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POVERTY TRAPS AND INFERIOR GOODS IN A DYNAMIC HECKSCHER–OHLIN MODEL

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We extend the dynamic Heckscher–Ohlin model in Bond et al. [*Economic Theory* (48, 171–204, 2011)] and show that if the labor-intensive good is inferior, then there may exist multiple steady states in autarky and poverty traps can arise. Poverty traps for the world economy, in the form of Pareto-dominated steady states, are also shown to exist. We show that the opening of trade can have the effect of pulling the initially poorer country out of a poverty trap, with both countries having steady state capital stocks exceeding the autarky level. However, trade can also pull an initially richer country into a poverty trap. These possibilities are a sharp contrast with dynamic Heckscher–Ohlin models with normality in consumption, where the country with the larger (smaller) capital stock than the other will reach a steady state where the level of welfare is higher (lower) than in the autarkic steady state.

Keywords: Dynamic Heckscher–Ohlin Model, Poverty Traps, Inferior Goods

1. INTRODUCTION

In dynamic general equilibrium models such as the Ramsey one, there is a negative relation between capital accumulation and the rental on capital: the rental rate decreases when capital stock (per unit of effective labor) increases. This guarantees the uniqueness and the saddlepoint stability of the steady state, where the rental rate is equal to the sum of the discount factor and the depreciation rate.

In this paper, we utilize the dynamic Heckscher–Ohlin (H-O) model in Bond et al. (2011) to show how inferiority in consumption of the labor-intensive good can lead to a nonmonotonic relationship between the capital stock and the return on capital, resulting in multiple steady states under autarky.[1] Specifically, assuming that a labor-intensive good is a necessity at low income levels and an inferior good at higher income levels, we obtain conditions on technologies and labor endowment under which there exist three steady state equilibria in autarky. When

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the labor-intensive good is inferior, increases in the capital stock will reduce the supply of the labor-intensive good but will also reduce the demand for the labor-intensive good. Multiplicity of steady states may arise if the latter effect is sufficiently large for some range of income levels. This multiplicity yields a "poverty trap," in the sense that economies with a sufficiently small initial capital stock will converge to an equilibrium with low steady state consumption.[2] This explanation of poverty traps is novel in that it arises in a model with complete markets, convex preferences and technology, and a constant discount factor.[3]

We also address the question of whether a country with a small initial stock of capital can avoid the poverty trap by opening trade with a country that has a larger initial stock of capital. We show that there exists a range of initial endowments such that the initially poorer country will be pulled out of a poverty trap autarkic equilibrium by the richer country. Furthermore, the richer country will also end up with higher steady state capital stock (and utility levels) under free trade than it would have had under autarky. On the other hand, we also show that there exist initial endowments of capital such that both countries end up with a lower steady state capital stock under free trade than under autarky. In this case, a country that would not be in the poverty trap under autarky is pulled down into a poverty trap by opening trade with an initially poorer trading partner.

The results on free trade with inferior goods result from a combination of two effects: a *poverty trap effect* and a *hysteresis effect* of initial endowments. The hysteresis effect was first identified by Chen (1992), who showed that the country with higher initial capital in a two-country trade model will also have the larger steady state capital stock with free trade when countries have identical and homothetic preferences with a constant intertemporal elasticity of substitution.[4] This assumption on preferences, combined with the factor price equalization property, leads to a continuum of steady state equilibria under free trade, all of which have the same level of world capital stock. Which steady state the world economy converges to is determined by the initial distribution of capital across countries. Because the rental rate on capital is lower in the country with the initially higher capital stock, free trade among the countries yields an increase (decrease) in the rental on capital for the initially richer (poorer) country, and encourages (discourages) its capital accumulation. So the initial ranking of factor endowment ratios among countries is maintained along the dynamic equilibrium path.

Bond et al. (2011) showed that the continuum of free trade steady states continues to exist when the assumption of homotheticity is relaxed, although the size of the world capital stocks may vary across steady states. In this paper we show that the poverty trap feature of the autarky equilibrium also applies to the world economy.[5] Specifically, for parameter values that generate a poverty trap for the autarkic economy, there will exist a continuum of steady states for the world economy that are saddle points and are Pareto dominated by other steady state equilibria that are saddle points. We refer to this as the poverty trap effect under free trade. The hysteresis effect operates because the marginal utility of income will grow at the same rate in each country on the path to the free trade equilibrium,

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so the relative ranking of country utility levels will be maintained on the path to the steady state. The combination of these two effects leads to the possibility that both countries have higher (or lower) utility levels in the free trade steady state than in their autarkic ones. We also show that a transfer of income from a country in the high-income steady state to a country in the poverty-trap steady state could lead to an increase in the steady state welfare of both countries.

A natural question to ask is whether the presence of inferior goods is an important phenomenon empirically. Recently Jensen and Miller (2008) have provided evidence that two staple commodities, rice and wheat, are Giffen goods in China. Although it is not necessary for the labor-intensive goods to be Giffen goods to result in multiple steady states in our model, this evidence does suggest that inferiority in consumption may play a significant role for staple goods in developing countries.

This paper is organized as follows. Section 2 presents the dynamic two-country H-O model. Section 3 derives the steady state equilibria in autarky and proves that the poverty trap can arise because of inferiority in consumption. Section 4 characterizes the free trade steady states, whereas Section 5 discusses the possibility of poverty traps with free trade and the effect of international transfers of income. Section 6 offers some concluding remarks.

2. THE DYNAMIC TWO-COUNTRY HECKSCHER–OHLIN MODEL

In this section we formulate the continuous-time version dynamic optimization problem for a representative country in a dynamic H-O model. By a dynamic H-O model, we mean that each country has access to the same technology for producing two goods using a fixed factor (labor, L) and a reproducible factor (capital, K) under conditions of perfect competition and constant returns to scale. Good 1 is a pure consumption good, and Good 2 is a consumable capital good. Factors of production are assumed to be mobile between sectors within a country, but immobile internationally, and there are no markets for international borrowing and lending. We refer to the representative country as the home country: the corresponding behavioral relations for the other (foreign) country will be denoted by an "$*$."

We assume that the home and foreign countries are symmetric except for the initial capital endowment in each country. They have the same population, normalized to be one, with each household having an endowment of labor, L, and a concave utility function u defined over consumption of goods 1 and 2, C_1 and C_2. We assume that the home country initially has a larger capital stock.

2.1. The Production Side

Letting F_i be the production function in sector i, we assume for simplicity[6]

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Assumption 1. Production technologies take the Cobb–Douglas form: $F_i(K, L) = A_i L^{a_i} K^{b_i}$, where $a_i, b_i > 0$ and $a_i + b_i = 1$. The pure consumption good 1 is labor-intensive: $a_1 > a_2$.

Letting w denote the wage rate and r the rental on capital, the technology in sector i can be characterized by the unit cost function $\chi_i(w, r)$, $i = 1, 2$. The competitive profit conditions require that

$$p \le \chi_1(w, r), \tag{1}$$

$$1 \le \chi_2(w, r), \tag{2}$$

where good 2 is chosen as numeraire. The stock of capital is denoted by K. Factor market equilibrium requires that

$$1 = v_1 + v_2, \tag{3}$$

$$k = v_1 \kappa_1(w/r) + v_2 \kappa_2(w/r), \tag{4}$$

where v_i is the fraction of labor devoted to sector i, $k \equiv K/L$, and $\kappa_i(w/r) \equiv \chi_{ir}(w, r)/\chi_{iw}(w, r)$.

Solving for w and r when (1) and (2) hold with equality, we obtain the factor prices $(w(p), r(p))$ that are consistent with production of both goods. Notice that we have $pw'(p)/w(p) > 1$ and $r'(p) < 0$ because of the Stolper–Samuelson theorem, and that the capital-labor ratio in sector i is given by[7]

$$k_i(p) \equiv \kappa_i\left(\frac{w(p)}{r(p)}\right) = \frac{b_i}{a_i}\left(\frac{A_1 a_1^{a_1} b_1^{b_1}}{A_2 a_2^{a_2} b_2^{b_2}} p\right)^{\frac{1}{a_1 - a_2}}, \quad i = 1, 2. \tag{5}$$

These factor prices will satisfy full employment for $k \in [k_1(p), k_2(p)]$. With incomplete specialization, we can express GDP as $[w(p) + r(p)k]L$. Applying the envelope theorem, we obtain the output of good i, Y_i, to be

$$Y_1(p, k) = \begin{cases} f_1(k)L, & \text{if } k < k_1(p) \Leftrightarrow p > p_1(k), \\ [w'(p) + r'(p)k]L, & \text{if } k \in [k_1(p), k_2(p)] \Leftrightarrow p \in [p_2(k), p_1(k)], \\ 0, & \text{if } k > k_2(p) \Leftrightarrow p < p_2(k), \end{cases} \tag{6}$$

$$Y_2(p, k) = \begin{cases} 0, & \text{if } k < k_1(p), \\ \{w(p) + r(p)k - p[w'(p) + r'(p)k]\}L, & \text{if } k \in [k_1(p), k_2(p)], \\ f_2(k)L, & \text{if } k > k_2(p), \end{cases} \tag{7}$$

where $f_i(k) \equiv F_i(K/L, 1)$ and p_i is the inverse function of k_i: $p_i(k_i(p)) = p$. The supply functions are linear in k with incomplete specialization.

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2.2. The Consumption Side

We analyze the optimization problem for a representative household that owns L units of labor. We will impose the following restrictions on this utility function:[8]

Assumption 2. The utility function is twice differentiable and strictly concave, with $u_{11} < 0$ and $D \equiv u_{11}u_{22} - u_{12}u_{21} > 0$ for any $(C_1, C_2) \in \{(C_1, C_2) \in \mathbf{R}_+^2 | u_i(C_1, C_2) > 0,\ i = 1, 2\}$, and satisfies $\lim_{C_i \to 0} u_i(C_1, C_2) = \infty$ $(i = 1, 2)$ for any C_j $(j \neq i)$.

The representative household is assumed to maximize the discounted sum of its utilities,

$$\max \int_0^\infty u(C_1, C_2) \exp(-\rho t) dt, \tag{8}$$

subject to its flow budget constraint

$$wL + rK = pC_1 + C_2 + \dot{K} + \delta K, \qquad K_0 \text{ given}, \tag{9}$$

where δ is the rate of depreciation on capital and ρ is the discount rate. The budget constraint reflects the assumed absence of an international capital market, because it requires that $pZ_1 + Z_2 = 0$, where $Z_1 = C_1 - Y_1$ $(Z_2 = C_2 + \dot{K} + \delta K - Y_2)$ is the excess demand for good 1 (2).

Solving the current-value Hamiltonian for this problem yields the necessary conditions for the choice of consumption levels, the differential equation describing the evolution of the costate variable, λ, and the transversality conditions

$$u_1(C_1, C_2) = \lambda p, \quad u_2(C_1, C_2) = \lambda, \tag{10}$$

$$\dot{\lambda} = \lambda(\rho + \delta - r), \tag{11}$$

$$\lim_{t \to \infty} K(t)\lambda(t) \exp(-\rho t) = 0. \tag{12}$$

It will be useful for the subsequent analysis to invert the necessary conditions for choice of consumption levels to obtain consumption relations $C_i(p, \lambda)$ for $i = 1, 2$ and an expenditure relation $E(p, \lambda) \equiv pC_1(p, \lambda) + C_2(p, \lambda)$. The following lemma from Bond et al. (2011) establishes some properties of these functions.

LEMMA 1.
- (i) $\lambda C_{1\lambda} = pC_{1p} + C_{2p}$.
- (ii) $E_\lambda = pC_{1\lambda} + C_{2\lambda} < 0$.
- (iii) $C_{1p} < 0$.
- (iv) $E_p = C_1 + \lambda C_{1\lambda}$.

Our expenditure relation differs from the standard expenditure function in that it holds constant the marginal utility of income, rather than the level of utility. Good i is normal if $C_{i\lambda} < 0$, so (ii) establishes that goods must be normal in total.

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Using (9), (11), and the expenditure function, we have

$$\dot{k} = w + rk - e(p, \lambda) - \delta k, \tag{13}$$

$$\dot{\lambda} = \lambda(\rho + \delta - r), \tag{14}$$

where $e(p, \lambda) \equiv E(p, \lambda)/L$. In the case of autarky, the system is closed by adding the market-clearing condition for good 1 at home,

$$z_1(p, k, \lambda) \equiv c_1(p, \lambda) - y_1(p, k) = 0, \tag{15}$$

where $c_1(p, \lambda) \equiv C_1(p, \lambda)/L$ and $y_1(p, k) \equiv Y_1(p, k)/L$.

Notice that both goods are produced in any autarkic steady state equilibrium, where $\dot{k} = \dot{\lambda} = 0$ and (15) hold, because $y_1 = 0$ implies $z_1 > 0$ from (15) and $y_2 = 0$ does $\dot{k} = -c_2 - \delta k < 0$ from $py_1 = w + rk$, (13), and (15). So the system can be expressed as

$$\dot{k} = w(p) + r(p)k - e(p, \lambda) - \delta k, \tag{16}$$

$$\dot{\lambda} = \lambda[\rho + \delta - r(p)], \tag{17}$$

$$0 = c_1(p, \lambda) - [w'(p) + r'(p)k] \tag{18}$$

around the autarkic steady state equilibrium, and these equations govern the evolution of (k, λ, p) under autarky.

2.3. The Foreign Country and World Market Equilibrium

The optimization problem for a foreign household is analogous to that for the home country and the solution of the foreign country's household optimization problem yields

$$\dot{k}^* = w^* + r^*k^* - e(p^*, \lambda^*) - \delta k^*, \tag{19}$$

$$\dot{\lambda}^* = \lambda^*(\rho + \delta - r^*). \tag{20}$$

In a free trade equilibrium, the price of good 1 will be equalized across countries and will be determined by the world market-clearing condition for good 1,

$$z_1(p, k, \lambda) + z_1(p, k^*, \lambda^*) = 0. \tag{21}$$

From (14) and (20), we have $r = r^*$ when $\dot{\lambda} = \dot{\lambda}^* = 0$. Therefore, in any free trade steady state equilibrium, we have one of the following cases:

(i) $k = k^* < k_1(p)$ and $r = r^* = pf_1'(k)$;
(ii) $k, k^* \in [k_1(p), k_2(p)]$ and $r = r^* = r(p)$;
(iii) $k = k^* > k_2(p)$ and $r = r^* = f_2'(k)$.

As in autarky, cases (i) and (iii) are inconsistent with the steady state equilibrium conditions: $\dot{k} + \dot{k}^* < 0$ in case (i) and $z_1 + z_1^* > 0$ in case (iii). So both countries

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are incompletely specialized at any free trade steady state and around it, and hence the free trade equilibrium can be solved for the evolution of (k, k^*, λ, p) using (16), (17),

$$\dot{k}^* = w(p) + r(p)k^* - e(p, \lambda^*) - \delta k^* \quad \textbf{(22)}$$

and

$$0 = c_1(p, \lambda) + c_1(p, \lambda^*) - [2w'(p) + r'(p)(k + k^*)], \quad \textbf{(23)}$$

where we have $\lambda^* = m\lambda$ for some $m > 0$. This is because any free trade equilibrium $\dot{\lambda}/\lambda = \dot{\lambda}^*/\lambda^*$ at each point in time as long as the conditions for factor price equalization are satisfied.

Let $\tilde{p}$ denote the price of good 1 equalizes the rental rate to $\theta \equiv \rho + \delta$: $\tilde{p} \equiv r^{-1}(\theta)$.[9] Notice that $\tilde{p}$ is a unique steady state price of good 1, for which all state variables can be constant.

Letting $\tilde{w} \equiv w(\tilde{p})$, we have

LEMMA 2. *The countries must be incompletely specialized in any autarkic or free trade steady state equilibrium. The prices of good 1 and factors consistent with steady states exist and are uniquely determined as* $p = \tilde{p}$, $w = \tilde{w}$, *and* $r = \theta$.

3. INFERIORITY IN CONSUMPTION AND MULTIPLE EQUILIBRIA IN AUTARKY

We will focus on the case in which the labor-intensive good is inferior for some range of income levels, which is a necessary condition for the autarkic economy to have multiple steady states. We assume that the labor-intensive good 1 is a necessity (i.e., the income elasticity of good 1 is less than one) for low income levels, and it becomes an inferior good when households' income goes up. Because the slope of an income expansion path, $dC_2/dC_1|_{dp=0}$, is given by

$$\left.\frac{dC_2}{dC_1}\right|_{dp=0} = \frac{\partial C_2/\partial\lambda}{\partial C_1/\partial\lambda} = \frac{C_{2\lambda}}{C_{1\lambda}},$$

the assumption implies that for any pair (p, λ), either

$$\text{(i)}\ \frac{\partial}{\partial\lambda}\left[\left.\frac{dC_2}{dC_1}\right|_{dp=0}\right] = \frac{C_{2\lambda\lambda}C_{1\lambda} - C_{2\lambda}C_{1\lambda\lambda}}{C_{1\lambda}^2} < 0 \text{ and } C_{1\lambda} < 0, \quad \text{or}$$

(ii) $C_{1\lambda} \geq 0$.

We require the slightly stronger property in the following assumption.

Assumption 3. For any price of good 1, there are two values of λ, denoted by $\lambda^1(p)$ and $\lambda^2(p)$, such that (i) $C_{2\lambda\lambda}C_{1\lambda} - C_{2\lambda}C_{1\lambda\lambda} \lesseqgtr 0$ if $\lambda \gtreqless \lambda^1(p)$; (ii) $C_{1\lambda} \lesseqgtr 0$ if $\lambda \gtreqless \lambda^2(p)$; and (iii) $0 < \lambda^1(p) < \lambda^2(p) < \infty$.

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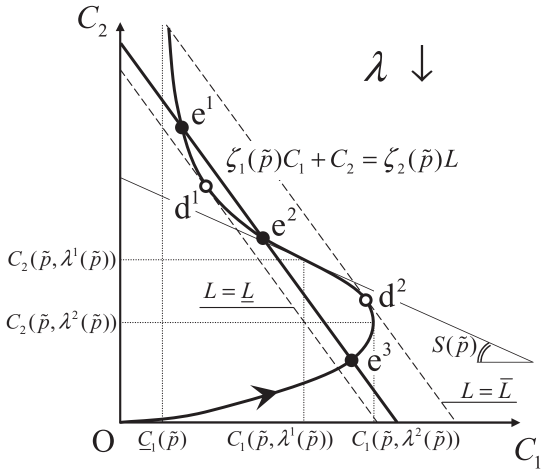

FIGURE 1. The income expansion path and the steady state Rybczynski line.

The implications of Assumption 3 for the income expansion path are illustrated by the curve $Oe^3d^2e^2d^1e^1$ in Figure 1, which is the income expansion path associated with the steady state price, $\tilde{p}$. Good 1 is a necessity for sufficiently low utility levels (i.e., $\lambda > \lambda^2$), is an inferior good whose marginal propensity to consume is decreasing in income for intermediate income levels ($\lambda \in (\lambda^1, \lambda^2)$), and is an inferior good whose marginal propensity to consume is increasing in income for sufficiently high income levels ($\lambda < \lambda^1$). In particular, (iii) implies that the path is asymptotic to the vertical line with $C_1 = \underline{C}_1(p) \equiv \lim_{\lambda\to 0} C_1(p, \lambda)$. One example of the utility function that satisfies Assumptions 2 and 3 is as follows:[10]

$$u(C_1, C_2) = \alpha \ln C_1 + \beta \ln C_2 - \gamma C_1 C_2, \qquad \textbf{(24)}$$

where all parameters are positive and $\alpha < \beta$.

3.1. Autarkic Steady State Equilibria

An autarkic steady state equilibrium will be a pair (λ, k) satisfying (18) and (16) at $p = \tilde{p}$ and $\dot{k} = 0$. Substituting (16) into (18), we have that the steady state marginal utility of income, $\tilde{\lambda}$, will be the solution to

$$\zeta_1(\tilde{p})c_1(\tilde{p}, \lambda) + c_2(\tilde{p}, \lambda) = \zeta_2(\tilde{p}), \qquad \textbf{(25)}$$

where $\zeta_1(p) \equiv p - [r(p) - \delta]/r'(p) > p$ and $\zeta_2(p) \equiv w(p) - [r(p) - \delta]w'(p)/r'(p) > 0$. From the market clearing condition (18) we have

$$k(\lambda) \equiv \frac{c_1(\tilde{p}, \lambda) - w'(\tilde{p})}{r'(\tilde{p})}.$$

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Thus, the pair $(k(\tilde{\lambda}), \tilde{\lambda}, \tilde{p})$ clearly satisfies (18) with $\dot{k} = \dot{\lambda} = 0$: $(k(\tilde{\lambda}), \tilde{\lambda})$ is the pair of steady state values of k and λ.

The solution for $\tilde{\lambda}$ can be illustrated by the use of Figure 1. The negatively sloped line given by

$$\zeta_1(\tilde{p})C_1 + C_2 = \zeta_2(\tilde{p})L \tag{26}$$

is the locus of values of C_1 and C_2 associated with steady state production and market clearing. Note that $\zeta_1(p) > p$ because of the Rybczynski effect: an increase in the capital stock must result in a more than proportional increase in the capital-intensive good and a reduction in the output of the labor-intensive good. Thus, higher steady state levels of C_2 correspond to higher steady state levels of capital stock in Figure 1. We refer to this as the steady state Rybczynski line.

It can be seen from (25) that an autarkic steady state will occur at the intersection of the income expansion path and the steady state Rybczynski line. Let $S(\tilde{p})$ denote the magnitude of the slope of the income expansion path at $(C_1, C_2) = (C_1(p, \lambda^1(p)), C_2(p, \lambda^1(p)))$. It then follows that if

$$S(\tilde{p}) < \tilde{p} - \frac{\rho}{r'(\tilde{p})} \tag{27}$$

holds,[11] then for some values of L, there exist three steady state equilibria in autarky.[12] This yields the following result.

LEMMA 3. *Let (27) hold. Then there is some range of values of* L, $(\underline{L}, \bar{L})$, *such that there exist three steady state equilibria in autarky for* $L \in (\underline{L}, \bar{L})$, *whereas there exists a unique one if* $L < \underline{L}$ *or* $L > \bar{L}$.

Remark 1. For a fixed steady state price of good 1, the slope of the steady state Rybczynski line becomes steep when the elasticity of the rental rate at the price, $\tilde{p}r'(\tilde{p})/r(\tilde{p})$, is small. So multiple steady states in autarky can arise under technologies with a sufficiently small elasticity of the rental on capital, even if the share of income spent on the inferior good is sufficiently small.

In the rest of the paper, we assume

Assumption 4. $S(\tilde{p}) < \zeta_1(\tilde{p})$ $(= \tilde{p} - \rho/r'(\tilde{p}))$ and $L \in (\underline{L}, \bar{L})$ hold.

Then, we denote the value of λ at each of the three autarkic steady states e^i in Figure 1 by $\tilde{\lambda}^i(L)$ ($i = 1, 2, 3$ and $\tilde{\lambda}^1 < \tilde{\lambda}^2 < \tilde{\lambda}^3$). $e^i = (C_1(\tilde{p}, \tilde{\lambda}^i), C_2(\tilde{p}, \tilde{\lambda}^i))$.[13] We will show later that the autarkic steady states with $\lambda = \tilde{\lambda}^1$ or $\tilde{\lambda}^3$ are saddlepoint stable, whereas the other is unstable.

3.2. Autarky Prices

To solve the system (16) and (17) for the time path of (λ, k), we need to derive the autarky price that solves (18) as a function of λ and k.

We first show that for given λ, there exists a critical capital stock $\underline{k}(\lambda)$ such that there is complete specialization in good 1 for $k \leq \underline{k}(\lambda)$ and incomplete

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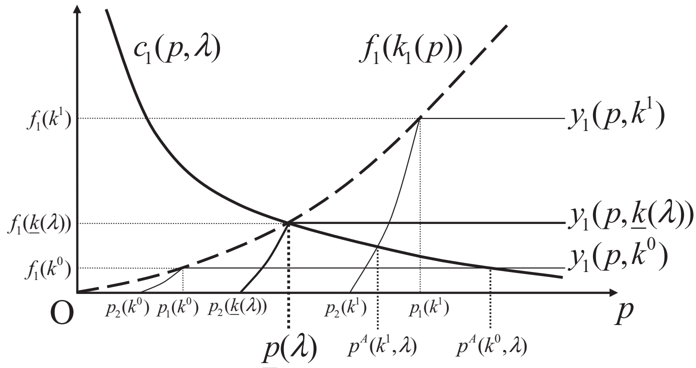

FIGURE 2. Determination of critical values of k and p.

specialization for $k > \underline{k}(\lambda)$. From (5) and (6), we have

$$\max_k y_1(p,k) = f_1(k_1(p)),$$
$$\frac{df_1(k_1(p))}{dp} > 0, \quad \lim_{p\to 0} f_1(k_1(p)) = 0, \text{ and } \quad \lim_{p\to\infty} f_1(k_1(p)) = \infty. \quad \textbf{(28)}$$

Let $\underline{p}(\lambda)$ be the solution to $c_1(p,\lambda) = f_1(k_1(p))$, which is the lowest price at which there is some value of capital stock that clears the market for good 1. Notice that for given λ, $\underline{p}(\lambda)$ exists and is unique because $c_{1p} < 0$ and (28). The solution for $\underline{p}(\lambda)$ is illustrated in Figure 2. The desired critical value is $\underline{k}(\lambda) \equiv k_1(\underline{p}(\lambda))$.

Figure 2 also illustrates the supply curve for good 1 for three values $k^0 < \underline{k}(\lambda) < k^1$. For $k < \underline{k}(\lambda)$, the equilibrium will be one of complete specialization in good 1, so the autarky price, p^A, is decreasing in k. For $k > \underline{k}(\lambda)$, the economy is incompletely specialized. Because the supply of good 1 is decreasing in k with incomplete specialization, the autarky price is increasing in k.

Based on this, we have a lemma as follows.

LEMMA 4. *The autarky price of good 1 is given by a continuous function of k and λ, and it has the following properties:*

$$\lim_{k\to 0} p^A(k,\lambda) = \lim_{k\to\infty} p^A(k,\lambda) = \infty, \ p^A(\underline{k}(\lambda),\lambda) = \underline{p}(\lambda), \ \textit{and}$$
$$\frac{\partial p^A(k,\lambda)}{\partial k} \begin{cases} < 0, & \textit{if } k < \underline{k}(\lambda), \\ > 0, & \textit{if } k > \underline{k}(\lambda). \end{cases} \quad \textbf{(29)}$$

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The autarky factor prices are also given by functions of k and λ as follows:

$$r^A(k,\lambda) = \begin{cases} p^A(k,\lambda) f_1'(k), & \textit{if } k < \underline{k}(\lambda), \\ r(p^A(k,\lambda)), & \textit{otherwise,} \end{cases} \tag{30}$$

$$w^A(k,\lambda) = \begin{cases} p^A(k,\lambda)[f_1(k) - k f_1'(k)], & \textit{if } k < \underline{k}(\lambda), \\ w(p^A(k,\lambda)), & \textit{otherwise.} \end{cases} \tag{31}$$

(29) and (30) together imply that r^A is continuous in k and λ and strictly decreasing in k with $\lim_{k\to 0} r^A(k,\lambda) = \infty$ *and* $\lim_{k\to\infty} r^A(k,\lambda) = 0$.

3.3. Stability of Autarkic Steady States

We conclude our analysis of the autarkic steady states by deriving the phase diagram for the closed economy. We show that the steady states e^1 and e^3 are saddle points, whereas e^2 is unstable.

Using Lemma 4, it can be seen that the $\dot{\lambda} = 0$ locus consists of two parts in the (k, λ) plane:

$$\lambda = 0 \ \text{ and } \ r^A(k,\lambda) = \theta.$$

For each $\lambda > 0$, there exists a unique value of k that satisfies $r^A(k,\lambda) = \theta$. Under incomplete specialization, the locus along which the equilibrium price of good 1 is constant is determined as

$$k = \frac{c_1(p,\lambda) - w'(p)}{r'(p)}, \tag{32}$$

which is derived from the market-clearing condition (18). So the $\dot{\lambda} = 0$ locus partly consists of the set

$$\left\{ (k,\lambda) \,\middle|\, k = \frac{c_1(\tilde{p},\lambda) - w'(\tilde{p})}{r'(\tilde{p})} \text{ and } k \geq \underline{k}(\lambda) \right\}.$$

On the other hand, we see from Lemma 4 that for a λ that violates[14]

$$\frac{c_1(\tilde{p},\lambda) - w'(\tilde{p})}{r'(\tilde{p})} \geq \underline{k}(\lambda), \tag{33}$$

the $\dot{\lambda} = 0$ locus is given by the set $\{(k,\lambda) | p^A(k,\lambda) f_1'(k) = \theta\}$. Based on this, the $\dot{\lambda} = 0$ locus can be drawn as in Figure 3 and $\dot{\lambda}$ is positive (negative) on the right (left) side of the locus.[15]

Next, we turn to the $\dot{k} = 0$ locus. Because the locus must lie in the region where $k > \underline{k}(\lambda)$, we have

$$\dot{k} = w\left(p^A(k,\lambda)\right) + r\left(p^A(k,\lambda)\right) k - e(p^A(k,\lambda),\lambda) - \delta k \tag{34}$$
$$= 0.$$

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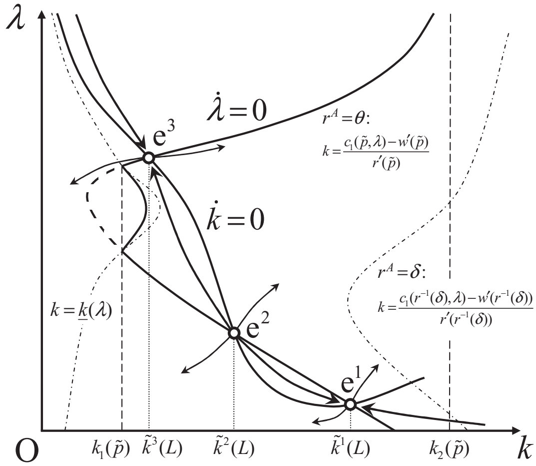

FIGURE 3. The phase diagram for the closed economy.

Then totally differentiating (34) yields

$$d\dot{k} = \left(r^A - \delta - \lambda c_{1\lambda}\frac{\partial p^A}{\partial k}\right)dk - \left(\lambda c_{1\lambda}\frac{\partial p^A}{\partial \lambda} + e_\lambda\right)d\lambda,$$

where use has been made of Lemma 1 (iv) and the market-clearing condition. It can easily be shown that the coefficient of $d\lambda$ is always positive,[16]

$$\lambda c_{1\lambda}(\partial p^A/\partial\lambda) + e_\lambda < 0, \tag{35}$$

and that of dk is positive when $r^A > \delta$ and $c_{1\lambda} < 0$.

For each $p > 0$, there necessarily exists an intersection of the line $\zeta_1(p)C_1 + C_2 = \zeta_2(p)L$ and the income expansion path with p, say (C_1', C_2'). From (25), we see that the intersection corresponds to the point in the (k, λ) plane where $\dot{k} = 0$ as follows:

$$C_1(p,\lambda) = C_1' \text{ and } k = \frac{C_1' - w'(p)L}{r'(p)L}.$$

Because production is completely specialized to good i when $k = k_i(p)$, we see from (6) and (7) that

$$k_1(p) = \frac{pw'(p) - w(p)}{-pr'(p) + r(p)} \text{ and } k_2(p) = -\frac{w'(p)}{r'(p)}. \tag{36}$$

Because k_i is strictly increasing in p, one can verify from (36) that ζ_2 and ζ_2/ζ_1 are both strictly increasing in p for $p < r^{-1}(\delta)$. So, as p increases from zero to $r^{-1}(\delta)$, the line $\zeta_1(p)C_1 + C_2 = \zeta_2(p)L$ sifts outward, whereas the income expansion path shifts inward.

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So there are two values of p, p_m^- and p_m^+, such that (i) $p_m^- \in (0, \tilde{p})$ and $p_m^+ \in (\tilde{p}, r^{-1}(\delta))$; (ii) the $\dot{k} = 0$ locus cuts exactly three times the curve

$$k = \frac{c_1(p, \lambda) - w'(p)}{r'(p)} \text{ with } p \in (p_m^-, p_m^+);$$

(iii) the locus cuts exactly once the curve with $p \in (0, p_m^-)$ or $p \in (p_m^+, r^{-1}(\delta)]$.[17] Also, notice that $\lim_{p \to 0} k_2(p) = 0$ implies that the $\dot{k} = 0$ locus is asymptotic to the vertical axis. Thus, we can derive the $\dot{k} = 0$ locus, above (below) which $\dot{k}$ is positive (negative), as in Figure 3.

The phase diagram shows that two autarkic steady state equilibria, e^1 and e^3, are saddlepoint stable, whereas the middle one is unstable, and we obtain the first main theorem as follows.

THEOREM 1. *If an initial stock of capital is greater (smaller) than* $\tilde{k}^2$*, the economy converges to the highest (lowest) autarkic steady state equilibrium* e^1 *(*e^3*): The poverty trap arises due to inferiority in consumption.*

Remark 2. Without the assumption of normality in consumption, the higher capital stock does not necessarily imply a lower rental on capital and there can be a nonmonotonic relation between capital stock and its rental rate.

This is because in our model, as capital stock in a country grows, the country's demand for the capital-intensive good will become sufficiently large.

Along the dynamic general equilibrium path where $\dot{k}$ is negative, the existing capital may be consumed as good 2. However, one may think that irreversible investment (or at least some costs of reversible investment) should be assumed, because we suppose that newly produced consumable capital is tradable but the existing one is internationally immobile. So, in the rest of the paper, we assume

Assumption 5. $L > \hat{L} \equiv \zeta_2(\tilde{p})\bar{L}/[\zeta_2(\tilde{p}) + \delta k_1(\tilde{p})]$.

Thus, we have

LEMMA 5. *Consuming the existing capital and complete specialization to produce good 1 do not occur along the path from* e^2 *to* e^3.

Proof. See the Appendix. ■

4. FREE TRADE STEADY STATE EQUILIBRIA

We now consider the determination of the steady state equilibria with free trade.

First, from (16) and (17), the steady state capital stock per unit of labor satisfy

$$\tilde{k}(\lambda) = \frac{e(\tilde{p}, \lambda) - \tilde{w}}{\rho}. \qquad \textbf{(37)}$$

Notice that $\tilde{k}$ is strictly decreasing in λ from Lemma 1 and that $\tilde{k}(\lambda) \in (k_1(\tilde{p}), k_2(\tilde{p}))$ for $\lambda \in [\tilde{\lambda}^1(L), \tilde{\lambda}^3(L)]$ because $\tilde{k}^i(L) \equiv \tilde{k}(\tilde{\lambda}^i(L)) \in$

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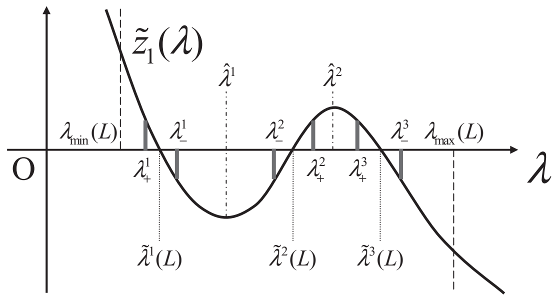

FIGURE 4. The nonmonotonic excess demand function.

$(k_1(\tilde{p}), k_2(\tilde{p})), i = 1, 2, 3$. Let $\lambda_{\min}(L)$ and $\lambda_{\max}(L)$ denote, respectively, the solutions to $\tilde{k}(\lambda) = k_2(\tilde{p})$ and $\tilde{k}(\lambda) = k_1(\tilde{p})$, which necessarily exist and are decreasing in L because $e(\tilde{p}, \lambda) = E(\tilde{p}, \lambda)/L$, $\lim_{\lambda\to 0} E(\tilde{p}, \lambda) = \infty$, $\lim_{\lambda\to\infty} E(\tilde{p}, \lambda) = 0$, and $E_\lambda < 0$.

Substituting (37) into the excess demand for good 1, $z_1(p, k, \lambda)$, we obtain a steady state (per unit of labor) excess demand function

$$\begin{aligned}\tilde{z}_1(\lambda) &= c_1(\tilde{p}, \lambda) - y_1(\tilde{p}, \tilde{k}(\lambda))\\ &= -\frac{r'(\tilde{p})}{\rho}[\zeta_1(\tilde{p})c_1(\tilde{p}, \lambda) + c_2(\tilde{p}, \lambda) - \zeta_2(\tilde{p})]\end{aligned} \quad \textbf{(38)}$$

for $\lambda \in [\lambda_{\min}, \lambda_{\max}]$. Then, it is clear from Figure 1 that

$$\tilde{z}_1(\lambda) \begin{cases} > 0, \text{ if } \lambda \in [\lambda_{\min}, \tilde{\lambda}^1) \text{ or } \lambda \in (\tilde{\lambda}^2, \tilde{\lambda}^3), \\ < 0, \text{ if } \lambda \in (\tilde{\lambda}^1, \tilde{\lambda}^2) \text{ or } \lambda \in (\tilde{\lambda}^3, \lambda_{\max}]. \end{cases}$$

Notice that the slope of the income expansion path with $p = \tilde{p}$ is equal to $\zeta_1(\tilde{p})$ at points d_1 and d_2 in Figure 1. So we see that there are exactly two values of λ, denoted by $\hat{\lambda}^i$ $(i = 1, 2)$, such that $\hat{\lambda}^1 \in (\tilde{\lambda}^1, \tilde{\lambda}^2)$, $\hat{\lambda}^2 \in (\tilde{\lambda}^2, \tilde{\lambda}^3)$, and $|C_{2\lambda}(\tilde{p}, \hat{\lambda}^i)/C_{1\lambda}(\tilde{p}, \hat{\lambda}^i)| = \zeta_1(\tilde{p})$, and hence we have

$$\tilde{z}_1'(\lambda) \begin{cases} > 0, \text{ if } \lambda \in (\hat{\lambda}^1, \hat{\lambda}^2), \\ < 0, \text{ if } \lambda \in [\lambda_{\min}, \hat{\lambda}^1) \text{ or } \lambda \in (\hat{\lambda}^2, \lambda_{\max}]. \end{cases}$$

Figure 4 is a graph of $\tilde{z}_1(\lambda)$ for some value of $L \in (\underline{L}, \bar{L})$.

Free trade steady state equilibria are obtained by solving $\tilde{z}_1(\lambda) + \tilde{z}_1(\lambda^*) = 0$. For example, each of the pairs $(\lambda, \lambda^*) = (\lambda_+^i, \lambda_-^j)$ $(i, j = 1, 2, 3)$ in Figure 4 corresponds to a free trade steady state equilibrium, where the home country imports good 1: $\tilde{z}_1(\lambda) > 0$. Because $\tilde{k}$ is decreasing in λ, we see that the capital abundant foreign exports the labor-intensive good 1 (the static H-O theorem is violated) at the steady state equilibria with $(\lambda, \lambda^*) = (\lambda_+^i, \lambda_-^j)$ $(i = 2, 3$ and $i \geq j)$. As stated in Bond et al. (2011), this occurs because the richer country

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demands less of the inferior labor-intensive good, and this effect dominates its relatively lower supply of the labor-intensive good at these equilibria.

Because both countries are incompletely specialized in the neighborhood of the steady states, from (16), (17), (22), and (23), we can reduce the dynamical system to a three-equation system in (k, k^*, λ) as follows:

$$\dot{k} = w(p(k, k^*, \lambda)) + r(p(k, k^*, \lambda))k - e(p(k, k^*, \lambda), \lambda) - \delta k, \quad \textbf{(39)}$$

$$\dot{k}^* = w(p(k, k^*, \lambda)) + r(p(k, k^*, \lambda))k^* - e(p(k, k^*, \lambda), m\lambda) - \delta k^*, \quad \textbf{(40)}$$

$$\dot{\lambda} = \lambda[\rho + \delta - r(p(k, k^*, \lambda))], \quad \textbf{(41)}$$

where $p(k, k^*, \lambda)$ is derived from the world market-clearing condition (23). We evaluate the elements of a Jacobian of the dynamical system, given m, to study the local dynamics around the stationary state. Then, we have its characteristic equation as follows [see Bond et al. (2011)]:

$$J(x) = \Gamma x^3 - 2\Gamma\rho x^2 + \left\{\Gamma\rho^2 - \frac{\lambda\rho}{r'}\left[\tilde{z}_1'(\lambda) + m\tilde{z}_1'(m\lambda)\right]\right\} x$$

$$+ \frac{\lambda\rho^2}{r'}[\tilde{z}_1'(\lambda) + m\tilde{z}_1'(m\lambda)],$$

where $\Gamma \equiv (-r'\frac{\partial p}{\partial k})^{-1} > 0$.

As proved in Lemma 3 in Bond et al. (2011), if $J(0) > 0$, then the characteristic equation has one negative root; if $J(0) < 0$ and $J(2\rho) > 0$, then the equation has no roots with negative real parts. Because $J(0) = \lambda\rho^2\left[\tilde{z}_1'(\lambda) + m\tilde{z}_1'(m\lambda)\right]/r'$ and $J(2\rho) = 2\Gamma\rho^3 - J(0)$, we have

LEMMA 6. *Free trade steady state equilibria are locally saddlepoint stable, if* $\tilde{z}_1'(\lambda) + m\tilde{z}_1'(\lambda^*)$ *is negative there, whereas they are unstable if it is positive.*

Figure 5 illustrates the set of equilibrium pairs of λ and λ^* for some $L > (\underline{L} + \bar{L})/2$.[18] The slope of these loci is given by

$$\left.\frac{d\lambda^*}{d\lambda}\right|_{d\tilde{z}_1 + d\tilde{z}_1^* = 0} = -\frac{\tilde{z}_1'(\lambda)}{\tilde{z}_1'(\lambda^*)}.$$

So we can see that for m sufficiently close to one, the ray from the origin, $\lambda^* = m\lambda$, cuts the loci exactly three times (e.g., points E^i or e^i, $i = 1, 2, 3$, in Figure 5), and that $\tilde{z}_1'(\lambda) + m\tilde{z}_1'(\lambda^*)$ is positive at the middle point, whereas it is negative at the others. So we have the following Proposition.

PROPOSITION 2. *There exists an open interval* $M(L)$ *such that for any* $m \in M(L)$, $\tilde{z}_1(\lambda) + \tilde{z}_1(m\lambda) = 0$ *has exactly three solutions for* λ. *The highest and lowest values of* λ *correspond to the free trade steady state equilibria that are locally saddlepoint stable, and the middle one to the unstable steady state equilibrium.*

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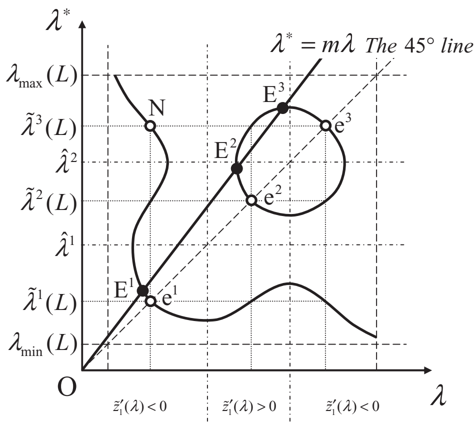

FIGURE 5. The set of equilibrium pairs of λ and λ^*.

5. POVERTY TRAPS AND TRADE

When both goods are normal, there is a unique autarky equilibrium. If trade is opened between two countries that have not reached the autarkic steady state, the country with the larger (smaller) initial capital stock will have a steady state welfare level that exceeds (is less than) that in the autarkic steady state. This effect of the initial position on steady state values, which we refer to as the hysteresis effect, results from the fact that with factor price equalization the marginal utility of income must grow at the same rate in each country. Thus, the initially richer country will remain the richer country along the path to the steady state.

The hysteresis effect will also operate in the case where the labor-intensive good is inferior, as illustrated in Figure 5. However, in this case we also have the possibility that the opening of trade switches the path away from (toward) the poverty trap equilibrium and toward (away from) the steady state with a higher capital stock. We refer to this as the poverty trap effect. We will show that the combination of these two effects leads to the possibility that the steady state capital stock is higher (or lower) in both countries under free trade than in autarky.

5.1. Dynamic Equilibrium Paths with Free Trade

Figure 6 illustrates the set of equilibrium pairs of k and k^*, which can be derived by using (37). The slope of these loci are given by

$$\left.\frac{dk^*}{dk}\right|_{d\tilde{z}_1+d\tilde{z}_1^*=0} = -\frac{\tilde{z}_1'(\lambda)e_\lambda(\tilde{p},\lambda^*)}{\tilde{z}_1'(\lambda^*)e_\lambda(\tilde{p},\lambda)}$$

from $\tilde{k}'(\lambda) = e_\lambda(\tilde{p},\lambda)/\rho$. Points e^i, E^i, N in Figure 6 correspond to those in Figure 5. Points e^i $(i = 1, 2, 3)$ and N are all autarkic free trade steady state equilibria, in the sense that the excess demand for good 1 is zero in both countries.

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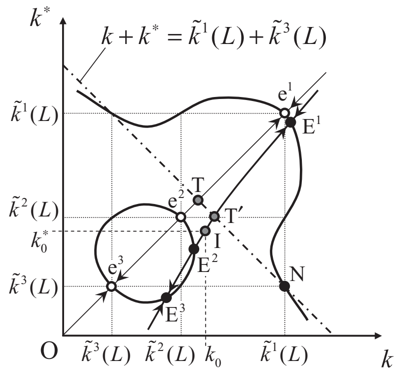

FIGURE 6. The set of equilibrium pairs of k and k^*.

Notice that if the initial capital stock in each country is the same, $k_0 = k_0^*$, then for all $t \geq 0$, $k = k^*$ and $z_1 = z_1^* = 0$ along the dynamic general equilibrium path, which is substantially the same as in autarky. Therefore, if $k_0 = k_0^* > \tilde{k}^2$ $(< \tilde{k}^2)$, the economy converges to e^1 (e^3), as shown in Figure 6.

So, from Proposition 2, if m is sufficiently close to one, a dynamic general equilibrium path where $\lambda^* = m\lambda$ holds for all $t \geq 0$ has the following properties: (i) there are three steady states on the path; (ii) the middle one is unstable; (iii) the economy converges to the highest (lowest) steady state when the capital stock in each country is initially higher (lower) than in the middle steady state (e.g., see locus E^{3}E^{2}IT′E^1 in Figure 6). Suppose that $k_0 > \tilde{k}^2 > k_0^*$ and the pair (k_0, k_0^*) is given by point I in Figure 6. Then it is apparent that capital stock and level of welfare are higher in both countries at the free trade steady state than at their autarkic steady states: $k > \tilde{k}^1$ and $k^* > \tilde{k}^3$ (equivalently, $\lambda < \tilde{\lambda}^1$ and $\lambda^* < \tilde{\lambda}^3$) hold at point E^1. Thus, we have the second main result as follows.

THEOREM 3. *There are two nonempty subsets of* $\{(k_0, k_0^*) | k_0 > \tilde{k}^2 > k_0^*\}$ *such that for each pair of one subset, both countries have higher capital stocks and levels of welfare at the free trade steady state than at their autarkic steady states, whereas they have lower ones for that of the other.*

Notice that the autarky rental rate is greater (smaller) than the steady state rental on capital θ in the capital abundant home country (in the capital scarce foreign country), and that both countries will have higher capital stocks and levels of welfare at the free trade steady state when the equalized rental on capital is greater than θ, and vice versa.

5.2. International Transfer of Income

Let us consider the case where the home and foreign countries had reached the highest and lowest autarkic steady states, respectively, before opening trade: the

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foreign country is in poverty trap. Because point N is an autarkic free trade steady state equilibrium, opening trade among the countries has no effect on their production patterns and levels of utility. Suppose that an adding free trade, there is foreign aid to overcome the poverty trap such that the richer home country transfers a part of its capital income to the foreign country forever. Because good 2 is a luxury good, this may yield an increase in demand for the capital-intensive good 2 in the foreign country, increase the rental on capital, and encourage capital accumulation in the foreign country. Moreover, we show that there is a possibility that both the donor and recipient countries have higher capital stocks and levels of welfare at the free trade steady state than at their autarkic steady states.

Let the amount of transfer at time t be

$$\mu(\tilde{k}^1 - \tilde{k}^3)r,$$

where μ is constant over time and $0 \leq \mu \leq 1/2$.[19] Notice that as long as production in each country is incompletely specialized along the dynamic equilibrium path with the scheme of transfer, it mimics the path that starts from

$$(k_0, k_0^*) = ((1-\mu)\tilde{k}^1 + \mu\tilde{k}^3, \mu\tilde{k}^1 + (1-\mu)\tilde{k}^3),$$

which is on the line NT in Figure 6.

Because the steady states close to point N are locally saddlepoint stable (see Figure 5 and Lemma 6), if μ (the ratio of transfer to home households' capital income) is small, then

$$k - \mu(\tilde{k}^1 - \tilde{k}^3) < \tilde{k}^1 \text{ and } k^* + \mu(\tilde{k}^1 - \tilde{k}^3) > \tilde{k}^3$$

will hold in the steady state: the level of welfare in the home country will be lower in steady state than at the autarkic one, and vice versa.

The following lemma, which is proven in the Appendix, establishes conditions under which incomplete specialization will hold in both countries along the dynamic general equilibrium path with $\mu = 1/2$.

LEMMA 7. *Suppose that* $\tilde{k}^1 + \tilde{k}^3 > 2\tilde{k}^2$ *and* $2C_1(\tilde{p}, \tilde{\lambda}^1) > C_1(\tilde{p}, \tilde{\lambda}^3)$ *hold. Then, along the dynamic general equilibrium path with* $\mu = 1/2$, *the economy converges to the steady state with* $k + k^* = 2\tilde{k}^1$ *and both countries are incompletely specialized on the path.*

Notice that $\tilde{k}^1 + \tilde{k}^3 > 2\tilde{k}^2$ necessarily holds for L that close to $\bar{L}$, because $\tilde{k}^1$ and $\tilde{k}^3$ are increasing in L because $\tilde{k}^2$ is decreasing, and $\lim_{L\to\bar{L}} \tilde{k}^1(L) > \lim_{L\to\bar{L}} \tilde{k}^2(L) = \lim_{L\to\bar{L}} \tilde{k}^3(L)$. The inequality $2C_1(\tilde{p}, \tilde{\lambda}^1) > C_1(\tilde{p}, \tilde{\lambda}^3)$ holds when the difference between $S(\tilde{p})$ and $\zeta_1(\tilde{p})$ is sufficiently small or $2\underline{C}_1(\tilde{p}) \geq C_1(\tilde{p}, \lambda^2(\tilde{p}))$ holds.

We see from Lemma 7 that both the donor and recipient countries will have higher levels of welfare at the steady state than at their autarkic steady states when μ is sufficiently close to one-half (see locus from T′ to E^1 in Figure 6).

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THEOREM 4. *Let the conditions in Lemma 7 hold and the economy initially stay in the steady state with* $(k, k^*) = (\tilde{k}^1, \tilde{k}^3)$. *Then, if the amount of transfer is sufficiently large* (μ *is sufficiently close to one-half), then the foreign country will overcome the poverty trap, and the welfare levels in both the donor and recipient countries will be higher in the steady state than in their autarkic steady states, whereas the steady state level of welfare in the home country will be lower when the amount of transfer is sufficiently small.*

6. CONCLUDING REMARKS

Our analysis has shown that when the labor-intensive good is inferior, there can be a nonmonotonic relation between capital stock and its rental rate, and hence multiple autarkic steady states and poverty traps can arise without any externality nor strategic complementarity, which are commonly assumed in the literature on poverty traps. We have also shown that there is a possibility that free trade between two countries, one of which had escaped from a poverty trap and the other of which was in it, will lead both countries out of a poverty trap or into it. In the former (latter) case, each country will reach a higher (lower) steady state level of welfare as a result of opening trade than in autarky. This contrasts sharply with the result in dynamic H-O models with normality in consumption: The country with the higher (lower) capital stock will reach a steady state where the level of welfare is higher (lower) than in the autarkic steady state.

NOTES

1. Kurz (1968) shows that multiple steady states may occur in the case where the stock of capital enters directly into the utility function. The stock of capital does not enter directly into the utility function in our analysis, although we assume that the capital good can either be consumed or invested.
2. In our model without any market failure, "poverty trap" does not imply that the trajectory to the steady state with low steady state consumption is not optimal. Any dynamic general equilibrium path to one of the steady states is Pareto-efficient.
3. Azariadis (1996) provides examples of poverty traps arising from nonconstant discount factors, subsistence consumption, and habit formation in convex one-sector economies with complete markets. Externalities and strategic complementarities, which are a common source of poverty traps, are also absent in our model [see also Matsuyama (2008) for a brief review of the poverty trap literature].
4. The hysteresis effect also plays a role in the analysis of Ventura (1997) and Atkeson and Kehoe (2000). Nishimura and Shimomura (2002) show that the dynamic H-O model exhibits indeterminacy in the presence of factoral externalities, which implies that the international ranking of factor endowment ratios can differ from the initial ranking. Nishimura and Shimomura (2006) and Doi et al. (2007) show that indeterminacy can arise in dynamic trade models without externalities, if one good is an inferior good in the steady state.
5. Bond et al. (2011) consider the role of inferior goods in creating multiple steady states, but attention is limited to preferences for which there are two autarkic steady states. Because one of these steady states is a saddle point and the other is unstable, the possibility of poverty traps did not arise.
6. We obtain all results in this paper without assuming Cobb–Douglas technologies, if the production function in each sector is linearly homogeneous, twice differentiable, and strictly quasi-concave with $F_{iKK} \equiv \partial^2 F_i/\partial K^2 < 0$ and $F_{iLL} \equiv \partial^2 F_i/\partial L^2 < 0$, and both factors are indispensable for producing.

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7. One can easily verify that

$$\chi_i(w, r) = \frac{1}{A_i}\left(\frac{w}{a_i}\right)^{a_i}\left(\frac{r}{b_i}\right)^{b_i}, \quad i = 1, 2,$$

$$w(p) = \left[\frac{\left(A_1 a_1^{a_1} b_1^{b_1} p\right)^{b_2}}{\left(A_2 a_2^{a_2} b_2^{b_2}\right)^{b_1}}\right]^{\frac{1}{a_1 - a_2}} \text{ and } r(p) = \left[\frac{\left(A_2 a_2^{a_2} b_2^{b_2}\right)^{a_1}}{\left(A_1 a_1^{a_1} b_1^{b_1} p\right)^{a_2}}\right]^{\frac{1}{a_1 - a_2}}.$$

8. This assumption allows the utility function to be nonhomothetic.

9. With general production technologies, we have to assume $\theta < \sup\{r | \chi_2(w, r) = 1\}$ to guarantee the existence of $\tilde{p}$.

10. It is proven in the Appendix and Doi et al. (2009).

11. With Cobb–Douglas technologies, $pr'(p)/r(p)$ is equal to $-a_2/(a_1 - a_2)$, and hence we have $\zeta_1(\tilde{p}) = (a_1\rho + a_2\delta)\tilde{p}/a_2\theta$. Suppose that households' preference is given by (24). Then we have $S(\tilde{p}) = \hat{s}(\alpha, \beta)\tilde{p}$, where $\hat{s}$ depends only on two parameters, α and β (see the Appendix). So (27) holds if and only if $\hat{s}(\alpha, \beta) < (a_1\rho + a_2\delta)/a_2\theta$. Notice that this condition is independent of the value of $\tilde{p}$.

12. In the case where both goods are normal, the income expansion path is upward-sloping and the solution must be unique.

13. To simplify the presentation, we suppress the dependence of steady state values on L when there is no ambiguity.

14. We will show in Appendix 7.3 that (33) holds for $^\forall\lambda > 0$ if $C_1(\tilde{p}, \lambda^2(\tilde{p}))$ is smaller than the C_1-intercept of the line $\zeta_1(\tilde{p})C_1 + C_2 = \zeta_2(\tilde{p})L$, as in Figure 1.

15. For λ that violates (33), we have $\underline{p}(\lambda) > \tilde{p}$, and therefore $p^A(k_1(\tilde{p}), \lambda) f_1'(k_1(\tilde{p})) > \tilde{p} f_1'(k_1(\tilde{p})) = r(\tilde{p}) > r(\underline{p}(\lambda))$, which implies that the locus with $r^A(k, \lambda) = \theta$ lies between $k_1(\tilde{p})$ and $\underline{k}(\lambda)$ for such λ.

16. See the Appendix.

17. Notice that p_m^+ must be smaller than $r^{-1}(\delta)$, because $\zeta_1(r^{-1}(\delta)) = p$ implies that the line $\zeta_1(r^{-1}(\delta))C_1 + C_2 = \zeta_2(r^{-1}(\delta))L$ cuts exactly once the income expansion path with $p = r^{-1}(\delta)$.

18. Notice that $L > (\underline{L} + \bar{L})/2$ implies $\tilde{z}_1(\hat{\lambda}^1) + \tilde{z}_1(\hat{\lambda}^2) < 0$, and hence there is only one value of λ^* that satisfies $\tilde{z}_1(\hat{\lambda}^1) + \tilde{z}_1(\lambda^*) = 0$, whereas there are three values of λ^* satisfying $\tilde{z}_1(\hat{\lambda}^2) + \tilde{z}_1(\lambda^*) = 0$, which yields the set of equilibrium pairs as in Figure 3. In Figure 3, we suppose that $\tilde{z}_1(\hat{\lambda}^1) + \tilde{z}_1(\lambda_{\min}) > 0$, $\tilde{z}_1(\hat{\lambda}^2) + \tilde{z}_1(\lambda_{\max}) < 0$, and $\tilde{z}_1(\lambda_{\min}) + \tilde{z}_1(\lambda_{\max}) > 0$, all of which do not matter in the following discussion.

19. Instead, one may assume that the home country transfers its ownership of a part of capital, $\mu(\tilde{k}^1 - \tilde{k}^3)$, to the foreign country at the beginning. We thank a referee for leading our attention to this point.

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APPENDIX

A.1. PROPERTIES OF THE UTILITY FUNCTION (24)

It can be easily shown that (24) satisfies Assumption 2 where the set $\{(C_1, C_2) \in \mathbf{R}_+^2 | u_i(C_1, C_2) > 0, i = 1, 2\}$ is given by $\{(C_1, C_2) \in \mathbf{R}_+^2 | \gamma C_1 C_2 < \alpha\}$.

The first-order conditions for the choice of consumption levels, (10), are

$$\frac{\alpha - \gamma C_1 C_2}{C_1} = \lambda p, \tag{A.1}$$

$$\frac{\beta - \gamma C_1 C_2}{C_2} = \lambda, \tag{A.2}$$

which yield

$$(\gamma C_1 C_2)^2 - \left(\alpha + \beta + \frac{\lambda^2 p}{\gamma}\right) \gamma C_1 C_2 + \alpha\beta = 0.$$

Therefore, we have

$$\gamma C_1 C_2 = \frac{\alpha + \beta + \frac{\lambda^2 p}{\gamma} - \left[\left(\alpha + \beta + \frac{\lambda^2 p}{\gamma}\right)^2 - 4\alpha\beta\right]^{\frac{1}{2}}}{2}, \tag{A.3}$$

because consumption bundles (C_1, C_2) satisfy $\gamma C_1 C_2 < \alpha$. Let us denote the right-hand side of (A.3) by $q(\lambda^2 p)$. Then we get

$$\lim_{\lambda^2 p \to 0} q(\lambda^2 p) = \alpha, \quad \lim_{\lambda^2 p \to \infty} q(\lambda^2 p) = 0, \text{ and } q'(\lambda^2 p) = -\frac{q(\lambda^2 p)^2}{\gamma[\alpha\beta - q(\lambda^2 p)^2]}.$$

Thus, for any positive p and λ, $q(\lambda^2 p)$ is between zero and α and strictly decreasing in $\lambda^2 p$.

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Utilizing this function, we obtain consumption relations $C_i(p, \lambda)$ $(i = 1, 2)$ from (A.1) and (A.2) as follows:

$$C_1(p, \lambda) = \frac{\alpha - q(\lambda^2 p)}{\lambda p} \text{ and } C_2(p, \lambda) = \frac{\beta - q(\lambda^2 p)}{\lambda},$$

where

$$\lim_{\lambda \to 0} C_1(p, \lambda) = 0, \ \lim_{\lambda \to 0} C_2(p, \lambda) = \infty, \text{ and } \lim_{\lambda \to \infty} C_i(p, \lambda) = 0 \ (i = 1, 2).$$

As a result of straightforward calculations, we get

$$C_{1\lambda}(p, \lambda) = -\frac{q(q^2 - 2\beta q + \alpha\beta)}{\gamma(\beta - q)(\alpha\beta - q^2)}, \tag{A.4}$$

$$C_{2\lambda}(p, \lambda) = -\frac{pq(q^2 - 2\alpha q + \alpha\beta)}{\gamma(\alpha - q)(\alpha\beta - q^2)}, \tag{A.5}$$

$$C_{1\lambda\lambda}(p, \lambda) = \frac{2\beta q(\alpha - q)\phi_1(q)}{\gamma(\beta - q)(\alpha\beta - q^2)^3 \lambda}, \tag{A.6}$$

$$C_{2\lambda\lambda}(p, \lambda) = \frac{2p\alpha q(\beta - q)\phi_2(q)}{\gamma(\alpha - q)(\alpha\beta - q^2)^3 \lambda}, \tag{A.7}$$

where

$$\phi_1(q) \equiv q^4 - 4\alpha q^3 + 6\alpha\beta q^2 - 4\alpha\beta^2 q + \alpha^2\beta^2,$$
$$\phi_2(q) \equiv q^4 - 4\beta q^3 + 6\alpha\beta q^2 - 4\alpha^2\beta q + \alpha^2\beta^2.$$

From (A.4)–(A.7), we have

$$C_{2\lambda\lambda}C_{1\lambda} - C_{2\lambda}C_{1\lambda\lambda} = \frac{2pq^3(\beta - \alpha)\psi(q)}{\gamma^2(\alpha - q)(\beta - q)(\alpha\beta - q^2)^3 \lambda},$$

where

$$\psi(q) \equiv q^4 - 6\alpha\beta q^2 + 4\alpha\beta(\alpha + \beta)q - 3\alpha^2\beta^2. \tag{A.8}$$

First, from (A.4) and (A.5), we see that

$$C_{1\lambda} \lesseqgtr 0 \text{ if } q(\lambda^2 p) \lesseqgtr \beta - \sqrt{\beta(\beta - \alpha)} \text{ and } C_{2\lambda} < 0 \text{ for } ^{\forall} q \in (0, \alpha),$$

where $\beta - \sqrt{\beta(\beta - \alpha)}$ is a solution to $q^2 - 2\beta q + \alpha\beta = 0$ and smaller than α.

Next, from (A.8), we have

$$\psi(0) = -3\alpha^2\beta^2 < 0, \psi(\alpha) = \alpha^2(\beta - \alpha)^2 > 0, \psi'(0) = 4\alpha\beta(\alpha + \beta) > 0,$$

and

$$\psi''(q) = 12(q^2 - \alpha\beta) < 0 \text{ for } q \in (0, \alpha).$$

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So we see that $\psi(q) = 0$ has a unique solution between zero and α, denoted by $\hat{q}(\alpha, \beta)$, and that $\psi(q)$ is negative (positive) if $q(\lambda^2 p)$ is smaller (greater) than $\hat{q}(\alpha, \beta)$, and hence

$$C_{2\lambda\lambda}C_{1\lambda} - C_{2\lambda}C_{1\lambda\lambda} \lesseqgtr 0 \quad \text{if } q(\lambda^2 p) \lesseqgtr \hat{q}(\alpha, \beta).$$

It can easily be shown that $\psi(\beta - \sqrt{\beta(\beta - \alpha)})$ is negative, which implies that $\hat{q}(\alpha, \beta) > \beta - \sqrt{\beta(\beta - \alpha)}$. Therefore, we see that $0 < \lambda^1(p) < \lambda^2(p) < \infty$ holds because

$$\lambda^1(p) = \left[\frac{q^{-1}(\hat{q}(\alpha, \beta))}{p}\right]^{\frac{1}{2}} \text{ and } \lambda^2(p) = \left[\frac{q^{-1}(\beta - \sqrt{\beta(\beta - \alpha)})}{p}\right]^{\frac{1}{2}}.$$

Thus, (24) satisfies Assumption 3.

Notice that the slope of the income expansion path at $(C_1, C_2) = (C_1(p, \lambda^1(p)), C_2(p, \lambda^1(p)))$, is given by

$$\frac{(\beta - \hat{q})(\hat{q}^2 - 2\alpha\hat{q} + \alpha\beta)}{(\alpha - \hat{q})(\hat{q}^2 - 2\beta\hat{q} + \alpha\beta)} p.$$

So we have

$$\hat{s}(\alpha, \beta) = -\frac{(\beta - \hat{q})(\hat{q}^2 - 2\alpha\hat{q} + \alpha\beta)}{(\alpha - \hat{q})(\hat{q}^2 - 2\beta\hat{q} + \alpha\beta)},$$

which is greater than one.

A.2. DERIVATION OF THE INEQUALITY (35)

Totally differentiating equations (10) with respect to C_1, C_2, p, and λ, we derive

$$\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \begin{bmatrix} dC_1 \\ dC_2 \end{bmatrix} = \begin{bmatrix} p \\ 1 \end{bmatrix} d\lambda + \begin{bmatrix} \lambda \\ 0 \end{bmatrix} dp.$$

Because the determinant of the coefficient matrix, $D = u_{11}u_{22} - u_{12}^2$, is positive at any point where $u_i(C_1, C_2) > 0$ (Assumption 2) and therefore invertible, we obtain

$$C_{1\lambda}(p, \lambda) \equiv \frac{\partial C_1}{\partial \lambda} = \frac{1}{D}(u_{22}p - u_{12}), \tag{A.9}$$

$$C_{2\lambda}(p, \lambda) \equiv \frac{\partial C_2}{\partial \lambda} = \frac{1}{D}(u_{11} - u_{12}p), \tag{A.10}$$

$$C_{1p}(p, \lambda) \equiv \frac{\partial C_1}{\partial p} = \frac{1}{D}\lambda u_{22} < 0, \tag{A.11}$$

$$C_{2p}(p, \lambda) \equiv \frac{\partial C_2}{\partial p} = -\frac{1}{D}\lambda u_{12}.$$

The results of Lemma 1 follow immediately from these comparative statics results.

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Totally differentiating (15) with respect to p and λ yields

$$\frac{\partial p^A}{\partial \lambda} = -\frac{c_{1\lambda}}{c_{1p} - y_{1p}}.$$

So, from (A.9)–(A.11) we have

$$\begin{aligned}
\lambda c_{1\lambda}\frac{\partial p^A}{\partial \lambda} + e_\lambda &= -\frac{\lambda c_{1\lambda}^2}{c_{1p} - y_{1p}} + pc_{1\lambda} + c_{2\lambda} \\
&\leq -\frac{\lambda c_{1\lambda}^2}{c_{1p}} + pc_{1\lambda} + c_{2\lambda} \\
&= c_{1\lambda}\left(\frac{u_{12}}{u_{22}} - p\right) + pc_{1\lambda} + c_{2\lambda} \\
&= \frac{1}{u_{22}} \\
&< 0.
\end{aligned}$$

A.3. PROOF OF LEMMA 5

From (32) and (36), production is incompletely specialized iff

$$k_1(p) < \frac{c_1(p,\lambda) - w'(p)}{r'(p)} < k_2(p) \qquad \textbf{(A.12)}$$

$$\Leftrightarrow \quad 0 < c_1(p,\lambda) < \frac{\zeta_2(p) + \delta k_1(p)}{\zeta_1(p)}. \qquad \textbf{(A.13)}$$

Therefore, if $C_1(p, \lambda^2(p))$ is smaller than the C_1-intercept of the line $\zeta_1(p)C_1 + C_2 = [\zeta_2(p) + \delta k_1(p)]L$, which must be true at least for $p = r^{-1}(\delta)$, then

$$k_1(p) < \frac{c_1(p,\lambda) - w'(p)}{r'(p)} < k_2(p) \text{ for } ^\forall\lambda > 0. \qquad \textbf{(A.14)}$$

So, if $C_1(\tilde{p}, \lambda^2(\tilde{p})) < [\zeta_2(\tilde{p}) + \delta k_1(\tilde{p})]L/\zeta_1(\tilde{p})$, then (A.14) holds with $p = \tilde{p}$, and hence

$$k = \frac{c_1(\tilde{p},\lambda) - w'(\tilde{p})}{r'(\tilde{p})} > \underline{k}(\lambda)$$

for $^\forall\lambda > 0$: Complete specialization to produce good 1 does not occur along the path from e^2 to e^3.

For $L > \hat{L}$, we have $[\zeta_2(\tilde{p}) + \delta k_1(\tilde{p})]L/\zeta_1(\tilde{p}) > \zeta_2(\tilde{p})\bar{L}/\zeta_1(\tilde{p})$, the right-hand side of which must be greater than $C_1(\tilde{p}, \lambda^2(\tilde{p}))$ (see Figure 1). Also, we see that if the line $\zeta_1(\tilde{p})C_1 + C_2 = [\zeta_2(\tilde{p}) - \dot{k}]L$ cuts the income expansion path with $p = \tilde{p}$ three times, then $\dot{k} > -\delta k_1(\tilde{p})$ holds. Therefore, $\dot{k}$ is greater than $-\delta k_1(\tilde{p})$ at the points in the (k, λ) plane that satisfy $\lambda \geq \tilde{\lambda}^1$ and $k = [c_1(\tilde{p}, \lambda) - w'(\tilde{p})]/r'(\tilde{p})$, and hence we have $\dot{k} > -\delta k$ (consuming the existing capital does not occur) along the path from e^2 to e^3. Q.E.D.

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A.4. PROOF OF LEMMA 7

Suppose $\mu = 1/2$ and $\tilde{k}^1 + \tilde{k}^3 > 2\tilde{k}^2$. Then, as long as both home and foreign are incompletely specialized, households' income, including transfer, is the same across the countries and it is initially greater than the income at the unstable autarkic steady state e^2, which implies that each of the countries will accumulate capital along the autarkic dynamic general equilibrium path to the steady state e^1. So we show that

$$k_1(p^A(k,\lambda)) < k + \frac{\tilde{k}^1 - \tilde{k}^3}{2} < k_2(p^A(k,\lambda)),$$

$$k_1(p^A(k,\lambda)) < k - \frac{\tilde{k}^1 - \tilde{k}^3}{2} < k_2(p^A(k,\lambda)),$$

hold along the path to e^1.

First, notice that $k_1(\tilde{p}) < \tilde{k}^3$ holds, and that along the path, $p^A(k,\lambda) \le \tilde{p}$, and hence $k_i(p^A(k,\lambda)) \le k_i(\tilde{p})$ $(i = 1, 2)$ holds, where $k \in [(\tilde{k}^1 + \tilde{k}^3)/2, \tilde{k}_1]$. Therefore, it suffices to prove that

$$k + \frac{\tilde{k}^1 - \tilde{k}^3}{2} < k_2(p^A(k,\lambda))$$

holds for $k \in [(\tilde{k}^1 + \tilde{k}^3)/2, \tilde{k}_1]$.

Second, we have $c_1(\tilde{p}, \tilde{\lambda}^1) \le c_1(\tilde{p}, \lambda) \le c_1(p^A(k,\lambda), \lambda)$, which implies that

$$w'(\tilde{p}) + r'(\tilde{p})\tilde{k}^1 \le w'(p^A) + r'(p^A)k \tag{A.15}$$

$$\Leftrightarrow r'(p^A)k - r'(\tilde{p})\tilde{k}^1 \ge w'(\tilde{p}) - w'(p^A). \tag{A.16}$$

Because $w'' > 0$, we have $w'(\tilde{p}) \ge w'(p^A)$, and then we see

$$r'(p^A) \ge \frac{\tilde{k}^1}{k} r'(\tilde{p}) > 2r'(\tilde{p}). \tag{A.17}$$

From (36), (A.15), and (A.17), we have

$$\begin{aligned}
k_2(p^A) - \left(k + \frac{\tilde{k}^1 - \tilde{k}^3}{2}\right) &= -\frac{1}{r'(p^A)}\left[w'(p^A) + r'(p^A)\left(k + \frac{\tilde{k}^1 - \tilde{k}^3}{2}\right)\right] \\
&\ge -\frac{1}{r'(p^A)}\left[w'(\tilde{p}) + r'(\tilde{p})\tilde{k}^1 + r'(p^A)\frac{\tilde{k}^1 - \tilde{k}^3}{2}\right] \\
&> -\frac{1}{r'(p^A)}\left[w'(\tilde{p}) + r'(\tilde{p})\tilde{k}^1 + r'(\tilde{p})(\tilde{k}^1 - \tilde{k}^3)\right] \\
&= -\frac{1}{r'(p^A)}\left\{2[w'(\tilde{p}) + r'(\tilde{p})\tilde{k}^1] - [w'(\tilde{p}) + r'(\tilde{p})\tilde{k}^3]\right\} \\
&= -\frac{2c_1(\tilde{p}, \tilde{\lambda}^1) - c_1(\tilde{p}, \tilde{\lambda}^3)}{r'(p^A)}.
\end{aligned}$$

Q.E.D.

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