde{X}, \widetilde{Y}) = \widetilde{R}(u(T), v(T))$. Hereby, (X, Y) fulfills only Assumptions 1 and 3, but $(\widetilde{X}, \widetilde{Y})$ fulfills also Assumption 2, in particular $\widetilde{R}$ and T are independent. We denote quantities with respect to $\widetilde{R}$ by a tilde: distribution $\widetilde{H}$, densities $\widetilde{h}$, $f_{\widetilde{R}T}$ and $f_{\widetilde{X}\widetilde{Y}}$, etc.

The distance between the corresponding distributions $\widetilde{H}$ of $\widetilde{R}$ and H_t from (10) is measured by $\delta\colon [-\pi/2, \pi/2] \times (0, \infty) \to \mathbb{R}$, $(t, r) \mapsto \delta_t(r)$ with

$$\delta_t = (\widetilde{H}^{\leftarrow} - H_t^{\leftarrow}) \circ H_t \quad \iff \quad H_t(r) = \widetilde{H}(r + \delta_t(r)). \tag{11}$$

The asymptotics of $\widetilde{H}$ should correspond to that of the H_t. Besides that, the choice of $\widetilde{H}$ is free; we assume for later considerations on densities that $\widetilde{h}$ is monotonically decreasing. Note that (R, T) as well as $(\widetilde{R}, T)$ fulfill Assumption 1(ii); hence, $\delta_t(r)$ is continuous in t and continuously differentiable in r, and the H_t possess densities h_t.

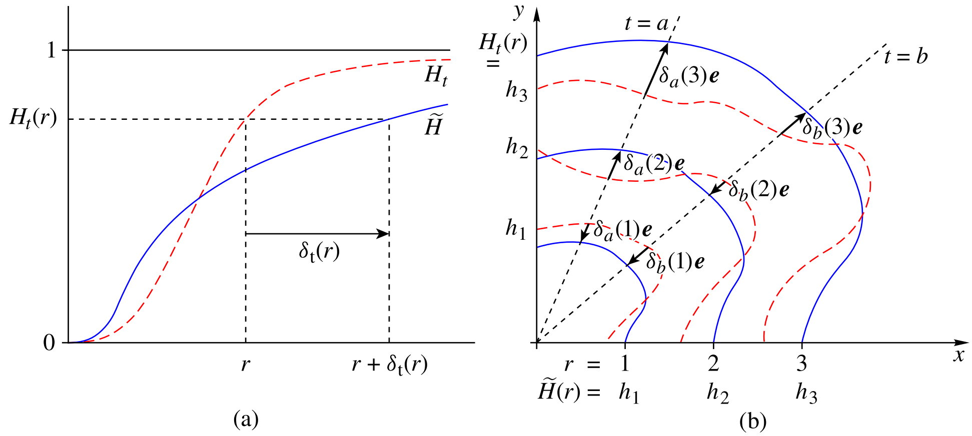

FIGURE 1: Meaning of δ_t: (a) distribution functions $\widetilde{H}$ (*solid*), H_t (*dashed*); (b) their level lines for $r = 1, 2, 3$.

In Figure 1(a) we illustrate the meaning of $\delta_t(r)$. In Figure 1(b) we illustrate δ_t in the (x, y)-plane (considering r as a function of x, y) as a radially directed vector field $\delta_t(r)\boldsymbol{e}$ with $\boldsymbol{e} = (1/r)(x, y) = (u(t), v(t))$. The sets $\{r = r_1\} = \{\widetilde{H}(r) = f_1\}$ with $f_1 = \widetilde{H}(r_1)$ do not coincide with $\{H_t(r) = f_1\}$ but differ from them by the shifts $\delta_t(r_1)\boldsymbol{e}$.

Each of the following two assumptions guarantees that all H_t are of type Γ with asymptotically equivalent auxiliary functions.

Assumption 4. *The following hold:*

(i) $\widetilde{\psi} \in \mathrm{RV}_\alpha(\infty)$*;*

(ii) $\delta_t'(r) \to 0$ *uniformly in t for $r \to \infty$.*

In the standard cases of elliptical vectors with $\widetilde{R}$ of type $\Gamma(\widetilde{\psi})$, e.g. bivariate normal, Kotz, and logistic distributions, the auxiliary function $\widetilde{\psi}$ is regularly varying and, hence, fulfills Assumption 4(i). However, if $\widetilde{H}$ cannot be chosen such that it possesses a regularly varying $\widetilde{\psi}$ then we have the following assumption.

Assumption 5. *The following hold:*

(i) $\delta_t(r)\widetilde{\psi}'(r)/\widetilde{\psi}(r) \to 0$ *for $r \to \infty$;*

(ii) $\delta_t'(r) \to 0$ *uniformly in t for $r \to \infty$.*

Note that Assumptions 4 and 5 do not exclude shifts $\delta_t(r) \to \infty$; see Example 2.

Geluk and de Haan (1987) investigated the class of Γ-varying functions and provided their major properties. We state one result (in their proof of Proposition 1.31(3)) in the following lemma.

Lemma 1. *Let $\widetilde{H}$ be a distribution function of type $\Gamma(\widetilde{\psi})$ and w a differentiable function with $w' \in \mathrm{RV}_\beta(\infty)$, $\beta > -1$. Then the composition $\widetilde{H} \circ w$ is of type $\Gamma((\widetilde{\psi} \circ w)/w')$.*

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Under Assumption 4(ii) or 5(ii) for (11), this lemma with $w_t(r) = r + \delta_t(r)$, i.e. $w_t' \in \mathrm{RV}_0(\infty)$, shows that all H_t are of type $\Gamma(\psi_t)$ with

$$\psi_t(r) = \frac{\widetilde{\psi}(r + \delta_t(r))}{1 + \delta_t'(r)} \sim \widetilde{\psi}(r + \delta_t(r)). \tag{12}$$

With Assumption 4(i) or 5(i), we have the following proposition.

Proposition 1. *Under Assumption 4 or 5, the conditional distribution functions H_t defined in (10) are of type $\Gamma(\widetilde{\psi})$ for all $t \in [-\pi/2, \pi/2]$.*

Proof. Under Assumption 4, with Remark 1(i) and $\lambda_t(r) := 1 + \delta_t(r)/r \to 1$ for $r \to \infty$, it follows that

$$\frac{\widetilde{\psi}(r + \delta_t(r))}{\widetilde{\psi}(r)} = \frac{\widetilde{\psi}(\lambda_t(r) r)}{\widetilde{\psi}(r)} \to 1.$$

Under Assumption 5, it follows that for $r \to \infty$,

$$\begin{aligned} \frac{\widetilde{\psi}(r + \delta_t(r))}{\widetilde{\psi}(r)} &= \exp(\ln \widetilde{\psi}(r + \delta_t(r)) - \ln \widetilde{\psi}(r)) \\ &\sim \exp(\delta_t(r)(\ln \widetilde{\psi}(r))') \\ &= \exp\bigg(\delta_t(r) \frac{\widetilde{\psi}'(r)}{\widetilde{\psi}(r)}\bigg) \\ &\to 1. \end{aligned}$$

Consequently, in both cases we have $\psi_t(r) \sim \widetilde{\psi}(r)$ for all $t \in [-\pi/2, \pi/2]$ and $r \to \infty$. Remark 2(i) yields that $\widetilde{\psi}$ is an auxiliary function for all H_t. □

Example 1. We consider $\widetilde{\psi}(r) = f(r)\exp(-\gamma r^\tau)$ for $\gamma, \tau > 0$ with $f' \in \mathrm{RV}_{\alpha-1}(\infty)$ and $\alpha \in \mathbb{R}$ as an example for a not regularly varying auxiliary function.

Let $\delta_t'(r) \to 0$ for $r \to \infty$. With $\lambda_t(r) := 1 + \delta_t(r)/r$ and (12), we obtain

$$\frac{\psi_t(r)}{\widetilde{\psi}(r)} \sim \frac{\widetilde{\psi}(\lambda_t(r) r)}{\widetilde{\psi}(r)} = \frac{f(\lambda_t(r) r)}{f(r)} \exp(-\gamma r^\tau [(\lambda_t(r))^\tau - 1]).$$

It holds that

$$\psi_t(r) \sim \widetilde{\psi}(r) \quad \Longleftrightarrow \quad -\gamma r^\tau((\lambda_t(r))^\tau - 1) = -\frac{\gamma r^\tau \tau \delta_t(r)}{r} + o\bigg(\frac{\delta_t(r)}{r}\bigg) \to 0,$$

or, equivalently, $\delta_t(r) = o(r^{1-\tau})$, i.e. the bounding condition for δ_t in Assumption 5(i) is *just* fulfilled:

$$\delta_t(r) \frac{\widetilde{\psi}'(r)}{\widetilde{\psi}(r)} = \delta_t(r) \bigg(\frac{f'(r)}{f(r)} - \gamma \tau r^{\tau-1}\bigg) \to 0.$$

Thus, weakening Assumption 5 is not possible, as it would violate $\psi_t \sim \widetilde{\psi}$.

Lemma 2. *Under Assumption 4(ii) or 5(ii) for (11), the functions $a_t(r)$ from the von Mises representation of $H_t(r)$ converge to $\widetilde{a}$ of $\widetilde{H}$ uniformly in t for $r \to \infty$.*

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Proof. Assumptions 4(ii) and 5(ii)) imply the existence of $D := \max(|\delta_t'(r)|)$ as well as of a monotonic sequence $\xi_n \to \infty$ such that for all t and $r > \xi_n$ it holds that $|\delta_t'(r)| \le D2^{-n}$; hence, $|\delta_t(r)| \le |\delta_t(\xi_n)| + D2^{-n}(r - \xi_n)$. Thus, all graphs of $|\delta_t|$ lie below a polygon of lines between ξ_n and ξ_{n+1} with slopes $D2^{-n} \to 0$ and, consequently, $\delta_t(r)/r \to 0$ uniformly in t for $r \to \infty$. Hence, for a_t and $\widetilde{a}$ from the von Mises representations of H_t and of $\widetilde{H}$ it holds that $a_t(r) = \widetilde{a}(r(1 + \delta_t(r)/r)) \to \widetilde{a}$ uniformly in t for $r \to \infty$. □

Proposition 1 and Lemma 2 show that for $\delta_t(r)$ fulfilling Assumption 4 or Assumption 5, we can apply Theorem 2 and obtain the following theorem.

Theorem 3. (Dependent polar components, asymptotically vanishing relative shifts.) *Let the reference model $(\widetilde{X}, \widetilde{Y})$ fulfill Assumptions 1 and 2 (with $f_{\widetilde{R}T}$ and $\widetilde{\psi}$), and let $(X, Y) = R(u(T), v(T))$ satisfy Assumptions 1, 3, and 4 or Assumptions 1, 3, and 5. Then the limit statements (8) and (9) hold.*

Example 2. We start with the elliptical normal distribution with correlation $\rho = 0.5$ as the reference model $(\widetilde{X}, \widetilde{Y})$, i.e. $\widetilde{H}(r) = 1 - \exp(-r^2/2)$. We choose the shifts

$$\delta_t(r) = \sqrt{\frac{r^3 \sin^2(t)}{(1 + r^2 \sin^2(t))}} = \sqrt{r}\,\sin(\arctan(r \sin(t)))$$

fulfilling the assumptions of Theorem 3. As $t_0 = 0$ and $\delta_0(r) = 0$, $\widetilde{H}$ coincides with H_0.

In Figure 2(a) we compare the level lines of the joint density f_{XY} of (X, Y) with the dependent polar components to those of the reference density $f_{\widetilde{X}\widetilde{Y}}$ (dashed ellipses) of $(\widetilde{X}, \widetilde{Y})$. Along some t-rays, the distance between the level lines of f_{XY} and $f_{\widetilde{X}\widetilde{Y}}$ becomes arbitrarily large as $\delta_t(r) \to \infty$, which is *not* excluded by $\delta_t'(r) \to 0$. We can also see that the level lines do not possess their maximal x-values on the ray $y = \rho x$ any longer, not even along any other fixed ray. However, Theorem 3 verifies that the limit results (8) and (9) hold with unchanged ρ, which means that the asymptotic behavior of (X, Y) is still determined by an arbitrarily small sector around the ray $\{y = \rho x\} = \{t = t_0\}$.

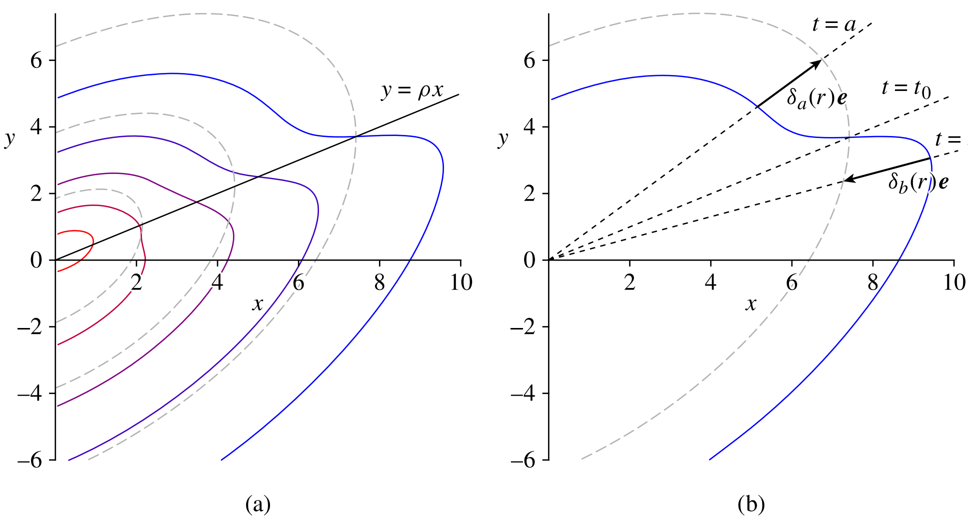

FIGURE 2: For Example 2. Level lines of the density of (X, Y) illustrating (a) Theorem 3 and (b) Corollary 2(i).

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Now we generalize Theorem 3 by considering relative shifts $\delta_t'(r)$ which do not vanish asymptotically but tend to a t-dependent limit, i.e. the auxiliary functions ψ_t are no longer asymptotically equivalent. For this purpose, we construct another polar representation for (X, Y) with a new radial component R^* counterbalancing the T-dependence and make the following assumptions.

Assumption 6. *The following hold:*

(i) $\widetilde{\psi} \in \mathrm{RV}_\alpha(\infty)$, $\alpha < 1$;

(ii) $\delta_t'(r) \to c(t) - 1$ *uniformly in* t *for* $r \to \infty$ *with some function* $c\colon [-\pi/2, \pi/2] \to (0, \infty)$.

Assumption 7. *Let* $u\colon [-\pi/2, \pi/2] \to [0, 1]$, $v\colon [-\pi/2, \pi/2] \to \mathbb{R}$ *be such that* $u^* := u/c$ *and* $v^* := v/c$ *fulfill Assumption 3 (with corresponding quantities* $u^*_{\max}$, l^*, t^*_0, κ^*, ρ^**).*

Proposition 2. *Under Assumption 6 for (11), the conditional distribution functions* H_t *are of type* $\Gamma(\psi_t)$ *with* $\psi_t(r) \sim c^{\alpha-1}(t)\widetilde{\psi}(r)$ *for* $r \to \infty$.

Proof. With Lemma 1 and $\lambda_t(r) := 1 + \delta_t(r)/r \to c(t)$ for $r \to \infty$, it follows that

$$\psi_t(r) = \frac{\widetilde{\psi}(r + \delta_t(r))}{1 + \delta_t'(r)} \sim \frac{\widetilde{\psi}(\lambda_t(r)r)}{\widetilde{\psi}(r)} \frac{\widetilde{\psi}(r)}{c(t)} \sim c^\alpha(t)\frac{\widetilde{\psi}(r)}{c(t)}. \qquad \square$$

Remark 5. Assumption 6 is the counterpart to Assumption 4 for $\widetilde{\psi} \in \mathrm{RV}_\alpha(\infty)$. There exists no analogue to Assumption 5 for $\widetilde{\psi} \notin \mathrm{RV}_\alpha(\infty)$ because of the following argument. If $\delta_t' \to c(t) - 1$ with a continuous nonconstant $c(t)$ and it holds that $\psi_t \sim a(t)\widetilde{\psi}$, $a(t) \in (0, \infty)$ as in Proposition 2, then $\widetilde{\psi}$ has to be regularly varying.

This can be shown by the following argument. We have $\delta_t(r) = (c(t) - 1)r + o(r)$; hence, with Lemma 1 it follows that $\psi_t(r)/\widetilde{\psi}(r) \sim (1/c(t))\widetilde{\psi}(c(t)r)/\widetilde{\psi}(r) \sim a(t)$. Remark 1(iv) implies that $\widetilde{\psi}$ is regularly varying.

What happens for not regularly varying $\widetilde{\psi}$ can be seen in this example. For $\widetilde{H}(r) = 1 - \exp(1 - \mathrm{e}^r)$, $\widetilde{\psi}(r) = \mathrm{e}^{-r}$, the quotient $\widetilde{\psi}(c(t)r)/\widetilde{\psi}(r) = \exp(-(c(t) - 1)r)$ tends to 0 or to ∞ for $c(t) \neq 1$.

Now we construct for $R(u(T), v(T))$ a new polar representation $R^*(u^*(T), v^*(T))$, keeping the angular component T but changing the radial component as follows.

Proposition 3. *Let be* $R^* = c(t)R$ *for* $T = t$*. Then, under Assumptions 6 and 7, the distributions* $H_t^*(r) = \mathbb{P}(R^* \leq r \mid T = t)$ *are of type* $\Gamma(\widetilde{\psi})$ *for all* $t \in [-\pi/2, \pi/2]$.

Proof. Lemma 1 with $w_t(r) = r/c(t)$ yields that $H_t^* \in \Gamma(\psi_t^*)$ with auxiliary functions $\psi_t^*(r) = c(t)\psi_t(r/c(t))$. Under Assumptions 6 and 7, and with Proposition 2, we have $\psi_t^*(r)/\widetilde{\psi}(r) = c(t)c^{\alpha-1}(t)\widetilde{\psi}(r/c(t))/\widetilde{\psi}(r) \sim c^\alpha(t)c^{-\alpha}(t) = 1$. Thus, all $H_t^* \in \Gamma(\widetilde{\psi})$. $\square$

Lemma 3. *Under Assumption 6(ii) for (11), the functions* $a_t^*(r)$ *from the von Mises representation of* $H_t^*(r)$ *converge to* $\widetilde{a}$ *of* $\widetilde{H}$ *uniformly in* t *for* $r \to \infty$.

Proof. From $H_t^*(r) = H_t(r/c(t))$ it follows that $\delta_t^*(r) = (1/c(t) - 1)r + \delta_t(r/c(t))$. We decompose $\delta_t(r) = {}_1\delta_t(r) + {}_2\delta_t(r)$ with ${}_1\delta_t(r) = (c(t) - 1)r$ and ${}_2\delta_t'(r) \to 0$. Then Lemma 2 implies that both $\delta_t(r)/r \to c(t) - 1$ and $\delta_t^*(r)/r \to 0$, and, hence, $a_t^*(r) \to \widetilde{a}$ uniformly in t for $r \to \infty$. $\square$

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Proposition 3 and Lemma 3 show that for $(X, Y) = R^*(u^*(T), v^*(T))$ we can apply Theorem 2 and obtain the following theorem.

Theorem 4. (Dependent polar components, asymptotically finite relative shifts.) *Let the reference model $(\widetilde{X}, \widetilde{Y})$ fulfill Assumptions 1 and 2 (with $f_{\widetilde{R}T}$ and $\widetilde{\psi}$), and let $(X, Y) = R(u(T), v(T))$ satisfy Assumptions 1, 6, and 7. Then the limit statements (8) and (9) hold with ρ^*, κ^*, and l^*.*

In the situation of Theorem 4, we demand Assumption 3 only for the *new* coordinate functions u^* and v^* so that the limit statements depend on their parameters.

Given a polar random vector as in Section 3 (fulfilling Assumptions 1, 2, and 3) for the original u and v, how much can we deviate from the independence of R and T such that the *original* limit statements remain valid? The next corollary gives an answer.

Corollary 1. *Let $(X, Y) = R(u(T), v(T))$ have coordinate functions u and v fulfilling Assumption 3 and dependent polar components fulfilling Assumptions 1 and 6 with $(c - 1) \in \mathrm{RV}^s_\beta(t_0)$ for some $\beta > \kappa$ and $u_{\max} = u^*_{\max}$. Then, the limit statements (8) and (9) hold with the original parameters ρ and κ.*

Proof. The condition $(c - 1) \in \mathrm{RV}^s_\beta(t_0)$ implies that $(1 - 1/c) \in \mathrm{RV}^s_\beta(t_0)$. Hence,

$$l^* = u^*_{\max} - u^* = u^*_{\max} - \frac{u}{c} = \left(u_{\max} - u + \left(1 - \frac{1}{c}\right)u\right) \in \mathrm{RV}^s_\kappa(t_0). \qquad \square$$

Example 3. Here we present an example for a polar random vector (X, Y) according to Theorem 4 and choose the same reference model as in Example 2, but now with

$$\delta_t(r) = r(c(t) - 1) + \frac{\sin(r)t}{\sqrt[4]{r + 2}}, \qquad c(t) = \frac{t}{2}\sin(-|3t|) + 1.$$

Since $(c - 1) \in \mathrm{RV}^s_1(0)$, the criterion of Corollary 1 is not fulfilled. The limit statements of Theorem 4 hold for $\kappa^* = 2$, $u^*_{\max} = 1.009\,95$, $t^*_0 = -0.289\,84$, and $\rho^* = 0.244\,13$.

In Figure 3(a) we display—analogously to Figure 2(a)—level lines of the joint density f_{XY} and those of the reference density $f_{\widetilde{X}\widetilde{Y}}$ (dotted ellipses). In Figure 3(b) the curve $(u^*(t), v^*(t))$ (solid line) is contrasted to the reference curve $(u(t), v(t))$ (dotted line). Note that the reference curve coincides with one of the level lines of $f_{\widetilde{X}\widetilde{Y}}$, while the curve $(u^*(t), v^*(t))$ in general does not coincide with any level line of f_{XY}.

The functions δ_t introduced as the shifts between the distributions H_t of R and $\widetilde{H}$ of $\widetilde{R}$ also provide information about the density f_{XY} of (X, Y). We compare f_{XY} with the density $f_{\widetilde{X}\widetilde{Y}}$ of the reference model $(\widetilde{X}, \widetilde{Y}) = \widetilde{R}(u(T), v(T))$. An interpretation of the first statement in the following corollary is given in Figure 2(b): in the context of Theorem 3, $\delta_t(r)$ displays the asymptotic distance between the level lines of f_{XY} and $f_{\widetilde{X}\widetilde{Y}}$ measured by the radial component R (not by the Euclidean distance).

Corollary 2. (i) *Let (X, Y) and $(\widetilde{X}, \widetilde{Y})$ fulfill the assumptions of Theorem 3. Then*

$$f_{XY}(r(u(t), v(t))) \sim f_{\widetilde{X}\widetilde{Y}}((r + \delta_t(r))(u(t), v(t))) \quad \textit{for } r \to \infty.$$

(ii) *Analogously, under the assumptions of Theorem 4 it holds that for $r \to \infty$,*

$$\begin{aligned} f_{XY}(r(u(t), v(t))) &\sim c^2(t) f_{\widetilde{X}\widetilde{Y}}((r + \delta_t(r))(u(t), v(t))) \\ &= f_{\widetilde{X}\widetilde{Y}}((r + \delta_t(r))(u^*(t), v^*(t))). \end{aligned}$$

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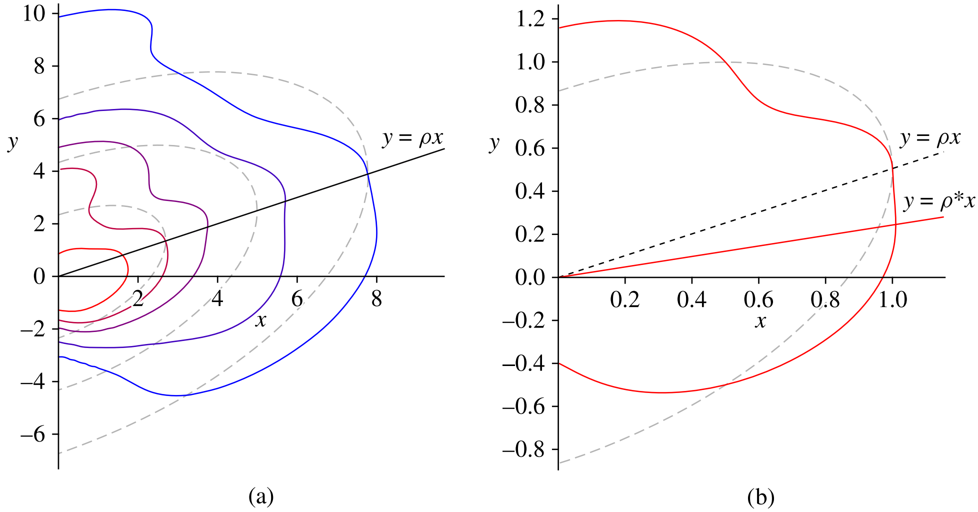

FIGURE 3: For Example 3. (a) Level lines of the density of (X, Y) illustrating Theorem 4 (b) Curve $(u^*(t), v^*(t))$ (*solid*) in contrast to $(u(t), v(t))$ (*dotted*).

In Examples 2 and 3 we point out the meaning of Theorems 3 and 4 for given shifts $\delta_t(r)$. But in concrete situations, one usually starts from the (estimated) distributions and calculates the shifts, which is considered in the following example.

Example 4. Consider a mixture of normal distributions with survival functions ${}_1\overline{H}_t(r) = \exp(-r^2/(2\sigma_t^2))$, where σ_t is continuous in t and other distributions with a quicker tail decay ${}_2\overline{H}_t(r) = \exp(-f_t(r))$, where $f\colon [-1, 1] \times (0, \infty) \to [0, \infty)$ is continuous in t and differentiable in r fulfilling $f_t(0) = 0$, $f_t'(r) > 0$, and $f_t(r)/r^2 \to \infty$ for $r \to \infty$,

$$H_t(r) = 1 - a(t)\exp\biggl(-\frac{r^2}{(2\sigma_t^2)}\biggr) - b(t)\exp(-f_t(r)),$$

where a and b are positive continuous functions with $a(t) + b(t) = 1$.

The natural choice of the reference model is $\widetilde{H}(r) = 1 - \exp(-r^2/2)$, and we derive the corresponding shifts from $\widetilde{H}(r + \delta_t(r)) = H_t(r)$ as

$$-\frac{(r + \delta_t(r))^2}{2} = \ln \overline{H}_t(r) \quad\Longrightarrow\quad \delta_t(r) = -r + \sqrt{-2\ln \overline{H}_t(r)} \geq -r. \tag{13}$$

For $r \to \infty$ it holds that

$$\begin{aligned}\overline{H}_t(r) &= \exp\biggl(-\frac{r^2}{(2\sigma_t^2)} + \ln a(t)\biggr)\biggl[1 + \exp\biggl(-f_t(r) + \frac{r^2}{(2\sigma_t^2)} + \ln\biggl(\frac{b(t)}{a(t)}\biggr)\biggr)\biggr]\\ &= \exp\biggl(-\frac{r^2}{(2\sigma_t^2)} + \ln a(t)\biggr)\biggl[1 + \biggl(\frac{b(t)}{a(t)}\biggr)\exp\biggl(-r^2\biggl(\frac{f_t(r)}{r^2} - \frac{1}{(2\sigma_t^2)}\biggr)\biggr)\biggr]\\ &\sim \exp\biggl(-\frac{r^2}{(2\sigma_t^2)} + \ln a(t)\biggr)\end{aligned}$$

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and, hence, with (13), we have

$$\delta_t(r) \sim -r + \sqrt{\frac{r^2}{\sigma_t^2} - 2\ln a(t)} \sim r\left(-1 + \left(\frac{1 - \sigma_t^2 \ln a(t)/r^2}{\sigma_t}\right)\right).$$

For $\sigma_t^2 = 1$ the shifts $\delta_t(r) \sim -\ln a(t)/r$ fulfill the assumptions from Theorem 3. For $\sigma_t^2 \neq 1$ it holds that $\delta_t(r) \sim r(1 - \sigma_t)/\sigma_t$ according to Theorem 4 with $c(t) = 1/\sigma_t$.

Remark 6. The result in Example 4 is as expected. The component of the distribution with the quicker tail decay is asymptotically negligible, the t-dependence of the variance can be removed with a change of the radial component. Theorems 2, 3, and 4 give a safe mathematical basis for such plausibility arguments.

Remark 7. If $\delta_t' \in \mathrm{RV}_\beta(\infty)$ with $\beta > 0$ then no scale transformation of R permits to apply Theorem 2 as Lemma 1 shows. If $\delta_t' \notin \mathrm{RV}_\beta(\infty)$ there exist cases with $\psi_t \sim \psi$ for all t although $\delta_t'(r)$ does not converge to 0, as the following example shows. We have $\overline{H}_t(r) = \exp(-r - t\sin(r)/\pi)$ are of type $\Gamma(1)$ for all t, but the shifts with respect to $\widetilde{H} = H_0$ yield $\delta_t'(r) = t\cos(r)/\pi$ not converging to 0 or to some other limit.

5. Comparison with Balkema and Embrechts' approach

Now we contrast our geometric approach for weakening the independence assumption to those of Balkema and Embrechts (2007). They do not explicitly use polar representations; however, in the bivariate case their model can be reformulated in terms of a polar representation for (X, Y) with $\kappa = 2$ and $R \in \Gamma(\psi)$. Balkema and Embrechts (2007) transfer the limit result in their Theorem 9.1 (p. 137) for some random vector with density $f_{\widetilde{XY}}$ to another one in their Theorem 11.2 (p. 158) with density

$$f_{XY}(ru(t), rv(t)) = Q(r, t) f_{\widetilde{XY}}(ru(t), rv(t)), \tag{14}$$

where Q is in $\mathcal{L}$, meaning that

$$\mathcal{Q}(r, t) := \frac{Q(r + r_0, t)}{Q(r, t)}, \qquad \lim_{r\to\infty} \mathcal{Q}(r, t) = 1 \quad \text{for all } r_0 > 0 \text{ and all } t.$$

Under Assumption 1, Q is also the quotient $f_{RT}/f_{\widetilde{RT}}$, cf. (7).

This paper extends the results of Balkema and Embrechts (2007) for the bivariate case in the sense that we consider more general polar distributions with arbitrary $\kappa > 0$. But even for $\kappa = 2$, we cover cases not included by Balkema and Embrechts (2007), i.e. with $\lim_{r\to\infty} \mathcal{Q}(r, t) \neq 1$, as it is shown in the following theorem.

Theorem 5. (Ratio of densities.) *Let (X, Y) fulfill the assumptions of Theorem 4, then for Q from (14) we obtain the following results, depending on the variation index α of the auxiliary function $\widetilde{\psi}$ and the values of the function c from Assumption 6.*

(a) *For $\alpha < 0$ (i.e. $\widetilde{\psi}(r) \to 0$): for t with $c(t) < 1, = 1, > 1$, we have $\lim_{r\to\infty} \mathcal{Q}(r, t) = 0, = 1, = \infty$, respectively.*

(b) *For $\alpha > 0$ (i.e. $\widetilde{\psi}(r) \to \infty$): we have $\lim_{r\to\infty} \mathcal{Q}(r, t) = 1$.*

(c) *For $\alpha = 0$: we have: if $\widetilde{\psi}(r) \to 0$ or $\widetilde{\psi}(r) \to \infty$, then (a) or (b) holds, respectively; if $\widetilde{\psi}(r) \to k \in (0, \infty)$ then it holds that $\lim_{r\to\infty} \mathcal{Q}(r, t) = \exp((1 - 1/c(t))r_0/k)$ for t with $c(t) \neq 1$ and $\lim_{r\to\infty} \mathcal{Q}(r, t) = 1$ for t with $c(t) = 1$.*

Proof. With (7), we obtain

$$Q(r,t) = \frac{f_{XY}(ru(t), rv(t))}{f_{\widetilde{X}\widetilde{Y}}(ru(t), rv(t))} = \frac{f_{RT}(r,t)}{f_{\widetilde{R}T}(r,t)} = \frac{h_t(r)}{\widetilde{h}(r)} = (1+\delta_t'(r))\frac{\widetilde{h}(r+\delta_t(r))}{\widetilde{h}(r)}.$$

Hence, we have

$$\begin{aligned}\mathcal{Q}(r,t) &= \frac{1+\delta_t'(r+r_0)}{1+\delta_t'(r)}\frac{\widetilde{h}(r+r_0+\delta_t(r+r_0))}{\widetilde{h}(r+r_0)}\frac{\widetilde{h}(r)}{\widetilde{h}(r+\delta_t(r))}\\ &\sim \frac{\widetilde{h}(r)}{\widetilde{h}(r+\delta_t(r))}\frac{\widetilde{h}(r+r_0+\delta_t(r+r_0))}{\widetilde{h}(r+r_0)}\\ &\sim \exp\biggl(\int_r^{r+\delta_t(r)}\frac{1}{\widetilde{\psi}(u)}\,\mathrm{d}u\biggr)\exp\biggl(-\int_{r+r_0}^{r+r_0+\delta_t(r+r_0)}\frac{1}{\widetilde{\psi}(u)}\,\mathrm{d}u\biggr). \qquad (15)\end{aligned}$$

For the last step, we exploit the fact that $1/(1-\widetilde{H})$ and $1/\widetilde{h}$ are $\Gamma(\widetilde{\psi})$-varying due to the assumed monotony of $\widetilde{h}$ (with l'Hospital in (3), $\widetilde{\psi}' \to 0$). Thus, we can apply the von Mises representation (4) for $\widetilde{h}$. As $\widetilde{\psi} \in \mathrm{RV}_\alpha(\infty)$ with $\alpha < 1$ (Assumption 6), it holds that $S(x) := \int_0^x 1/\widetilde{\psi}(u)\,\mathrm{d}u \in \mathrm{RV}_{1-\alpha}(\infty)$ with a positive variation index; hence,

$$\begin{aligned}\exp\biggl(\int_r^{r+\delta_t(r)}\frac{1}{\widetilde{\psi}(u)}\,\mathrm{d}u\biggr) &= \exp\biggl(S(r)\biggl(\frac{S((1+\delta_t(r)/r)r)}{S(r)}-1\biggr)\biggr)\\ &\sim \exp(S(r)(c^{1-\alpha}(t)-1)).\end{aligned}$$

Putting the last expression into (15), we finally obtain

$$\mathcal{Q}(r,t) \sim \exp((c^{1-\alpha}(t)-1)(S(r)-S(r+r_0))) = (\exp(c^{1-\alpha}(t)-1))^{S(r)-S(r+r_0)}. \qquad (16)$$

The limit of (16) results from the behavior of S: for $\alpha < 0$ the variation index of S is $1-\alpha > 1$, and for $\alpha > 0$ it is $1-\alpha < 1$, and for $\alpha = 0$ it is $1-\alpha = 1$. □

Remark 8. (i) For the situation considered in Theorem 3 with $\delta_t' \to 0$, i.e. $c \equiv 1$, the density quotient Q is in $\mathcal{L}$ as in Balkema and Embrechts (2007).

(ii) The case distinction in Theorem 5 corresponds to the tail behavior of $\widetilde{H}$. If $1-\widetilde{H}$ decreases at least exponentially fast ('light tail'), we are in case (a) or (c) and generally $Q \notin \mathcal{L}$. If $1-\widetilde{H}$ decreases slower than any exponential but faster than any power function ('mildly heavy tail'), we are in case (b) and $Q \in \mathcal{L}$.

(iii) In Example 4 (for $\sigma_t^2 \neq 1$), the corresponding Q is not in $\mathcal{L}$, the quotient $\mathcal{Q}(r,t)$ tends to 0 if $\sigma_t^2 < 1$ or to ∞ if $\sigma_t^2 > 1$. Thus, this example is not covered by the theorem of Balkema and Embrechts (2007).

Our approach to measure the dependence between the polar components R and T with the shifts $\delta_t(r)$ is intuitive and is based primarily on distributions and not on densities. Both criteria for $\delta_t(r)$ provided in Theorem 3 and in Theorem 4 possess a geometrical interpretation, while Balkema and Embrechts (2007, p. 158) 'warn the reader that functions from the class $\mathcal{L}$ are not as tame as they may seem'.

To sum it up, describing random vectors using a polar representation $R(u(T), v(T))$ permits a lot of freedom in modeling the asymptotic behavior as it requires only weak and local assumptions on u and v. Allowing a certain dependence between R and T, we show validity of the limit results, which is of importance for empirical applications.

6. Proof of Theorem 2

To prove Theorem 2 we follow the strategy of the proof for independent R and T in Fougères and Soulier (2010), cf. Seifert (2014). The additionally required steps are presented in the following. Under the assumptions of Theorem 2 for any $p > 0, \alpha > 1$ there exist t-independent constants $C_p, C_{p,\alpha}$ such that for x large enough and $z \geq 0$,

$$\frac{\overline{H}_t(x + z\psi(x))}{\overline{H}_t(x)} \leq C_p(1+z)^{-p}, \qquad \frac{\overline{H}_t(\alpha x)}{\overline{H}_t(x)} \leq C_{p,\alpha}\left(\frac{\psi(x)}{x}\right)^p, \tag{17}$$

and for $\breve{k} \in \mathrm{RV}_\alpha(0)$ with $\alpha > -1$ bounded on compact subsets of $(0, \infty]$ it holds that, locally uniformly in $d \geq 0$,

$$\lim_{x\to\infty} \int_d^\infty \frac{\overline{H}_{t(z)}(x + z\psi(x))}{\overline{H}_{t(z)}(x)} \frac{\breve{k}(z\psi(x)/x)}{\breve{k}(\psi(x)/x)} \,\mathrm{d}z = \int_d^\infty \mathrm{e}^{-z} z^\alpha \,\mathrm{d}z. \tag{18}$$

For $\overline{H}$ instead of $\overline{H}_t$, this is shown in Fougères and Soulier (2010, Lemmas 5.1 and 5.2). So it remains to prove that the constants can be chosen independent of t. The condition $a_t(x) \to a_t$ uniformly in t implies that for all $A > 1$ and x large enough it holds that $1/\sqrt{A} \leq a_t(x)/a_t \leq \sqrt{A}$ and $1/\sqrt{A} \leq a_t(x + z\psi(x))/a_t \leq \sqrt{A}$. Thus, we have

$$\frac{1}{A} \leq \left(\frac{\overline{H}_t(x + z\psi(x))}{\overline{H}_t(x)}\right) \exp\left(\int_x^{x+z\psi(x)} \frac{1}{\psi(u)} \,\mathrm{d}u\right) = \frac{a_t(x + z\psi(x))}{a_t(x)} \leq A.$$

Now we sketch the proof of Theorem 2. The probability of a set $B_{xy} := \{X > x, Y > y\}$ with $y > 0$ is calculated by integrating the conditional survival function $\overline{H}_t$ over the boundary of B_{xy} parameterized by $t \in (t_-, \pi/2)$, where $t_- := \tau^{-1}(0)$ with τ from Assumption 1(iii) and g denoting the continuous and positive marginal density of T. Thus,

$$\begin{aligned} \mathbb{P}\{X > x, Y > y\} &= \int_{t_-}^{\pi/2} \overline{H}_t\left(\max\left(\frac{x}{u(t)}, \frac{y}{v(t)}\right)\right) g(t) \,\mathrm{d}t \\ &= \int_{t_0-\varepsilon}^{t_1(x)} \overline{H}_t\left(\frac{y}{v(t)}\right) g(t) \,\mathrm{d}t \\ &\quad + \int_{t_1(x)}^{t_0+\varepsilon} \overline{H}_t\left(\frac{x}{u(t)}\right) g(t) \,\mathrm{d}t + \int_{|t-t_0|>\varepsilon} \overline{H}_t\left(\max\left(\frac{x}{u(t)}, \frac{y}{v(t)}\right)\right) g(t) \,\mathrm{d}t \\ &=: I(x) + J(x) + \mathrm{rem}(x) \end{aligned} \tag{19}$$

with an arbitrary $\varepsilon \in (0, t_-)$, $y = \rho x + xk(\psi(x)/x)\zeta$ with k from (8), and $t_1(x) := (v/u)^{-1}(y/x) = \arctan(\rho + k(\psi(x)/x)\zeta) \to \arctan(\rho) = t_0$ for $x \to \infty$.

We treat the case $\rho > 0$, $t_0 \leq t_1(x) < t_0 + \varepsilon$, from which the other cases can be deduced as in Seifert (2014). For dependent R and T the mean value theorem is required:

$$J(x) = \overline{H}_{q(x)}(x) \int_{t_1(x)}^{t_0+\varepsilon} \left(\frac{\overline{H}_t(x/u(t))}{\overline{H}_t(x)}\right) g(t) \,\mathrm{d}t =: \overline{H}_{q(x)}(x) L(x) \tag{20}$$

for some mean value $q(x) \in (t_1(x), t_0 + \varepsilon)$. For $x \to \infty$ it holds that $J(x) \sim \overline{H}_{t_0}(x) L(x)$, as we show later, cf. (21). Analogously to the proof presented in Seifert (2014, Theorem 1

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with $\tau = 0$), a substitution $t \mapsto z$ is made such that the argument of $\overline{H}$ becomes $x + z\psi(x)$ and (18) can be applied such that for $x \to \infty$, we have

$$J(x) \sim \overline{H}_{t_0}(x)\kappa^{1/\kappa-1}k_J\left(\frac{\psi(x)}{x}\right)\int_{|\zeta|^\kappa/\kappa}^{\infty} \mathrm{e}^{-z}z^{1/\kappa-1}\,\mathrm{d}z,$$
$$I(x) \sim \overline{H}_{t_0}(x)k_I\left(\frac{\psi(x)}{x}\right)f_\zeta(x)\int_{|\zeta|^\kappa/\kappa}^{\infty} \mathrm{e}^{-z}\,\mathrm{d}z$$

with $k_J \in \mathrm{RV}_{1/\kappa}(0)$ and $k_I \in \mathrm{RV}_{1+1/\kappa}(0)$ for $\rho = 0$, $k_I \in \mathrm{RV}_1(0)$ for $\rho \neq 0$, and some bounded function f_ζ. Since for $\kappa > 1$ as well as for $\rho = 0$ the variation index of k_J is smaller than that of k_I, and it follows that $I(x) = o(J(x))$ for $x \to \infty$.

Since u has a unique global maximum 1 at t_0, for all $\varepsilon > 0$ there exists an $\eta_\varepsilon \in (0, 1)$ with $u(t) < 1 - \eta_\varepsilon$ for all $|t - t_0| > \varepsilon$. The mean value theorem for some $|\breve{q} - t_0| > \varepsilon$ yields

$$\mathrm{rem}(x) \le \int_{|t-t_0|>\varepsilon} \overline{H}_t\left(\frac{x}{u(t)}\right)g(t)\,\mathrm{d}t \le \overline{H}_{\breve{q}}\left(\frac{x}{1-\eta_\varepsilon}\right)\int_{|t-t_0|>\varepsilon} g(t)\,\mathrm{d}t \le \overline{H}_{\breve{q}}\left(\frac{x}{1-\eta_\varepsilon}\right).$$

The second statement of (17) implies that for $\alpha > 1$, $p > 0$, and for all q there exists a $C_{q,p,\alpha}$ with $\overline{H}_q(\alpha x)/\overline{H}_{t_0}(x) \le C_{q,p,\alpha}(\psi(x)/x)^p$. Choosing $\alpha = 1/(1 - \eta_\varepsilon)$, it follows that $\mathrm{rem}(x) = o(\overline{H}_{t_0}(x)(\psi(x)/x)^p)$ for all $p > 0$ and, hence, $\mathrm{rem}(x) = o(J(x))$ for $x \to \infty$. Consequently, for $\kappa > 1$ and $\rho = 0$, J determines the asymptotics of (19). The proof of Theorem 2 can be completed as in Seifert (2014), where it is also shown how to deduce the statement (9) for random norming from the case $\rho = 0$.

Now we prove $J(x) \sim \overline{H}_{t_0}(x)L(x)$, cf. (20). For any $\varepsilon_1 \in (0, \varepsilon)$, we can decompose

$$J(x) = \int_{t_1(x)}^{t_0+\varepsilon_1} \overline{H}_t\left(\frac{x}{u(t)}\right)g(t)\,\mathrm{d}t + \int_{t_0+\varepsilon_1}^{t_0+\varepsilon} \overline{H}_t\left(\frac{x}{u(t)}\right)g(t)\,\mathrm{d}t =: J_1(x) + J_2(x). \qquad (21)$$

Analogously to $\mathrm{rem}(x) = o(J(x))$ as proved above, it follows that $J_2(x) = o(J_1(x))$ and, hence, $J(x) \sim J_1(x) = \overline{H}_{q_1(x)}(x)L(x)$ for some mean value $q_1(x) \in (t_0, t_0 + \varepsilon_1)$. Since this holds for any arbitrarily small ε_1, it follows that $J(x) \sim \overline{H}_{t_0}(x)L(x)$.

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