J. Austral. Math. Soc. Ser. B 26 (1984), 108–118

THE DIFFRACTION OF LONG ELASTIC WAVES BY ELLIPTIC CYLINDRICAL CAVITIES

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(Received 5 October 1982; revised 10 March 1983)

Abstract

The method of asymptotic matching introduced by Buchwald [1] is adapted to the case of the diffraction of plane longitudinal and shear waves by cylindrical cavities with elliptic cross-sections. It is assumed that the dimensions of the cross section are small compared with the wavelength of the incident waves. Asymptotic formulae for the scattered wave potentials are obtained.

The method is valid when the cavity reduces to a two-dimensional stress free crack whose length is small compared with the wavelength. Formulae for the scattered waves, and for the stress-concentrations at the crack tips are obtained.

1. Introduction

A previous paper (Buchwald [1]) established a relationship between the equations of plane elastodynamics and elastostatics which is a suitable basis for using an aysmptotic matching technique to solve low frequency diffraction problems. The scattering of plane P waves by a circular cylindrical cavity of small radius was used as an example of the method.

In this paper we undertake an extension of the method to low frequency scattering by an elliptic cylindrical hole. Muskhelishvili's [5] conformal mapping method is used to solve the appropriate elastostatic boundary value problem, and asymptotic formulae giving the scattered field for both P and S waves are obtained.

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https://doi.org/10.1017/S0334270000004367 Published online by Cambridge University Press

The method is valid when the minor axis of the ellipse becomes zero, and the scattering is by a stress-free crack. Asymptotic formulae for the stress-concentrations at the crack tips are easily determined by considering the results of the elastostatic problem.

2. Summary of basic results

Following [1], the displacement u, v referred to axes $0x'$, $0y'$, in the plane strain of an elastic body may be expressed in the form

$$\mu u = \mu' \frac{\partial \phi}{\partial x'} - \frac{\partial \psi}{\partial y'}, \tag{2.1a}$$

$$\mu v = \mu' \frac{\partial \phi}{\partial y'} + \frac{\partial \psi}{\partial x'}, \tag{2.1b}$$

where x', y' are Cartesian coordinates; $\phi(x', y')$, $\psi(x', y')$ are displacement functions and λ, μ are the Lamé coefficients of the material. The constant λ', μ' are defined by

$$\mu' = \mu/(\lambda + 2\mu), \quad \lambda' = \lambda/(\lambda + 2\mu). \tag{2.2}$$

Let L be a typical length and $x = x'/L$, $y = y'/L$. The equations of a harmonically vibrating elastic body may then be expressed as

$$\nabla^2\left(\frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial y}\right) + k^2\left(\mu'\frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial y}\right) = 0, \tag{2.3a}$$

$$\nabla^2\left(\frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial x}\right) + k^2\left(\mu'\frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial x}\right) = 0. \tag{2.3b}$$

In these equations we assume an implicit time factor $\exp(-i\sigma t)$, and

$$k^2 = \rho\sigma^2 L^2/\mu, \tag{2.4}$$

where ρ is the density. We assume the perturbation expansions for small k,

$$\phi = \phi_0 + k\phi_1 + k^2\phi_2 + \cdots, \tag{2.5a}$$

$$\psi = \psi_0 + k\psi_1 + k^2\psi_2 + \cdots. \tag{2.5b}$$

Substitution in (2.3) yields, after equating powers of k,

$$\nabla^4\phi_l = -\mu'\nabla^2\phi_{l-2}; \qquad \nabla^4\psi_l = -\nabla\psi_{l-2}; \qquad l = 0, 1, 2, \ldots, \tag{2.6}$$

with the convention that ϕ_{l-2} and ψ_{l-2} are zero for $l = 0$ and $l = 1$. In particular, ϕ_0, ψ_0, ϕ_1, ψ_1 satisfy the biharmonic equation, solutions of which may be obtained using Muskhelishvili's techniques as follows. Let

$$w_l = \phi_l(x, y) + i\psi_l(x, y). \tag{2.7}$$

It is shown in [1] that for $l = 0, 1$, a general expression for w_l is given by

$$w_l = \bar{z}\Omega_l(z) + \int \omega_l(z)\,dz; \qquad l = 0, 1, \tag{2.8}$$

where $\Omega_l(z)$, $\omega_l(z)$ are functions of the complex variable $z = x + iy$ which are analytic in the appropriate domain, and $\bar{z} = x - iy$. Substitution in (2.1) and the stress-strain relations yields, for both $l = 0$ and $l = 1$,

$$\Theta = \tau_{xx} + \tau_{yy} = 2(1-\mu')\nabla^2\phi = 4(1-\mu')\left[\Omega'(z) + \bar{\Omega}'(\bar{z})\right], \tag{2.9}$$

$$\begin{aligned}\Phi &= \tau_{xx} - \tau_{yy} + 2i\tau_{xy}\\ &= -8(1-\mu')\frac{\partial^2\phi}{\partial\bar{z}^2} = -4(1-\mu')\left[z\bar{\Omega}''(\bar{z}) + \bar{\omega}'(\bar{z})\right],\end{aligned} \tag{2.10}$$

and

$$\mu(u + iv)/(1-\mu') = \kappa\Omega(z) - z\bar{\Omega}'(\bar{z}) - \bar{\omega}(\bar{z}), \tag{2.11}$$

where $\Omega'(z)$, $\Omega''(z)$ are the first and second derivatives of Ω with respect to z, τ_{xx}, τ_{xy}, τ_{yy} are the stresses and $\kappa = (1+\mu')/(1-\mu')$. It may then be shown that in the rotated system of coordinates $n + is = ze^{-i\alpha}$,

$$\tau_{nn} + i\tau_{ns} = \tfrac{1}{2}\Theta + \tfrac{1}{2}\Phi e^{-2i\alpha}, \tag{2.12}$$

where α is the angle of rotation. It should also be noted that for constant C, E,

$$\Omega_R(z) = Eiz, \qquad \omega_R(z) = C, \qquad w_R = Eiz\bar{z} + Cz, \tag{2.13}$$

correspond to a rigid body displacement with zero stresses. Also note that, given any analytic function $\chi(z)$, substitution in (2.1) shows that

$$w^* = \phi^* + i\psi^* = \kappa\chi(z) + \bar{\chi}(\bar{z}) \tag{2.14}$$

corresponds to zero displacements and stresses.

Equations (2.7) to (2.14) define, effectively, the plane elastostatic problem, whose solution is obtained by using boundary values of the stresses and displacements to determine $\Omega(z)$, $\omega(z)$ in the domain being considered.

Returning to the original dynamic problem, it may be shown that when $k \neq 0$, and ϕ, ψ satisfy the Helmholtz equations

$$(\nabla^2 + \mu' k^2)\phi = 0; \quad (\nabla^2 + k^2)\psi = 0, \tag{2.15}$$

then ϕ, ψ are also solutions of (2.3).

A general representation of waves scattered by a finite obstacle is given by the solution of (2.15)

$$\phi_s = \sum_{j=0}^{\infty}\left(A_j\cos j\theta + A'_j\sin j\theta\right)H_j^{(1)}(k_1 r), \tag{2.16a}$$

$$\psi_s = \sum_{j=0}^{\infty}\left(B_j\sin j\theta + B'_j\cos j\theta\right)H_j^{(1)}(kr), \tag{2.16b}$$

https://doi.org/10.1017/S0334270000004367 Published online by Cambridge University Press

where $k_1^2 = \mu' k^2$, $z = re^{i\theta}$, and the $H_j^{(1)}$ are Hankel functions of the first kind, so that the expressions represent outward travelling cylindrical waves.

The remainder of this paper is concerned with the determination of the constants A_j, B_j by asymptotically matching solutions ϕ_0, ϕ_1, ψ_0, ψ_1, of the biharmonic equation for the given boundary value problems with ϕ_s, ψ_s, in (2.16), as $kr \to 0$.

3. Diffraction of a P wave by an elliptic hole

Let the hole have as boundary the ellipse

$$\left[\frac{x'}{(1+m)R}\right]^2 + \left[\frac{y'}{(1-m)R}\right]^2 = 1, \qquad 0 \leqslant m < 1, \tag{3.1}$$

and assume an incident P wave of the form

$$\phi_I = \exp[i(Kx' - \sigma t)]; \qquad \psi_I = 0; \tag{3.2}$$

where $K^2 = \rho\sigma^2/(\lambda + 2\mu)$, which leaves the boundary of the hole stress free. In this case the appropriate scale transformation

$$z = re^{i\theta} = x + iy, \quad x = x'/R, \quad y = y'/R, \quad k_1 = k\sqrt{\mu'} = RK, \tag{3.3}$$

reduces the problem to the format of Section 2. When α is the angle the normal to the ellipse makes with the x axis, we find, in (2.12), that

$$e^{-2i\alpha} = \frac{2mz - (1+m^2)\bar{z}}{2m\bar{z} - (1+m^2)z}, \tag{3.4}$$

whence it may be shown from (2.9) that the incident wave in (3.2) gives rise to boundary stress on the ellipse given by

$$K^{-2}(\tau_{nn} + i\tau_{ns}) = \tau_I = \tau_I^{(0)} + ik_1\tau_I^{(1)} + O(k^2), \tag{3.5}$$

where

$$\tau_I^{(0)} = \tau_I^{(1)}/x = -(1-\mu')\left[1 + \gamma_1 e^{-2i\alpha}\right], \tag{3.6}$$

$$\gamma_1 = \mu'/(1-\mu') = \mu/(\lambda + \mu), \tag{3.7}$$

and $e^{-2i\alpha}$ is given in (3.4).

In order to solve the static problem we use the conformal transformation

$$z = \zeta + m/\zeta \tag{3.8}$$

which maps the exterior of the ellipse on to the exterior of the circle $|\zeta| = 1$ in the ζ plane. The actual determination of the appropriate functions $\Omega(z)$, $\omega(z)$, may be performed by using the methods of either Muskhelishvili [5] or Green and

Zerna [3]. It is then straightforward to confirm that

$$\Omega_0(z) = \beta_1/\zeta, \tag{3.9}$$

$$\omega_0(z) = (1+m^2)\Omega_0(z)/(\zeta^2 - m) + \beta_2/\zeta, \tag{3.10}$$

with

$$\beta_1 = \tfrac{1}{2}(m - \gamma_1)k_1^2; \qquad \beta_2 = \tfrac{1}{2}(1 + m^2 - 2m\gamma_1)k_1^2, \tag{3.11}$$

are analytic for $|\zeta| > 1$, and satisfy

$$\tau_{nn} + i\tau_{ns} = -K^2\tau_I^{(0)} \tag{3.12}$$

on $|\zeta| = 1$.

Similarly,

$$\Omega_1(z) = \delta_1 \log\zeta + \delta_2\zeta^{-2}, \tag{3.13}$$

$$\begin{aligned}\omega_1(z) = \delta_3 \log\zeta + \delta_4 + \delta_5\zeta^{-2} + \delta_6(\zeta^2 - m)^{-1}\\ + \delta_7\zeta^{-2}(\zeta^2 - m)^{-1},\end{aligned} \tag{3.14}$$

where

$$\begin{aligned}\delta_1 = \tfrac{1}{8}k_1^2(1 - m^2); \quad \delta_2 = \tfrac{1}{4}(1 + m)\beta_1; \quad \delta_3 = -(1 + 2\gamma_1)\delta_1;\\ \delta_4 = -m\delta_1; \qquad \delta_5 = \tfrac{1}{8}k_1^2(1+m)(1 + 2m^2 - 3m\gamma_1);\\ \delta_6 = -(1+m^2)\delta_1; \qquad \delta_7 = 2(1+m^2)\delta_2;\end{aligned} \tag{3.15}$$

are also analytic for $|\zeta| > 1$, and yield the solution to the problem for which

$$\tau_{nn} + i\tau_{ns} = -K^2\tau_I^{(1)}, \tag{3.16}$$

on $|\zeta| = 1$, and for which the displacement $u + iv$, given in (2.10), is single valued.

We now substitute the expressions for $\Omega(z)$, $\omega(z)$, determined in (3.9), (3.10), (3.13) and (3.14) into (2.7) and (2.8). Noting that the inverse of (3.7) is

$$2\zeta = z + (z^2 - 4m)^{1/2} \tag{3.17}$$

it is found that, as $z \to \infty$,

$$\begin{aligned}\zeta &= z - mz^{-1} - m^2z^{-3} + O(z^{-5}),\\ \zeta^{-1} &= z^{-1} + mz^{-3} + 2m^2z^{-5} + O(z^{-7}),\\ (\zeta^2 - m)^{-1} &= z^{-2} + 3mz^{-4} + O(z^{-6}).\end{aligned}$$

Whence, on replacing z by $re^{i\theta}$, we obtain, for $r >> 1$,

$$\phi_0 = C_1 \log r \rightarrow (C_2 + C_3r^{-2})\cos 2\theta + C_4r^{-2}\cos 4\theta + O(r^{-4}), \tag{3.18}$$

$$\psi_0 = C_1\theta - (C_2 + C_3r^{-2})\sin 2\theta - C_4r^{-2}\sin 4\theta + O(r^{-4}), \tag{3.19}$$

where

$$C_1 = \beta_2, \quad C_4 = mC_2 = m\beta_1, \quad C_3 = -\tfrac{1}{2}\left[m\beta_2 + (1+m^2)\beta_1\right], \tag{3.20}$$

and

$$\phi_1 = D_1 r\theta \sin\theta + \left(\mu' D_2 r \log r + D_3 r + D_4 r^{-1}\right)\cos\theta$$
$$+ \left(D_5 r^{-1} + D_6 r^{-3}\right)\cos 3\theta + D_7 r^{-3}\cos 5\theta + O(r^{-5}), \tag{3.21}$$

$$\psi_1 = -\mu' D_1 r\theta\cos\theta + \left(D_2 r\log r + D_3 r - D_4 r^{-1}\right)\sin\theta$$
$$- \left(D_5 r^{-1} + D_6 r^{-3}\right)\sin 3\theta - D_7 r^{-3}\sin 5\theta + O(r^{-5}), \tag{3.22}$$

where

$$\begin{gathered} D_1 = -D_2 = 2(1+\gamma_1)\delta_1; \qquad D_3 = (1 + 2\gamma_1 - m)\delta_1; \\ D_4 = m\delta_3 - \delta_5 - \delta_6; \qquad D_5 = \delta_2 - m\delta_1; \\ D_6 = \tfrac{1}{6}\left(3m^2\delta_3 - 4m\delta_5 - 6m\delta_6 - 2\delta_7\right); \qquad D_7 = \tfrac{1}{2}m(4\delta_2 - 3m\delta_1). \end{gathered} \tag{3.23}$$

Comparison of these results with the expressions for the scattered field in (2.16) indicates that, to this approximation, we may express the scattered field as

$$\begin{aligned} \phi_s &= \sum_{j=0}^{3} A_j H_j^{(1)}(k_1 r)\cos j\theta, \\ \psi_s &= \sum_{j=1}^{3} B_j H_j^{(1)}(kr)\sin j\theta, \end{aligned} \tag{3.24}$$

where the asymptotic values of the Hankel function are, for small ξ,

$$H_0(\xi) = 1 + \frac{2i}{\pi}[\gamma + \log(\xi/2)] + O(\xi^2\log\xi), \tag{3.25a}$$

$$H_1(\xi) = -\frac{2i}{\pi\xi} + [2\gamma - \pi i - 1 + 2\log(\xi/2)]\frac{i\xi}{2\pi} + O(\xi^3\log\xi), \tag{3.25b}$$

$$H_j(\xi) = -\frac{i(j-2)!}{\pi}\left(\frac{2}{\xi}\right)^j\left(j - 1 + \frac{\xi^2}{4}\right) + O(\xi^{4-j}), \qquad j = 2, 3, \tag{3.25c}$$

where γ is Euler's constant, and ξ is the argument of the Hankel functions in (3.24). In (3.24) we take advantage of the symmetry of the problem, and note that for $j \geqslant 4$ the terms are small in this approximation.

Following [1], we match the expressions in (3.24), as $kr \to 0$, with the corresponding expressions in (3.18) to (3.22), with the result that, approximately,

$$\phi_s = \phi_0 + ik_1\phi_1 + \phi^* + \phi_R; \qquad \psi_s = \psi_0 + ik_1\psi_1 + \psi^* + \psi_R; \tag{3.26}$$

https://doi.org/10.1017/S0334270000004367 Published online by Cambridge University Press

where ϕ^*, ψ^*, ϕ_R, ψ_R, give zero stresses, as in (2.13) and (2.14), and are given by

$$\phi^* = X_0 \log r + X_1(r\cos\theta\log r - r\theta\sin\theta) + \left(Y_1 r^{-1} + Z_1 r\right)\cos\theta + X_2 r^{-2}\cos 2\theta + X_3 r^{-3}\cos 3\theta, \qquad (3.27)$$

$$\psi^* = \mu'\left[X_0\theta + X_1(r\sin\theta\log r + r\theta\cos\theta) - \left(Y_1 r^{-1} - Z_1 r\right)\sin\theta - X_2 r^{-2}\sin 2\theta - X_3 r^{-3}\sin 3\theta\right], \qquad (3.28)$$

$$\phi_R = C_1 r\cos\theta, \qquad \psi_R = C_1 r\sin\theta, \qquad (3.29)$$

where the constants X_0, X_1, X_2, X_3, Y_1, Z_1, C_1, are arbitrary. The matching procedures provide enough equations to determine these constants, and the A_j, B_j, in (3.24), in order that (3.26) is satisfied. The result is:

$$\gamma_1 A_0 = \tfrac{1}{4}\pi i k_1^2\left(1 + m^2 - 2m\gamma_1\right), \qquad (3.30)$$

$$A_1 k_1 = -B_1 k = \tfrac{1}{4}\pi k_1^3(1 - m^2), \qquad (3.31)$$

$$A_2 = -B_2 = \tfrac{1}{2}\pi i k_1^2(m - \gamma_1), \qquad (3.32)$$

$$A_3 = -B_3 k_1/k = \pi k_1^4(1 + m)\left[\gamma_1 - m^2\right]/16, \qquad (3.33)$$

with

$$\pi(1 - \mu')X_0 = 2iA_0; \qquad \pi(1 - \mu')X_1 = ik_1A_1;$$

$$X_2 = 2(m - \gamma_1); \qquad \pi k_1^3 X_3 = -16iA_3;$$

$$Y_1 = \tfrac{1}{2}ik_1(m^2 - 1); \qquad 4(1 - \mu')Z_1 = ik_1^3(1 - m^2)(\log kk_1 - b);$$

and

$$C_1 = \tfrac{1}{8}ik_1^3(1 - m^2)\left[(1 + \gamma_1)\{b(1 + \mu') - 2\log k - 2\mu'\log k_1\} - (1 + 2\gamma_1 - m)\right]. \qquad (3.34)$$

We note that A_3, B_3 are $O(k^4)$, and are, therefore, to be neglected, to the order of approximations that have been made. The result is that, to the smallest order, the scattered wave is

$$\phi_s = \sum_{j=0}^{2} A_j \cos j\theta\, H_j^{(1)}(k_1 r); \qquad \psi_s = \sum_{j=1}^{2} B_j \sin j\theta\, H_j^{(1)}(kr); \qquad (3.35)$$

where the constants A_0, A_1, A_2, B_1, B_2 are $O(k^2)$, and are given in (3.30) to (3.32).

There are two special cases. Firstly, the ellipse is a circle when $m = 0$, and (3.35) then reduces to the scattered field determined in [1] for a circular cavity. The second case $m = 1$ is more interesting, since the obstacle becomes a stress free crack of length 4 in the nondimensional coordinates, the orientation of the crack being parallel to the wave direction. Note that when $m = 1$, in (3.30)–(3.32),

$$A_1 = B_1 = 0, \qquad \gamma_1 A_0 = A_2 = -B_2 = \tfrac{1}{2}\pi i k_1^2(1-\gamma_1), \tag{3.36}$$

where $1-\gamma_1 = \lambda/(\lambda+\mu)$.

Calculation of the stress singularities at $z = \pm 2$ proceeds as follows. We let $m = 1$ in (3.9), (3.10), (3.13), (3.14), whereupon the substitution (3.17) yields

$$\Omega_0(z) = \tfrac{1}{2}\nu\left[z - (z^2-4)^{1/2}\right], \qquad \Omega_1(z) = \tfrac{1}{4}\nu\left[z^2 - 2 - z(z^2-4)^{1/2}\right], \tag{3.37}$$

$$\omega_0(z) = 2\nu(z^2-4)^{-1/2},$$
$$\omega_1(z) = \tfrac{1}{4}\nu\left[4z(z^2-4)^{-1/2} + z(z^2-4)^{1/2} - 2 - z^2\right], \tag{3.38}$$

where

$$\nu = \tfrac{1}{2}(1-\gamma_1)k_1^2,$$

and we take a branch cut on $|x| < 2$, $y = 0$, such that $(z^2-4)^{1/2} \sim z$, as $|z| \to \infty$. Note that $\Omega(z)$, $\omega(z)$ have singularities at the crack tips $z = \pm 2$. Substitution in (2.11) shows, however, that the displacement is finite at $z = \pm 2$. Suppose also that near the crack tips

$$z = \pm 2 \pm \varepsilon e^{i\alpha}$$

where $0 < \varepsilon << 1$, and $|\alpha| < \pi$. It may then be shown that the singular parts of the stresses near the crack tips are given by

$$\tau_{xx} + \tau_{yy} = -4\nu_{\pm}\varepsilon^{-1/2}\cos\tfrac{1}{2}\alpha, \quad \tau_{xx} - \tau_{yy} = 2\nu_{\pm}\varepsilon^{-1/2}\sin\alpha\sin(3\alpha/2), \tag{3.39}$$

$$\tau_{xy} = -\nu_{\pm}\varepsilon^{-1/2}\sin\alpha\cos(3\alpha/2),$$

where

$$\nu_{\pm} = k_1^2(1 \pm ik_1)(1-2\mu')/2,$$

and the signs refer to the cases $z = \pm 2$, respectively.

4. Additional diffraction problems

In this section we consider examples of other orientations of the ellipse, and diffraction of shear waves. We first consider the problem of the diffraction of the P wave

$$\phi_I = \exp[i(Ky' - \sigma t)], \qquad \psi_I = 0, \tag{4.1}$$

travelling in the positive y direction, with the elliptic obstacle as in (3.1). We follow the analysis of the previous section, but now, in (3.5),

$$\tau_I^{(0)} = \tau_I^{(1)}/y = (1-\mu')\left(\gamma_1 e^{-2i\alpha} - 1\right), \tag{4.2}$$

with γ_1 and $e^{-2i\alpha}$ given in (3.4) and (3.7). The subsequent calculation is similar to that in Section 3, with a number of minor algebraic changes. The final results are that the scattered wave is given by

$$\phi_s = A_0 H_0^{(1)}(k_1 r) + A_1' H_1^{(1)}(k_1 r)\sin\theta + A_2 H_2^{(1)}(k_1 r)\cos 2\theta, \tag{4.3}$$

$$\psi_s = B_1' H_1^{(1)}(kr)\cos\theta + B_2 H_2^{(1)}(kr)\sin 2\theta, \tag{4.4}$$

where

$$\gamma_1 A_0 = \tfrac{1}{4}\pi i k_1^2\left(1 + m^2 + 2m\gamma_1\right), \tag{4.5}$$

$$A_1' k_1 = B_1' k = \tfrac{1}{4}\pi k_1^3(1-m^2), \tag{4.6}$$

$$A_2 = -B_2 = \tfrac{1}{2}\pi i k_1^2(m+\gamma_1), \tag{4.7}$$

and terms of $O(k^4 \log k)$ are neglected.

When $m = 1$ there is a crack at right angles to the direction of wave propagation. It may be shown that in this case $\Omega_1(z)$, $\omega_1(z)$ vanish, and

$$(1-\mu')\Omega_0(z) = \tfrac{1}{4}k_1^2\left[z - (z^2-4)^{1/2}\right], \qquad (1-\mu')\omega_0(z) = k_1^2(z^2-4)^{-1/2}. \tag{4.8}$$

Substitution of these expressions in (2.9), (2.10) and (2.11) again yields the necessary result, that the displacements at the crack tips $z = \pm 2$ are finite. The stress concentration at $z = 2$ is given by

$$\tau_{xx} + \tau_{yy} = -2k_1^2\varepsilon^{-1/2}\cos\tfrac{1}{2}\alpha, \qquad \tau_{xx} - \tau_{yy} = +k_1^2\varepsilon^{-1/2}\sin\alpha\sin(3\alpha/2),$$
$$\tau_{xy} = -\tfrac{1}{2}k_1^2\varepsilon^{-1/2}\sin\alpha\cos(3\alpha/2), \tag{4.9}$$

when $z = 2 + \varepsilon e^{i\alpha}$, and $0 < \varepsilon << 1$. Equivalent formulae for the stress singularity at $z = -2$ are obtained by noting the symmetry about the y axis.

The second example is the propagation of the shear wave

$$\phi_I = 0, \psi_I = \exp[i(K'x' - \sigma t)], \tag{4.10}$$

parallel to the x axis. Noting that in this case $K' = \rho\sigma^2/\mu$, $\tau_{xx} = \tau_{yy} = 0$, and $\tau_{xy} = -K'^2\exp[i(K'x' - \sigma t)]$, the analysis follows the previous examples, with some algebraic differences. The results are that the scattered wave is given by (2.16), the only non-zero coefficients being

$$A_2 = B_2 = -\tfrac{1}{2}\pi i k^2(1-\mu')^{-1}, \tag{4.11}$$

$$A_1' k_1 = B_1' k = \tfrac{1}{4}\pi k^3(m^2-1), \tag{4.12}$$

and A_3', B_3' are $O(k^4)$. When $m = 1$ it may be shown that

$$\Omega_0(\zeta) = 2\nu_2/\zeta, \qquad \Omega_1(\zeta) = \nu_2/\zeta^2, \tag{4.13}$$

$$\omega_0(\zeta) = 4\nu_2\zeta^{-1}(\zeta^2 - 1)^{-1}, \qquad \omega_1(\zeta) = \nu_2\zeta^{-2}\left[1 + 4(\zeta^2 - 1)^{-1}\right], \tag{4.14}$$

where

$$\nu_2 = -\tfrac{1}{4}ik^2/(1 - \mu').$$

It may then be shown from these expressions that the singular parts of the stresses at the crack tips $z = \pm 2$ are given by

$$\tau_{xx} + \tau_{yy} = 2k^2(1 \pm ik)\varepsilon^{-1/2}\sin(\alpha/2),$$
$$\tau_{xx} - \tau_{yy} = \tfrac{1}{2}k^2(1 \pm ik)\varepsilon^{-1/2}[\sin(5\alpha/2) + 3\sin(\alpha/2)],$$
$$\tau_{xy} = -\tfrac{1}{4}k^2(1 \pm ik)\varepsilon^{-1/2}[\cos(5\alpha/2) + 3\cos(\alpha/2)].$$

Finally, in the case where the direction of the incident shear wave is perpendicular to the crack, we take

$$\phi_I = 0, \qquad \psi_I = \exp[i(K'y' - \sigma t)], \tag{4.15}$$

when the scattered wave is given by (2.16), with all coefficients zero except

$$A_2 = B_2 = \tfrac{1}{2}\pi ik^2/(1 - \mu'), \tag{4.16}$$

$$A_1k_1 = -B_1k = \tfrac{1}{4}\pi k^3(m^2 - 1). \tag{4.17}$$

When $m = 1$, the stress field near $z = 2$ is

$$\tau_{xx} = -\tfrac{1}{4}k^2\varepsilon^{-1/2}(7\sin(\alpha/2) + \sin(5\alpha/2)),$$
$$\tau_{yy} = \tfrac{1}{4}k^2\varepsilon^{-1/2}(\sin(5\alpha/2) - \sin(\alpha/2)),$$
$$\tau_{xy} = \tfrac{1}{4}k^2\varepsilon^{-1/2}(3\cos(\alpha/2) + \cos(5\alpha/2)).$$

5. Conclusion

Other methods of using asymptotic matching techniques for elastic wave scattering are described by Datta [2], as well as by Gautesen [3] and Viswanathan and Chandra [6]. The main point of the work described in this paper is that it allows the use of Muskhelishvili's complex variable and conformal mapping techniques for low frequency two-dimensional applications. For a finite, stress free crack, the determination of the leading singular terms in the stresses is a particular feature of the method.

Acknowledgement

One of us (T. Tran Cong) has been supported by a post-doctoral fellowship provided by the Australian Research Grants Scheme under grant number F 8115691 I.

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