Proceedings of the Royal Society of Edinburgh: Section A Mathematics, 1–27, *2025*
DOI:10.1017/prm.2024.134

Well-posedness for strongly damped abstract Cauchy problems of fractional order

João Aquino
Departamento de Matemática, Instituto de Biociências, Letras e Ciências Exatas, Universidade Estadual Paulista "Júlio de Mesquita Filho" - UNESP, CEP 15054-000 São José do Rio Preto, SP, Brazil (joao.aquino@unesp.br)

Carlos Lizama
Universidad de Santiago de Chile, Facultad de Ciencias, Departamento de Matemática y Ciencia de la Computación, Las Sophoras 173, Santiago, 9170022 Chile (carlos.lizama@usach.cl) (corresponding author)

Andréa Prokopczyck
Departamento de Matemática, Instituto de Biociências, Letras e Ciências Exatas, Universidade Estadual Paulista "Júlio de Mesquita Filho" - UNESP, CEP 15054-000 São José do Rio Preto, SP, Brazil (andrea.prokopczyck@unesp.br)

(Received 11 April 2024; revised 25 November 2024; accepted 2 December 2024)

Let X be a complex Banach space and B be a closed linear operator with domain $\mathcal{D}(B) \subset X$, $a, b, c, d \in \mathbb{R}$, and $0 < \beta < \alpha$. We prove that the problem

$$u(t) - (aB + bI)(g_{\alpha-\beta} * u)(t) - (cB + dI)(g_\alpha * u)(t) = h(t), \quad t \geq 0,$$

where $g_\alpha(t) = t^{\alpha-1}/\Gamma(\alpha)$ and $h : \mathbb{R}_+ \to X$ is given, has a unique solution for any initial condition on $\mathcal{D}(B) \times X$ as long as the operator B generates an ad-hoc Laplace transformable and strongly continuous solution family $\{R_{\alpha,\beta}(t)\}_{t\geq 0} \subset \mathcal{L}(X)$. It is shown that such a solution family exists whenever the pair (α, β) belongs to a subset of the set $(1, 2] \times (0, 1]$ and B is the generator of a cosine family or a C_0-semigroup in X. In any case, it also depends on certain compatibility conditions on the real parameters a, b, c, d that must be satisfied.

Keywords: C_0-semigroup; cosine family; mild solution; solution family; well-posedness

2020 *Mathematics subject classification:* Primary: 47D09
Secondary: 35R11; 47D06

1. Introduction

This article deals with the study of well-posedness and existence of solutions for the following inhomogeneous strongly damped abstract Cauchy problem of fractional order

$$\begin{cases} \tau D_t^{\alpha} u(t) - (aB + bI) D_t^{\beta} u(t) - (cB + dI) u(t) = f(t), \\ u(0) = x \in X, \\ u'(0) = y \in X, \end{cases} \quad (1.1)$$

with $\tau \neq 0$, $a, b, c, d \in \mathbb{R}$ and $0 < \beta \leqslant 1 < \alpha \leqslant 2$, where B is a closed linear operator defined in a complex Banach space X and D_t^{γ} denotes the Caputo fractional derivative of order $\gamma > 0$.

In the integer case $\alpha = 2$ and $\beta = 1$, the well-posedness for the homogeneous Cauchy problem (1.1), that is the existence of a strongly continuous family of bounded and linear operators $\{W(t)\}_{t \geq 0}$ that solves (1.1) in case $f \equiv 0$, was settled by Neubrander [28, theorem 11]. In such case, it was proved in [28, corollary 18] that for $x, y \in \mathcal{D}(B)$ the unique strict solution of (1.1) is given by

$$u(t) = W'(t)x + W(t)[y - \frac{1}{\tau}(aB + bI)x] + \frac{1}{\tau}\int_0^t W(t-s)f(s)ds \quad (1.2)$$

where $W(t)$ is formally the inverse Laplace transform of

$$\lambda \to \frac{\tau}{a\lambda + c}\left(\frac{\tau\lambda^2 - b\lambda - d}{a\lambda + c} - B\right)^{-1}.$$

Several works in the literature have dealt with the investigation of Eq. (1.1) in the integer case $(\alpha, \beta) = (2, 1)$. It corresponds to a second order Cauchy problem and is named, in applications, as the strongly damped linear Klein–Gordon equation, or strongly damped wave equation, among others. See, e.g., [5–7, 12, 28, 29, 32]. For example, for $\tau = 1$, $d = 0$, it is the linear part of the perturbed sine-equation [10] and for $\tau = 1$, $b = d = 0$ it corresponds to the linear part of the viscous Cahn–Hilliard equation [10] or the Kuznetsov equation [22].

In the fractional case, Kirane and Tatar [20] consider a semilinear model that include (1.1) for $\alpha = 2, a = d = 0, c = 1, b = -1$, and $0 < \beta < 2$. Agarwal et al. [2] and more recently Zhou and He [36] studied Eq. (1.1) for $b = d = 0$ and $0 < \beta < 1 < \alpha < 2$. An interesting review can be found for instance in [33]. Equation (1.1) with $\alpha = 2\beta$ is known as the time-fractional telegraph equation. The special case $\beta = 1/2$ can be interpreted as the heat equation subject to a damping effect represented by the 1/2-order time-derivative. In all of these cases, $B = \Delta$ the Laplacian operator.

In the abstract case, Eq. (1.1) with $(\alpha, \beta) = (2, 1)$ has been studied by Ikehata, Todorova, and Yordanov [14]. These authors consider (1.1) with $b = d = 0$, $f \equiv 0$ and B being a nonnegative self-adjoint operator defined in a Hilbert space with a dense domain. They prove existence and uniqueness of mild solutions based on semigroup theory [14, proposition 2.1]. Several authors have proposed generalizations to models more general than (1.1) in the integer case, see, e.g., [27, 28, 34]. Typically, these authors consider the model

$$u''(t) + Bu'(t) + Au(t) = f(t),$$

where A and B are closed linear operators defined in a complex Banach space. However, due to its generality, this approach lacks the possibility of distinguishing its dynamics by means of an eventual combination of the physical parameters of the equation, and it also loses the special features that could be obtained by an explicit description of the solution in terms of a unique strongly continuous family of operators and by means of a kind of variation of parameter formula like (1.2), which is very useful for exploring associated semi linear problems. The same happens in the fractional model proposed in this article, and of which the authors are unaware of previous studies.

Let $\alpha, \beta > 0$ be given and we define

$$W_{\alpha,\beta}(t) := \int_0^t R_{\alpha,\beta}(s)ds, \tag{1.3}$$

where $R_{\alpha,\beta}(t)$ is formally the Laplace transform of

$$\lambda \to \frac{\tau\lambda^{\alpha-1}}{a\lambda^\beta + c}\left(\frac{\tau\lambda^\alpha - b\lambda^\beta - d}{a\lambda^\beta + c} - B\right)^{-1}.$$

Consider $g_\alpha(t) := t^{\alpha-1}/\Gamma(\alpha)$, $t > 0$, which is the Gelfand–Shilov function. Using Laplace transform methods, we can prove that $\{R_{\alpha,\beta}(t)\}_{t\geq 0}$ is a solution family of (1.1), in the sense that a formal solution of (1.1) must have the form

$$\begin{aligned} u(t) = W'_{\alpha,\beta}(t)x + \int_0^t R_{\alpha,\beta}(t-s)[y - \frac{1}{\tau}g_{\alpha-\beta}(s)(aB+bI)x]ds \\ + \frac{1}{\tau}\int_0^t R_{\alpha,\beta}(t-s)(g_{\alpha-1} * f)(s)]ds \end{aligned} \tag{1.4}$$

which exactly coincides with (1.2) in case $\alpha = 2$ and $\beta = 1$. However, because of the new terms $g_{\alpha-1}(t)$ and $g_{\alpha-\beta}(t)$ appearing in (1.4), we cannot expect differentiability of $u(t)$ in general, and therefore (1.4) fails to be a strict solution for (1.1). The following natural question arises:

(Q) Is (1.4) the solution of (1.1) in some extent?

In this article, we are able to solve this problem proving that, for any $x \in \mathcal{D}(B)$ and $y \in X$, (1.4) is a solution of the following integrated version of (1.1)

$$\begin{aligned} \tau u(t) - (aB+bI)(g_{\alpha-\beta} * u)(t) - (cB+dI)(g_\alpha * u)(t) \\ = (g_\alpha * f)(t) + \tau(x+ty) - g_{\alpha-\beta+1}(t)(aB+bI)x, \end{aligned} \tag{1.5}$$

under appropriate conditions.

We note that solutions for integrated versions of (1.1) in the integer case $\alpha = 2$, $\beta = 1$ were previously considered by Neubrander [28, proposition 19]. The study of this kind of solutions is already known because, in applications, it is useful to find a notion of a weaker solution where x, y can be less regular than when considering strict solutions. See also [33, Section 2.2 and Section 2.3] for an explicit fundamental formula of integrated solutions of (1.1) with $a = d = 0$ and $B = \Delta$, the Laplacian operator.

In this article, assuming that B is the generator of a solution family $\{R_{\alpha,\beta}(t)\}_{t\geq 0}$, we show that existence of a unique solution for the integrated problem (1.5) can be assured in the sector $0 < \beta < \alpha$, $1 < \alpha \leq 2$, see theorem 5.4.

In some special subsets of the sector described above, we are able to show that the existence of solutions for the integrated problem can be guaranteed, under the stronger hypothesis that B is the generator of a C_0-semigroup or a strongly continuous cosine family. However, some restrictions are needed. More precisely, we first consider the set

$$\Omega_{\alpha,\beta} := \{(\alpha, \beta) : 0 < \beta \leq 1 < \alpha < 2\}, \tag{1.6}$$

and we show that, for $c = 0$ in (1.5), there exists a solution family, and hence we have well-posedness, if B is the generator of a cosine family. See theorem 3.7 (b). Then, we decompose $\Omega_{\alpha,\beta}$ in two complementary subsets:

$$\Omega^1_{\alpha,\beta} := \{(\alpha, \beta) : 0 < \beta \leq 1 < \alpha < 1 + \beta \leq 2\},$$
$$\Omega^2_{\alpha,\beta} := \{(\alpha, \beta) : 0 < \beta \leq 1 < 1 + \beta < \alpha \leq 2\}.$$

For $\Omega^1_{\alpha,\beta}$ we need $c = 0, a \neq 0,\ \tau \neq 0\,, b, d \in \mathbb{R}$, and B generator of a C_0-semigroup for well-posedness, while for $\Omega^2_{\alpha,\beta}$ we need $b = d = 0$, $a, c \geq 0$, $\tau > 0$ and B generator of a cosine family. See theorem 3.7 (a) and theorem 3.9, respectively.

To achieve our results, we will use subordination methods and Laplace transform theory, together with a strong application of a criterion due to Prüss [31, theorem 4.3, p. 104] for the existence of resolvent families under the assumption B is the generator of a cosine family, plus certain specific conditions in the associated kernel.

This article is organized as follows: In §2, we give the necessary preliminaries useful for following the main text of the article. Section 3 is dedicated to defining the notion of solution family that we will use and its relationship with the best known theory of (a, k)-regularized families, proving that solutions families are a particular case of the latter when the parameters satisfy $b \geq 0$ and $\tau > 0$. In §4, we define our notions of mild and well-posed solution for the homogeneous problem, proving that under the hypothesis of existence of a solution family, well-posedness can be guaranteed. Then, the existence and uniqueness of mild solutions for the inhomogeneous problem can be proved. In §5, we prove our main results in this article. First, we observe that in the entire case, that is, $(\alpha, \beta) = (2, 1)$, well-posedness follows under the subordination hypothesis that A is the generator of a C_0-semigroup. Then, we show that under certain constraints on the pair (α, β) we can distinguish two situations: $1 < \alpha < 1+\beta \leq 2, 0 < \beta \leq 1$ and $1+\beta < \alpha \leq 2, 0 < \beta \leq 1$. In the first case, if $c = 0$, the leading term in the equation (1.1) is $D^\alpha_t u$ and the solution family is subordinate to B being the generator of a C_0-semigroup. In the second case, if $b = d = 0$, the leading term is $D^\beta_t u$ and the solution family is subordinated to B being a generator of a cosine family. These conclusions provide new insights into how fractional terms influence the class of abstract evolution equation (1.1). Finally, some examples to illustrate our abstract results are given.

2. Preliminaries

In this section, we provide some of the fundamental concepts that we will need.

DEFINITION 2.1. ([3], definition 1, p. 5) *Let X be a complex Banach space. Let $f:\mathbb{R}_+ \longrightarrow X$ be an integrable function (as a Bochner integral) and let $T:\mathbb{R}_+ \longrightarrow \mathcal{L}(X,Y)$ be strongly continuous. Then the convolution of T and f is defined by*

$$(T * f)(t) = \int_0^t T(t-s)f(s)\,ds,\ t \in \mathbb{R}_+.$$

THEOREM 2.2 (Titchmarsh's Theorem). ([35], theorem VIII, p. 286), ([4], corollary 2.8.4) *Let $k \in L^1[0,T]$ with $0 \in supp(k)$ and $f \in L^1([0,T],X)$. If*

$$(k * f)(t) = \int_0^t k(t-s)f(s)ds = 0$$

on $[0,T]$, then $f \equiv 0$.

Let $u \in C(\mathbb{R}_+;X)$ and $v \in C^1(\mathbb{R}_+,X)$. Then for every $t \geqslant 0$,

$$\frac{d}{dt}[(u * v)(t)] = u(t)v(0) + (u * v')(t). \tag{2.1}$$

The Laplace transform of a function $f \in L^1(\mathbb{R}_+,X)$ is defined by

$$\mathcal{L}(f)(\lambda) = \widehat{f}(\lambda) := \int_0^\infty e^{-\lambda t}f(t)dt = \lim_{T\to\infty}\int_0^T e^{-\lambda t}f(t)dt,\ \mathrm{Re}(\lambda) > w.$$

Let $\alpha > 0$, $m = \lceil \alpha \rceil$ and $u:[0,\infty) \longrightarrow C^m(\mathbb{R}_+;X)$ be a function. The Caputo fractional derivative of u of order α is defined by

$$D_t^\alpha u(t) := \int_0^t g_{m-\alpha}(t-s)u^{(m)}(s)ds,\ t > 0$$

where

$$g_\beta(t) := \frac{t^{\beta-1}}{\Gamma(\beta)},\ t > 0,\ \beta > 0,$$

is the Gelfand–Shilov function. If $\alpha = 0$, we denote $D_t^0 u(t) := u(t)$. In particular,

$$\widehat{D_t^\alpha f}(\lambda) = \lambda^\alpha \widehat{f}(\lambda) - \sum_{k=0}^{m-1} f^{(k)}(0)\lambda^{\alpha-1-k}. \tag{2.2}$$

The Riemann–Liouville fractional integral of order $\alpha > 0$ is defined as follows

$$J_t^\alpha f(t) := (g_\alpha * f)(t),\ f \in L^1_{loc}(\mathbb{R}_+),\ t > 0.$$

Moreover, the Riemann–Liouville fractional derivative of order $\alpha > 0$ is given by

$$\mathbf{D}_t^\alpha f(t) := D_t^m (g_{m-\alpha} * f)(t) = D_t^m J_t^{m-\alpha} f(t).$$

The two-parametric Mittag–Leffler function [13, Section 4.1, p. 64] is defined as follows

$$E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \ \mathrm{Re}(\alpha) > 0, \ \beta \in \mathbb{C}, \ z \in \mathbb{C}.$$

Some interesting properties of these functions can be found in the book by Bateman and Erdelyi [8, Section 18.1, p. 206].

The following Laplace transform that involves the Mittag–Leffler function is obtained in [30, Section 1.2.2, p. 21]:

$$\int_0^\infty e^{-pt} t^{\alpha k + \beta - 1} E_{\alpha,\beta}^{(k)}(\pm a t^\alpha) dt = \frac{k! p^{\alpha - \beta}}{(p^\alpha \mp a)^{k+1}}, \quad \alpha > 0, \ \beta > 0, \ \mathrm{Re}(p) > |a|^{\frac{1}{\alpha}}. \tag{2.3}$$

We recall the following definition that corresponds to a slight modification of [25, definition 2.1].

Definition 2.3. *Let $k \in C(\mathbb{R}_+)$, $k \neq 0$ and let $a \in L^1_{loc}(\mathbb{R}_+)$, $a \neq 0$. Let A be a linear operator with domain $\mathcal{D}(A) \subset X$. A strongly continuous family $\{R(t)\}_{t \geqslant 0} \subset \mathcal{L}(X)$ is called an (a,k)-regularized resolvent family on X (or simply (a,k)-regularized family) having A as a generator if the following properties hold.*

(i) $\lim_{t \to 0} \frac{R(t)}{k(t)} = I$ *if* $k(0) \in \overline{\mathbb{C}} \backslash \{0\}$ *and* $R(0) = 0$ *if* $k(0) = 0$;

(ii) $R(t)x \in \mathcal{D}(A)$ *and* $R(t)Ax = AR(t)x$, *for all* $x \in \mathcal{D}(A)$ *and* $t \geqslant 0$;

(iii) $R(t)x = k(t)x + \int_0^t a(t-s)AR(s)x\, ds$, $t \geqslant 0$, $x \in \mathcal{D}(A)$.

Remark 2.4. Note that in case $k(t) = g_\beta(t)$, the above definition coincides with [23, definition 1.9]. And, in case $k(t) \equiv 1$, definition 2.3 coincides with the definition of a resolvent family [31, definition 1.3, p. 32].

Remark 2.5. If $\{R(t)\}_{t \geq 0}$ is an (a,k)-regularized family having A as a generator, then by [24, lemma 2.2, p. 281], if $x \in X$ then $\int_0^t a(t-s)R(s)xds \in \mathcal{D}(A)$ and $R(t)x = k(t)x + A\int_0^t a(t-s)R(s)xds$.

Remark 2.6. Let A be a closed linear operator and let $\{R(t)\}_{t \geq 0}$ be an exponentially bounded and strongly continuous operator family in $\mathcal{L}(X)$ such that the Laplace transform $\widehat{R}(\lambda)$ exists for $\lambda > \omega$. It was proved in [25, p. 3] that $R(t)$ is an (a,k)-regularized family with generator A if and only if for every $\lambda > \omega$,

https://doi.org/10.1017/prm.2024.134 Published online by Cambridge University Press

$(I - \widehat{a}(\lambda)A)^{-1}$ exists in $\mathcal{L}(X)$ and

$$\frac{\widehat{k}(\lambda)}{\widehat{a}(\lambda)}\left(\frac{1}{\widehat{a}(\lambda)} - A\right)^{-1} = \int_0^\infty e^{-\lambda s} R(s)x\, ds.$$

In the particular case of definition 2.3 with $a(t) = g_\alpha(t)$ and $k(t) = g_\beta(t)$, we obtain the following equivalent definition.

DEFINITION 2.7. ([23], definition 1.9, p. 4) *Let X be a Banach space and $\alpha > 0$, $\beta > 0$. A one parameter family $\{S_{\alpha,\beta}(t)\}_{t\geq 0} \subset \mathcal{L}(X)$ is called an (α,β)-resolvent family if the following conditions are satisfied:*

(a) $\lim_{t\to 0} t^{1-\beta} S_{\alpha,\beta}(t) = \frac{1}{\Gamma(\beta)} I$ *if* $0 < \beta < 1$, $S_{\alpha,1}(0) = I$ *and* $S_{\alpha,\beta}(0) = 0$ *if* $\beta > 1$*;*
(b) $S_{\alpha,\beta}(s)S_{\alpha,\beta}(t) = S_{\alpha,\beta}(t)S_{\alpha,\beta}(s)$*, for all* $s, t > 0$*;*
(c) *The functional equation*

$$S_{\alpha,\beta}(s)J_t^\alpha S_{\alpha,\beta}(t) - J_s^\alpha S_{\alpha,\beta}(s)S_{\alpha,\beta}(t) = g_\beta(s)J_t^\alpha S_{\alpha,\beta}(t) - g_\beta(t)J_s^\alpha S_{\alpha,\beta}(s),$$

holds for all $t, s > 0$.

We recall [25, p. 3] that a family $\{R(t)\}_{t\geq 0} \subset \mathcal{L}(X)$ is said to be exponentially bounded or of type (M,ω) if there exist constants $M > 0$ and $\omega \in \mathbb{R}$ such that

$$\|R(t)\| \leq M e^{\omega t} \quad \text{for all } t \geq 0.$$

We recall the following subordination result.

THEOREM 2.8. [23], theorem 2.5 *Let $0 < \eta_1 \leq 2$, $0 < \eta_2$. If A generates an exponentially bounded (η_1, η_2)-resolvent family $\{S_{\eta_1,\eta_2}(t)\}_{t>0}$, then for each $\beta' \geq 0$ and $0 < \alpha' < 1$ we have that A generates an exponentially bounded $(\alpha'\eta_1, \alpha'\eta_2 + \beta')$-resolvent family given by*

$$S_{\alpha'\eta_1,\alpha'\eta_2+\beta'}(t) = \int_0^\infty \psi_{\alpha',\beta'}(t,s) S_{\eta_1,\eta_2}(s) ds, \;\; t > 0,$$

where

$$\psi_{\alpha',\beta'}(t,s) := t^{\beta'-1} \sum_{n=0}^\infty \frac{(-st^{-\alpha'})^n}{n!\Gamma(-\alpha' n + \beta')}$$

is called the scaled Wright function [1].

We recall the following definition.

DEFINITION 2.9. ([31], definition 4.4, p. 94) *A function $a : (0,\infty) \to \mathbb{R}$ is called a creep function if $a(t)$ is nonnegative, nondecreasing, and concave. A creep function $a(t)$ has the standard form*

https://doi.org/10.1017/prm.2024.134 Published online by Cambridge University Press

$$a(t) = a_0 + a_\infty t + \int_0^t a_1(\tau)d\tau,\ t > 0, \tag{2.4}$$

where $a_0 = a(0+) \geq 0$, $a_\infty = \lim_{t\to\infty} \frac{a(t)}{t} = \inf_{t>0} \frac{a(t)}{t} \geq 0$ *and* $a_1(t) = a'(t) - a_\infty$ *is nonnegative, nonincreasing,* $\lim_{t\to\infty} a_1(t) = 0$.

We introduce the following definition.

Definition 2.10. *Let* $\beta > 0,\ \alpha \geq 1,\ a, b, c, d, \tau \in \mathbb{R},\ \tau \neq 0,$ *where* $(a, c) \neq (0, 0)$. *A closed linear operator* B *is said to generate a solution family* $\{R_{\alpha,\beta}(t)\}_{t\geqslant 0} \subseteq \mathcal{L}(X)$ *if* $R_{\alpha,\beta}(t)$ *is strongly continuous of type* (M, ω), *the set* $\left\{\mu \in \mathbb{C} : \mu = \frac{\tau\lambda^\alpha - b\lambda^\beta - d}{a\lambda^\beta + c}\right\} \subset \rho(B)$ *and*

$$\widehat{R_{\alpha,\beta}}(\lambda) = \frac{\tau\lambda^{\alpha-1}}{a\lambda^\beta + c}\left(\frac{\tau\lambda^\alpha - b\lambda^\beta - d}{a\lambda^\beta + c} - B\right)^{-1}, \quad Re(\lambda) > \omega.$$

Some examples of solution families are given in the following.

Example 2.11.

(a) If $\beta = 1$, $a = b = d = 0$, and $c = \tau = 1$, then for each $\alpha \geq 1$, $\{R_{\alpha,1}(t)\}_{t\geq 0}$ is an $(\alpha, 1)$-resolvent family for the equation $D_t^\alpha u(t) = Bu(t)$ because in this case

$$\widehat{R_{\alpha,1}}(\lambda) = \lambda^{\alpha-1}(\lambda^\alpha - B)^{-1}, \quad Re(\lambda) > \omega,$$

see [9, p. 215]. If $\alpha = 2$, $\{R_{2,1}(t)\}_{t\geq 0}$ is a strongly continuous cosine family, see [4, proposition 3.14.4]. In the border case $\alpha = 1$, $\{R_{1,1}(t)\}_{t\geq 0}$ is a C_0-semigroup, see [4, theorem 3.1.7].

(b) If $\alpha \geq 1,\ 0 < \beta \leq 1,\ a = d = 0, \tau = c = 1$, and $b \leq 0$, then $\{R_{\alpha,\beta}(t)\}_{t\geq 0}$ is a $(\alpha - 1, \beta)_{-b}$-regularized family as defined in [18, definition 2.4].

(c) If $\alpha = 1$, $0 < \beta < 1$, $a = \tau = 1$, $b = c = 0$, and $d = -1$, then $\{R_{\alpha,\beta}(t)\}_{t\geq 0}$ is a (a, k) regularized resolvent family with $a(t) = t^{-\beta}E_{1,1-\beta}(t)$ and $k(t) = e^{-t}$. In this case

$$\widehat{R_{1,\beta}}(\lambda) = \frac{1}{\lambda^\beta}\left(\frac{\lambda + 1}{\lambda^\beta} - B\right)^{-1}, \quad Re(\lambda) > \omega,$$

see [26, p. 138].

The following result is contained in [31, theorem 4.3, p. 104].

Theorem 2.12. *Let* A *generate a cosine family and assume that* $a(t)$ *is a creep function with* $a_1(t)$ *log-convex. Then* A *generates an* $(a, 1)$*-resolvent family.*

We finish this section with an interesting property of (a, k)-regularized families that will be repeatedly used.

LEMMA 2.13. *Let $\{R(t)\}_{t\geq 0}$ be an (a,k)-regularized family having A as a generator and satisfying with $(1*a)(t)\neq 0$, for all $t>0$. If $b\in L^1_{loc}(\mathbb{R}_+)$, then $(b*R)(t)x\in \mathcal{D}(A)$ for all $x\in X,\ t\geq 0$.*

Proof. Since $R(t)$ is an (a,k)-regularized family by [24, remark 2.4 (4)], $(b*R)(t)$ is an $(a,b*k)$-regularized family. Define $S(t):=(b*R)(t)$. By [24, remark 2.2], $(a*S)(t)x\in\mathcal{D}(A)$ for every $x\in X$. Let $\lambda\in\rho(A)$ and $x\in X$ be fixed. Define $y=(\lambda-A)^{-1}x\in\mathcal{D}(A)$ and let $z:=(b*R)(t)x$. We have

$$z=(b*R)(t)(\lambda-A)y=\lambda(b*R)(t)y-(b*R)(t)Ay.$$

Then, convolving with the kernel $a(t)$, we obtain

$$(1*a)(t)z=\lambda(a*b*R)(t)y-(a*b*R)(t)Ay=\lambda(a*S)(t)y \\ -(a*S)(t)Ay\in\mathcal{D}(A).$$

□

3. Sufficient conditions for existence of solution families

In this section, we want to give conditions on a closed operator B so that it is a generator of a solution family.

PROPOSITION 3.1. *Let $0<\beta<\alpha<2+\beta$ and B generator of a cosine family $\{C(t)\}_{t>0}$. Then B generates an exponentially bounded $(\alpha-\beta,\alpha)$-resolvent family.*

Proof. Recall that B being a generator of a cosine family $\{C(t)\}_{t>0}$ is equivalent to saying that B is generator of an exponentially bounded $(2,1)$-resolvent family. By theorem 2.8, choosing $\eta_1=2,\ \eta_2=1$ and $\alpha'=\dfrac{\alpha-\beta}{2},\ \beta'=\dfrac{\alpha+\beta}{2}$, and using the hypothesis, we obtain $0<\dfrac{\alpha-\beta}{2}<1$. We conclude that B is the generator of an exponentially bounded $(\alpha'\eta_1,\alpha'\eta_2+\beta')=(\alpha-\beta,\alpha)$-resolvent family $\{S_{\alpha-\beta,\alpha}(t)\}_{t>0}$. Furthermore,

$$S_{\alpha-\beta,\alpha}(t)x=\int_0^\infty \psi_{\frac{\alpha-\beta}{2},\frac{\alpha+\beta}{2}}(t,s)C(s)x\,ds.$$

□

Our next result assumes B as generator of a C_0-semigroup instead a cosine family.

PROPOSITION 3.2. *Let $0<\beta<\alpha<1+\beta$ and B generator of a C_0-semigroup $\{T(t)\}_{t>0}$. Then B generates an exponentially bounded $(\alpha-\beta,\alpha)$-resolvent family.*

Proof. Since B generates a C_0-semigoup $\{T(t)\}_{t>0}$, it is an exponentially bounded $(1,1)$-resolvent family $\{S_{1,1}(t)\}_{t>0}$. Define $\eta_1=\eta_2=1$ and $\alpha'=\alpha-\beta,\ \beta'=\beta$. By hypothesis, we obtain $0<\alpha-\beta<1$. Therefore, by theorem 2.8, we obtain that B

generates an exponentially bounded $(\alpha'\eta_1, \alpha'\eta_2 + \beta') = (\alpha - \beta, \alpha)$-resolvent family $\{S_{\alpha-\beta,\alpha}(t)\}_{t>0}$. Furthermore,

$$S_{\alpha-\beta,\alpha}(t)x = \int_0^\infty \psi_{\alpha-\beta,\beta}(t,s)T(s)x ds.$$

□

REMARK 3.3. We recall from [4, theorem 3.14.17, p. 215] that if an operator B generates a cosine family, then it generates a C_0-semigroup.

COROLLARY 3.4. *Let $0 < \beta < \alpha < 1 + \beta$ and B generator of a cosine family. Then B generates an exponentially bounded $(\alpha - \beta, \alpha)$-resolvent family.*

REMARK. By [1, theorem 12], the following equality holds:

$$S_{\alpha-\beta,\alpha}(t)x = (g_\beta * S_{\alpha-\beta,\alpha-\beta})(t)x.$$

The following is an auxiliary result for our main result of this section.

LEMMA 3.6. *Suppose that $a \neq 0,\ \tau \neq 0,\ \dfrac{a}{\tau} > 0,\ c = 0,\ \tau, a, b, d \in \mathbb{R}$, and assume any of the following.*

(i) *$0 < \beta < \alpha < 1 + \beta$ and B generator of a C_0-semigroup;*
(ii) *$0 < \beta < \alpha < 2 + \beta$ and B generator of a cosine family.*

Then the set $\left\{\mu \in \mathbb{C}\ :\ \mu = \dfrac{\tau\lambda^\alpha - b\lambda^\beta - d}{a\lambda^\beta}\right\} \subseteq \rho(B)$ and there exist a strongly continuous family $\{Q(t)\}_{t\geq 0} \subset \mathcal{L}(X)$, exponentially bounded, such that

$$\widehat{Q(\lambda)} = (\tau\lambda^\alpha - (aB + bI)\lambda^\beta - d)^{-1}, \quad Re(\lambda) > \omega.$$

Proof. Since B generates a C_0-semigroup (resp. cosine family) then $\dfrac{a}{\tau}B + \dfrac{b}{\tau}I$ also generates a C_0-semigroup (resp. cosine family), see [4]. Then, by proposition 3.2 (resp. proposition 3.1), we obtain in any case that $\dfrac{a}{\tau}B + \dfrac{b}{\tau}I$ generates an exponentially bounded $(\alpha - \beta, \alpha)$-resolvent family. Then, we conclude that $\left\{\mu \in \mathbb{C}\ :\ \mu = \dfrac{\tau\lambda^\alpha - b\lambda^\beta}{a\lambda^\beta}\right\} \subseteq \rho(B)$. Furthermore, since $\{S_{\alpha-\beta,\alpha}(t)\}_{t\geq 0}$ is an exponentially bounded family, say $\|S_{\alpha-\beta,\alpha}(t)\| \leq Me^{\omega t},\ \omega \in \mathbb{R}_+,\ t \geq 0$, then using induction we easily obtain

$$\|S^{*(k+1)}_{\alpha-\beta,\alpha}(t)\| \leq M^{k+1}e^{\omega t}\frac{t^k}{k!},\ k \in \mathbb{N}.$$

Now, for each $t \geq 0$ and $x \in X$, we define

$$Q(t)x := \sum_{k=0}^\infty d^k S^{*(k+1)}_{\alpha-\beta,\alpha}(t)x.$$

It is clear that $Q(t)$ is strongly continuous. Moreover,

$$\|Q(t)\| = \left\| \sum_{k=0}^{\infty} d^k S^{*(k+1)}_{\alpha-\beta,\alpha}(t) \right\| \le \sum_{k=0}^{\infty} |d^k| \, \|S^{*(k+1)}_{\alpha-\beta,\alpha}(t)\| \le \sum_{k=0}^{\infty} |d^k| M^{k+1} e^{\omega t} \frac{t^k}{k!}$$
$$= e^{\omega t} \sum_{k=0}^{\infty} |d^k| M \, M^k \frac{t^k}{k!} = M e^{\omega t} \sum_{k=0}^{\infty} \frac{(|d|Mt)^k}{k!} = M e^{\omega t} e^{|d|Mt} = M e^{(\omega+|d|M)t}.$$

It proves that $Q(t)$ is well-defined and exponentially bounded. Taking the Laplace transform of $Q(t)$, we obtain by continuity of the Laplace transform

$$\widehat{Q}(\lambda) = \mathcal{L}\left(\sum_{k=0}^{\infty} d^k S^{*(k+1)}_{\alpha-\beta,\alpha}(t) \right)(\lambda) = \sum_{k=0}^{\infty} d^k \mathcal{L}\big(S^{*(k+1)}_{\alpha-\beta,\alpha}(t)\big)(\lambda)$$
$$= \sum_{k=0}^{\infty} d^k \mathcal{L}\big(S^{*(k)}_{\alpha-\beta,\alpha}(t)\big)(\lambda) \widehat{S_{\alpha-\beta,\alpha}}(\lambda) = \widehat{S_{\alpha-\beta,\alpha}}(\lambda) \sum_{k=0}^{\infty} \big[d\widehat{S_{\alpha-\beta,\alpha}}(\lambda)\big]^k.$$

Since $\widehat{S_{\alpha-\beta,\alpha}}(\lambda) = \dfrac{\lambda^{-\beta}}{\tau}(\lambda^{\alpha-\beta} - (\frac{a}{\tau}B + \frac{b}{\tau}I))^{-1} = \lambda^{-\beta}(\tau\lambda^{\alpha-\beta} - (aB + bI))^{-1}$, and $\alpha > \beta$, we obtain, for $\mathrm{Re}(\lambda)$ sufficiently large, that

$$\widehat{S_{\alpha-\beta,\alpha}}(\lambda) \sum_{k=0}^{\infty} \big[d\widehat{S_{\alpha-\beta,\alpha}}(\lambda)\big]^k = \lambda^{-\beta}(\tau\lambda^{\alpha-\beta} - (aB + bI))^{-1}$$
$$\times \sum_{k=0}^{\infty} \big[d\lambda^{-\beta}(\tau\lambda^{\alpha-\beta} - (aB + bI))^{-1}\big]^k. \tag{3.1}$$

Since $\|d\lambda^{-\beta}(\tau\lambda^{\alpha-\beta} - (aB + bI))^{-1}\| < 1$ for λ large enough we obtain, using the Neumann series, that $\left\{\mu; \mu = \dfrac{\tau\lambda^{\alpha} - b\lambda^{\beta} - d}{a\lambda^{\beta}}\right\} \subseteq \rho(B)$. Moreover, (3.1) equals to

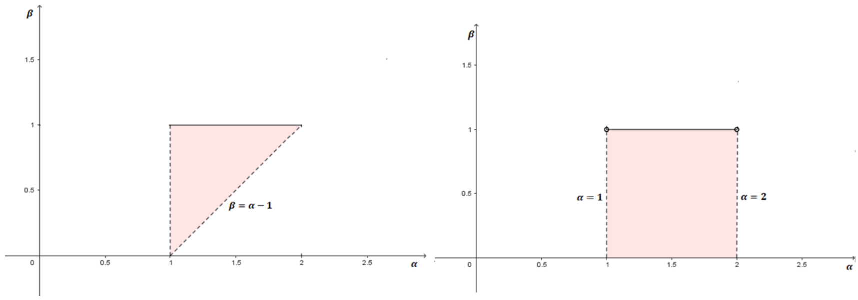
$$\lambda^{-\beta}(\tau\lambda^{\alpha-\beta} - (aB + bI))^{-1}\big[1 - d\lambda^{-\beta}\big(\tau\lambda^{\alpha-\beta} - (aB + bI)\big)^{-1}\big]^{-1}$$
$$= \frac{1}{\lambda^{\beta}}(\tau\lambda^{\alpha-\beta} - (aB + bI) - d\lambda^{\beta})^{-1} = (\tau\lambda^{\alpha} - (aB + bI)\lambda^{\beta} - d)^{-1},$$

proving the theorem. □

The following main result give our first practical condition for generation of a solution family. It focuses on two sectors illustrated below:

THEOREM 3.7. *Let $a, b, c, d, \tau \in \mathbb{R}$ where $a \neq 0,\ \tau \neq 0,\ c = 0$, and assume any of the following.*
(a) $0 < \beta \le 1 < \alpha < 1 + \beta \le 2$ *and B generator of a C_0-semigroup;*
(b) $0 < \beta \le 1 < \alpha < 2$ *and B generator of a cosine family.*
Then B generates a solution family $\{R_{\alpha,\beta}(t)\}_{t\ge 0}$.

https://doi.org/10.1017/prm.2024.134 Published online by Cambridge University Press



(a) $0 < \beta \le 1 < \alpha < 1 + \beta \le 2$. (b) $0 < \beta \le 1 < \alpha < 2$.

Figure 1. Sectors for (a) and (b) as given in Theorem 3.7.

Proof. By lemma 3.6, there exists a strongly continuous family $Q(t)$ satisfying $\widehat{Q(\lambda)} = (\tau\lambda^{\alpha} - (aB + bI)\lambda^{\beta} - d)^{-1}$, $Re(\lambda) > \omega$. Note that

$$\widehat{Q}(\lambda) = (\tau\lambda^{\alpha} - a\lambda^{\beta}B - b\lambda^{\beta} - d)^{-1} = \frac{1}{a\lambda^{\beta}}\left(\frac{\tau\lambda^{\alpha} - b\lambda^{\beta} - d}{a\lambda^{\beta}} - B\right)^{-1}.$$

Since $1 < \alpha < 2$ we can define $R_{\alpha,\beta}(t) := \frac{d}{dt}(g_{2-\alpha} * Q)(t)$, which corresponds to the fractional derivative of order $\alpha - 1$ in the sense of Riemann Liouville for $Q(t)$. We obtain that

$$\widehat{R_{\alpha,\beta}}(\lambda) = \lambda\frac{1}{\lambda^{2-\alpha}}\widehat{Q}(\lambda) = \frac{\tau\lambda^{\alpha-1}}{a\lambda^{\beta}}\left(\frac{\tau\lambda^{\alpha} - b\lambda^{\beta} - d}{a\lambda^{\beta}} - B\right)^{-1}.$$

Finally, definition 2.10 shows that $R_{\alpha,\beta}(t)$ is a solution family generated by B. □

We will repeatedly use the following remark.

Remark 3.8. The function $g_{\gamma}(t)$ is nonincreasing for $0 < \gamma < 1$ and increasing for $\gamma > 1$. Moreover, $g'_{\gamma}(t) = \dfrac{\gamma - 1}{t}g_{\gamma}(t)$, $t > 0$.

In case $c \neq 0$, we have the next theorem that aims to the sector illustrated below.

Theorem 3.9. *Let $a, b, c, d, \tau \in \mathbb{R}$ where $b = d = 0$, $a, c \geq 0$, $\tau > 0$ and let B be the generator of a cosine family. Suppose that $0 < \beta \le 1 < 1 + \beta < \alpha \le 2$. Then B generates a solution family $\{R_{\alpha,\beta}(t)\}_{t\geq 0}$.*

Proof. First, note that since $b = d = 0$, then if such solution family exists, it must verify

$$\widehat{R_{\alpha,\beta}}(\lambda) = \frac{1}{\lambda\widehat{a}(\lambda)}\left(\frac{1}{\widehat{a}(\lambda)} - B\right)^{-1} = \frac{\tau\lambda^{\alpha-1}}{a\lambda^{\beta} + c}\left(\frac{\tau\lambda^{\alpha}}{a\lambda^{\beta} + c} - B\right)^{-1}.$$

Therefore, $\{R_{\alpha,\beta}(t)\}_{t\geq 0}$ is an $(a, 1)$-regularized family, that is, a resolvent family (see remark 2.4) with $a(t) = \dfrac{a}{\tau}g_{\alpha-\beta}(t) + \dfrac{c}{\tau}g_{\alpha}(t)$. Our strategy for the proof is to use theorem 2.12.

https://doi.org/10.1017/prm.2024.134 Published online by Cambridge University Press

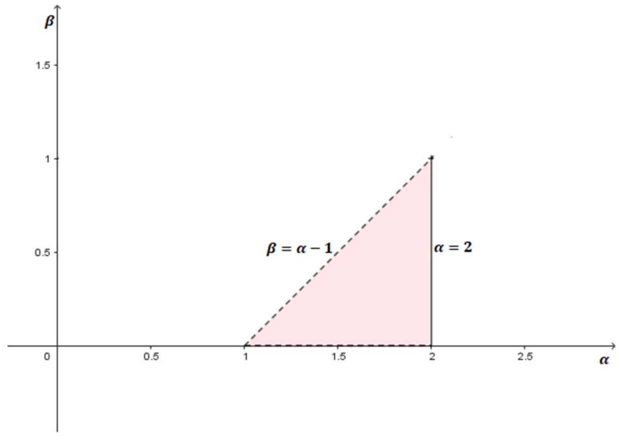


Figure 2. Sector $0 < \beta \le 1 < 1 + \beta < \alpha \le 2$.

We claim that $a(t)$ is a creep function (see definition 2.9). In fact, let us define $a_1(t) := \frac{a}{\tau} g_{\alpha-\beta-1}(t) + \frac{c}{\tau} g_{\alpha-1}(t)$. Note that we can rewrite $a_1(t)$ as

$$a(t) = \frac{a}{\tau} g_{\alpha-\beta}(t) + \frac{c}{\tau} g_{\alpha}(t) = a_0 + a_\infty t + \int_0^t a_1(s)ds,$$

where $a_0 = 0$, $a_\infty := \lim_{t\to\infty} \frac{a(t)}{t} = \lim_{t\to\infty} \frac{1}{t}\Big[\frac{a}{\tau} g_{\alpha-\beta}(t) + \frac{c}{\tau} g_\alpha(t)\Big] = \lim_{t\to\infty}\Big[\frac{a}{\tau}\frac{t^{\alpha-\beta-2}}{\Gamma(\alpha-\beta)} + \frac{c}{\tau}\frac{t^{\alpha-2}}{\Gamma(\alpha)}\Big] = 0$ and $a_1(t) = a'(t) - a_\infty$. Moreover, since $a, c \ge 0$ and $\tau > 0$ it is clear that $a_1(t)$ is nonnegative, $\lim_{t\to\infty} a_1(t) = 0$ and since by hypothesis $0 < \alpha - \beta - 1 < 1$ and $0 < \alpha - 1 < 1$, by remark 3.8 we obtain that $a_1(t)$ is nonincreasing. By definition 2.9, we conclude that $a(t)$ is a creep function, proving the claim.

Next, we claim that $a_1(t) := \frac{a}{\tau} g_{\alpha-\beta-1}(t) + \frac{c}{\tau} g_{\alpha-1}(t)$ is log-convex. Indeed, consider $f(t) = \log(a_1(t))$ then $f''(t) = \frac{a_1''(t)a_1(t) - a_1'(t)a_1'(t)}{a_1^2(t)}$ where, using remark 3.8, we obtain

$$a_1'(t) = \frac{a}{\tau}(\alpha-\beta-2)t^{-1} g_{\alpha-\beta-1}(t) + \frac{c}{\tau}(\alpha-2)t^{-1} g_{\alpha-1}(t)$$

and

$$a_1''(t) = \frac{a}{\tau}(\alpha-\beta-2)(\alpha-\beta-3)t^{-2} g_{\alpha-\beta-1}(t) + \frac{c}{\tau}(\alpha-2)(\alpha-3)t^{-2} g_{\alpha-1}(t).$$

We will to show that $a_1''(t)a_1(t) \ge a_1'(t)a_1'(t)$. We have that

https://doi.org/10.1017/prm.2024.134 Published online by Cambridge University Press

$$a_1''(t)a_1(t) = \left[\frac{a}{\tau}(\alpha-\beta-2)(\alpha-\beta-3)t^{-2}g_{\alpha-\beta-1}(t) + \frac{c}{\tau}(\alpha-2)(\alpha-3)t^{-2}g_{\alpha-1}(t)\right] \times \times \left[\frac{a}{\tau}g_{\alpha-\beta-1}(t) + \frac{c}{\tau}g_{\alpha-1}(t)\right]$$

$$= \frac{a^2}{\tau^2}(\alpha-\beta-2)(\alpha-\beta-3)t^{-2}g^2_{\alpha-\beta-1}(t) + \frac{ac}{\tau^2}(\alpha-\beta-2)(\alpha-\beta-3)t^{-2} \times g_{\alpha-\beta-1}(t)g_{\alpha-1}(t)$$

$$+ \frac{ac}{\tau^2}(\alpha-2)(\alpha-3)t^{-2}g_{\alpha-1}(t)g_{\alpha-\beta-1}(t) + \frac{c^2}{\tau^2}(\alpha-2)(\alpha-3)t^{-2}g^2_{\alpha-1}(t)$$

$$= \frac{a^2}{\tau^2}(\alpha-\beta-2)(\alpha-\beta-3)t^{-2}g^2_{\alpha-\beta-1}(t) + \frac{ac}{\tau^2}t^{-2}g_{\alpha-\beta-1}(t)g_{\alpha-1}(t)\big[(\alpha-\beta-2)(\alpha-\beta-3) + (\alpha-2)(\alpha-3)\big] + \frac{c^2}{\tau^2}(\alpha-2)(\alpha-3)t^{-2}g^2_{\alpha-1}(t) \quad (3.2)$$

and

$$a_1'(t)a_1'(t) = \left[\frac{a}{\tau}(\alpha-\beta-2)t^{-1}g_{\alpha-\beta-1}(t) + \frac{c}{\tau}(\alpha-2)t^{-1}g_{\alpha-1}(t)\right]^2$$

$$= \frac{a^2}{\tau^2}(\alpha-\beta-2)^2t^{-2}g^2_{\alpha-\beta-1}(t) + \frac{ac}{\tau^2}t^{-2}g_{\alpha-\beta-1}(t)g_{\alpha-1}(t)[2(\alpha-\beta-2)(\alpha-2)] + \frac{c^2}{\tau^2}(\alpha-2)^2t^{-2}g^2_{\alpha-1}(t). \quad (3.3)$$

By hypothesis, we immediately have $(\alpha-\beta-2)(\alpha-\beta-3) > (\alpha-\beta-2)^2$, $(\alpha-2)(\alpha-3) > (\alpha-2)^2$. On the other hand, from the identity

$$(\alpha-\beta-2)(\alpha-\beta-3) + (\alpha-2)(\alpha-3) = 2(\alpha-2)^2 - 2\beta(\alpha-2) - 2(\alpha-2) + \beta^2 + \beta$$

and since by hypothesis $-2(\alpha-2) > 0$, we obtain

$$\begin{aligned} 2(\alpha-2)^2 - 2\beta(\alpha-2) - 2(\alpha-2) + \beta^2 + \beta &\geq 2(\alpha-2)^2 - 2\beta(\alpha-2) \\ &= 2(\alpha-\beta-2)(\alpha-2). \end{aligned}$$

Comparing (3.2) with (3.3), and taking into account that $ac \geq 0$, we obtain that $a_1''(t)a_1(t) \geq a_1'(t)a_1'(t)$ and hence $f''(t) \geq 0$ for all $t > 0$. Therefore, $a_1(t)$ is log-convex, proving the claim and the theorem. □

https://doi.org/10.1017/prm.2024.134 Published online by Cambridge University Press

4. A particular (a, k)-regularized family

The following main theorem shows that under some conditions on the parameters, a solution family $\{R_{\alpha,\beta}(t)\}_{t\geq 0}$ is a particular case of an (a,k)-regularized family, and therefore many properties can be available from the general theory, see, e.g., [21, 24, 25].

THEOREM 4.1. *Let $a, b, c, d, \tau \in \mathbb{R}$ be given with $b \geq 0$, $\tau > 0$, $(a, c) \neq (0, 0)$ and let B be the generator of a solution family $\{R_{\alpha,\beta}(t)\}_{t\geq 0}$. Then $\{R_{\alpha,\beta}(t)\}_{t\geqslant 0}$ is an (a, k)-regularized family with*

(I) $a(t) = aD_t^{\beta} p_{\alpha,\beta}(t) + cp_{\alpha,\beta}(t)$ *and* $k(t) = \tau D_t^{\alpha-1} p_{\alpha,\beta}(t)$, *if* $0 < \beta \leq 1 < \alpha < 2$.

(II) $a(t) = aD_t^{\beta} p_{1,\beta}(t) + \dfrac{a}{\tau} g_{1-\beta}(t) + cp_{1,\beta}(t)$ *and* $k(t) = \tau p_{1,\beta}(t)$, *if* $0 < \beta \leq 1 = \alpha$,

where

$$p_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{d^k}{\tau^{k+1}} \left[t^{\alpha-1} E_{\alpha-\beta,\alpha}\left(\frac{b}{\tau} t^{\alpha-\beta}\right)\right]^{*(k+1)}. \tag{4.1}$$

Proof. From definition 2.10, $\{R_{\alpha,\beta}(t)\}_{t\geqslant 0}$ is an exponentially bounded family of strongly continuous operators. From remark 2.6, it follows that $\{R_{\alpha,\beta}(t)\}_{t\geqslant 0}$ is a (a, k)-regularized family with generator B if we can find $a(t)$ and $k(t)$ Laplace transformable functions such that

$$\widehat{R_{\alpha,\beta}}(\lambda) = \frac{\tau\lambda^{\alpha-1}}{a\lambda^{\beta}+c}\left(\frac{\tau\lambda^{\alpha} - b\lambda^{\beta} - d}{a\lambda^{\beta}+c} - B\right)^{-1} = \frac{\widehat{k}(\lambda)}{\widehat{a}(\lambda)}\left(\frac{1}{\widehat{a}(\lambda)} - B\right)^{-1}, \tag{4.2}$$

for all $\lambda \in \mathbb{C}$, $\mathrm{Re}(\lambda) > \omega$.

For $0 < \beta \leq 1 < \alpha < 2$, we define $p_{\alpha,\beta}(t)$ as in (4.1). We first prove that $p_{\alpha,\beta}(t)$ is well-defined and exponentially bounded. In fact, by [30, theorem 1.6, p. 35], for all $t > 0$ and since $b \geq 0$, $\tau > 0$, we have

$$\left| t^{\alpha-1} E_{\alpha-\beta,\alpha}\left(\frac{b}{\tau} t^{\alpha-\beta}\right)\right| \leq t^{\alpha-1} \frac{C}{1 + \dfrac{b}{\tau} t^{\alpha-\beta}} = t^{\alpha-1} C \frac{\tau}{\tau + bt^{\alpha-\beta}} \leq t^{\alpha-1} C.$$

Note that $t^{\alpha-1} = g_{\alpha}(t)\Gamma(\alpha)$ and since $1 \leq \alpha < 2$, $\Gamma(\alpha) \leq 1$. Then

$$\left| t^{\alpha-1} E_{\alpha-\beta,\alpha}\left(\frac{b}{\tau} t^{\alpha-\beta}\right)\right| \leq g_{\alpha}(t) C.$$

Using induction, we obtain

$$\left| \frac{d^k}{\tau^{k+1}} \left[t^{\alpha-1} E_{\alpha-\beta}\left(\frac{b}{\tau} t^{\alpha-\beta}\right)\right]^{*(k+1)} \right| \leq \frac{|d|^k C^{k+1}}{\tau^{k+1}} g_{(k+1)\alpha}(t), \text{ for all } k \in \mathbb{N}_0.$$

Hence,

$$
\begin{aligned}
|p_{\alpha,\beta}(t)| &= \left| \sum_{k=0}^{\infty} \frac{d^k}{\tau^{k+1}} \left[t^{\alpha-1} E_{\alpha-\beta,\alpha}\Big(\frac{b}{\tau} t^{\alpha-\beta}\Big)\right]^{*(k+1)} \right| \\
&\leq \frac{C}{\tau} \sum_{k=0}^{\infty} \frac{|d|^k C^k}{\tau^k} \frac{t^{\alpha k+\alpha-1}}{\Gamma(\alpha k+\alpha)} = \frac{C}{\tau} t^{\alpha-1} \sum_{k=0}^{\infty} \frac{\left(\frac{|d|C}{\tau} t^{\alpha}\right)^k}{\Gamma(\alpha k+\alpha)} \\
&= \frac{C}{\tau} t^{\alpha-1} E_{\alpha,\alpha}\Big(\frac{|d|C}{\tau} t^{\alpha}\Big).
\end{aligned}
$$

It proves that $p_{\alpha,\beta}(t)$ is well-defined for each $t>0$, and Laplace transformable. Taking the Laplace transform of $p_{\alpha,\beta}(t)$, we have that

$$
\begin{aligned}
\widehat{p_{\alpha,\beta}}(\lambda) &= \mathcal{L}\left(\sum_{k=0}^{\infty} \frac{d^k}{\tau^{k+1}} \left[t^{\alpha-1} E_{\alpha-\beta,\alpha}\Big(\frac{b}{\tau} t^{\alpha-\beta}\Big)\right]^{*(k+1)} \right)(\lambda) \\
&= \sum_{k=0}^{\infty} \frac{d^k}{\tau^{k+1}} \left[\mathcal{L}\left(t^{\alpha-1} E_{\alpha-\beta,\alpha}\Big(\frac{b}{\tau} t^{\alpha-\beta}\Big)\right)(\lambda)\right]^{k+1} \\
&= \sum_{k=0}^{\infty} d^k \left[\frac{1}{\tau}\mathcal{L}\left(t^{\alpha-1} E_{\alpha-\beta,\alpha}\Big(\frac{b}{\tau} t^{\alpha-\beta}\Big)\right)(\lambda)\right]^{k+1} = \sum_{k=0}^{\infty} d^k \left[\frac{\lambda^{-\beta}}{\tau\left(\lambda^{\alpha-\beta} - \frac{b}{\tau}\right)}\right]^{k+1} \\
&= \sum_{k=0}^{\infty} d^k \left[\frac{\lambda^{-\beta}}{\tau\lambda^{\alpha-\beta} - b}\right]^{k+1} = \sum_{k=0}^{\infty} \frac{d^k \lambda^{-\beta k-\beta}}{(\tau\lambda^{\alpha-\beta} - b)^{k+1}} \\
&= \frac{\lambda^{-\beta}}{(\tau\lambda^{\alpha-\beta} - b)} \sum_{k=0}^{\infty} \frac{d^k \lambda^{-\beta k}}{(\tau\lambda^{\alpha-\beta} - b)^{k}}.
\end{aligned}
$$

Since $\frac{d\lambda^{-\beta}}{\tau\lambda^{\alpha-\beta}-b} \to 0$ as $\lambda \to \infty$, we obtain for λ sufficiently large

$$
\widehat{p_{\alpha,\beta}}(\lambda) = \frac{\lambda^{-\beta}}{(\tau\lambda^{\alpha-\beta} - b)} \frac{1}{\left[1 - \dfrac{d\lambda^{-\beta}}{\tau\lambda^{\alpha-\beta} - b}\right]} = \frac{1}{\tau\lambda^{\alpha} - b\lambda^{\beta} - d}. \tag{4.3}
$$

An application of [31, theorem 0.4] with $k=1$ and $g(\lambda) = \dfrac{1}{\tau\lambda^{\alpha} - b\lambda^{\beta} - d}$, together with the uniqueness of the Laplace transform, shows that $p'_{\alpha,\beta}(t)$ exists. We distinguish the following cases:

(I) $0 < \beta \leq 1 < \alpha < 2$. We define $a(t) = a(g_{1-\beta} * p'_{\alpha,\beta})(t) + cp_{\alpha,\beta}(t)$ and $k(t) = \tau(g_{2-\alpha} * p'_{\alpha,\beta})(t)$. From (4.3) and using the initial value theorem for the Laplace transform, we obtain

$$
p_{\alpha,\beta}(0) = \lim_{|\lambda|\to\infty} \lambda\widehat{p_{\alpha,\beta}}(\lambda) = \lim_{|\lambda|\to\infty} \frac{1}{\lambda^{\alpha-1}\left(\tau - \dfrac{b}{\lambda^{\alpha-\beta}} - \dfrac{d}{\lambda^{\alpha}}\right)} = 0.
$$

https://doi.org/10.1017/prm.2024.134 Published online by Cambridge University Press

Therefore, taking the Laplace transform of $a(t)$, we have

$$\widehat{a}(\lambda) = a\lambda^{\beta}\widehat{p_{\alpha,\beta}}(\lambda) - \lambda^{\beta-1}p_{\alpha,\beta}(0) + c\widehat{p_{\alpha,\beta}}(\lambda) = \frac{a\lambda^{\beta}+c}{\tau\lambda^{\alpha}-b\lambda^{\beta}-d}.$$

Analogously, taking the Laplace transform of $k(t)$, we obtain

$$\widehat{k}(\lambda) = \tau\lambda^{\alpha-1}\widehat{p_{\alpha,\beta}}(\lambda) - \lambda^{\alpha-2}p_{\alpha,\beta}(0) = \frac{\tau\lambda^{\alpha-1}}{\tau\lambda^{\alpha}-b\lambda^{\beta}-d}.$$

By (4.2), we conclude that $\{R_{\alpha,\beta}(t)\}_{t\geq 0}$ is an (a,k)-regularized family.

(II) $0<\beta<\alpha=1$. We define $a(t) = a(g_{1-\beta} * p'_{1,\beta})(t) + \dfrac{a}{\tau}g_{1-\beta}(t) + cp_{1,\beta}(t)$ and $k(t) = \tau p_{1,\beta}(t)$. In this case, using the initial value theorem of Laplace transform, we obtain

$$p_{1,\beta}(0) = \lim_{|\lambda|\to\infty}\lambda\widehat{p_{1,\beta}}(\lambda) = \lim_{|\lambda|\to\infty}\frac{1}{\left(\tau - \dfrac{b}{\lambda^{1-\beta}} - \dfrac{d}{\lambda}\right)} = \frac{1}{\tau}.$$

Therefore, taking the Laplace transform of $a(t)$, we have

$$\begin{aligned}\widehat{a}(\lambda) &= a\lambda^{\beta}\widehat{p_{1,\beta}}(\lambda) - a\lambda^{\beta-1}p_{1,\beta}(0) + \frac{a}{\tau}\frac{1}{\lambda^{1-\beta}} + c\widehat{p_{1,\beta}}(\lambda)\\ &= a\lambda^{\beta}\widehat{p_{1,\beta}}(\lambda) + c\widehat{p_{1,\beta}}(\lambda) = \frac{a\lambda^{\beta}+c}{\tau\lambda-b\lambda^{\beta}-d}.\end{aligned}$$

Furthermore, taking the Laplace transform of $k(t)$, we have

$$\widehat{k}(\lambda) = \tau\widehat{p_{1,\beta}}(\lambda) = \tau\frac{1}{\tau\lambda-b\lambda^{\beta}-d} = \frac{\tau}{\tau\lambda-b\lambda^{\beta}-d}.$$

It proves the claim and finishes the proof. □

We finish this section with the following lemma that will be useful in the next section.

LEMMA 4.2. *Let* $0<\beta\leq 1\leq\alpha<2$, $a,b,c,d,\tau\in\mathbb{R}$, $\tau\neq 0$, *where* $(a,c)\neq(0,0)$. *Let* B *be the generator of a solution family* $\{R_{\alpha,\beta}(t)\}_{t\geq 0}$. *Then for each* $\gamma>0$, *the family* $\{(g_\gamma * R_{\alpha,\beta})(t)\}_{t\geq 0}$ *is exponentially bounded and, for each* $x\in X$, $(g_\gamma * R_{\alpha,\beta})(t)x\in\mathcal{D}(B)$.

Proof. By definition 2.10, the family $\{R_{\alpha,\beta}(t)\}_{t\geq 0}$ is exponentially bounded and then for $t>0$ and $x\in X$ we have

$$\begin{aligned}\|(g_\gamma * R_{\alpha,\beta})(t)x\| &\leq \int_0^t |g_\gamma(t-s)|\,\|R_{\alpha,\beta}(s)\|\|x\|ds \leq \int_0^t g_\gamma(t-s)Me^{\omega s}ds\|x\|\\ &= M\int_0^t g_\gamma(s)e^{\omega(t-s)}ds\|x\| \leq Me^{\omega t}\int_0^\infty g_\gamma(s)e^{-\omega s}ds\|x\|\\ &= \frac{M}{\omega^\gamma}e^{\omega t}\|x\|.\end{aligned}$$

https://doi.org/10.1017/prm.2024.134 Published online by Cambridge University Press

Since by theorem 4.1, $\{R_{\alpha,\beta}(t)\}_{t\geq 0}$ is an (a,k)-regularized family, then by lemma 2.13 we have that $(g_\gamma * R_{\alpha,\beta})(t)x \in \mathcal{D}(B)$ for each $x \in X$ and $\gamma > 0$. □

We continue with the following result.

LEMMA 4.3. *Let* $0 < \beta \leq 1 \leq \alpha < 2$, $a,b,c,d,\tau \in \mathbb{R}$, $\tau \neq 0$, *where* $(a,c) \neq (0,0)$, *and let* B *be the generator of a solution family* $\{R_{\alpha,\beta}(t)\}_{t\geqslant 0}$. *Then for all* $x \in X$

$$(g_{\alpha-1} * R_{\alpha,\beta})(t)x = \frac{(aB+bI)}{\tau}(g_{2\alpha-\beta-1} * R_{\alpha,\beta})(t)x + \frac{(cB+dI)}{\tau} \times (g_{2\alpha-1} * R_{\alpha,\beta})(t)x + g_\alpha(t)x. \tag{4.4}$$

Proof. By theorem 4.1, $\{R_{\alpha,\beta}(t)\}_{t\geq 0}$ is an (a,k)-regularized family then by lemma 2.13, $(g_\gamma * R_{\alpha,\beta})(t)x \in \mathcal{D}(B)$ for each $x \in X$ and $\gamma > 0$. In addition, since $R_{\alpha,\beta}(t)$ and $(g_\gamma * R_{\alpha,\beta})(t)$, $\gamma > 0$ are exponentially bounded by definition 2.10 and lemma 4.2, taking the Laplace transform we obtain for any $x \in X$ and λ sufficiently large:

$$\begin{aligned}
&\widehat{R_{\alpha,\beta}}(\lambda)\widehat{g_{\alpha-1}}(\lambda)x - \frac{1}{\tau}(aB+bI)\widehat{g_{2\alpha-\beta-1}}(\lambda)\widehat{R_{\alpha,\beta}}(\lambda)x - \frac{1}{\tau}(cB+dI)\widehat{g_{2\alpha-1}}(\lambda) \\
&\times \widehat{R_{\alpha,\beta}}(\lambda)x \\
&= \frac{1}{\lambda^{\alpha-1}}\widehat{R_{\alpha,\beta}}(\lambda)x - \frac{(aB+bI)}{\tau}\frac{1}{\lambda^{2\alpha-\beta-1}}\widehat{R_{\alpha,\beta}}(\lambda)x - \frac{(cB+dI)}{\tau}\frac{1}{\lambda^{2\alpha-1}}\widehat{R_{\alpha,\beta}}(\lambda)x \\
&= \frac{\tau\lambda^\alpha}{\tau\lambda^\alpha}\frac{1}{\lambda^{\alpha-1}}\widehat{R_{\alpha,\beta}}(\lambda)x - (aB+bI)\frac{\lambda^\beta}{\tau\lambda^\alpha\lambda^{\alpha-1}}\widehat{R_{\alpha,\beta}}(\lambda)x - (cB+dI)\frac{1}{\tau\lambda^\alpha\lambda^{\alpha-1}} \\
&\times \widehat{R_{\alpha,\beta}}(\lambda)x \\
&= \frac{1}{\lambda^\alpha}\frac{1}{\tau\lambda^{\alpha-1}}\left[\tau\lambda^\alpha - \lambda^\beta(aB+bI) - (cB+dI)\right]\widehat{R_{\alpha,\beta}}(\lambda)x \\
&= \frac{1}{\lambda^\alpha}\frac{1}{\tau\lambda^{\alpha-1}}\left[\tau\lambda^\alpha - b\lambda^\beta - d - a\lambda^\beta B - cB\right]\widehat{R_{\alpha,\beta}}(\lambda)x \\
&= \frac{1}{\lambda^\alpha}\frac{1}{\tau\lambda^{\alpha-1}}\left(\tau\lambda^\alpha - b\lambda^\beta - d - (a\lambda^\beta + c)B\right)\widehat{R_{\alpha,\beta}}(\lambda)x \\
&= \frac{1}{\lambda^\alpha}\frac{(a\lambda^\beta+c)}{\tau\lambda^{\alpha-1}}\left(\frac{\tau\lambda^\alpha - b\lambda^\beta - d}{a\lambda^\beta+c} - B\right)\widehat{R_{\alpha,\beta}}(\lambda)x = \frac{x}{\lambda^\alpha} = \widehat{g_\alpha}(\lambda)x.
\end{aligned}$$

Hence, by uniqueness of the Laplace transform, we obtain (4.4). □

We have the next lemma.

LEMMA 4.4. *Let* $0 < \beta \leq 1 \leq \alpha < 2$, $a,b,c,d,\tau \in \mathbb{R}$, $\tau \neq 0$, *where* $(a,c) \neq (0,0)$, *and let* B *be the generator of a solution family* $\{R_{\alpha,\beta}(t)\}_{t\geqslant 0}$. *Then for all* $x \in \mathcal{D}(B)$

$$\begin{aligned}
&\tau R_{\alpha,\beta}(t)x - (aB+bI)\big(g_{\alpha-\beta} * R_{\alpha,\beta}\big)(t)x - \tau x - (aB+bI)\big(g_{\alpha-\beta} * R_{\alpha,\beta}\big)(t)x \\
&+ \frac{1}{\tau}(aB+bI)\big(g_{2\alpha-2\beta} * R_{\alpha,\beta}\big)(t)(aB+bI)x + g_{\alpha-\beta+1}(t)(aB+bI)x \qquad (4.5) \\
&- (cB+dI)\big(g_\alpha * R_{\alpha,\beta}\big)(t)x + \frac{1}{\tau}(cB+dI)\big(g_{2\alpha-\beta} * R_{\alpha,\beta}\big)(t)(aB+bI)x = 0.
\end{aligned}$$

https://doi.org/10.1017/prm.2024.134 Published online by Cambridge University Press

Proof. Note that by theorem 4.1 and lemma 2.13, for each $x \in X$, $(g_\gamma * R_{\alpha,\beta})(t)x \in \mathcal{D}(B)$, $\gamma > 0$, and since $x \in \mathcal{D}(B)$ we have that $(aB + bI)x \in X$ and $(g_\gamma * R_{\alpha,\beta})(t)(aB + bI)x \in \mathcal{D}(B)$, for each $x \in \mathcal{D}(B)$. Proceeding as in lemma 4.3, and taking the Laplace transform of (4.5), we obtain

$$
\begin{aligned}
&\tau\widehat{R_{\alpha,\beta}}(\lambda)x - (aB+bI)\widehat{R_{\alpha,\beta}}(\lambda)\widehat{g_{\alpha-\beta}}(\lambda)x - \frac{\tau}{\lambda}x - (aB+bI)\widehat{R_{\alpha,\beta}}(\lambda)\widehat{g_{\alpha-\beta}}(\lambda)x\\
&\quad + \frac{1}{\tau}(aB+bI)\widehat{R_{\alpha,\beta}}(\lambda)\widehat{g_{2\alpha-2\beta}}(\lambda)(aB+bI)x + (aB+bI)\widehat{g_{2\alpha-\beta+1}}(\lambda)x\\
&\quad - (cB+dI)\widehat{g_\alpha}(\lambda)\widehat{R_{\alpha,\beta}}(\lambda)x + \frac{1}{\tau}(cB+dI)\widehat{g_{2\alpha-\beta}}(\lambda)\widehat{R_{\alpha,\beta}}(\lambda)(aB+bI)x\\
&= \tau\widehat{R_{\alpha,\beta}}(\lambda)x - \frac{(aB+bI)}{\lambda^{\alpha-\beta}}\widehat{R_{\alpha,\beta}}(\lambda)x - \frac{\tau}{\lambda}x - \frac{1}{\lambda^{\alpha-\beta}}\widehat{R_{\alpha,\beta}}(\lambda)(aB+bI)x\\
&\quad + \frac{1}{\tau}(aB+bI)\frac{1}{\lambda^{2\alpha-2\beta}}\widehat{R_{\alpha,\beta}}(\lambda)(aB+bI)x + (aB+bI)\frac{1}{\lambda^{\alpha-\beta+1}}x\\
&\quad - (cB+dI)\frac{1}{\lambda^\alpha}\widehat{R_{\alpha,\beta}}(\lambda)x + \frac{1}{\tau}\frac{(cB+dI)}{\lambda^{2\alpha-\beta}}\widehat{R_{\alpha,\beta}}(\lambda)(aB+bI)x\\
&= \left[\tau - \frac{1}{\lambda^{\alpha-\beta}}(aB+bI) - (cB+dI)\frac{1}{\lambda^\alpha}\right]\widehat{R_{\alpha,\beta}}(\lambda)x - \frac{\tau}{\lambda}x + \frac{(aB+bI)}{\lambda^{\alpha-\beta+1}}x\\
&\quad + \left[-\frac{1}{\lambda^{\alpha-\beta}} + \frac{1}{\tau}\frac{1}{\lambda^{2\alpha-2\beta}}(aB+bI) + \frac{1}{\tau\lambda^{2\alpha-\beta}}(cB+dI)\right]\widehat{R_{\alpha,\beta}}(\lambda)(aB+bI)x\\
&= \left[\tau - \frac{\lambda^\beta}{\lambda^\alpha}(aB+bI) - (cB+dI)\frac{1}{\lambda^\alpha}\right]\widehat{R_{\alpha,\beta}}(\lambda)x - \frac{\tau}{\lambda}x + \frac{(aB+bI)}{\lambda^{\alpha-\beta+1}}x\\
&\quad + \left[-\frac{1}{\lambda^{\alpha-\beta}} + \frac{1}{\tau}\frac{\lambda^\beta}{\lambda^{2\alpha-\beta}}(aB+bI) + \frac{1}{\tau\lambda^{2\alpha-\beta}}(cB+dI)\right]\widehat{R_{\alpha,\beta}}(\lambda)(aB+bI)x\\
&= \frac{1}{\lambda^\alpha}\left[\tau\lambda^\alpha - \lambda^\beta(aB+bI) - (cB+dI)\right]\widehat{R_{\alpha,\beta}}(\lambda)x - \frac{\tau}{\lambda}x + (aB+bI)\frac{1}{\lambda^{\alpha-\beta+1}}x\\
&\quad + \left[-\frac{1}{\lambda^{\alpha-\beta}}\frac{\tau\lambda^{\alpha-1}}{\tau\lambda^{\alpha-1}} + \frac{1}{\tau}\frac{\lambda^\beta\lambda^{-1}}{\lambda^{\alpha-1}\lambda^{\alpha-\beta}}(aB+bI)\right.\\
&\qquad \left. + \frac{\lambda^{-1}}{\tau\lambda^{\alpha-1}\lambda^{\alpha-\beta}}(cB+dI)\right]\widehat{R_{\alpha,\beta}}(\lambda)(aB+bI)x\\
&= \frac{1}{\lambda^\alpha}\left[\tau\lambda^\alpha - b\lambda^\beta - d - a\lambda^\beta B - cB\right]\widehat{R_{\alpha,\beta}}(\lambda)x - \frac{\tau}{\lambda}x + (aB+bI)\frac{1}{\lambda^{\alpha-\beta+1}}x\\
&\quad - \frac{1}{\lambda^{\alpha-\beta+1}}\frac{1}{\tau\lambda^{\alpha-1}}\left[\tau\lambda^\alpha - \lambda^\beta(aB+bI) - (cB+dI)\right]\widehat{R_{\alpha,\beta}}(\lambda)(aB+bI)x\\
&= \frac{1}{\lambda^\alpha}\left[\tau\lambda^\alpha - b\lambda^\beta - d - a\lambda^\beta B - cB\right]\widehat{R_{\alpha,\beta}}(\lambda)x - \frac{\tau}{\lambda}x + (aB+bI)\frac{1}{\lambda^{\alpha-\beta+1}}x\\
&\quad - \frac{1}{\lambda^{\alpha-\beta+1}}\frac{1}{\tau\lambda^{\alpha-1}}\left[\tau\lambda^\alpha - b\lambda^\beta - d - a\lambda^\beta B - cB\right]\widehat{R_{\alpha,\beta}}(\lambda)(aB+bI)x\\
&= \frac{1}{\lambda^\alpha}\left[\tau\lambda^\alpha - b\lambda^\beta - d - (a\lambda^\beta + c)B\right]\widehat{R_{\alpha,\beta}}(\lambda)x - \frac{\tau}{\lambda}x + (aB+bI)\frac{1}{\lambda^{\alpha-\beta+1}}x\\
&\quad - \frac{1}{\lambda^{\alpha-\beta+1}}\frac{1}{\tau\lambda^{\alpha-1}}\left[\tau\lambda^\alpha - b\lambda^\beta - d - (a\lambda^\beta + c)B\right]\widehat{R_{\alpha,\beta}}(\lambda)(aB+bI)x
\end{aligned}
$$

https://doi.org/10.1017/prm.2024.134 Published online by Cambridge University Press

$$= \frac{\tau\lambda^{-1}}{\tau\lambda^{-1}}\frac{(a\lambda^\beta + c)}{\lambda^\alpha}\left[\frac{\tau\lambda^\alpha - b\lambda^\beta - d}{(a\lambda^\beta + c)} - B\right]\widehat{R_{\alpha,\beta}}(\lambda)x - \frac{\tau}{\lambda}x + (aB + bI)\frac{1}{\lambda^{\alpha-\beta+1}}x$$
$$- \frac{1}{\lambda^{\alpha-\beta+1}}\frac{(a\lambda^\beta + c)}{\tau\lambda^{\alpha-1}}\left[\frac{\tau\lambda^\alpha - b\lambda^\beta - d}{(a\lambda^\beta + c)} - B\right]\widehat{R_{\alpha,\beta}}(\lambda)(aB + bI)x$$
$$= \tau\lambda^{-1}x - \frac{\tau}{\lambda}x + (aB + bI)\frac{1}{\lambda^{\alpha-\beta+1}}x - \frac{1}{\lambda^{\alpha-\beta+1}}(aB + bI)x = 0.$$

Hence, the claim follows by uniqueness of the Laplace transform. □

We finally prove the following result.

LEMMA 4.5. *Let* $0 < \beta \le 1 \le \alpha < 2$, $a, b, c, d, \tau \in \mathbb{R}$, $\tau \neq 0$, *where* $(a, c) \neq (0, 0)$, *and let* B *be the generator of a solution family* $\{R_{\alpha,\beta}(t)\}_{t\geqslant 0}$. *Then for all* $x \in X$, *we have*

$$\tau(g_1 * R_{\alpha,\beta})(t)x - t\tau x - (aB + bI)(g_{\alpha-\beta} * g_1 * R_{\alpha,\beta})(t)x - (cB + dI)$$
$$\times (g_\alpha * g_1 * R_{\alpha,\beta})(t)x = 0. \tag{4.6}$$

Proof. We have

$$\frac{\tau}{\lambda}\widehat{R_{\alpha,\beta}}(\lambda)x - \frac{\tau}{\lambda^2}x - (aB + bI)\widehat{g_{\alpha-\beta}}(\lambda)\frac{1}{\lambda}\widehat{R_{\alpha,\beta}}(\lambda)x - (cB + dI)\widehat{g_\alpha}(\lambda)\frac{1}{\lambda}\widehat{R_{\alpha,\beta}}(\lambda)x$$
$$= \frac{\tau}{\lambda}\widehat{R_{\alpha,\beta}}(\lambda)x - \frac{\tau}{\lambda^2}x - (aB + bI)\frac{1}{\lambda^{\alpha-\beta}}\frac{1}{\lambda}\widehat{R_{\alpha,\beta}}(\lambda)x - (cB + dI)\frac{1}{\lambda^\alpha}\frac{1}{\lambda}\widehat{R_{\alpha,\beta}}(\lambda)x$$
$$= \left[\frac{\tau}{\lambda} - \frac{\lambda^\beta}{\lambda^\alpha}\frac{1}{\lambda}(aB + bI) - \frac{1}{\lambda^\alpha}\frac{1}{\lambda}(cB + dI)\right]\widehat{R_{\alpha,\beta}}(\lambda)x - \frac{\tau}{\lambda^2}x$$
$$= \frac{1}{\lambda^\alpha}\frac{1}{\lambda}\left[\tau\lambda^\alpha - \lambda^\beta(aB + bI) - (cB + dI)\right]\widehat{R_{\alpha,\beta}}(\lambda)x - \frac{\tau}{\lambda^2}x$$
$$= \frac{1}{\lambda^\alpha}\frac{1}{\lambda}\frac{\tau\lambda^{-1}}{\tau\lambda^{-1}}\left[\tau\lambda^\alpha - b\lambda^\beta - d - a\lambda^\beta B - cB\right]\widehat{R_{\alpha,\beta}}(\lambda)x - \frac{\tau}{\lambda^2}x$$
$$= \frac{\tau\lambda^{-1}}{\lambda\tau\lambda^{\alpha-1}}\left[\tau\lambda^\alpha - b\lambda^\beta - d - (a\lambda^\beta + c)B\right]\widehat{R_{\alpha,\beta}}(\lambda)x - \frac{\tau}{\lambda^2}x$$
$$= \frac{\tau\lambda^{-1}}{\lambda}\frac{(a\lambda^\beta + c)}{\tau\lambda^{\alpha-1}}\left[\frac{\tau\lambda^\alpha - b\lambda^\beta - d}{(a\lambda^\beta + c)} - \right]\widehat{R_{\alpha,\beta}}(\lambda)x - \frac{\tau}{\lambda^2}x$$
$$= \frac{\tau}{\lambda^2}x - \frac{\tau}{\lambda^2}x = 0.$$

□

https://doi.org/10.1017/prm.2024.134 Published online by Cambridge University Press

5. Well-posedness and sufficient conditions

Let B be a closed operator on a complex Banach space X. We consider the abstract Cauchy problem

$$\begin{cases} \tau D_t^{\alpha} u(t) - (aB + bI) D_t^{\beta} u(t) - (cB + dI) u(t) = 0, \\ u(0) = x \in X, \\ u'(0) = y \in X, \end{cases} \tag{5.1}$$

with $\tau \neq 0$, $a, b, c, d \in \mathbb{R}$ and $0 < \beta \leqslant 1 < \alpha \leqslant 2$, where D_t^{γ} denotes the Caputo derivative of order $\gamma > 0$.

By a strict solution of (5.1), we understand a function $u \in C^2(\mathbb{R}_+; X) \cap C^1(\mathbb{R}_+; \mathcal{D}(B))$ such that $u(t) \in \mathcal{D}(B)$ for all $t \geq 0$ and (5.1) holds. If a strict solution exists, then it follows that $x, y \in \mathcal{D}(B)$. In applications, it is useful to find a weaker notion of solution where x, y may be arbitrary. This can be done by integrating the equation. Assume that u is a strict solution. Since B is closed, it follows from [4, proposition 1.1.7] that $(g_{\alpha-\beta} * u)(t) \in \mathcal{D}(B)$, $(g_{\alpha} * u)(t) \in \mathcal{D}(B)$ and

$$\tau u(t) - (aB + bI)(g_{\alpha-\beta} * u)(t) - (cB + dI)(g_{\alpha} * u)(t) = \tau(x + ty) - g_{\alpha-\beta+1}(t) \times (aB + bI)x, \; t > 0. \tag{5.2}$$

We introduce the following definition.

Definition 5.1. *Let $0 < \beta < \alpha$. A function $u \in C(\mathbb{R}_+; X)$ is called a mild solution of (5.1) if $(g_{\alpha-\beta} * u)(t) \in \mathcal{D}(B)$, $(g_{\alpha} * u)(t) \in \mathcal{D}(B)$ for all $t > 0$ and (5.2) holds.*

It is clear that mild and strict solutions differ merely by regularity.

Definition 5.2. *We say that (5.1) is well-posed if for each $x \in \mathcal{D}(B)$ and each $y \in X$ there exists a unique mild solution.*

We observe that this notion of well-posedness has been considered by other authors, see, e.g., references [4, 16, 17] where it is named mildly well-posedness. Our first main result in this section is the following.

Theorem 5.3. *Let $0 < \beta < \alpha$, $(a, c) \neq (0, 0)$. Let B the generator of a solution family $\{R_{\alpha,\beta}(t)\}_{t \geqslant 0}$ on X. Then (5.1) is well-posed.*

Proof. Uniqueness: Let $u_1, u_2 \in C(\mathbb{R}_+; X)$ be two mild solutions of (5.1). Then $u := u_1 - u_2 \in C(\mathbb{R}_+; X)$ and $\tau u(t) - (aB + bI)(g_{\alpha-\beta} * u)(t) - (cB + dI)(g_{\alpha} * u)(t) = 0$, for all $t \geq 0$. Hence

$$(\tau g_{\beta} - (aB + bI) g_{\alpha} - (cB + dI) g_{\alpha+\beta}) * u(t) = 0.$$

Therefore, by Titchmarch's theorem, it follows that $u \equiv 0$.

https://doi.org/10.1017/prm.2024.134 Published online by Cambridge University Press

Existence: Let $x \in \mathcal{D}(B)$ and $y \in X$. We define

$$u(t) = R_{\alpha,\beta}(t)x + \int_0^t R_{\alpha,\beta}(t-s)\Big[y - \frac{1}{\tau}g_{\alpha-\beta}(s)(aB+bI)x\Big]ds, \quad t \geq 0. \tag{5.3}$$

We divide the proof in three steps.

Step 1: When $y=0$ we define

$$u_1(t) = R_{\alpha,\beta}(t)x - \frac{1}{\tau}(g_{\alpha-\beta} * R_{\alpha,\beta})(t)(aB+bI)x. \tag{5.4}$$

Now, we will to show that (5.4) satisfies the expression (5.2). In fact, it is enough to show that

$$\tau\big(u_1(t) - x\big) - (aB+bI)\big[(g_{\alpha-\beta} * u_1)(t) - g_{\alpha-\beta+1}(t)x\big] - (cB+dI)(g_\alpha * u_1)(t) = 0.$$

By theorem 4.1 and lemma 2.13, for each $x \in X$, $(g_\gamma * R_{\alpha,\beta})(t)x \in \mathcal{D}(B)$, $\gamma > 0$, and since $x \in \mathcal{D}(B)$ for $a \neq 0$, we have that $(aB+bI)x \in X$ and $(g_\gamma * R_{\alpha,\beta})(t)(aB+bI)x \in \mathcal{D}(B)$, for each $x \in \mathcal{D}(B)$. Hence $(g_{\alpha-\beta} * u_1)(t) \in \mathcal{D}(B)$ and $(g_\alpha * u_1)(t) \in \mathcal{D}(B)$. Using (5.4) and lemma 4.4, we obtain that

$$\begin{aligned}
&\tau R_{\alpha,\beta}(t)x - \big(g_{\alpha-\beta} * R_{\alpha,\beta}\big)(t)(aB+bI)x - \tau x - (aB+bI)\big(g_{\alpha-\beta} * R_{\alpha,\beta}\big)(t)x \\
&\quad + \frac{1}{\tau}(aB+bI)\big(g_{2\alpha-2\beta} * R_{\alpha,\beta}\big)(t)(aB+bI)x + (aB+bI)g_{\alpha-\beta+1}(t)x \\
&\quad - (cB+dI)\big(g_\alpha * R_{\alpha,\beta}\big)(t)x + \frac{1}{\tau}(cB+dI)\big(g_{2\alpha-\beta} * R_{\alpha,\beta}\big)(t)(aB+bI)x \\
&= \Big[\tau R_{\alpha,\beta}(t) - \big(g_{\alpha-\beta} * R_{\alpha,\beta}\big)(t)(aB+bI) - \tau - (aB+bI)\big(g_{\alpha-\beta} * R_{\alpha,\beta}\big)(t) \\
&\quad + \frac{1}{\tau}(aB+bI)\big(g_{2\alpha-2\beta} * R_{\alpha,\beta}\big)(t)(aB+bI) + (aB+bI)g_{\alpha-\beta+1}(t) \\
&\quad - (cB+dI)\big(g_\alpha * R_{\alpha,\beta}\big)(t) + \frac{1}{\tau}(cB+dI)\big(g_{2\alpha-\beta} * R_{\alpha,\beta}\big)(t)(aB+bI)\Big]x = 0,
\end{aligned}$$

proving that $u_1(t)$ satisfies (5.2).

Step 2: When $x=0$ we define

$$u_2(t) = (g_1 * R_{\alpha,\beta})(t)y. \tag{5.5}$$

By lemma 2.13, it is clear that $(g_{\alpha-\beta} * u_2)(t) \in \mathcal{D}(B)$ and $(g_\alpha * u_2)(t) \in \mathcal{D}(B)$. We will show that

$$\tau\big(u_2(t) - ty\big) - (aB+bI)\big[(g_{\alpha-\beta} * u_2)(t)\big] - (cB+dI)(g_\alpha * u_2)(t) = 0.$$

In fact, using (5.5), we obtain that the left hand side of the above identity equals to

$$\begin{aligned}
&\tau(g_1 * R_{\alpha,\beta})(t)y - t\tau y - (aB+bI)(g_{\alpha-\beta} * g_1 * R_{\alpha,\beta})(t)y - (cB+dI) \\
&\quad \times (g_\alpha * g_1 * R_{\alpha,\beta})(t)y
\end{aligned}$$

being this expression zero by lemma 4.5, as desired.

https://doi.org/10.1017/prm.2024.134 Published online by Cambridge University Press

Step 3: By steps 1 and 2, we obtain that

$$u(t) = u_1(t) + u_2(t)$$

satisfy (5.2). This proves the theorem. □

Now we consider the non-homogeneous abstract Cauchy problem. Let B be a closed operator and let $f \in L^1([0,T],X)$ where $T > 0$. For $0 < \beta \leq 1 < \alpha \leq 2$ we consider the problem

$$\begin{cases} \tau D_t^\alpha u(t) - (aB + bI)D_t^\beta u(t) - (cB + dI)u(t) = f(t), \quad t \in [0,T], \\ u(0) = x, \\ u'(0) = y, \end{cases} \tag{5.6}$$

where $x \in \mathcal{D}(B)$ and $y \in X$. A function $u \in C([0,T],X)$ is called a *mild solution* of (5.6) if $(g_{\alpha-\beta} * u)(t) \in \mathcal{D}(B)$, $(g_\alpha * u)(t) \in \mathcal{D}(B)$ and

$$\begin{aligned} \tau u(t) - (aB + bI)(g_{\alpha-\beta} * u)(t) - (cB + dI)(g_\alpha * u)(t) &= (g_\alpha * f)(t) + \tau(x + ty) \\ - g_{\alpha-\beta+1}(t)(aB + bI)x, \quad t \in [0,T]. \end{aligned} \tag{5.7}$$

Our main theorem in this section show that in the case when B generates a solution family there always exists a mild solution.

THEOREM 5.4. *Let $0 < \beta < \alpha$, $\alpha > 1$, $a, b, c, d, \tau \in \mathbb{R}$, $\tau \neq 0$, where $(a,c) \neq (0,0)$. Let B be the generator of a solution family $\{R_{\alpha,\beta}(t)\}_{t \geqslant 0}$ on X. Then for every $f \in L^1([0,T],X)$ the problem (5.6) has a unique mild solution u given by*

$$\begin{aligned} u(t) = R_{\alpha,\beta}(t)x &+ \int_0^t R_{\alpha,\beta}(t-s)\Big[y - \frac{1}{\tau}g_{\alpha-\beta}(s)(aB + bI)x\Big]ds \\ &+ \frac{1}{\tau}\int_0^t R_{\alpha,\beta}(t-s)(g_{\alpha-1} * f)(s)]ds. \end{aligned} \tag{5.8}$$

Proof. Uniqueness is proven as in theorem 5.3. For existence, we have seen that

$$R_{\alpha,\beta}(t)x + \int_0^t R_{\alpha,\beta}(t-s)\Big[y - \frac{1}{\tau}g_{\alpha-\beta}(s)(aB + bI)x\Big]ds$$

is a mild solution of the homogeneous problem. It remains to show that

$$u_1(t) = \frac{1}{\tau}(g_{\alpha-1} * R_{\alpha,\beta} * f)(t) \tag{5.9}$$

is a mild solution of (5.6) with initial value $x = y = 0$. Extending f by 0 to $\mathbb{R}_+$, we have that $u_1 \in C([0,T];X)$. Note that $(g_{\alpha-\beta} * u_1)(t) \in \mathcal{D}(B)$ and $(g_\alpha * u_1)(t) \in \mathcal{D}(B)$ thanks to lemma 2.13. Using (5.9) and lemma 4.3, we obtain that

https://doi.org/10.1017/prm.2024.134 Published online by Cambridge University Press

$$\begin{aligned}
&\tau u_1(t) - (aB+bI)(g_{\alpha-\beta}*u_1)(t) - (cB+dI)(g_\alpha * u_1)(t) - (g_\alpha * f)(t)\\
&= \tau\big(\frac{1}{\tau}(g_{\alpha-1}*R_{\alpha,\beta}*f)(t)\big) - (aB+bI)\big[(g_{\alpha-\beta}*\frac{1}{\tau}g_{\alpha-1}*R_{\alpha,\beta}*f)(t)\big]\\
&\quad - (cB+dI)(g_\alpha * \frac{1}{\tau}g_{\alpha-1}*R_{\alpha,\beta}*f)(t) - (g_\alpha * f)(t)\\
&= (g_{\alpha-1}*R_{\alpha,\beta}*f)(t) - \frac{(aB+bI)}{\tau}(g_{2\alpha-\beta-1}*R_{\alpha,\beta}*f)(t)\\
&\quad - \frac{(cB+dI)}{\tau}(g_{2\alpha-1}*R_{\alpha,\beta}*f)(t) - (g_\alpha * f)(t)\\
&= \Bigg(\Bigg[(g_{\alpha-1}*R_{\alpha,\beta})(t) - \frac{(aB+bI)}{\tau}(g_{2\alpha-\beta-1}*R_{\alpha,\beta})(t)\\
&\quad - \frac{(cB+dI)}{\tau}(g_{2\alpha-1}*R_{\alpha,\beta})(t)\Bigg]*f\Bigg)(t) - (g_\alpha * f)(t)\\
&= (g_\alpha * f)(t) - (g_\alpha * f)(t) = 0
\end{aligned}$$

proving that $u_1(t)$ satisfies (5.7). This proves the claim. □

6. Examples

We finish this work illustrating some special cases where our abstract results apply.

6.1. Linear Kuznetsov equation

The Kuznetsov equation [22] models propagation of non-linear acoustic waves in thermoviscous elastic media. This equation is treated in different works, for example, see [11, 15].

We consider the following linear version of de Kuznetsov equation, with fractional order in time, and Dirichlet boundary conditions

$$\begin{cases} u_{tt}(t,x) - c^2\Delta u(t,x) - \dfrac{\delta}{p_0}\Delta D_t^\beta u(t,x) = f(t,x), \;\; t>0, \;\; x\in(0,1)\\ u(t,0) = u(t,1) = 0, \;\; u(0,x) = u_0(x), \;\; u_t(0,x) = u_1(x), \end{cases} \tag{6.1}$$

where $0<\beta<1$, where $u(x,0)=u_0(x)$, $u_t(x,0)=u_1(x)$ and c, p_0, γ, δ are the velocity of the sound, the density, the ratio of the specific heats, and the viscosity of the medium, respectively.

Choose $X = L^p(0,1)$, $1\le p<\infty$. Consider $B=\Delta$ be the Dirichlet Laplacian on $L^p(0,1)$, with

$$\mathcal{D}(B) := W_0^{2,p}(0,1) = \{v\in W^{2,p}(0,1);\, v(0)=v(1)=0\}.$$

Then, by [19, lemma 2.3], for every $1\le p<\infty$, B generates a bounded cosine family in X. Therefore, using theorems 3.9 and 5.4, we obtain the following result.

THEOREM 6.1. *Let $0<\beta<1$, $\le p<\infty$, $f\in L^1_{loc}(\mathbb{R}_+;L^p(0,1))$ be such that $\int_0^\infty e^{-\omega t}\|f(t)\|dt<\infty$ for some $\omega>0$. Then for all $u_0\in W_0^{2,p}(0,1)$ and $u_1\in$*

$L^p(0,1)$, *there is a unique* $u \in C(\mathbb{R}_+; L^p(0,1))$ *satisfying*

$$\tau\big(u(t,x) - u_0(x) - tu_1(x)\big) - \frac{\delta}{p_0\Gamma(2-\beta)}\Delta \int_0^t (t-s)^{1-\beta} u(s,x)ds + \frac{\delta}{p_0\Gamma(3-\beta)} t^{2-\beta}\Delta u_0(x) - c^2\Delta \int_0^t (t-s)u(s,x)ds = \int_0^t (t-s)f(s,x)ds,\ t \geq 0,\ x \in (0,1).$$

Proof. Note that Eq. (6.1) is a particular case of (1.1) when we consider $\tau = 1$, $\alpha = 2$, $0 < \beta < 1$, $a = \dfrac{\delta}{p_0} > 0$, $c = c^2 > 0$, $B = \Delta$, $b = d = 0$, and $f \in L^1_{loc}(\mathbb{R}_+; L^2(\Omega))$. □

Consider the abstract model

$$u''(t) - aBD_t^\beta u(t) - cBu(t) = f(t), \qquad t \geq 0, \tag{6.2}$$

where $0 < \beta < 1$, $a, c > 0$, $u(0) \in \mathcal{D}(B)$ and $u'(0) \in X$. Using theorems 3.9 and 5.4, we obtain the following general result.

THEOREM 6.2. *Let* B *be the generator of a cosine family, then, for each* $f \in L^1_{loc}(\mathbb{R}_+; X)$ *such that* $\int_0^\infty e^{-\omega t}\|f(t)\|dt < \infty$, $\omega > 0$, *the model (6.2) has a unique mild solution.*

6.2. Linear Klein–Gordon equation

Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with sufficiently smooth boundary $\partial\Omega$. We consider the following fractional version of the Klein–Gordon equation

$$\begin{cases} D_t^\alpha \varphi(t,x) - a\Delta D_t^\beta \varphi(t,x) - bD_t^\beta \varphi(t,x) - \Delta\varphi(t,x) - d\varphi(t,x) = f(t,x), \\ \quad (t,x) \in \mathbb{R}_+ \times \Omega \\ \varphi(t,x) = 0,\ (t,x) \in \mathbb{R}_+ \times \partial\Omega \\ \varphi(0,x) = \varphi_0(x),\ \varphi_t(0,x) = \varphi_1(x),\ x \in \Omega, \end{cases} \tag{6.3}$$

where $a > 0$, $b < 0$ and $d \in \mathbb{R}$, see [5, Section 1] in case $\alpha = 2$ and $\beta = 1$.

Let $X = L^2(\Omega)$. We define $(Bv)(x) = (\Delta v)(x)$, $x \in \Omega$, $v \in \mathcal{D}(B)$ and

$$\mathcal{D}(B) := \{v \in H_0^1(\Omega); Bv \in L^2(\Omega)\}. \tag{6.4}$$

Using theorem 5.4, we obtain the following result.

THEOREM 6.3. *Let* $a > 0$, $b < 0$, $0 < \beta < \alpha$, $\alpha > 1$, *and* $f \in L^1([0,T], L^2(\Omega))$. *Suppose that the Laplacian operator* $B := \Delta$ *generates a solution family* $\{R_{\alpha,\beta}(t)\}_{t\geq 0}$ *on* $L^2(\Omega)$. *Then, for all* $\varphi_0 \in \mathcal{D}(B)$ *and* $\varphi_1 \in L^2(\Omega)$, *there exists*

https://doi.org/10.1017/prm.2024.134 Published online by Cambridge University Press

a unique $u \in C([0,T], L^2(\Omega))$ *such that*

$$\varphi(t,x) - \varphi_0(x) - t\varphi_1(x) - (a\Delta + bI)\left[\int_0^t g_{\alpha-\beta}(t-s)u(s,x)ds - g_{\alpha-\beta+1}(t)x\right]$$
$$-(\Delta + dI)\int_0^t g_\alpha(t-s)u(s,x)ds = \int_0^t g_\alpha(t-s)f(s,x)ds,\ t \geq 0.$$

For example, in case $\alpha = 2$ and $\beta = 1$, we know by [4, example 7.2.1, p. 424] that B is the generator of a C_0-semigroup on $L^2(\Omega)$ and therefore, by [28, corollary 13], a solution family $\{R_{2,1}(t)\}_{t\geq 0}$ on $L^2(\Omega)$ always exists and is given by

$$R_{2,1}(t)v := W'(t)v, \quad v \in \mathcal{D}(B), \tag{6.5}$$

where $\{W(t)\}_{t\geq 0} \subset \mathcal{L}(L^2(\Omega))$ is a strongly differentiable family, see [28, corollary 13].

Acknowledgements

J. A. is partially supported by National Council for Scientific Technological Development (CNPq, Brazil) process number 200454/2022-6 and partially supported by CAPES. C. L. is partially supported by CONICYT under FONDECYT grant number 1220036.

References

1 L. Abadias and P. J. Miana. A subordination principle on Wright functions and regularized resolvent families. *J. Funct. Spaces* **9** (2015), 158145.

2 R. P. Agarwal, M. Jleli and B. Samet. Nonexistence of global solutions for a time-fractional damped wave equation in a k-times halved space. *Comp. & Math. Appl.* **78** (2019), 1608–1620.

3 E. Alvarez and C. Lizama. The super-diffusive singular perturbation problem. *Mathematics* **8** (2020), 1–14.

4 W. Arendt, C. J. K. Batty, M. Hieber and F. Neubrander. *Vector-valued Laplace transforms and Cauchy problems*, Monographs in Mathematics, 2nd edition, 96 (Birkhäuser/Springer Basel AG, Basel, 2011).

5 P. Aviles and J. Sandefur. Nonlinear second order equations with applications to partial differential equations. *J. Diff. Equ.* **58** (1985), 404–427.

6 J. D. Avrin. Convergence of the strongly damped nonlinear Klein-Gordon equation in $\mathbb{R}^d$ with radial symmetry. *Proc. Roy. Soc. Edinburgh Sect. A* **107** (1987), 169–174.

7 J. D. Avrin. Convergence properties of the strongly damped nonlinear Klein-Gordon equation. *J. Diff. Equ.* **67** (1987), 243–255.

8 H. Bateman and A. Erdélyi. *Higher transcendental functions* (McGraw-Hill, New York, 1955).

9 E. Bazhlekova. Subordination principle for fractional evolution equations. *Fract. Calc. Appl. Anal.* **3** (2000), 213–230.

10 A. N. Carvalho, J. W. Cholewa and T. Dlotko. Strongly damped wave equations: Bootstrapping and regularity of solutions. *J. Diff. Equ.* **244** (2008), 2310–2333.

11 A. Dekkers and A. Rozanova-Pierrat. Cauchy problem for the Kuznetsov equation. *Discret. Cont. Dyn. Syst.* **39** (2019), 277–307.

12 P. Ding and Z. Yang. Complete regularity and strong attractor for the strongly damped wave equation with critical nonlinearities on $\mathbb{R}^3$. *J. Evol. Equ.* **23** (2023), 42.

13 G. Gorenflo, A. A. Kilbas and F. Mainardi. *Mittag-Leffler functions, related topics and applications*, Monographs in Mathematics, Vol. 2 (Springer, Berlin, 2020).

14 R. Ikehata, G. Todorova and B. Yordanov. Wave equations with strong damping in Hilbert spaces. *J. Diff. Equ.* **254** (2013), 3352–3368.

15 B. Kaltenbacher, I. Lasiecka and R. Marchand. Well-posedness and exponential decay rates for the Moore–Gibson–Thompson equation arising in high intensity ultrasound. *Control Cybern.* **40** (2011), 971–988.

16 V. Keyantuo and C. Lizama. *Mild well-posedness of abstract differential equations*, Functional Analysis and Evolution Equations (editors H. Amann, W. Arendt, M. Hieber, F. M. Neubrander, S. Nicaise, J. von Below), (Birkhäuser Verlag, Basel, 2008), 371–387.

17 V. Keyantuo, C. Lizama and V. Poblete. Periodic solutions of integro-differential equations in vector-valued function spaces. *J. Diff. Equ.* **246** (2009), 1007–1037.

18 V. Keyantuo, C. Lizama and M. Warma. Asymptotic behavior of fractional order semilinear evolution equations. *Diff. Integral Equ.* **26** (2013), 757–780.

19 V. Keyantuo and M. Warma. The wave equation with Wentzell-Robin boundary conditions on Lp-spaces. *J. Diff. Equ.* **229** (2006), 680–697.

20 M. Kirane and N. E. Tatar. Exponential growth for a fractionally damped wave equation. *Z. Anal. Anwend.* **22** (2003), 167–177.

21 M. Kostic. (a, k)-regularized C-resolvent families: Regularity and local properties. *Abstr. Appl. Anal.* **2009** (2009), 858242.

22 V. Kuznetsov. Equations of nonlinear acoustics. *Soviet Phys. Acoust.* **16** (1971), 467–470.

23 C. Lizama. *Abstract linear fractional evolution equations*, Handbook of Fractional Calculus With Applications. Vol. 2: Fractional Differential Equations (Ed. A. Kochubei Y. Luchko), pp. 465–498 (De Gruyter, Berlin, 2019).

24 C. Lizama. Regularized solutions for abstract Volterra equations. *J. Math. Anal. Appl.* **243** (2000), 278–292.

25 C. Lizama and G. M. N'Guérékata. Mild solutions for abstract fractional differential equations. *Appl. Anal.* **92** (2013), 1731–1754.

26 C. Lizama and H. Prado. Fractional relaxation equations on Banach spaces. *Appl. Math. Lett.* **23** (2010), 137–142.

27 I. V. Melnikova and A.I. Filinkov. Well-posedness of differential-operator problems II: The Cauchy problem for complete second-order equations in Banach spaces. *J. Math. Sci.* **93** (1999), 22–41.

28 F. Neubrander. Well-posedness of higher order abstract Cauchy problems. *Trans. Amer. Math. Soc.* **295** (1986), 257–290.

29 Y. Pang and Y. Yang. A note on finite time blowup for dissipative Klein-Gordon equation. *Nonlinear Anal.* **195** (2020), 111729.

30 I. Podlubny. *Fractional differential equations*, Mathematics in Science and Engineering 198 (Academic Press, San Diego, 1999).

31 J. Prüss. *Evolutionary integral equations and applications*, Monographs in Mathematics 87 (Birkhäuser, Switzerland, 1993).

32 J. T. Sandefur. Existence and uniqueness of solutions of second order nonlinear differential equations. *SIAM J. Math. Anal.* **14** (1983), 477–487.

33 M. Stojanovic and R. Gorenflo. Nonlinear two-term time fractional diffusion-wave problem. *Nonlinear Anal. Real World Appl.* **11** (2010), 3512–3523.

34 X. Tijun and L. Jin. On complete second order linear differential equations in Banach spaces. *Pacific J. Math.* **142** (1990), 175–195.

35 E. C. Titchmarsh. The zeros of certain integral functions. *Proc. London Math. Soc.* **2** (1925), 283–302.

36 Y. Zhou and J. W. He. Well-posedness and regularity for fractional damped wave equations. *Monatshefte für Mathematik.* **194** (2021), 425–458.