WHY IS KEMENY'S CONSTANT A CONSTANT?

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Abstract

In their 1960 book on finite Markov chains, Kemeny and Snell established that a certain sum is invariant. The value of this sum has become known as *Kemeny's constant*. Various proofs have been given over time, some more technical than others. We give here a very simple physical justification, which extends without a hitch to continuous-time Markov chains on a finite state space. For Markov chains with denumerably infinite state space, the constant may be infinite and even if it is finite, there is no guarantee that the physical argument will hold. We show that the physical interpretation does go through for the special case of a birth-and-death process with a finite value of Kemeny's constant.

Keywords: Kemeny's constant; discrete-time Markov chain; continuous-time Markov chain; passage time; deviation matrix

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1. Introduction

Consider a discrete-time, irreducible and aperiodic Markov chain $\{X_t \colon t = 0, 1, \ldots\}$ on a finite state space $\mathcal{S}$, with transition matrix P and stationary probability vector $\boldsymbol{\pi}$ such that $\boldsymbol{\pi}^\top P = \boldsymbol{\pi}^\top$ and $\boldsymbol{\pi}^\top \mathbf{1} = 1$. For $i \in \mathcal{S}$, define the *first passage times*

$$T_i = \inf\{t \geq 1 \colon X_t = i\}.$$

Denoting by $\mathbb{E}_i[\cdot]$ the conditional expectation given that $X_0 = i$, Kemeny and Snell [10, Theorem 4.4.10] proved that

$$\sum_{j \in \mathcal{S}} \pi_j \mathbb{E}_i[T_j] = K, \tag{1.1}$$

independently of the initial state i. The value K is known as *Kemeny's constant*.

A prize was offered to the first person to give an intuitively plausible reason for the sum in (1.1) to be independent of i (Grinstead and Snell [7, p. 469]). The prize was won by Doyle [5] with an argument given in the next section. We prove in Theorem 3.1 that (1.1) results from the obvious fact that a discrete-time Markov chain takes n steps during an interval of time of length n, independently of the initial state i. We move on to extend the argument to finite-state *continuous-time* Markov chains; see (3.5). In Section 3 we also discuss an important connection between K and the deviation matrix of the Markov chain.

In Section 4 we consider Markov chains with a denumerably infinite state space $\mathcal{S}$. Here, the situation becomes more complex because the sum in (1.1) might not converge. We show that it is independent of i in the sense that it is infinite for all i or a constant independent of i.

In Section 5 we restrict our discussion to positive recurrent birth-and-death processes. We show that K is infinite in discrete time, and in continuous time it is finite if transitions from state i occur sufficiently fast as i approaches ∞. Furthermore, our physical explanation holds for birth-and-death processes if $K < \infty$.

2. A simple algebraic proof

The simplest proof goes as follows: define $\omega_i = \sum_{j \in \mathcal{S}} \pi_j \mathbb{E}_i[T_j]$ and $\boldsymbol{\omega} = [\omega_i]_{i \in \mathcal{S}}$, condition on X_1, and write

$$
\begin{aligned}
\omega_i &= 1 + \sum_{j \in \mathcal{S}} \pi_j \sum_{k \in \mathcal{S}, k \neq j} P_{ik} \mathbb{E}_k[T_j] \\
&= 1 + \sum_{j \in \mathcal{S}} \pi_j \sum_{k \in \mathcal{S}} P_{ik} \mathbb{E}_k[T_j] - \sum_{j \in \mathcal{S}} P_{ij} \quad (\text{using } \pi_j = 1/\mathbb{E}_j[T_j]) \\
&= \sum_{j \in \mathcal{S}} \pi_j \sum_{k \in \mathcal{S}} P_{ik} \mathbb{E}_k[T_j] \\
&= \sum_{k \in \mathcal{S}} P_{ik} \omega_k
\end{aligned}
$$

so that $\boldsymbol{\omega} = P \boldsymbol{\omega}$ (see, for example, Hunter [9]). Doyle [5] argued from the maximum principle that all components of $\boldsymbol{\omega}$ must be equal. Alternatively, one may conclude from the Perron–Frobenius theorem that $\boldsymbol{\omega}$ must be proportional to the eigenvector $\mathbf{1}$ of P.

Instead of the passage times T_j, we shall use the *first hitting times* $\{\theta_i : i \in \mathcal{S}\}$ with

$$
\theta_i = \inf\{t \geq 0 \colon X_t = i\}.
$$

The only difference is that $\theta_i = 0 < T_i$ if $X_0 = i$, otherwise $\theta_i = T_i \geq 1$. Using θ_j instead of T_j, we obtain another version of Kemeny's constant:

$$
\sum_{j \in \mathcal{S}} \pi_j \mathbb{E}_i[\theta_j] = K', \tag{2.1}
$$

where $K' = K - 1$. We prefer to work with this version of Kemeny's constant because the equality (2.1) holds in continuous time as well; furthermore, using $\mathbb{E}_i[\theta_j]$ helps us establish a direct connection with the deviation matrix $\mathcal{D}$ of the Markov chain. We shall discuss this in the next section.
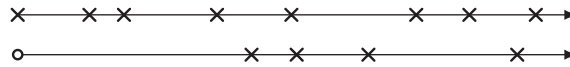
FIGURE 1: Visits to j starting in j (*upper*) and i (*lower*).

3. The case when $\mathscr{S}$ is finite

Our physical justification is based on the following argument. We start from

$$\sum_{j \in \mathscr{S}} \pi_j \mathbb{E}_i[\theta_j] = \sum_{j \in \mathscr{S}} \frac{\mathbb{E}_i[\theta_j]}{\mathbb{E}_j[T_j]} \tag{3.1}$$

which we transform to

$$\sum_{j \in \mathscr{S}} \pi_j \mathbb{E}_i[\theta_j] = \sum_{j \in \mathscr{S}} \lim_{n \to \infty} (\mathbb{E}_j[N_j(n)] - \mathbb{E}_i[N_j(n)]), \tag{3.2}$$

where

$$N_j(n) = \sum_{0 \leq t \leq n} 1_{\{X_t = j\}}$$

is the total number of visits to j during the interval of time $[0, n]$.

The formal justification for the transition from (3.1) to (3.2) is given in Lemma 3.1 below, but we provide a heuristic argument first, explained with the help of Figure 1. The upper line is a representation of a trajectory of the renewal process $\{\theta_j^{(k)} : k \geq 0\}$ of successive visits to j, starting from $X_0 = j$; the $\theta_j^{(k)}$s are marked with a cross $\times$. The lower line represents a trajectory of the delayed renewal process of visits to j, starting from $X_0 = i \neq j$.

Now, the jth term in the right-hand side of (3.2) is the expected difference between the total number of events in the two processes. We observe a smaller expected number of visits to j if the process starts from $i \neq j$ because of the initial delay. The expected length of this delay is $\mathbb{E}_i[\theta_j]$ and $\mathbb{E}_j[T_j]$ is the expected length of intervals between visits to j. The ratio $\mathbb{E}_i[\theta_j]/\mathbb{E}_j[T_j]$ is the expected number of visits that are missed over the whole history of the process by starting from i instead of j. The formal argument is as follows.

Lemma 3.1. *For all i and j,*

$$\frac{\mathbb{E}_i[\theta_j]}{\mathbb{E}_j[T_j]} = \lim_{n \to \infty} (\mathbb{E}_j[N_j(n)] - \mathbb{E}_i[N_j(n)]).$$

Proof. The statement is obvious if $i = j$ since then $\mathbb{E}_j[\theta_j] = 0$ by the definition of θ_j. We assume now that i and j are different, arbitrary but fixed, and to simplify the notation we define $\widetilde{N}_i(n) = \mathbb{E}_i[N_j(n)]$ and $f_i(t) = \mathbb{P}_i[\theta_j = t]$, with $f_i(0) = 0$. We have

$$\widetilde{N}_j(n) = \sum_{0 \leq \nu \leq n} \mathbb{P}_j[X_\nu = j].$$

Furthermore, conditioning on the first visit to state j, we can write, for $n \geq 0$,

$$
\begin{aligned}
\widetilde{N}_i(n) &= \sum_{0 \leq t \leq n} f_i(t) \widetilde{N}_j(n - t) \\
&= \sum_{0 \leq t \leq n} f_i(t) \sum_{0 \leq v \leq n-t} \mathbb{P}_j[X_v = j] \\
&= \sum_{0 \leq v \leq n} \mathbb{P}_j[X_v = j] \sum_{0 \leq t \leq n-v} f_i(t)
\end{aligned}
$$

and so

$$
\widetilde{N}_j(n) - \widetilde{N}_i(n) = \sum_{0 \leq v \leq n} \mathbb{P}_j[X_v = j] \mathbb{P}_i[\theta_j > n - v].
$$

Finally,

$$
\lim_{n \to \infty} (\widetilde{N}_j(n) - \widetilde{N}_i(n)) = \lim_{n \to \infty} \sum_{0 \leq v \leq n} \mathbb{P}_j[X_{n-v} = j] \mathbb{P}_i[\theta_j > v] = \frac{1}{\mathbb{E}_j[T_j]} \sum_{v \geq 0} \mathbb{P}_i[\theta_j > v]
$$

by the key renewal theorem (Resnick [13, Section 3.8]), and is equal to $\pi_j \mathbb{E}_i[\theta_j]$. This completes the proof. $\qquad \square$

Lemma 3.1 leads immediately to an understanding of the reason why the left-hand side of (1.1) is independent of i.

Theorem 3.1. *For an irreducible and aperiodic discrete-time Markov chain with finite state space $\mathscr{S}$ and stationary distribution $\boldsymbol{\pi}$, $\sum_{j \in \mathscr{S}} \pi_j \mathbb{E}_i[T_j]$ is independent of i.*

Proof. Since $\mathscr{S}$ is finite, we can interchange the limit and sum in (3.2) to obtain

$$
\begin{aligned}
\sum_{j \in \mathscr{S}} \pi_j \mathbb{E}_i[\theta_j] &= \sum_{j \in \mathscr{S}} \lim_{n \to \infty} (\mathbb{E}_j[N_j(n)] - \mathbb{E}_i[N_j(n)]) \\
&= \lim_{n \to \infty} \left(\sum_{j \in \mathscr{S}} \mathbb{E}_j[N_j(n)] - \sum_{j \in \mathscr{S}} \mathbb{E}_i[N_j(n)] \right) \\
&= \lim_{n \to \infty} \left(\sum_{j \in \mathscr{S}} \mathbb{E}_j[N_j(n)] - (n + 1) \right) \tag{3.3}
\end{aligned}
$$

independently of i. $\qquad \square$

We see that the change of perspective from first hitting times to numbers of visits brings a different physical interpretation from those previously suggested. Indeed, as mentioned above, the second term under the limit in (3.3) just counts the number of steps that the Markov chain has taken, independently of its starting state.

The deviation matrix lends itself beautifully to such a change of point of view. It is defined as

$$
\mathscr{D} = \sum_{n \geq 0} (P^n - \mathbf{1} \cdot \boldsymbol{\pi}^\top)
$$

if the series converges. From [4, Theorem 4.1], and [15, Proposition 3.2], the series converges if and only if $\mathbb{E}_{\boldsymbol{\pi}}[\theta_j] < \infty$ for some state $j \in \mathscr{S}$, and then it is finite for every j, where

$\mathbb{E}_{\pi}[\cdot]$ denotes the conditional expectation, given that X_0 has the distribution π. An equivalent condition is that $\mathbb{E}_j[\theta_j^2] < \infty$ for some j. When $|\mathcal{S}| < \infty$, the series always converges and $\mathcal{D} = (I - P)^{\#}$, the group inverse of $I - P$ (see [3]).

Obviously,

$$\mathcal{D}_{ij} = \lim_{n \to \infty} \sum_{t=0}^{n} ([P^t]_{ij} - \pi_j)$$

$$= \lim_{n \to \infty} \sum_{t=0}^{n} (\mathbb{E}_i[1_{\{X_t = j\}}] - \mathbb{E}_{\pi}[1_{\{X_t = j\}}])$$

$$= \lim_{n \to \infty} (\mathbb{E}_i[N_j(n)] - \mathbb{E}_{\pi}[N_j(n)]).$$

In addition, $\mathcal{D}_{jj} = \pi_j \mathbb{E}_{\pi}[\theta_j]$ using [15, Proposition 3.3] with the discrete-time analogue of the argument in [4, Equation (5.7)], and so

$$\sum_{j \in \mathcal{S}} \mathcal{D}_{jj} = \sum_{j \in \mathcal{S}} \pi_j \mathbb{E}_{\pi}[\theta_j]$$

$$= \sum_{j \in \mathcal{S}} \pi_j \sum_{i \in \mathcal{S}} \pi_i \mathbb{E}_i[\theta_j]$$

$$= \sum_{i \in \mathcal{S}} \pi_i \sum_{j \in \mathcal{S}} \pi_j \mathbb{E}_i[\theta_j]$$

$$= \sum_{i \in \mathcal{S}} \pi_i K'$$

$$= K'. \tag{3.4}$$

Equation (3.4) provides a convenient representation for Kemeny's constant in terms of the trace of the deviation matrix.

The point of view that we have taken in this section extends to finite-state continuous-time Markov chains. For such a chain with irreducible generator Q, define the first hitting time and the first passage time as

$$\theta_i = \inf\{t \geq 0 \colon X_t = i\}, \qquad T_i = \inf\{t \geq J_1 \colon X_t = i\},$$

where J_1 is the first jump time of the Markov chain; if $X_0 = i$ then $\theta_i = 0 < T_i$, otherwise $\theta_i = T_i > 0$. Lemma 3.1 then becomes our next lemma.

Lemma 3.2. *For a continuous-time Markov chain,*

$$\pi_j \mathbb{E}_i[\theta_j] = \lim_{t \to \infty} (\mathbb{E}_j[M_j(t)] - \mathbb{E}_i[M_j(t)]),$$

where $M_j(t) = \int_0^t 1_{\{X(u) = j\}} \, du$ is the total time spent in j until time t.

Proof. We follow the same steps as in Lemma 3.1 with the only difference that here

$$\mathbb{E}_j[M_j(t)] = \int_0^t \mathbb{P}_j[X(u) = j] \, du \quad \text{and} \quad \mathbb{E}_i[M_j(t)] = \int_0^t dG_i(v) \mathbb{E}_j[M_j(t - v)],$$

where $G_i(t) = \mathbb{P}_i[\theta_j \leq t]$. $\qquad \square$

From this, we obtain

$$\sum_{j \in \mathcal{S}} \pi_j \mathbb{E}_i[\theta_j] = \lim_{t \to \infty} \left(\sum_{j \in \mathcal{S}} \mathbb{E}_j[M_j(t)] - t \right) \tag{3.5}$$

independently of i. By an argument similar to that leading to (3.4), we also have

$$\sum_{j \in \mathcal{S}} \pi_j \mathbb{E}_i[\theta_j] = \sum_{j \in \mathcal{S}} \pi_j \mathbb{E}_{\boldsymbol{\pi}}[\theta_j] = \sum_{j \in \mathcal{S}} \mathcal{D}_{jj}$$

with the continuous-time deviation matrix defined by $\mathcal{D} = \int_0^{\infty} (\mathrm{e}^{Qt} - \mathbf{1} \cdot \boldsymbol{\pi}^{\top}) \, \mathrm{d}t$.

4. The case when $\mathcal{S}$ is infinite

If $\mathcal{S}$ is denumerably infinite, it is not easy to see in general how the exchange of limit and sum inherent in the step between the first and second equations of (3.3) can be justified. However, $\sum_{j \in \mathcal{S}} \pi_j \mathbb{E}_i[\theta_j]$ is still independent of i in the sense that it is either finite and constant with respect to i, or infinite for all i.

Theorem 4.1. *For an irreducible, positive-recurrent discrete or continuous-time Markov chain* $\{X_t\}$ *with a countably-infinite state space* $\mathcal{S}$, *either*

(i) $\sum_{j \in \mathcal{S}} \pi_j \mathbb{E}_i[\theta_j]$ *is equal to a finite constant that is independent of i, or*

(ii) $\sum_{j \in \mathcal{S}} \pi_j \mathbb{E}_i[\theta_j]$ *is infinite for all $i \in \mathcal{S}$.*

Proof. The argument presented in Section 2 due to [5] and [9] holds even when the state space is infinite. Writing the expressions in terms of $\mathbb{E}_i[\theta_j]$ rather than $\mathbb{E}_i[T_j]$, for a discrete-time Markov chain with transition matrix P, we have

$$\begin{aligned}
\xi_i &\equiv \sum_{j \in \mathcal{S}} \pi_j \mathbb{E}_i[\theta_j] \\
&= \sum_{j \neq i} \pi_j \left[1 + \sum_{k \in \mathcal{S}} P_{ik} \mathbb{E}_k[\theta_j] \right] \\
&= 1 - \pi_i + \sum_{k \in \mathcal{S}} P_{ik} \sum_{j \neq i} \pi_j \mathbb{E}_k[\theta_j] \\
&= 1 - \pi_i + \sum_{k \in \mathcal{S}} P_{ik} \left(\sum_{j \in \mathcal{S}} \pi_j \mathbb{E}_k[\theta_j] - \pi_i \mathbb{E}_k[\theta_i] \right) \tag{4.1} \\
&= 1 - \pi_i + \sum_{k \in \mathcal{S}} P_{ik} \sum_{j \in \mathcal{S}} \pi_j \mathbb{E}_k[\theta_j] - \pi_i \sum_{k \in \mathcal{S}} P_{ik} \mathbb{E}_k[\theta_i], \tag{4.2}
\end{aligned}$$

where the series in (4.1) and the first series in (4.2) both converge or both diverge, and the second series in (4.2) converges to $\mathbb{E}_i[T_i] - 1 < \infty$ by assumption. Thus, we may write

$$\xi_i = 1 - \pi_i + \sum_{k \in \mathcal{S}} P_{ik} \xi_k - \pi_i [\mathbb{E}_i[T_i] - 1] = \sum_{k \in \mathcal{S}} P_{ik} \xi_k. \tag{4.3}$$

Since we have assumed that $\{X_t\}$ is recurrent, it follows from [14, Theorem 5.4^D] that, if $\boldsymbol{\xi}$ is entrywise finite, then it must be a multiple of $\mathbf{1}$. On the other hand, if ξ_k is infinite for some k

then (4.3) implies that ξ_i must be infinite for any i such that $P_{ik} > 0$. It follows by irreducibility that ξ_i must be infinite for all $i \in \mathcal{S}$.

For a continuous-time Markov chain with transition matrix Q, similar reasoning holds with P the transition matrix of the jump chain with entries $P_{ij} = q_{ij} 1_{\{i \neq j\}}/q_i$. $\qquad\square$

Remark 4.1. In Section 5 we shall show that, for a discrete-time birth-and-death process with state space $\{0, 1, \ldots\}$, $K' = \sum_{j \in \mathcal{S}} \pi_j \mathbb{E}_i[\theta_j]$ is infinite for all i.

For a general m-state discrete-time Markov chain, Hunter [8, Theorem 4.2] used a spectral argument to show that $K \geq \frac{1}{2}(m + 1)$ which implies that $K' \geq \frac{1}{2}(m - 1)$. We do not see how to extend this argument to show that K' is infinite for a general infinite-state discrete-time Markov chain, but we do not know of an example of such a chain with finite K'. We conjecture that K' is infinite for all infinite-state, discrete-time Markov chains.

On the other hand, it is possible for K' to be finite for an infinite-state continuous-time Markov chain. We shall present some examples in Section 5.

Remark 4.2. In Section 3 we showed that, when $\mathcal{S}$ is finite, K' is equal to the trace of the deviation matrix. The argument in Appendix A shows that this is the case when $\mathcal{S}$ is infinite and $\mathcal{D}$ exists. Specifically, K' is finite and equal to $\sum_{j \in \mathcal{S}} \mathcal{D}_{jj}$ if this sum is finite, and infinite when $\sum_{j \in \mathcal{S}} \mathcal{D}_{jj}$ is infinite.

When $\mathcal{D}$ does not exist, $\mathbb{E}_{\boldsymbol{\pi}}[\theta_j]$ must be infinite for all $j \in \mathcal{S}$, by [4, Theorem 4.1] and [15, Proposition 3.2]. Then

$$\sum_{i \in \mathcal{S}} \pi_i \sum_{j \in \mathcal{S}} \pi_j \mathbb{E}_i[\theta_j] = \sum_{j \in \mathcal{S}} \pi_j \sum_{i \in \mathcal{S}} \pi_i \mathbb{E}_i[\theta_j] = \sum_{j \in \mathcal{S}} \pi_j \mathbb{E}_{\boldsymbol{\pi}}[\theta_j] = \infty.$$

Since $\sum_{j \in \mathcal{S}} \pi_j \mathbb{E}_i[\theta_j]$ must be independent of i if it is finite, the last equality can occur only if $K' = \sum_{j \in \mathcal{S}} \pi_j \mathbb{E}_i[\theta_j] = \infty$.

5. Birth-and-death processes

Let us assume now that $\{X_t\}$ is a birth-and-death process on the infinite state space $\{0, 1, \ldots\}$. Choosing $X_0 = 0$ without loss of generality, we have

$$K' = \sum_{j \geq 0} \pi_j \mathbb{E}_0[\theta_j]. \tag{5.1}$$

We shall examine continuous and discrete-time processes simultaneously. In continuous time, we denote by λ_n and μ_n the transition rates from n to $n + 1$ and from n to $n - 1$, respectively; in discrete time, these are the one-step transition probabilities. We assume that $\lambda_n > 0$ for all $n \geq 0$, $\mu_n > 0$ for all $n \geq 1$, so that the process is irreducible. We further assume that the birth-and-death process is positive recurrent, so that $B = \sum_{n \geq 0} \beta_n$ is finite, where

$$\beta_0 = 1, \qquad \beta_n = \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n} \quad \text{for } n \geq 1, \tag{5.2}$$

and the stationary distribution is given by $\pi_n = B^{-1} \beta_n$ (see [4, Equation (6.3)]), which is still valid in the discrete-time case.

Theorem 5.1. *For an irreducible, positive recurrent, birth-and-death process on* $\{0, 1, \ldots\}$, *the constant* K' *is finite if and only if*

$$\Theta = \sum_{k \geq 0} (\lambda_k \pi_k)^{-1} \sum_{j \geq k+1} \pi_j < \infty. \tag{5.3}$$

In that case, $K' = \Theta - \mathbb{E}_\pi[\theta_0]$, with

$$\mathbb{E}_\pi[\theta_0] = \sum_{k\geq 0}(\lambda_k\pi_k)^{-1}\left(\sum_{j\geq k+1}\pi_j\right)^2 < \Theta.$$

Proof. We start from (5.1) and write

$$\begin{aligned}
K' &= \sum_{j\geq 1}\pi_j\sum_{0\leq k\leq j-1}(\lambda_k\pi_k)^{-1}\sum_{0\leq \ell\leq k}\pi_\ell \\
&= \sum_{k\geq 0}(\lambda_k\pi_k)^{-1}\left(\sum_{j\geq k+1}\pi_j\right)\left(1 - \sum_{\ell\geq k+1}\pi_\ell\right) \\
&= \sum_{k\geq 0}(\lambda_k\pi_k)^{-1}\sum_{j\geq k+1}\pi_j - \sum_{k\geq 0}(\lambda_k\pi_k)^{-1}\left(\sum_{j\geq k+1}\pi_j\right)^2 \qquad (5.4)
\end{aligned}$$

if both series converge, with the first equation following from [4, Equation (6.4)], which is still valid in the discrete-time case. The first series is Θ by definition, the second is equal to $\mathbb{E}_\pi[\theta_0]$ by [4, Equation (6.6)]. It is obvious that $\mathbb{E}_\pi[\theta_0] \leq \Theta$.

If $\Theta < \infty$ then $\mathbb{E}_\pi[\theta_0] < \infty$, the deviation matrix exists, and $K' = \Theta - \mathbb{E}_\pi[\theta_0]$ by (5.4). If $\Theta = \infty$ and $\mathbb{E}_\pi[\theta_0] < \infty$ then $K' = \infty$ by (5.4) again. Finally, if $\mathbb{E}_\pi[\theta_0] = \infty$ then we have already seen in Remark 4.2 that $K' = \infty$. $\qquad\square$

Remark 5.1. First, let us deal with discrete-time birth-and-death processes. From [4, Equation (6.6)], we see that $\Theta = \lim_{n\to\infty}\mathbb{E}_n[\theta_0]$ and we may interpret Theorem 5.1 as saying that, for Kemeny's constant to be finite, it is necessary (and sufficient) that having ventured to any state n, no matter how far from the origin, the process will reach state 0 in bounded expected time.

In discrete time, every transition from a state to one of its neighbours requires at least one unit of time, so that $\mathbb{E}_n[\theta_0] \geq n$ is unbounded. This tells us that (5.1) diverges for all discrete-time birth-and-death processes.

Remark 5.2. For the continuous-time birth-and-death process with birth rates λ_n and death rates μ_n, the right-hand side of (5.3) is the 'D series'; see [1, p. 261] or [11, p. 245]. A recurrent continuous-time birth-and-death process with a finite D series is said to have an *entrance boundary at ∞*, a classification that goes back to Feller [6]. Bansaye *et al.* [2] described this as *instantaneously coming down from infinity*.

A number of authors have looked at the consequences of the D series being finite. For example, when there is an absorbing state at -1, this condition is equivalent to the existence of a unique quasistationary distribution (see [16, Theorem 3.2]). The condition is also equivalent to *strong ergodicity* of the birth-and-death process in the sense that $\lim_{t\to\infty}\sup_i|p_{ij}(t)-\pi_j| = 0$; see [12, Theorem 3.1], [17, Theorem 3.1], and [18, Corollary 2.4].

For a continuous-time birth-and-death process with finite K', we can show that the change of limit and sum in (3.3) can be justified, and so the physical interpretation given in Section 3 holds in this case as well. The details are given in the following lemma.

Corollary 5.1. *Consider a continuous-time, irreducible, positive recurrent, birth-and-death process on $\{0, 1, \ldots\}$. If $\Theta < \infty$ then*

$$K' = \lim_{t\to\infty}\left(\sum_{j\in\mathcal{S}}\mathbb{E}_j[M_j(t)] - t\right).$$

Proof. We fix $i = 0$ and note that

$$K' = \sum_{j \in \mathcal{S}} \pi_j \mathbb{E}_0[\theta_j] = \sum_{j \in \mathcal{S}} \lim_{t \to \infty} (\mathbb{E}_j[M_j(t)] - \mathbb{E}_0[M_j(t)])$$

by Lemma 3.2. We plan to use the Fatou–Lebesgue dominated convergence theorem to justify the change of limit and sum. To that end, we construct a bound m_j for $\mathbb{E}_j[M_j(t)] - \mathbb{E}_0[M_j(t)]$ such that $\sum_j m_j < \infty$.

Denote by $^{(0)}m_j(t)$ the expected sojourn time in j during the interval $(0, t)$, starting from j, under taboo of state 0 and let $^{(0)}m_j(\infty) = \lim_{t \to \infty}(^{(0)}m_j(t))$. Then

$$\begin{aligned}
\mathbb{E}_j[M_j(t)] - E_0[M_j(t)] &= {}^{(0)}m_j(t) + \int_0^t \mathbb{E}_0[M_j(t-u)]\,\mathrm{d}\mathbb{P}_j[\theta_0 \leq u] - \mathbb{E}_0[M_j(t)] \\
&\leq {}^{(0)}m_j(t) - (1 - \mathbb{P}_0[\theta_0 \leq t])\mathbb{E}_0[M_j(t)] \\
&\leq {}^{(0)}m_j(t) \\
&\leq {}^{(0)}m_j(\infty).
\end{aligned}$$

It is a simple matter to show that $^{(0)}m_j(\infty) = \pi_j \sum_{1 \leq k \leq j} 1/(\pi_k \mu_k)$ and that $\sum_{j \in \mathcal{S}} {}^{(0)}m_j(\infty) = \Theta$, which is finite by assumption. Thus,

$$K' = \lim_{t \to \infty} \sum_{j \in \mathcal{S}} (\mathbb{E}_j[M_j(t)] - E_0[M_j(t)]) = \lim_{t \to \infty} \sum_{j \in \mathcal{S}} (\mathbb{E}_j[M_j(t)] - t])$$

by dominated convergence, and this concludes the proof. $\qquad\square$

Example 5.1. For the M/M/1 queue, $\lambda_n = \lambda$ and $\mu_n = \mu$, independently of n. The process is positive recurrent if and only if the ratio $\rho = \lambda/\mu$ is strictly less than 1, and $\pi_n = (1 - \rho)\rho^n$. Equation (5.3) becomes

$$\Theta = \sum_{k \geq 0} \lambda^{-1} \rho^{-k} \sum_{j \geq k+1} \rho^j = \sum_{k \geq 0} \frac{1}{\mu - \lambda} = \infty$$

so that Kemeny's constant is infinite. However, it may be finite for a process that we name the *sped-up* M/M/1 queue: we take an arbitrary sequence $\{\lambda_n\}$ and define $\mu_n = \rho\lambda_{n-1}$, with $\rho < 1$. Here, $\beta_n = \rho^n$ so that the process is positive recurrent, $\pi_n = (1 - \rho)\rho^n$, and

$$\Theta = \sum_{k \geq 0} \lambda_k^{-1} \rho^{-k} \sum_{j \geq k+1} \rho^j = \sum_{k \geq 0} \frac{\lambda_k^{-1}\rho}{1 - \rho},$$

which converges if $\lambda_n \to \infty$ sufficiently fast. In that case, $\mu_n \to \infty$ also.

The sped-up M/M/1 queue example illustrates that for Kemeny's constant to be finite, transitions have to occur faster as the process is further away from 0. Actually, as we show in the next lemma, it is necessary that transitions from n to $n-1$ occur sufficiently fast, transition rates from n to $n+1$ being less critical.

Lemma 5.1. *For Θ to be finite, it is necessary, but not sufficient, that the series $\sum_{j \geq 1} 1/\mu_j$ converges.*

Proof. We write (5.3) as

$$\Theta = \sum_{j \geq 1} f_j \tag{5.5}$$

with

$$
\begin{aligned}
f_j &= \pi_j \sum_{0 \leq k \leq j-1} (\lambda_k \pi_k)^{-1} \\
&= \pi_{j-1} \frac{\lambda_{j-1}}{\mu_j} \left(\sum_{0 \leq k \leq j-2} (\lambda_k \pi_k)^{-1} + (\lambda_{j-1} \pi_{j-1})^{-1} \right) \\
&= \frac{\lambda_{j-1} f_{j-1} + 1}{\mu_j} \quad \text{for } j \geq 1, \tag{5.6}
\end{aligned}
$$

if we define $f_0 = 0$.

This shows that $f_j \geq 1/\mu_j$, so that the series (5.5) diverges if $\sum_{j \geq 1} 1/\mu_j$ diverges. The proof that this is not a sufficient condition is given in Example 5.3 below. $\qquad\square$

Remark 5.3. Lemma 5.1 gives a different justification for the fact that (5.1) diverges for all discrete-time birth-and-death processes: here, $\mu_n \leq 1 - \lambda_n < 1$ by assumption, and the series $\sum_{j \geq 1} 1/\mu_j$ diverges.

Example 5.2. A direct consequence of Lemma 5.1 is that K' is infinite for the M/M/∞ queue for which $\mu_n = n\mu$: the transition rates from n to $n-1$ are not large enough. We may, however, use Lemma 5.1 to *design* processes for which Kemeny's constant is finite. To that end, we choose a sequence $\{f_j\}$ such that the series (5.5) converges, use (5.6) to define the sequence

$$\mu_j = \frac{\lambda_{j-1} f_{j-1} + 1}{f_j},$$

and then find a sequence $\{\lambda_j\}$ such that the process is positive recurrent, that is, such that $\sum_{n \geq 0} \beta_n$ converges, with β_n defined in (5.2). Two such examples follow. For the first,

- $f_1 = 0$, $f_j = 1/j^2$ for $j \geq 1$,
- $\mu_1 = 1$, $\mu_j = j^2(1 + 1/(j-1)^2)$ for $j \geq 2$, and
- $\lambda_j = 1$ for all j.

We easily see that

$$\beta_n = \frac{1}{\prod_{2 \leq j \leq n} j^2(1 + 1/(j-1)^2)} < \frac{1}{(n!)^2}.$$

For the second,

- $f_j = \gamma^j$, with $\gamma < 1$,
- $\mu_j = \gamma^{-j} + \lambda_{j-1} \gamma^{-1}$, and
- $\{\lambda_j\}$ is arbitrary.

Here,

$$\beta_n = \frac{\prod_{0 \leq j \leq n-1} \lambda_j}{\prod_{0 \leq j \leq n-1}(\lambda_j \gamma^{-1} + \gamma^{-(j+1)})} < \gamma^n.$$

Example 5.3. In this last example we show that Lemma 5.1 is not a necessary *and sufficient* condition. Take

- $\mu_j = j^{1+\alpha}$, with $0 < \alpha < 1$, and

- $\lambda_j = \mu_j$ for $j \geq 1$, $\lambda_0 = 1$.

With these parameters, $\beta_n = 1/\mu_n$, so that both $\sum_{n \geq 1} 1/\mu_n$ and $\sum_{n \geq 1} \beta_n$ converge.

By (5.6), we have $f_j \mu_j = 1 + f_{j-1} \mu_{j-1} = j$, so that

$$\sum_{j \geq 1} f_j = \sum_{j \geq 1} \frac{j}{\mu_j} = \sum_{j \geq 1} \frac{1}{j^\alpha}$$

diverges.

Appendix A. Proof that when $\mathcal{D}$ exists, K' is given by its trace

Here we show that, whenever the deviation matrix $\mathcal{D}$ exists, which occurs if and only if $\mathbb{E}_\pi[\theta_j] < \infty$ for some state $j \in \mathcal{S}$, then K' is finite and equal to $\sum_{j \in \mathcal{S}} \mathcal{D}_{jj}$ if this sum is finite, and infinite when $\sum_{j \in \mathcal{S}} \mathcal{D}_{jj}$ is infinite.

Since the deviation matrix exists, [4, Equation (5.5)] implies that

$$\mathcal{D}_{ij} = \pi_j(\mathbb{E}_\pi[\theta_j] - \mathbb{E}_i[\theta_j])$$

which yields

$$\pi_j \mathbb{E}_i[\theta_j] = \mathcal{D}_{jj} - \mathcal{D}_{ij}, \tag{A.1}$$

observing that $\mathbb{E}_j[\theta_j] = 0$. Summing (A.1) over j, we see that

$$\sum_{j \in \mathcal{S}} \pi_j \mathbb{E}_i[\theta_j] = \sum_{j \in \mathcal{S}} (\mathcal{D}_{jj} - \mathcal{D}_{ij}) = \lim_{K \to \infty} \sum_{j \in \mathcal{S}_K} (\mathcal{D}_{jj} - \mathcal{D}_{ij}) = \lim_{K \to \infty} \left(\sum_{j \in \mathcal{S}_K} \mathcal{D}_{jj} - \sum_{j \in \mathcal{S}_K} \mathcal{D}_{ij} \right),$$

where $\{\mathcal{S}_K\}$ is a monotone sequence of finite subsets converging to $\mathcal{S}$.

By [4, Theorem 5.2], $\sum_{j \in \mathcal{S}} \mathcal{D}_{ij} = 0$ for all $i \in \mathcal{S}$. Therefore, for any $\varepsilon > 0$, there exists $K_0(i)$ such that

$$-\frac{\varepsilon}{2} \leq \sum_{j \in \mathcal{S}_K} \mathcal{D}_{ij} \leq \frac{\varepsilon}{2} \quad \text{for all } K \geq K_0(i).$$

If $\sum_{j \in \mathcal{S}} \mathcal{D}_{jj} = \infty$ then, for any $M > 0$, there exists K_1 such that $\sum_{j \in \mathcal{S}_K} \mathcal{D}_{jj} \geq M$ for all $K > K_1$. It follows that, if $K > \max(K_0(i), K_1)$ then

$$\sum_{j \in \mathcal{S}_K} \mathcal{D}_{jj} - \sum_{j \in \mathcal{S}_K} \mathcal{D}_{ij} \geq M - \frac{\varepsilon}{2}$$

and so

$$\sum_{j \in \mathcal{S}} \pi_j \mathbb{E}_i[\theta_j] = \sum_{j \in \mathcal{S}} (\mathcal{D}_{jj} - \mathcal{D}_{ij}) = \infty$$

independently of i.

On the other hand, if $\sum_{j \in \mathscr{S}} \mathcal{D}_{jj} = L < \infty$ then, for any $\varepsilon > 0$, there exists K_1 such that $L - \frac{1}{2}\varepsilon \leq \sum_{j \in \mathscr{S}_K} \mathcal{D}_{jj} \leq L + \frac{1}{2}\varepsilon$ for all $K > K_1$. Then, again for fixed i, taking $K > \max(K_0(i), K_1)$,

$$L - \varepsilon \leq \sum_{j \in \mathscr{S}_K} \mathcal{D}_{jj} - \sum_{j \in \mathscr{S}_K} \mathcal{D}_{ij} \leq L + \varepsilon$$

and so

$$\sum_{j \in \mathscr{S}} \pi_j \mathbb{E}_i[\theta_j] = \sum_{j \in \mathscr{S}} \mathcal{D}_{jj} = L.$$

This argument holds for all i, and we see that $\sum_{j \in \mathscr{S}} \pi_j \mathbb{E}_i[\theta_j]$ is independent of i.

Acknowledgements

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