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Existence and conditional energetic stability of solitary gravity–capillary water waves with constant vorticity

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We present an existence and stability theory for gravity–capillary solitary waves with constant vorticity on the surface of a body of water of finite depth. Exploiting a rotational version of the classical variational principle, we prove the existence of a minimizer of the wave energy $\mathcal{H}$ subject to the constraint $\mathcal{I} = 2\mu$, where $\mathcal{I}$ is the wave momentum and $0 < \mu \ll 1$. Since $\mathcal{H}$ and $\mathcal{I}$ are both conserved quantities, a standard argument asserts the stability of the set D_μ of minimizers: solutions starting near D_μ remain close to D_μ in a suitably defined energy space over their interval of existence. In the applied mathematics literature solitary water waves of the present kind are described by solutions of a Korteweg–de Vries equation (for strong surface tension) or a nonlinear Schrödinger equation (for weak surface tension). We show that the waves detected by our variational method converge (after an appropriate rescaling) to solutions of the appropriate model equation as $\mu \downarrow 0$.

Keywords: water waves; solitary waves; vorticity; calculus of variations

2010 *Mathematics subject classification:* Primary 76B15

1. Introduction

1.1. Variational formulation of the hydrodynamic problem

1.1.1. *The water-wave problem*

In this paper we consider a two-dimensional perfect fluid bounded below by a flat rigid bottom $\{y = 0\}$ and above by a free surface $\{y = d + \eta(x, t)\}$. The fluid has unit density and flows under the influence of gravity and surface tension with constant vorticity ω so that the velocity field $(u(x, y, t), v(x, y, t))$ in the fluid domain $\Sigma_\eta = \{0 < y < d + \eta(x, t)\}$ satisfies $v_x - u_y = \omega$. We study waves that are perturbations of underlying shear flows given by $\eta = 0$ and $(u, v) = (\omega(d - y), 0)$

(which may be a good description of tidal currents: see Constantin [12, ch. 2.3.2]) and are evanescent as $x \to \pm\infty$. In terms of a generalized velocity potential ϕ such that $(u, v) = (\phi_x + \omega(d-y), \phi_y)$ and stream function ψ such that $(u, v) = (\psi_y, -\psi_x)$, the governing equations are

$$
\begin{aligned}
\Delta\phi &= 0, && 0 < y < d + \eta, \\
\phi_y &= 0, && y = 0, \\
\eta_t &= \phi_y - \eta_x\phi_x + \omega\eta\eta_x, && y = d + \eta, \\
\phi_t &= -\tfrac{1}{2}|\nabla\psi|^2 - \omega\psi - g\eta + \beta\left[\frac{\eta_x}{\sqrt{1+\eta_x^2}}\right]_x, && y = d + \eta,
\end{aligned}
$$

with $\eta(x, t), \phi(x, y, t), \psi(x, y, t) + \frac{1}{2}\omega(d - y)^2 \to 0$ as $x \to \pm\infty$, where g and β are the acceleration due to gravity and the (positive) coefficient of surface tension, respectively (see Constantin *et al*. [14]).

At this point it is convenient to introduce dimensionless variables

$$
(x', y') = \frac{1}{d}(x, y), \qquad t' = \left(\frac{g}{d}\right)^{1/2} t,
$$

$$
\eta'(x', t') = \frac{1}{d}\eta(x, t), \qquad \phi'(x', t') = \frac{1}{(gd^3)^{1/2}}\phi(x, t), \qquad \psi'(x', t') = \frac{1}{(gd^3)^{1/2}}\psi(x, t)
$$

and parameters $\omega' = \omega(d/g)^{1/2}$, $\beta' = \beta/gd^2$; one obtains the equations

$$
\begin{aligned}
\Delta\phi &= 0, && 0 < y < 1 + \eta, && (1.1)\\
\phi_y &= 0, && y = 0, && (1.2)\\
\eta_t &= \phi_y - \eta_x\phi_x + \omega\eta\eta_x, && y = 1 + \eta, && (1.3)\\
\phi_t &= -\tfrac{1}{2}|\nabla\psi|^2 - \omega\psi - \eta + \beta\left[\frac{\eta_x}{\sqrt{1+\eta_x^2}}\right]_x, && y = 1 + \eta, && (1.4)
\end{aligned}
$$

in which the primes have been dropped for notational simplicity. In particular, we seek *solitary-wave solutions* of (1.1)–(1.4), that is, waves of permanent form that propagate from right to left with constant (dimensionless) speed ν, so that $\eta(x, t) = \eta(x + \nu t)$ (and of course $\eta(x + \nu t) \to 0$ as $x + \nu t \to \pm\infty$).

1.1.2. *Formulation as a Hamiltonian system*

We proceed by reducing the hydrodynamic problem to a pair of non-local coupled evolutionary equations for the variables η and $\xi = \phi|_{y=1+\eta}$. For fixed η and ξ, let ϕ denote the unique solution to the boundary-value problem

$$
\begin{aligned}
\Delta\phi &= 0, \quad 0 < y < 1 + \eta, \\
\phi &= \xi, \quad y = 1 + \eta, \\
\phi_y &= 0, \quad y = 0,
\end{aligned}
$$

and denote the harmonic conjugate of ϕ by $\tilde{\psi}$. We define the *Hilbert transform* $H(\eta)$ and *Dirichlet–Neumann operator* $G(\eta)$ for this boundary-value problem by

$$
H(\eta)\xi = \tilde{\psi}|_{y=1+\eta}, \qquad G(\eta)\xi = (\phi_y - \eta_x\phi_x)|_{y=1+\eta},
$$

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so that $G(\eta) = -\partial_x H(\eta)$ and note that the boundary conditions (1.3), (1.4) can be written as

$$\begin{aligned}\eta_t &= G(\eta)\xi + \omega\eta\eta_x,\\ \xi_t &= -\frac{1}{2(1+\eta_x^2)}(\xi_x^2 - (G(\eta)\xi)^2 - 2\eta_x\xi_x G(\eta)\xi)\\ &\quad + \omega\eta\xi_x - \omega H(\eta)\xi - \eta + \beta\left[\frac{\eta_x}{\sqrt{1+\eta_x^2}}\right]_x.\end{aligned}$$

Wahlén [25] observed that the above equations can be formulated as the Hamiltonian system

$$\begin{pmatrix}\eta_t\\ \xi_t\end{pmatrix} = \begin{pmatrix}0 & 1\\ -1 & \omega\partial_x^{-1}\end{pmatrix}\begin{pmatrix}\delta_\eta\mathcal{H}\\ \delta_\xi\mathcal{H}\end{pmatrix}, \tag{1.5}$$

in which

$$\mathcal{H}(\eta,\xi) = \int_{-\infty}^{\infty}(\tfrac{1}{2}\xi G(\eta)\xi + \omega\xi\eta\eta_x + \tfrac{1}{6}\omega^2\eta^3 + \tfrac{1}{2}\eta^2 + \beta(\sqrt{1+\eta_x^2}-1))\,\mathrm{d}x \tag{1.6}$$

(note that the well-known formulation of the water-wave problem by Zakharov [26] is recovered in the irrotational case $\omega = 0$). This Hamiltonian system has the conserved quantities $\mathcal{H}(\eta,\xi)$ (total energy) and

$$\mathcal{I}(\eta,\xi) = \int_{-\infty}^{\infty}(\xi\eta_x + \tfrac{1}{2}\omega\eta^2)\,\mathrm{d}x \tag{1.7}$$

(total horizontal momentum), which satisfies the equation

$$\begin{pmatrix}\eta_x\\ \xi_x\end{pmatrix} = \begin{pmatrix}0 & 1\\ -1 & \omega\partial_x^{-1}\end{pmatrix}\begin{pmatrix}\delta_\eta\mathcal{I}\\ \delta_\xi\mathcal{I}\end{pmatrix}; \tag{1.8}$$

these quantities are associated with its independence of t and x, respectively. According to (1.5) and (1.8), a solution of the form $\eta(x,t) = \eta(x+\nu t)$, $\xi(x,t) = \xi(x+\nu t)$ is characterized as a critical point of the total energy subject to the constraint of fixed momentum (cf. Benjamin [4]). It is therefore a critical point of the functional $\mathcal{H} - \nu\mathcal{I}$, where the speed of the wave is given by the Lagrange multiplier ν. This functional depends on the single independent variable $x+\nu t$, which we now abbreviate to x.

A similar variational principle for waves of permanent form with a general distribution of vorticity has been used by Groves and Wahlén [16] in an existence theory for solitary waves. Groves and Wahlén interpreted their variational functional as an action functional and derived a formulation of the hydrodynamic problem as an infinite-dimensional spatial Hamiltonian system; a rich solution set is found using a centre-manifold reduction technique to convert it into a Hamiltonian system with a finite number of degrees of freedom.

In this paper we present a direct existence theory for minimizers of $\mathcal{H}$ subject to the constraint $\mathcal{I} = 2\mu$ for $0 < \mu < \mu_0$, where μ_0 is a fixed positive constant chosen small enough for the validity of our calculations. We seek constrained minimizers in a two-step approach.

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(1) *Fix* $\eta \neq 0$ *and minimize* $\mathcal{H}(\eta, \cdot)$ *over* $T_\mu = \{\xi \colon \mathcal{I}(\eta, \xi) = 2\mu\}$*.* This problem (of minimizing a quadratic functional over a linear manifold) admits a unique global minimizer ξ_η.

(2) *Minimize* $\mathcal{J}_\mu(\eta) := \mathcal{H}(\eta, \xi_\eta)$ *over* $\eta \in U \setminus \{0\}$*.* Here U is a fixed ball centred upon the origin in a suitable function space. Because ξ_η minimizes $\mathcal{H}(\eta, \cdot)$ over T_μ, there exists a Lagrange multiplier ν_η such that

$$G(\eta)\xi_\eta + \omega\eta\eta' = \nu_\eta\eta',$$

and straightforward calculations show that

$$\xi_\eta = \nu_\eta G(\eta)^{-1}\eta' - \tfrac{1}{2}\omega G(\eta)^{-1}(\eta^2)',$$

$$\nu_\eta = \left(\frac{1}{2}\int_{-\infty}^{\infty} \eta' G(\eta)^{-1}\eta' \,\mathrm{d}x\right)^{-1} \left(\mu - \frac{\omega}{4}\int_{-\infty}^{\infty} \eta^2 \,\mathrm{d}x + \frac{\omega}{4}\int_{-\infty}^{\infty} (\eta^2)' G(\eta)^{-1}\eta' \,\mathrm{d}x\right)$$

so that

$$\mathcal{J}_\mu(\eta) = \mathcal{K}(\eta) + \frac{(\mu + \mathcal{G}(\eta))^2}{\mathcal{L}(\eta)}, \tag{1.9}$$

where

$$\mathcal{G}(\eta) = \frac{\omega}{4}\int_{-\infty}^{\infty} \eta^2 K(\eta)\eta \,\mathrm{d}x - \frac{\omega}{4}\int_{-\infty}^{\infty} \eta^2 \,\mathrm{d}x, \tag{1.10}$$

$$\begin{aligned} \mathcal{K}(\eta) = {} & \int_{-\infty}^{\infty} \left(\frac{\eta^2}{2} + \beta[\sqrt{1 + \eta'^2} - 1]\right) \mathrm{d}x \\ & - \frac{\omega^2}{2}\int_{-\infty}^{\infty} \frac{\eta^2}{2} K(\eta) \frac{\eta^2}{2} \,\mathrm{d}x + \frac{\omega^2}{6}\int_{-\infty}^{\infty} \eta^3 \,\mathrm{d}x, \end{aligned} \tag{1.11}$$

$$\mathcal{L}(\eta) = \frac{1}{2}\int_{-\infty}^{\infty} \eta K(\eta)\eta \,\mathrm{d}x \tag{1.12}$$

and $K(\eta) = -\partial_x G(\eta)^{-1}\partial_x$. This computation also shows that the dimensionless speed of a solitary wave corresponding to a constrained minimizer η of $\mathcal{H}$ is

$$\nu = \frac{\mu + \mathcal{G}(\eta)}{\mathcal{L}(\eta)}.$$

This two-step approach to the constrained minimization problem was introduced in a corresponding theory for irrotational solitary waves by Buffoni [5] who used a conformal mapping due to Babenko [1,2] to transform $\mathcal{J}_\mu$ into another functional $\tilde{\mathcal{J}}_\mu$ depending only upon $H(0)$, and hence simplified the necessary variational analysis. Buffoni established the existence of a (non-zero) minimizer of $\tilde{\mathcal{J}}_\mu$ for strong surface tension (see Buffoni [5]) and obtained partial results in this direction for weak surface tension (see Buffoni [6, 7]). A method for completing his results for weak surface tension was sketched in a short note by Groves and Wahlén [17]; in the present paper we give complete details, including non-zero vorticity in our treatment and working directly with the original physical variables. Although versions of the Babenko transformation for non-zero constant vorticity have been published (see Constantin and Varvaruca [13] and Martin [23]), finding minimizers of $\mathcal{J}_\mu$ over $U \setminus \{0\}$ has the advantage of immediately yielding precise information on solutions to the original water-wave equations (1.1)–(1.4).

1.1.3. *Functional-analytic framework*

An appropriate functional-analytic framework for the above variational problem is introduced in § 2. We work with the function spaces

$$H^r(\mathbb{R}) = \overline{(\mathcal{S}(\mathbb{R}), \|\cdot\|_r)}, \qquad \|\eta\|_r^2 := \int_{-\infty}^{\infty} (1+k^2)^r |\hat{\eta}|^2 \, \mathrm{d}k,$$

for $r \in \mathbb{R}$ (the standard Sobolev spaces) and

$$H_\star^{1/2}(\mathbb{R}) = \overline{(\mathcal{S}(\mathbb{R}), \|\cdot\|_{H_\star^{1/2}(\mathbb{R})})}, \qquad \|\eta\|_{H_\star^{1/2}(\mathbb{R})}^2 := \int_{-\infty}^{\infty} (1+k^2)^{-1/2} k^2 |\hat{\eta}|^2 \, \mathrm{d}k,$$

$$H_\star^{-1/2}(\mathbb{R}) = \overline{(\bar{\mathcal{S}}(\mathbb{R}), \|\cdot\|_{H_\star^{-1/2}(\mathbb{R})})}, \qquad \|\eta\|_{H_\star^{-1/2}(\mathbb{R})}^2 := \int_{-\infty}^{\infty} (1+k^2)^{1/2} k^{-2} |\hat{\eta}|^2 \, \mathrm{d}k;$$

here $\overline{(\mathcal{S}(\mathbb{R}), \|\cdot\|)}$ denotes the completion of the inner product space constructed by equipping the Schwartz class $\mathcal{S}(\mathbb{R})$ (or the subclass $\bar{\mathcal{S}}(\mathbb{R})$ of Schwartz-class functions with zero mean) with the norm $\|\cdot\|$, and $\hat{\eta} = \mathcal{F}[\eta]$ is the Fourier transform of η.

The mathematical analysis of $G(\eta)$ and $K(\eta)$ is complicated by the fact that they are defined in terms of boundary-value problems in the variable domain Σ_η. Lannes [20, ch. 2 and 3] has presented a comprehensive theory for handling such boundary-value problems by transforming them into serviceable nonlinear elliptic problems in the fixed domain Σ_0, and here we adapt Lannes's methods to our specific requirements. Our main results are stated in the following theorem, according to which equations (1.10)–(1.12) define analytic functionals $\mathcal{G}, \mathcal{K}, \mathcal{L} \colon W^{s+3/2} \to \mathbb{R}$ for $s > 0$. In accordance with this theorem we take $U = B_M(0) \subseteq H^2(\mathbb{R})$, where $M > 0$ is chosen small enough so that $\bar{B}_M(0) \subseteq H^2(\mathbb{R})$ lies in $W^{s+3/2}$ and for the validity of our calculations.

Theorem 1.1. *Choose $h_0 \in (0,1)$ and define $W = \{\eta \in W^{1,\infty}(\mathbb{R}) \colon 1 + \inf \eta > h_0\}$ and $W^r = H^r \cap W$ for $r \geqslant 0$.*

(i) *The Dirichlet–Neumann operator $G(\eta)$ is an isomorphism*

$$H_\star^{1/2}(\mathbb{R}) \to H_\star^{-1/2}(\mathbb{R})$$

for each $\eta \in W$.

(ii) *The Dirichlet–Neumann operator $G \colon W \to \mathcal{L}(H_\star^{1/2}(\mathbb{R}), H_\star^{-1/2}(\mathbb{R}))$ and Neumann–Dirichlet operator $G^{-1} \colon W \to \mathcal{L}(H_\star^{-1/2}(\mathbb{R}), H_\star^{1/2}(\mathbb{R}))$ are analytic.*

(iii) *The operator $K \colon W^{s+3/2} \to \mathcal{L}(H^{s+3/2}(\mathbb{R}), H^{s+1/2}(\mathbb{R}))$ is analytic for each $s > 0$.*

1.2. Heuristics

The existence of small-amplitude solitary waves is predicted by studying the dispersion relation for the linearized version of (1.1)–(1.4). Linear waves of the form $\eta(x,t) = \cos k(x + \nu t)$ exist whenever

$$1 + \beta k^2 - \omega\nu - \nu^2 f(k) = 0, \quad f(k) = |k| \coth |k|,$$

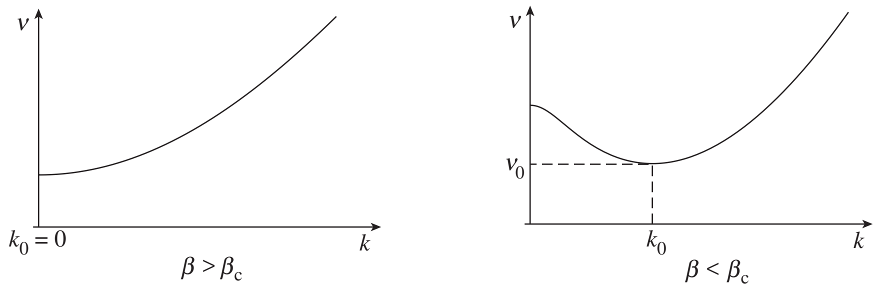

Figure 1. Dispersion relation for linear water waves.

that is, whenever

$$\nu = -\frac{\omega}{2f(k)} + \frac{1}{2}\left(\frac{\omega^2}{f(k)^2} + \frac{4(1+\beta k^2)}{f(k)}\right)^{1/2}.$$

The function $k \mapsto \nu(k)$, $k \geqslant 0$, has a unique global minimum $\nu_0 = \nu(k_0)$, and one finds that $k_0 > 0$ for $\beta < \beta_c$ and $k_0 = 0$ (with $\nu_0 = \nu(0) = \frac{1}{2}(-\omega + \sqrt{\omega^2+4})$) for $\beta > \beta_c$, where

$$\beta_c = \tfrac{1}{6}(\omega^2 + 2 - \omega\sqrt{\omega^2+4})$$

(see figure 1). For later use let us also note that

$$g(k) := 1 + \beta k^2 - \omega\nu_0 - \nu_0^2 f(k) \geqslant 0, \quad k \in \mathbb{R},$$

with equality precisely when $k = \pm k_0$.

Bifurcations of nonlinear solitary waves are expected whenever the linear group and phase speeds are equal, so that $\nu'(k) = 0$ (see Dias and Kharif [15, § 3]). We therefore expect the existence of small-amplitude solitary waves with speed near ν_0; the waves bifurcate from laminar flow when $\beta > \beta_c$ and from a linear periodic wave train with frequency $k_0\nu(k_0)$ when $\beta < \beta_c$. Model equations for both types of solution have been derived by Johnson [19, §§ 4 and 5].

Case 1 ($\beta > \beta_c$). The appropriate model equation is the Korteweg–de Vries equation

$$-2u_T - \left(\beta - \frac{\nu_0^2}{3}\right)u_{XXX} + (\omega^2+3)uu_X = 0, \tag{1.13}$$

in which

$$\eta = \mu^{2/3}u(X,T) + O(\mu^{4/3}), \qquad X = \mu^{1/3}(x + \nu_0 t), \quad T = 2(\omega^2+4)^{-1/2}\mu^{2/3}t.$$

At this level of approximation, a solution to (1.13) of the form $u(X,T) = \phi(X + \nu_{\mathrm{KdV}}T)$ with $\phi(X) \to 0$ as $X \to \pm\infty$ corresponds to a solitary water wave with speed

$$\nu = \nu_0 + 2(\omega^2+4)^{-1/2}\mu^{2/3}\nu_{\mathrm{KdV}} = -\tfrac{1}{2}\omega + \tfrac{1}{2}(\omega^2+4)^{1/2} + 2(\omega^2+4)^{-1/2}\mu^{2/3}\nu_{\mathrm{KdV}}.$$

The following lemma gives a variational description of the set of such solutions; the corresponding solitary waves are sketched in figure 2.

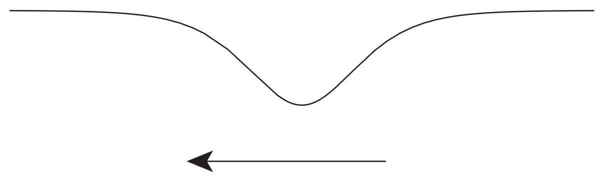

Figure 2. Korteweg–de Vries theory predicts the existence of small-amplitude solitary waves of depression for strong surface tension.

LEMMA 1.2.

(i) *The set of solutions to the ordinary differential equation*

$$-(\beta - \tfrac{1}{3}\nu_0^2)\phi'' - 2\nu_{\mathrm{KdV}}\phi + \tfrac{3}{2}(\tfrac{1}{3}\omega^2 + 1)\phi^2 = 0$$

satisfying $\phi(X) \to 0$ *as* $X \to \infty$ *is* $D_{\mathrm{KdV}} = \{\phi_{\mathrm{KdV}}(\cdot + y) \colon y \in \mathbb{R}\}$, *where*

$$\nu_{\mathrm{KdV}} = -\frac{2(\frac{3}{16})^{2/3}(\frac{1}{3}\omega^2 + 1)^{4/3}}{(\beta - \frac{1}{3}\nu_0^2)^{1/3}(\omega^2 + 4)^{1/3}},$$

$$\phi_{\mathrm{KdV}}(x) = -\frac{\sqrt{3}(\frac{3}{16})^{1/6}(\frac{1}{3}\omega^2 + 1)^{1/3}}{(\beta - \frac{1}{3}\nu_0^2)^{1/3}(\omega^2 + 4)^{1/3}} \operatorname{sech}^2\left(\frac{(\frac{3}{16})^{1/3}(\frac{1}{3}\omega^2 + 1)^{2/3}x}{(\beta - \frac{1}{3}\nu_0^2)^{2/3}(\omega^2 + 4)^{1/6}}\right).$$

These functions are precisely the minimizers of the functional $\mathcal{E}_{\mathrm{KdV}} \colon H^1(\mathbb{R}) \to \mathbb{R}$ *given by*

$$\mathcal{E}_{\mathrm{KdV}}(\phi) = \frac{1}{2}\int_{-\infty}^{\infty} ((\beta - \tfrac{1}{3}\nu_0^2)(\phi')^2 + (\tfrac{1}{3}\omega^2 + 1)\phi^3)\,\mathrm{d}x$$

over the set $N_{\mathrm{KdV}} = \{\phi \in H^1(\mathbb{R}) \colon \|\phi\|_0^2 = 2\alpha_{\mathrm{KdV}}\}$; *the constant* $2\nu_{\mathrm{KdV}}$ *is the Lagrange multiplier in this constrained variational principle and*

$$c_{\mathrm{KdV}} := \inf\{\mathcal{E}_{\mathrm{KdV}}(\phi) \colon \phi \in N_{\mathrm{KdV}}\} = -\frac{\frac{9}{5}(\frac{2}{3})^{1/3}(\frac{1}{3}\omega^2 + 1)^{4/3}}{(\beta - \frac{1}{3}\nu_0^2)^{1/3}(\omega^2 + 4)^{5/6}}.$$

Here the numerical value $\alpha_{\mathrm{KdV}} = 2(\omega^2 + 4)^{-1/2}$ *is chosen for compatibility with an estimate (see proposition 5.4) in the following water-wave theory.*

(ii) *Suppose that* $\{\phi_m\} \subset N_{\mathrm{KdV}}$ *is a minimizing sequence for* $\mathcal{E}_{\mathrm{KdV}}$. *There exists a sequence* $\{x_m\}$ *of real numbers with the property that a subsequence of* $\{\phi_m(\cdot + x_m)\}$ *converges in* $H^1(\mathbb{R})$ *to an element of* D_{KdV}.

CASE 2 ($\beta < \beta_{\mathrm{c}}$). The appropriate model equation is the cubic nonlinear Schrödinger equation

$$2\mathrm{i}A_T - \tfrac{1}{4}g''(k_0)A_{XX} + \tfrac{3}{2}(\tfrac{1}{2}A_3 + A_4)|A|^2 A = 0, \tag{1.14}$$

in which

$$\eta = \tfrac{1}{2}\mu(A(X,T)\mathrm{e}^{\mathrm{i}k_0(x+\nu_0 t)} + \mathrm{c.c.}) + O(\mu^2),$$
$$X = \mu(x + \nu_0 t), \qquad T = 4k_0(\omega + 2\nu_0 f(k_0))^{-1}\mu^2 t$$

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and A_3, A_4 are functions of β and ω that are given in corollary 4.25 and proposition 4.28; the abbreviation 'c.c.' denotes the complex conjugate of the preceding quantity. (It is demonstrated in Appendix B that $A_3 + 2A_4$ is negative.) At this level of approximation, a solution to (1.14) of the form $A(X,T) = \mathrm{e}^{\mathrm{i}\nu_{\mathrm{NLS}}T}\phi(X)$ with $\phi(X) \to 0$ as $X \to \pm\infty$ corresponds to a solitary water wave with speed

$$\nu = \nu_0 + 4(\omega + 2\nu_0 f(k_0))^{-1}\mu^2\nu_{\mathrm{NLS}}.$$

The following lemma gives a variational description of the set of such solutions (see Cazenave [10, § 8]); the corresponding solitary waves are sketched in figure 3.

Lemma 1.3.

(i) *The set of complex-valued solutions to the ordinary differential equation*

$$-\tfrac{1}{4}g''(k_0)\phi'' - 2\nu_{\mathrm{NLS}}\phi + \tfrac{3}{2}(\tfrac{1}{2}A_3 + A_4)|\phi|^2\phi = 0$$

satisfying $\phi(X) \to 0$ *as* $X \to \infty$ *is*

$$D_{\mathrm{NLS}} = \{\mathrm{e}^{\mathrm{i}\omega}\phi_{\mathrm{NLS}}(\cdot + y) \colon \omega \in [0, 2\pi),\ y \in \mathbb{R}\},$$

where

$$\nu_{\mathrm{NLS}} = -\frac{9\alpha_{\mathrm{NLS}}^2}{8g''(k_0)}\left(\frac{A_3}{2} + A_4\right)^2,$$

$$\phi_{\mathrm{NLS}}(x) = \alpha_{\mathrm{NLS}}\left(-\frac{3}{g''(k_0)}\left(\frac{A_3}{2} + A_4\right)\right)^{1/2} \operatorname{sech}\left(-\frac{3\alpha_{\mathrm{NLS}}}{g''(k_0)}\left(\frac{A_3}{2} + A_4\right)x\right)$$

These functions are precisely the minimizers of the functional $\mathcal{E}_{\mathrm{NLS}} \colon H^1(\mathbb{R}) \to \mathbb{R}$ *given by*

$$\mathcal{E}_{\mathrm{NLS}}(\phi) = \int_{-\infty}^{\infty} (\tfrac{1}{8}g''(k_0)|\phi'|^2 + \tfrac{3}{8}(\tfrac{1}{2}A_3 + A_4)|\phi|^4)\,\mathrm{d}x$$

over the set $N_{\mathrm{NLS}} = \{\phi \in H^1(\mathbb{R}) \colon \|\phi\|_0^2 = 2\alpha_{\mathrm{NLS}}\}$*; the constant* $2\nu_{\mathrm{NLS}}$ *is the Lagrange multiplier in this constrained variational principle and*

$$c_{\mathrm{NLS}} := \inf\{\mathcal{E}_{\mathrm{NLS}}(\phi) \colon \phi \in N_{\mathrm{NLS}}\} = -\frac{3\alpha_{\mathrm{NLS}}^3}{4g''(k_0)}\left(\frac{A_3}{2} + A_4\right)^2.$$

Here the numerical value $\alpha_{\mathrm{NLS}} = \frac{1}{2}(\frac{1}{4}\nu_0 f(k_0) + \frac{1}{8}\omega)^{-1}$ *is chosen for compatibility with an estimate (see proposition 5.10) in the following water-wave theory.*

(ii) *Suppose that* $\{\phi_n\} \subset N_{\mathrm{NLS}}$ *is a minimizing sequence for* $\mathcal{E}_{\mathrm{NLS}}$*. There exists a sequence* $\{x_m\}$ *of real numbers with the property that a subsequence of* $\{\phi_m(\cdot + x_m)\}$ *converges in* $H^1(\mathbb{R})$ *to an element of* D_{NLS}*.*

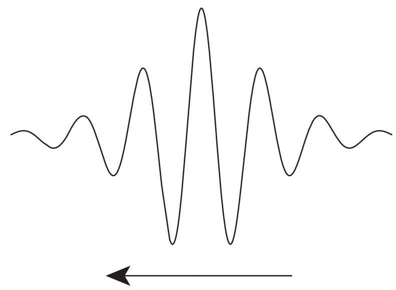

Figure 3. Nonlinear Schrödinger theory predicts the existence of small-amplitude envelope solitary waves for weak surface tension.

1.3. The main results

In this paper we establish the existence of minimizers of the functional $\mathcal{J}_\mu$ over $U \setminus \{0\}$ and confirm that the corresponding solitary water waves are approximated by suitable scalings of the functions ϕ_{KdV} (for $\beta > \beta_{\mathrm{c}}$) and ϕ_{NLS} (for $\beta < \beta_{\mathrm{c}}$). The following theorem states these results more precisely.

THEOREM 1.4.

(i) *The set B_μ of minimizers of $\mathcal{J}_\mu$ over $U \setminus \{0\}$ is non-empty.*

(ii) *Suppose that $\{\eta_m\}$ is a minimizing sequence for $\mathcal{J}_\mu$ on $U \setminus \{0\}$ that satisfies*

$$\sup_{m \in \mathbb{N}} \|\eta_m\|_2 < M.$$

There exists a sequence $\{x_m\} \subset \mathbb{R}$ with the property that a subsequence of $\{\eta_m(x_m + \cdot)\}$ converges in $H^r(\mathbb{R})$, $r \in [0, 2)$, to a function $\eta \in B_\mu$.

(iii) *Suppose that $\beta > \beta_{\mathrm{c}}$. The set B_μ of minimizers of $\mathcal{J}_\mu$ over $U \setminus \{0\}$ satisfies*

$$\sup_{\eta \in B_\mu} \inf_{x \in \mathbb{R}} \|\phi_\eta - \phi_{\mathrm{KdV}}(\cdot + x)\|_1 \to 0$$

as $\mu \downarrow 0$, where we write

$$\eta_1(x) = \mu^{2/3} \phi_\eta(\mu^{1/3} x)$$

and η_1 is obtained from η by multiplying its Fourier transform by the characteristic function of the interval $[-\delta_0, \delta_0]$ with $\delta_0 > 0$. Furthermore, the speed ν_μ of the corresponding solitary water waves satisfies

$$\nu_\mu = \nu_0 + 2(\omega^2 + 4)^{-1/2} \nu_{\mathrm{KdV}} \mu^{2/3} + o(\mu^{2/3})$$

uniformly over $\eta \in B_\mu$.

(iv) *Suppose that $\beta < \beta_{\mathrm{c}}$. The set B_μ of minimizers of $\mathcal{J}_\mu$ over $U \setminus \{0\}$ satisfies*

$$\sup_{\eta \in B_\mu} \inf_{\substack{\omega \in [0, 2\pi], \\ x \in \mathbb{R}}} \|\phi_\eta - \mathrm{e}^{\mathrm{i}\omega} \phi_{\mathrm{NLS}}(\cdot + x)\|_1 \to 0$$

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as $\mu \downarrow 0$, where we write

$$\eta_1^+(x) = \tfrac{1}{2}\mu\phi_\eta(\mu x)\mathrm{e}^{\mathrm{i}k_0 x}$$

and η_1^+ is obtained from η by multiplying its Fourier transform by the characteristic function of the interval $[k_0 - \delta_0, k_0 + \delta_0]$ with $\delta_0 \in (0, k_0/3)$. Furthermore, the speed ν_μ of the corresponding solitary water waves satisfies

$$\nu_\mu = \nu_0 + 4(\omega + 2\nu_0 f(k_0))^{-1}\nu_{\mathrm{NLS}}\mu^2 + o(\mu^2)$$

uniformly over $\eta \in B_\mu$.

The first part of theorem 1.4 is proved by reducing it to a special case of the second. We proceed by introducing the coercive penalized functional $\mathcal{J}_{\rho,\mu}\colon H^2(\mathbb{R}) \to \mathbb{R} \cup \{\infty\}$ defined by

$$\mathcal{J}_{\rho,\mu}(\eta) = \begin{cases} \mathcal{K}(\eta) + \dfrac{(\mu + \mathcal{G}(\eta))^2}{\mathcal{L}(\eta)} + \rho(\|\eta\|_2^2), & \eta \in U \setminus \{0\}, \\ \infty, & \eta \notin U \setminus \{0\}, \end{cases}$$

where $\rho\colon [0, M^2) \to \mathbb{R}$ is a smooth increasing 'penalization' function that explodes to infinity as $t \uparrow M^2$ and vanishes for $0 \leqslant t \leqslant \tilde{M}^2$; the number $\tilde{M}$ is chosen very close to M. Minimizing sequences $\{\eta_m\}$ for $\mathcal{J}_{\rho,\mu}$, which clearly satisfy $\sup_{m\in\mathbb{N}} \|\eta_m\|_2 < M$, are studied in detail in § 3 with the help of the concentration-compactness principle (see Lions [21,22]). The main difficulty here lies in discussing the consequences of 'dichotomy'.

On the one hand the functionals $\mathcal{G}$, $\mathcal{K}$ and $\mathcal{L}$ are non-local and therefore do not act linearly when applied to the sum of two functions with disjoint supports. They are, however, 'pseudo-local' in the sense that

$$\left\{\begin{matrix}\mathcal{G}\\ \mathcal{K}\\ \mathcal{L}\end{matrix}\right\}(\eta_m^{(1)} + \eta_m^{(2)}) - \left\{\begin{matrix}\mathcal{G}\\ \mathcal{K}\\ \mathcal{L}\end{matrix}\right\}(\eta_m^{(1)}) - \left\{\begin{matrix}\mathcal{G}\\ \mathcal{K}\\ \mathcal{L}\end{matrix}\right\}(\eta_m^{(2)}) \to 0$$

as $m \to \infty$, where $\{\eta_m^{(1)}\}$, $\{\eta_m^{(2)}\}$ have the properties that

$$\operatorname{supp}\eta_m^{(1)} \subset [-R_m, R_m], \quad \operatorname{supp}\eta_m^{(2)} \subset \mathbb{R} \setminus (-S_m, S_m)$$

for sequences $\{R_m\}$, $\{S_m\}$ of positive real numbers with $R_m, S_m \to \infty$, $R_m/S_m \to 0$ as $m \to \infty$ (see lemma 3.9(iii)). This result is established in § 2.2.2 by a new method that involves studying the weak formulation of the boundary-value problems defining the terms in the power-series expansion of K about $\eta_0 \in W^{s+3/2}$. On the other hand, no *a priori* estimate is available to rule out 'dichotomy' at this stage; proceeding iteratively, we find that minimizing sequences can theoretically have profiles with infinitely many 'bumps'. In particular, we show that $\{\eta_m\}$ asymptotically lies in the region unaffected by the penalization and construct a special minimizing sequence $\{\tilde{\eta}_m\}$ for $\mathcal{J}_{\rho,\mu}$ that lies in a neighbourhood of the origin with radius $O(\mu^{1/2})$ in $H^2(\mathbb{R})$ and satisfies $\|\mathcal{J}_\mu'(\tilde{\eta}_m)\|_0 \to 0$ as $n \to \infty$. The fact that the construction is independent of the choice of $\tilde{M}$ allows us to conclude that $\{\tilde{\eta}_m\}$ is also a minimizing sequence for $\mathcal{J}_\mu$ over $U \setminus \{0\}$.

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The special minimizing sequence $\{\tilde{\eta}_m\}$ is used in § 4 to establish the *strict subadditivity* of the infimum c_μ of $\mathcal{J}_\mu$ over $U \setminus \{0\}$, that is, the inequality

$$c_{\mu_1+\mu_2} < c_{\mu_1} + c_{\mu_2}, \quad 0 < \mu_1, \mu_2, \mu_1 + \mu_2 < \mu_0.$$

The strict subadditivity of c_μ follows from the fact that the function

$$a \mapsto a^{-q}\mathcal{M}_{a^2\mu}(a\tilde{\eta}_m), \quad a \in [1, a_0], \tag{1.15}$$

is decreasing and strictly negative for some $q > 2$ and $a_0 \in (1, 2]$, where

$$\mathcal{M}_\mu(\eta) := \mathcal{J}_\mu(\eta) - \mathcal{K}_2(\eta) - \frac{(\mu + \mathcal{G}_2(\eta))^2}{\mathcal{L}_2(\eta)}$$

is the 'nonlinear' part of $\mathcal{J}_\mu(\eta)$ (see § 4.4). We proceed by approximating $\mathcal{M}_\mu(\eta_m)$ with its dominant term and showing that this term has the required property.

The heuristic arguments given above suggest firstly that the spectrum of minimizers of $\mathcal{J}_\mu$ over $U\setminus\{0\}$ (that is, the support of their Fourier transform) is concentrated near wavenumbers $k = \pm k_0$, and secondly that they have the Korteweg–de Vries or nonlinear Schrödinger length-scales; the same should be true of the functions $\tilde{\eta}_m$, which approximate minimizers. We therefore decompose $\tilde{\eta}_m$ into the sum of a function $\tilde{\eta}_{m,1}$, whose spectrum is compactly supported near $k = \pm k_0$, and a function $\tilde{\eta}_{m,2}$, whose spectrum is bounded away from these points, and study $\tilde{\eta}_{m,1}$ using the weighted norm

$$|||\eta|||_\alpha^2 := \int_{-\infty}^{\infty} (1 + \mu^{-4\alpha}(|k| - k_0)^4)|\hat{\eta}(k)|^2 \, \mathrm{d}k.$$

A careful analysis of the equation $\mathcal{J}_\mu'(\tilde{\eta}_m) = O(\mu^N)$ in $L^2(\mathbb{R})$ shows that $|||\tilde{\eta}_{m,1}|||_\alpha^2 = O(\mu)$ and $\|\tilde{\eta}_{m,2}\|_2 = O(\mu^{2+\alpha})$ for $\alpha < \frac{1}{3}$ when $\beta > \beta_\mathrm{c}$ and for $\alpha < 1$ when $\beta < \beta_\mathrm{c}$. Using these estimates on the size of $\tilde{\eta}_n$, we find that

$$\mathcal{M}_\mu(\tilde{\eta}_m) = \begin{cases} c\displaystyle\int_{-\infty}^{\infty} \tilde{\eta}_{m,1}^3 \, \mathrm{d}x + o(\mu^{5/3}), & \beta > \beta_\mathrm{c}, \\ -c\displaystyle\int_{-\infty}^{\infty} \tilde{\eta}_{m,1}^4 \, \mathrm{d}x + o(\mu^3), & \beta < \beta_\mathrm{c}. \end{cases}$$

That the function (1.15) is decreasing and strictly negative follows from the above estimate and the fact that $\mathcal{M}_\mu(\eta_m)$ is negative for any minimizing sequence $\{\eta_m\}$ for $\mathcal{J}_\mu$ over $U \setminus \{0\}$.

Knowledge of the strict subadditivity property of c_μ (and general estimates for general minimizing sequences) reduces the proof of theorem 1.4(ii) to a straightforward application of the concentration-compactness principle (see § 5.1). Parts (iii) and (iv) are derived from lemmas 1.2(ii) and 1.3(ii) by means of a scaling and contradiction argument from the estimates

$$\|\phi_\eta\|_0^2 = 2\begin{Bmatrix}\alpha_{\mathrm{KdV}} \\ \alpha_{\mathrm{NLS}}\end{Bmatrix} + o(1), \qquad \begin{Bmatrix}\mathcal{E}_{\mathrm{KdV}} \\ \mathcal{E}_{\mathrm{NLS}}\end{Bmatrix}(\phi_\eta) = \begin{Bmatrix}c_{\mathrm{KdV}} \\ c_{\mathrm{NLS}}\end{Bmatrix} + o(1), \quad \eta \in B_\mu,$$

which emerge as part of the proof of theorem 1.4(i) (see § 5.2).

Some of the techniques used in the present paper were developed by Buffoni *et al*. [9] in an existence theory for three-dimensional irrotational solitary waves. While we make reference to relevant parts of that paper, many aspects of our construction differ significantly from theirs. In particular, our treatment of non-local analytic operators is more comprehensive. Their version of theorem 1.1 (see Buffoni *et al*. [9, lemmas 1.1 and 1.4]) is obtained using a less sophisticated 'flattening' transformation and shows only that the operators are analytic at the origin. Correspondingly, 'pseudo-localness' in the sense described above is established there only for constant-coefficient boundary-value problems (using an explicit representation of the solution by means of Green functions). Our treatment of the consequences of 'dichotomy' in the concentration-compactness principle (see § 3) is on the other hand similar to that given by Buffoni *et al*. [9] and we omit proofs that are straightforward modifications of theirs; the main difference here is that negative values of the parameter μ emerge in our iterative construction of the special minimizing sequence (see the remarks below lemma 3.8).

1.4. Conditional energetic stability

Our original problem of finding minimizers of $\mathcal{H}(\eta,\xi)$ subject to the constraint $\mathcal{I}(\eta,\xi)=2\mu$ is also solved as a corollary to theorem 1.4(ii); one follows the two-step minimization procedure described in § 1.1 (see § 5.1).

Theorem 1.5.

(i) *The set D_μ of minimizers of $\mathcal{H}$ on the set*

$$S_\mu = \{(\eta,\xi)\in U\times H_\star^{1/2}(\mathbb{R})\colon \mathcal{I}(\eta,\xi)=2\mu\}$$

is non-empty.

(ii) *Suppose that $\{(\eta_m,\xi_m)\}\subset S_\mu$ is a minimizing sequence for $\mathcal{H}$ with the property that $\sup_{m\in\mathbb{N}}\|\eta_m\|_2 < M$. There exists a sequence $\{x_m\}\subset\mathbb{R}$ with the property that a subsequence of $\{(\eta_m(x_m+\cdot),\xi_m(x_m+\cdot))\}$ converges in $H^r(\mathbb{R})\times H_\star^{1/2}(\mathbb{R})$, $r\in[0,2)$, to a function in D_μ.*

It is a general principle that the solution set of a constrained minimization problem constitutes a stable set of solutions of the corresponding initial-value problem (see, for example, Cazenave and Lions [11]). The usual informal interpretation of the statement that a set X of solutions to an initial-value problem is 'stable' is that a solution that begins close to a solution in X remains close to a solution in X at all subsequent times. Implicit in this statement is the assumption that the initial-value problem is globally well posed, that is, every pair (η_0,Φ_0) in an appropriately chosen set is indeed the initial datum of a unique solution $t\mapsto(\eta(t),\Phi(t))$, $t\in[0,\infty)$. At present there is no global well-posedness theory for gravity–capillary water waves with constant vorticity (although there is a large and growing body of literature concerning well-posedness issues for water-wave problems in general). Assuming the existence of solutions, we obtain the following stability result as a corollary of theorem 1.5 using the argument given by Buffoni *et al*. [9, theorem 5.5]. (The only property of a solution (η,ξ) to the initial-value problem that is relevant to stability theory is that $\mathcal{H}(\eta(t),\xi(t))$ and $\mathcal{I}(\eta(t),\xi(t))$ are constant; we therefore adopt this property as the definition of a solution.)

THEOREM 1.6. *Suppose that* $(\eta, \xi)\colon [0,T] \to U \times H_\star^{1/2}(\mathbb{R})$ *has the properties that*

$$\mathcal{H}(\eta(t), \xi(t)) = \mathcal{H}(\eta(0), \xi(0)), \qquad \mathcal{I}(\eta(t), \xi(t)) = \mathcal{I}(\eta(0), \xi(0)), \quad t \in [0,T],$$

and

$$\sup_{t\in[0,T]} \|\eta(t)\|_2 < M.$$

Choose $r \in [0,2)$ *and let* 'dist' *denote the distance in* $H^r(\mathbb{R}) \times H_\star^{1/2}(\mathbb{R})$. *For each* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$\mathrm{dist}((\eta(0), \xi(0)), D_\mu) < \delta \quad \Longrightarrow \quad \mathrm{dist}((\eta(t), \xi(t)), D_\mu) < \varepsilon$$

for $t \in [0,T]$.

This result is a statement of the *conditional energetic stability of the set* D_μ. Here *energetic* refers to the fact that the distance in the statement of stability is measured in the 'energy space' $H^r(\mathbb{R}) \times H_\star^{1/2}(\mathbb{R})$, while *conditional* alludes to the well-posedness issue. Note that the solution $t \mapsto (\eta(t), \xi(t))$ may exist in a smaller space over the interval $[0,T]$, at each instant of which it remains close (in energy space) to a solution in D_μ. Furthermore, theorem 1.6 is a statement of the stability of the *set* of constrained minimizers D_μ; establishing the uniqueness of the constrained minimizer would imply that D_μ consists of translations of a single solution, so that the statement that D_μ is stable is equivalent to classical orbital stability of this unique solution (see Benjamin [3]). The phrase 'conditional energetic stability' was introduced by Mielke [24] in his study of the stability of irrotational solitary water waves with strong surface tension using dynamical-systems methods.

2. The functional-analytic setting

2.1. Non-local operators

The goal of this section is to introduce rigorous definitions of the Dirichlet–Neumann operator $G(\eta)$, its inverse $N(\eta)$ and the operator $K(\eta) := -\partial_x(N(\eta)\partial_x)$.

2.1.1. *Function spaces*

Choose $h_0 \in (0,1)$. We consider the class

$$W = \{\eta \in W^{1,\infty}(\mathbb{R})\colon 1 + \inf \eta > h_0\}$$

of surface profiles and denote the fluid domain by

$$\Sigma_\eta = \{(x,y) \in \mathbb{R}^2\colon 0 < y < 1 + \eta(x)\}, \quad \eta \in W.$$

The observation that velocity potentials are unique only up to additive constants leads us to introduce the completion $H_\star^1(\Sigma_\eta)$ of

$$\mathcal{S}(\Sigma_\eta) = \{\phi \in C^\infty(\bar{\Sigma}_\eta)\colon |x|^m |\partial_x^{\alpha_1} \partial_y^{\alpha_2} \phi| \text{ is bounded for all } m, \alpha_1, \alpha_2 \in \mathbb{N}_0\}$$

with respect to the Dirichlet norm as an appropriate function space for ϕ. The corresponding space for the trace $\phi|_{y=1+\eta}$ is the space $H_\star^{1/2}(\mathbb{R})$ defined in § 1.1.3.

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Proposition 2.1. *Fix $\eta \in W$. The trace map $\phi \mapsto \phi|_{y=1+\eta}$ defines a continuous operator $H^1_\star(\Sigma_\eta) \to H^{1/2}_\star(\mathbb{R})$ with a continuous right inverse $H^{1/2}_\star(\mathbb{R}) \to H^1_\star(\Sigma_\eta)$.*

We also use anisotropic function spaces for functions defined in the strip $\Sigma_0 = \mathbb{R} \times (0,1)$.

Definition 2.2. Suppose that $r \in \mathbb{R}$ and $n \in \mathbb{N}_0$.

(i) The Banach space $(L^\infty H^r, \|\cdot\|_{r,\infty})$ is defined by

$$L^\infty H^r = L^\infty((0,1), H^r(\mathbb{R})), \qquad \|u\|_{r,\infty} = \operatorname*{ess\,sup}_{y\in(0,1)} \|u(\cdot,y)\|_{H^r(\mathbb{R})}.$$

(ii) The Banach space $(H^{r,m}, \|\cdot\|_{r,m})$ is defined by

$$H^{r,m} = \bigcap_{j=0}^{m} H^j((0,1), H^{r-j}(\mathbb{R})), \qquad \|u\|_{r,m} = \sum_{j=0}^{m} \|\Lambda^{r-j}\partial_y^j u\|_{L^2(\Sigma)},$$

where $\Lambda f = \mathcal{F}^{-1}[(1+k^2)^{1/2}\hat{f}(k)]$.

The following propositions state some properties of these function spaces that are used in the subsequent analysis; they are deduced from results for standard Sobolev spaces (see Hörmander [18, theorem 8.3.1] for proposition 2.4).

Proposition 2.3.

(i) *The space $C_0^\infty(\bar{\Sigma})$ is dense in $H^{r,1}$ for each $r \in \mathbb{R}$.*

(ii) *For each $r \in \mathbb{R}$ the mapping $u \mapsto u|_{y=1}$, $u \in C_0^\infty(\bar{\Sigma})$, extends continuously to an operator $H^{r+1,1} \to H^{r+1/2}(\mathbb{R})$.*

(iii) *The space $H^{r+1,1}$ is continuously embedded in $L^\infty H^{r+1/2}$ for each $r \in \mathbb{R}$.*

(iv) *The space $H^{r+1,1}$ is a Banach algebra for each $r > 0$.*

Proposition 2.4. *Suppose that r_0, r_1 and r_2 satisfy $r_0 \leqslant r_1$, $r_0 \leqslant r_2$, $r_1 + r_2 \geqslant 0$ and $r_0 < r_1 + r_2 - \frac{1}{2}$. The product $u_1 u_2$ of each $u_1 \in L^\infty H^{r_1}$ and $u_2 \in H^{r_2,0}$ lies in $H^{r_0,0}$ and satisfies*

$$\|u_1 u_2\|_{r_0,0} \leqslant c\|u_1\|_{r_1,\infty}\|u_2\|_{r_2,0}.$$

Proposition 2.5. *For each bounded linear function $L\colon L^2(\mathbb{R}) \to L^\infty H^0$ the formula $(\eta, w) \mapsto L(\eta)w$ defines a bounded bilinear function $L^2(\mathbb{R}) \times H^1(\Sigma) \to L^2(\Sigma)$ that satisfies the estimate*

$$\|L(\eta)w\|_0 \leqslant c\|L\|\|w\|_0^{1/2}\|w\|_1^{1/2}\|\eta\|_0.$$

The assertion remains valid when Σ is replaced by $\{|x| < M\}$ or $\{|x| > M\}$ and the estimate holds uniformly over all values of M greater than unity.

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2.1.2. *The Dirichlet–Neumann operator*

The Dirichlet–Neumann operator $G(\eta)$ for the boundary-value problem

$$\Delta\phi = 0, \quad 0 < y < 1 + \eta, \tag{2.1}$$

$$\phi = \xi, \quad y = 1 + \eta, \tag{2.2}$$

$$\phi_y = 0, \quad y = 0, \tag{2.3}$$

is defined formally as follows: fix $\xi = \xi(x)$, solve (2.1)–(2.3) and set

$$G(\eta)\xi = (\phi_y - \eta'\phi_x)|_{y=1+\eta}.$$

Our rigorous definition of $G(\eta)$ is given in terms of weak solutions to (2.1)–(2.3) (see Lannes [20, proposition 2.9] for the proof of lemma 2.7).

Definition 2.6. Suppose that $\xi \in H^{1/2}_\star(\mathbb{R})$ and $\eta \in W$. A *weak solution* of (2.1)–(2.3) is a function $\phi \in H^1_\star(\Sigma_\eta)$ with $\phi|_{y=1+\eta} = \xi$ that satisfies

$$\int_{\Sigma_\eta} \nabla\phi \cdot \nabla\psi \,\mathrm{d}x\,\mathrm{d}y = 0$$

for all $\psi \in H^1_\star(\Sigma_\eta)$ with $\psi|_{y=1+\eta} = 0$.

Lemma 2.7. *For each $\xi \in H^{1/2}_\star(\mathbb{R})$ and $\eta \in W$ there exists a unique weak solution ϕ of (2.1)–(2.3). The solution satisfies the estimate*

$$\|\phi\|_{H^1_\star(\Sigma_\eta)} \leqslant C\|\xi\|_{H^{1/2}_\star(\mathbb{R})},$$

where $C = C(\|\eta\|_{1,\infty})$.

Definition 2.8. Suppose that $\eta \in W$ and $\xi \in H^{1/2}_\star(\mathbb{R})$. The *Dirichlet–Neumann operator* is the bounded linear operator $G(\eta)\colon H^{1/2}_\star(\mathbb{R}) \to H^{-1/2}_\star(\mathbb{R})$ defined by

$$\int_{-\infty}^{\infty} (G(\eta)\xi_1)\xi_2 \,\mathrm{d}x = \int_{\Sigma_\eta} \nabla\phi_1 \cdot \nabla\phi_2 \,\mathrm{d}x\,\mathrm{d}y,$$

where $\phi_j \in H^1_\star(\Sigma_\eta)$ is the unique weak solution of (2.1)–(2.3) with $\xi = \xi_j$, $j = 1, 2$.

2.1.3. *The Neumann–Dirichlet operator*

The Neumann–Dirichlet operator $N(\eta)$ for the boundary-value problem

$$\Delta\phi = 0, \quad 0 < y < 1 + \eta, \tag{2.4}$$

$$\phi_y - \eta'\phi_x = \xi, \quad y = 1 + \eta, \tag{2.5}$$

$$\phi_y = 0, \quad y = 0, \tag{2.6}$$

is defined formally as follows: fix $\xi = \xi(x)$, solve (2.4)–(2.6) and set

$$N(\eta)\xi = \phi|_{y=1+\eta}.$$

Our rigorous definition of $N(\eta)$ is also given in terms of weak solutions; lemma 2.10 is proved in the same fashion as lemma 2.7.

Definition 2.9. Suppose that $\xi \in H_\star^{-1/2}(\mathbb{R})$ and $\eta \in W$. A *weak solution* of (2.4)–(2.6) is a function $\phi \in H_\star^1(\Sigma_\eta)$ that satisfies

$$\int_{\Sigma_\eta} \nabla\phi \cdot \nabla\psi \,\mathrm{d}x\,\mathrm{d}y = \int_{-\infty}^{\infty} \xi\psi|_{y=1+\eta}\,\mathrm{d}x$$

for all $\psi \in H_\star^1(\Sigma_\eta)$.

Lemma 2.10. *For each $\xi \in H_\star^{-1/2}(\mathbb{R})$ and $\eta \in W$ there exists a unique weak solution ϕ of (2.4)–(2.6). The solution satisfies the estimate*

$$\|\phi\|_{H_\star^1(\Sigma_\eta)} \leqslant C\|\xi\|_{H_\star^{-1/2}(\mathbb{R})},$$

where $C = C(\|\eta\|_{1,\infty})$.

Definition 2.11. Suppose that $\eta \in W$ and $\xi \in H_\star^{-1/2}(\mathbb{R})$. The *Neumann–Dirichlet operator* is the bounded linear operator $N(\eta)\colon H_\star^{-1/2}(\mathbb{R}) \to H_\star^{1/2}(\mathbb{R})$ defined by

$$N(\eta)\xi = \phi|_{y=1+\eta},$$

where $\phi \in H_\star^1(\Sigma_\eta)$ is the unique weak solution of (2.4)–(2.6).

The relationship between $G(\eta)$ and $N(\eta)$ is clarified by the following result, which follows from the definitions of these operators.

Lemma 2.12. *Suppose that $\eta \in W$. The operator $G(\eta) \in \mathcal{L}(H_\star^{1/2}(\mathbb{R}), H_\star^{-1/2}(\mathbb{R}))$ is invertible with $G(\eta)^{-1} = N(\eta)$.*

2.1.4. *Analyticity of the operators*

Let us begin by recalling the definition of analyticity given by Buffoni and Toland [8, definition 4.3.1] together with a precise formulation of our result in their terminology.

Definition 2.13. Let X and Y be Banach spaces, let U be a non-empty open subset of X and let $\mathcal{L}_\mathrm{s}^k(X,Y)$ be the space of bounded k-linear symmetric operators $X^k \to Y$ with norm

$$|||m||| := \inf\{c\colon \|m(\{f\}^{(k)})\|_Y \leqslant c\|f\|_X^k \text{ for all } f \in X\}.$$

A function $F\colon U \to Y$ is *analytic at a point* $x_0 \in U$ if there exist real numbers $\delta, r > 0$ and a sequence $\{m_k\}$, where $m_k \in \mathcal{L}_\mathrm{s}^k(X,Y)$, $k \in \mathbb{N}_0$, with the properties that

$$F(x) = \sum_{k=0}^{\infty} m_k(\{x - x_0\}^{(k)}), \quad x \in B_\delta(x_0),$$

and

$$\sup_{k \geqslant 0} r^k |||m_k||| < \infty.$$

The function is *analytic* if it is analytic at each point $x_0 \in U$.

THEOREM 2.14.

(i) *The Dirichlet–Neumann operator* $G\colon W \to \mathcal{L}(H_\star^{1/2}(\mathbb{R}), H_\star^{-1/2}(\mathbb{R}))$ *is analytic.*

(ii) *The Neumann–Dirichlet operator* $N\colon W \to \mathcal{L}(H_\star^{-1/2}(\mathbb{R}), H_\star^{1/2}(\mathbb{R}))$ *is analytic.*

To prove this theorem we study the dependence of solutions to the boundary-value problems (2.1)–(2.3) and (2.4)–(2.6) on η by transforming them into equivalent problems in the fixed domain $\Sigma := \Sigma_0$. For this purpose we define a change of variable $(x, y) = F^\delta(x, y')$ in the following way. Choose $\delta > 0$ and an even function $\chi \in C_0^\infty(\mathbb{R})$ with $\chi(k) \in [0, 1]$ for $k \in \mathbb{R}$, $\operatorname{supp}\chi \in [-2, 2]$ and $\chi(x) \equiv 1$ for $|x| \leqslant 1$, write

$$\eta^\delta(x, y') = \mathcal{F}^{-1}[\chi(\delta(1 - y')k)\hat{\eta}(k)](x)$$

and define

$$F^\delta(x, y') = (x, y'(1 + \eta^\delta(x, y'))) = (x, y' + f^\delta(x, y')),$$

in which $f^\delta(x, y') = y'\eta^\delta(x, y')$.

LEMMA 2.15. *Suppose that* $\eta \in W$. *The mapping* F^δ *is a bijection* $\Sigma \to \Sigma_\eta$ *and* $\bar{\Sigma} \to \bar{\Sigma}_\eta$ *with* $y \in C_\mathrm{b}^1(\Sigma)$, $y' \in C_\mathrm{b}^1(\Sigma_\eta)$ *and*

$$\inf_{(x,y')\in\bar{\Sigma}} y_{y'}(x, y') = \inf_{(x,y')\in\bar{\Sigma}} (1 + f^\delta_{y'}(x, y)) > 0$$

for each $\delta \in (0, \delta_{\max})$, *where* $\delta_{\max} = \delta_{\max}(\|\eta'\|_\infty^{-1})$.

Proof. Writing

$$\eta^\delta(x, y') = \int_{-\infty}^{\infty} K(s)\eta(x - \delta(1 - y')s)\,\mathrm{d}s,$$

where $K = (2\pi)^{-1/2}\mathcal{F}^{-1}[\chi] \in \mathcal{S}(\mathbb{R})$, one finds that $\eta^\delta \in C^\infty(\Sigma) \cap C_\mathrm{b}^1(\Sigma)$ with $\|\eta^\delta\|_\infty \leqslant c\|\eta\|_\infty$, $\|\eta_x^\delta\|_\infty \leqslant c\|\eta'\|_\infty$, $\|\eta^\delta_{y'}\|_\infty \leqslant c\delta\|\eta'\|_\infty$. It follows that $F^\delta \in C^\infty(\Sigma)$ and $y \in C_\mathrm{b}^1(\Sigma)$. Furthermore, $y(x, 0) = 0$, $y(x, 1) = 1 + \eta(x)$ and

$$\begin{aligned}
\partial_{y'} y &= 1 + y'\eta^\delta_{y'} + \eta^\delta \\
&= 1 + y'\eta^\delta_{y'} + \eta - \int_{y'}^1 \eta^\delta_{y'} \\
&\geqslant h_0 - c\delta\|\eta'\|_\infty \\
&\geqslant \tfrac{1}{2}h_0 \\
&> 0
\end{aligned}$$

for sufficiently small δ (depending only upon $\|\eta'\|_\infty^{-1}$), so that F^δ is a bijection $\Sigma \to \Sigma_\eta$ and $\bar{\Sigma} \to \bar{\Sigma}_\eta$. It follows from the inverse function theorem that $(F^\delta)^{-1} \in C^\infty(\Sigma_\eta)$; the estimate

$$\det \mathrm{d}F^\delta[x, y'] = \partial_{y'} y(x, y') \geqslant \tfrac{1}{2}h_0$$

and the fact that $\mathrm{d}F^\delta$ is bounded on Σ imply that $\mathrm{d}(F^\delta)^{-1} \in C_\mathrm{b}(\Sigma_\eta)$, whereby $y' \in C_\mathrm{b}^1(\Sigma_\eta)$. □

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The change of variable $(x,y) = F^\delta(x,y')$ transforms the boundary-value problem (2.4)–(2.6) into

$$\nabla\cdot((I+Q)\nabla u) = 0, \quad 0<y<1, \tag{2.7}$$
$$(I+Q)\nabla u\cdot(0,1) = \xi, \quad y=1, \tag{2.8}$$
$$(I+Q)\nabla u\cdot(0,-1) = 0, \quad y=0, \tag{2.9}$$

where

$$Q = \begin{pmatrix} f_y^\delta & -f_x^\delta \\ -f_x^\delta & \dfrac{-f_y^\delta + (f_x^\delta)^2}{1+f_y^\delta} \end{pmatrix}$$

and the primes have been dropped for notational simplicity.

Lemma 2.16. *The mapping $W\to (L^\infty(\bar\Sigma))^{2\times 2}$ given by $\eta\mapsto Q(\eta)$ is analytic.*

It is helpful to consider the more general boundary-value problem

$$\nabla\cdot((I+Q)\nabla u) = \nabla\cdot G, \qquad 0<y<1, \tag{2.10}$$
$$(I+Q)\nabla u\cdot(0,1) = \xi + G\cdot(0,1), \quad y=1, \tag{2.11}$$
$$(I+Q)\nabla u\cdot(0,-1) = G\cdot(0,-1), \qquad y=0, \tag{2.12}$$

where $I+Q\in(L^\infty(\bar\Sigma))^{2\times 2}$ is uniformly positive definite, that is, there exists a constant $p_0>0$ such that

$$(I+Q)(x,y)\nu\cdot\nu \geqslant p_0|\nu|^2$$

for all $(x,y)\in\bar\Sigma$ and all $\nu\in\mathbb{R}^2$.

Definition 2.17. Suppose that $\xi\in H_\star^{-1/2}(\mathbb{R})$ and $G\in(L^2(\Sigma))^2$. A *weak solution* of (2.10)–(2.12) is a function $u\in H_\star^1(\Sigma)$ that satisfies

$$\int_\Sigma (I+Q)\nabla u\cdot\nabla w\,\mathrm{d}x\,\mathrm{d}y = \int_\Sigma G\cdot\nabla w\,\mathrm{d}x\,\mathrm{d}y + \int_{-\infty}^{\infty} \xi w|_{y=1}\,\mathrm{d}x$$

for all $w\in H_\star^1(\Sigma)$.

Lemma 2.18. *For each $\xi\in H_\star^{-1/2}(\mathbb{R})$ and $G\in(L^2(\Sigma))^2$ the boundary-value problem (2.10)–(2.12) has a unique weak solution $u\in H_\star^1(\Sigma)$. The solution satisfies the estimate*

$$\|u\|_{H_\star^1(\Sigma)} \leqslant C(\|\xi\|_{H_\star^{-1/2}(\mathbb{R})} + \|G\|_{L^2(\mathbb{R})}),$$

where $C = C(p_0^{-1})$.

Lemma 2.18 applies in particular to (2.7)–(2.9) for each fixed $\eta\in W$ (the matrix $I+Q$ is uniformly positive definite since it is uniformly bounded above, its determinant is unity and its upper left entry is positive). The next theorem shows that its unique weak solution depends analytically upon η.

Theorem 2.19. *The mapping $W\to\mathcal{L}(H_\star^{-1/2}(\mathbb{R}), H_\star^1(\Sigma))$ given by $\eta\mapsto(\xi\mapsto u)$, where $u\in H_\star^1(\Sigma)$ is the unique weak solution of (2.7)–(2.9), is analytic.*

Proof. Choose $\eta_0 \in W$ and write $\tilde{\eta} = \eta - \eta_0$ and

$$Q(x,y) = \sum_{n=0}^{\infty} Q^n(x,y), \qquad Q^n = \tilde{m}_n(\tilde{\eta}^{\{(n)\}}),$$

where $\tilde{m}_n(\tilde{\eta}^{\{(n)\}}) \in \mathcal{L}_s^n(W^{1,\infty}(\mathbb{R}), (L^\infty(\bar{\Sigma}))^{2\times 2})$ satisfies

$$|||\tilde{m}^n||| \leqslant C_2 r^{-n} \|\tilde{\eta}\|_{1,\infty}^n$$

(see lemma 2.16). We proceed by seeking a solution of (2.7)–(2.9) of the form

$$u(x,y) = \sum_{n=0}^{\infty} u^n(x,y), \qquad u^n = m_1^n(\{\tilde{\eta}\}^{(n)}), \tag{2.13}$$

where $m_1^n \in \mathcal{L}_\mathrm{s}^n(W^{1,\infty}(\mathbb{R}), H_\star^1(\Sigma))$ is linear in ξ and satisfies

$$|||m_1^n||| \leqslant C_1 B^n \|\xi\|_{H_\star^{-1/2}(\mathbb{R})}$$

for some constant $B > 0$.

Substituting the *ansatz* (2.13) into the equations, one finds that

$$\nabla \cdot ((I + Q^0)\nabla u^0) = 0, \quad 0 < y < 1, \tag{2.14}$$
$$(I + Q^0)\nabla u^0 \cdot (0,1) = \xi, \quad y = 1, \tag{2.15}$$
$$(I + Q^0)\nabla u^0 \cdot (0,-1) = 0, \quad y = 0, \tag{2.16}$$

and

$$\nabla \cdot ((I + Q^0)\nabla u^n) = \nabla \cdot G^n, \qquad 0 < y < 1, \tag{2.17}$$
$$(I + Q^0)\nabla u^n \cdot (0,1) = G^n \cdot (0,1), \qquad y = 1, \tag{2.18}$$
$$(I + Q^0)\nabla u^n \cdot (0,-1) = G^n \cdot (0,-1), \quad y = 0, \tag{2.19}$$

for $n \in \mathbb{N}$, where

$$G^n = -\sum_{k=1}^{n} Q^k \nabla u^{n-k}.$$

The estimate for m^0 follows directly from lemma 2.18. Proceeding inductively, suppose that the result for m^n is true for all $k < n$. Estimating

$$\begin{aligned} \|G^n\|_0 &\leqslant \sum_{k=1}^{n} \|Q^k\|_\infty \|\nabla u^{n-k}\|_0 \\ &\leqslant C_1 C_2 B^n \|\xi\|_{H_\star^{-1/2}(\mathbb{R})} \|\tilde{\eta}\|_{1,\infty}^n \sum_{k=1}^{n} (Br)^{-k} \end{aligned} \tag{2.20}$$

and using lemma 2.18 again, we find that

$$\begin{aligned} \|u^n\|_{H_\star^1(\Sigma)} &\leqslant C_1 C_2 C_3 B^n \|\xi\|_{H_\star^{-1/2}(\mathbb{R})} \|\tilde{\eta}\|_{1,\infty}^n \sum_{k=1}^{\infty} (Br)^{-k} \\ &\leqslant C_1 B^n \|\xi\|_{H_\star^{-1/2}(\mathbb{R})} \|\tilde{\eta}\|_{1,\infty}^n \end{aligned}$$

for sufficiently large values of B (independently of n).

A straightforward supplementary argument shows that (2.13) defines a weak solution u of (2.17)–(2.19). □

Theorem 2.14(ii) follows from theorem 2.19, the equation $N(\eta)\xi = u|_{y=1}$ and the continuity of the trace operator $H^1_\star(\Sigma) \to H^{1/2}_\star(\mathbb{R})$, while theorem 2.14(i) follows from the inverse function theorem for analytic functions.

Finally, we record another useful result.

THEOREM 2.20. *For each $\eta \in W$ the norms*

$$\xi \mapsto \left(\int_{-\infty}^{\infty} \xi G(\eta)\xi \,\mathrm{d}x \right)^{1/2}, \qquad \kappa \mapsto \left(\int_{-\infty}^{\infty} \kappa N(\eta)\kappa \,\mathrm{d}x \right)^{1/2}$$

are equivalent to the usual norms for $H^{1/2}_\star(\mathbb{R})$ and $H^{-1/2}_\star(\mathbb{R})$, respectively.

Proof. Let $T\colon H^{-1/2}_\star(\mathbb{R}) \mapsto H^{1/2}_\star(\mathbb{R})$ be the isometric isomorphism $\eta \mapsto \mathcal{F}^{-1}[(1 + k^2)^{1/2}k^{-2}\hat{\eta}]$, which has the property that

$$\int_{-\infty}^{\infty} \psi\xi \,\mathrm{d}x = \langle T\psi, \xi \rangle_{H^{1/2}_\star(\mathbb{R})}, \qquad \psi \in H^{-1/2}_\star(\mathbb{R}),\ \xi \in H^{1/2}_\star(\mathbb{R}).$$

It follows from definition 2.8, lemma 2.12 and the calculation

$$\langle TG(\eta)\xi, \xi \rangle_{H^{1/2}_\star(\mathbb{R})} = \int_{-\infty}^{\infty} (G(\eta)\xi)\xi \,\mathrm{d}x = \int_{\Sigma_\eta} |\nabla\phi|^2 \,\mathrm{d}x\,\mathrm{d}y \geqslant 0,$$

where ϕ is the unique weak solution of (2.1)–(2.3), that $TG(\eta)$ is a self-adjoint positive isomorphism $H^{1/2}_\star(\mathbb{R}) \to H^{1/2}_\star(\mathbb{R})$. The spectral theory for bounded self-adjoint operators shows that

$$\xi \mapsto \langle TG(\eta)\xi, \xi \rangle^{1/2}_{H^{1/2}_\star(\mathbb{R})}, \qquad \xi \mapsto \langle N(\eta)T^{-1}\xi, \xi \rangle^{1/2}_{H^{1/2}_\star(\mathbb{R})}$$

are both equivalent to the usual norm for $H^{1/2}_\star(\mathbb{R})$, so that

$$\kappa \mapsto \langle N(\eta)\kappa, T\kappa \rangle^{1/2}_{H^{1/2}_\star(\mathbb{R})}$$

is equivalent to the usual norm for $H^{-1/2}_\star(\mathbb{R})$. The assertion now follows from the first equality in the previous equation and the calculation

$$\langle N(\eta)\kappa, T\kappa \rangle_{H^{1/2}_\star(\mathbb{R})} = \int_{-\infty}^{\infty} (N(\eta)\kappa)\kappa \,\mathrm{d}x.$$

□

2.1.5. *The operator $K(\eta) = -\partial_x(N(\eta)\partial_x)$*

Our first result for this operator is obtained from the material presented above for N.

THEOREM 2.21.

(i) *The operator $K\colon W \to \mathcal{L}(H^{1/2}(\mathbb{R}), H^{-1/2}(\mathbb{R}))$ is analytic.*

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(ii) *For each $\eta \in W$ the operator $K(\eta)\colon H^{1/2}(\mathbb{R}) \to H^{-1/2}(\mathbb{R})$ is an isomorphism and the norm*

$$\zeta \mapsto \left(\int_{-\infty}^{\infty} \zeta K(\eta)\zeta \, \mathrm{d}x \right)^{1/2}$$

is equivalent to the usual norm for $H^{1/2}(\mathbb{R})$.

Proof. (i) This result follows from the definition of K and the continuity of the operators $\partial_x\colon H^{1/2}(\mathbb{R}) \to H_\star^{-1/2}(\mathbb{R})$ and $\partial_x\colon H_\star^{1/2}(\mathbb{R}) \to H^{-1/2}(\mathbb{R})$.

(ii) This result is obtained by writing

$$\begin{aligned}\int_{-\infty}^{\infty} \zeta K(\eta)\zeta \, \mathrm{d}x &= \int_{-\infty}^{\infty} \zeta' N(\eta)\zeta' \, \mathrm{d}x \\ &\geqslant c\|\zeta'\|^2_{H_\star^{-1/2}(\mathbb{R})} \\ &= c\|\zeta\|^2_{1/2},\end{aligned}$$

in which theorem 2.20 has been used. □

In the remainder of this section we establish the following result concerning the analyticity of K in higher-order Sobolev spaces, using the symbol W^r as an abbreviation for $W \cap H^r(\mathbb{R})$.

THEOREM 2.22. *The operator $K\colon W^{s+3/2} \to \mathcal{L}(H^{s+3/2}(\mathbb{R}), H^{s+1/2}(\mathbb{R}))$ is analytic for each $s > 0$.*

To prove theorem 2.22 it is necessary to establish additional regularity of the weak solutions u^n, $n \in \mathbb{N}_0$, of the boundary-value problems given by (2.14)–(2.16) and (2.17)–(2.19). We proceed by examining the general boundary-value problem (2.10)–(2.12) under additional regularity assumptions on ζ and G. Our result is stated in lemma 2.25, the proof of which requires an *a priori* estimate and a commutator estimate (see Lannes [20, Proposition B.10(2)] for a derivation of the latter).

LEMMA 2.23. *Suppose that $Q \in (H^{s+1,2})^{2\times 2}$ and $G \in (H^{t,1})^2$ for some $t \in (\frac{1}{2} - s, s+1]$. The weak solution u to (2.10)–(2.12) satisfies the* a priori *estimate*

$$\|\nabla u\|_{t,1} \leqslant C(\|G\|_{t,1} + \|\nabla u\|_{t,0}),$$

where $C = C(p_0^{-1}, \|Q\|_{s+1,2})$.

Proof. Note that

$$\begin{aligned}\|\nabla u\|_{t,1} &= \|u_x\|_{t,1} + \|u_y\|_{t,1} \\ &= \|u_x\|_{t,0} + \|u_{xy}\|_{t-1,0} + \|u_y\|_{t,0} + \|u_{yy}\|_{t-1,0} \\ &\leqslant C(\|\nabla u\|_{t,0} + \|u_{yy}\|_{t-1,0})\end{aligned}$$

because $\|u_{xy}\|_{t-1,0} \leqslant \|u_y\|_{t,0}$, and to estimate $\|u_{yy}\|_{t-1,0}$ we use (2.10), which we write in the form

$$(1+q_{22})u_{yy} = \nabla\cdot G - \partial_x[(1+q_{11})u_x + q_{12}u_y] - \partial_y(q_{12}u_x) - q_{22y}u_y.$$

Denoting the right-hand side of this equation by H, one finds that

$$\begin{aligned}\|u_{yy}\|_{t-1,0} &= \|(1+q_{22})^{-1}H\|_{t-1,0}\\ &\leqslant \|H\|_{t-1,0} + \|\tilde{q}_{22}H\|_{t-1,0}\\ &\leqslant (1+\|\tilde{q}_{22}\|_{s+1/2,\infty})\|H\|_{t-1,0}\\ &\leqslant C\|H\|_{t-1,0},\end{aligned}$$

where $\tilde{q}_{22} = -q_{22}(1+q_{22})^{-1}$ and we have used the interpolation estimate

$$\left\|\frac{p}{1+p}\right\|_r \leqslant C_1(p_0^{-1}, \|p\|_\infty)\|p\|_r \leqslant C_2(p_0^{-1}, \|p\|_r)$$

for $p \in H^r(\mathbb{R})$, $r > \frac{1}{2}$, with $1 + p(x) \geqslant p_0$ for all $x \in \mathbb{R}$.

It remains to estimate $\|H\|_{t-1,0}$. Observe that $\|\nabla\cdot G\|_{t-1,0} \leqslant \|G\|_{t,1}$, $\|u_{xx}\|_{t-1,0} \leqslant \|\nabla u\|_{t,0}$ and

$$\begin{aligned}\|q_{ij}\nabla u_x\|_{t-1,0} &\leqslant C\|Q\|_{s+1/2,\infty}\|\nabla u_x\|_{t-1,0}\\ &\leqslant C\|Q\|_{s+1,1}\|\nabla u\|_{t,0}.\end{aligned} \tag{2.21}$$

The terms in H involving derivatives of Q are treated differently.

Suppose first that $t \leqslant s + \frac{1}{2}$. Combining the estimate

$$\begin{aligned}\left\|\begin{Bmatrix}\partial_x\\ \partial_y\end{Bmatrix} q_{ij}\nabla u\right\|_{t-1,0} &\leqslant C\left\|\begin{Bmatrix}\partial_x\\ \partial_y\end{Bmatrix} q_{ij}\right\|_{s-1/2,\infty}\|\nabla u\|_{t,0}\\ &\leqslant C\|Q\|_{H^{s+1,2}}\|\nabla u\|_{t,0}\end{aligned}$$

(see proposition 2.4) and the estimate (2.21), one obtains the required result

$$\|u_{yy}\|_{t-1,0} \leqslant \|H\|_{t-1,0} \leqslant C(\|G\|_{t,1} + \|\nabla u\|_{t,0}).$$

For $t \in (s + \frac{1}{2}, s+1]$ we instead estimate

$$\begin{aligned}\left\|\begin{Bmatrix}\partial_x\\ \partial_y\end{Bmatrix} q_{ij}\nabla u\right\|_{t-1,0} &\leqslant C\left\|\begin{Bmatrix}\partial_x\\ \partial_y\end{Bmatrix} q_{ij}\right\|_{s,0}\|\nabla u\|_{t-1/2-\varepsilon,\infty}\\ &\leqslant C\|Q\|_{s+1,1}\|\nabla u\|_{t-\varepsilon,1}\end{aligned}$$

with $0 < \varepsilon < \min\{\frac{1}{2}, s\}$ by proposition 2.4 to find that

$$\begin{aligned}\|u_{yy}\|_{t-1,0} &\leqslant C(\|G\|_{t,1} + \|\nabla u\|_{t,0} + \|\nabla u\|_{t-\varepsilon,1})\\ &\leqslant C(\|G\|_{t,1} + \|\nabla u\|_{t,0} + \|u_{yy}\|_{t-1-\varepsilon,0}).\end{aligned}$$

The result follows by repeating this argument a finite number of times and using the already established result for $t = s + \frac{1}{2}$. □

LEMMA 2.24. *Suppose that $r_0 > \frac{1}{2}$, $\Delta \in [0,1]$ and $r \in (-\frac{1}{2}, r_0 + \Delta]$ and define $\Lambda_\varepsilon^r = \Lambda^r\chi(\varepsilon\Lambda)$ for $\varepsilon \in [0, \varepsilon_0)$. The estimate*

$$\|[\Lambda_\varepsilon^r, u]v\|_0 \leqslant c\|u\|_{r_0+\Delta}\|v\|_{r-\Delta}$$

holds for each $u \in H^{r_0+\Delta}$ and each $v \in H^{r-\Delta}$, where the constant c does not depend upon ε.

LEMMA 2.25. *Suppose that* $Q \in (H^{s+1,2})^{2\times 2}$ *and* $\zeta \in H^{t+3/2}(\mathbb{R})$, $G \in (H^{t+1,1})^2$ *for some* $t \in [0,s]$. *The weak solution* u *of (2.10)–(2.12) with* $\xi = \zeta'$ *satisfies* $\nabla u \in H^{t+1,1}$ *with*

$$\|\nabla u\|_{t+1,1} \leqslant C(\|G\|_{t+1,1} + \|\zeta\|_{t+3/2}),$$

where $C = C(p_0^{-1}, \|Q\|_{s+1,2})$.

Proof. Choose $r \in (0, t+1]$, $\varepsilon > 0$ and note that Λ_ε^r is well defined as an operator on $H_\star^1(\Sigma)$. Writing $w = (\Lambda_\varepsilon^r)^2 u$ in definition 2.17, we find that

$$\int_\Sigma \Lambda_\varepsilon^r(P\nabla u)\cdot\nabla\Lambda_\varepsilon^r u \,\mathrm{d}x\,\mathrm{d}y = \int_\Sigma \Lambda_\varepsilon^r G\cdot\nabla\Lambda_\varepsilon^r u\,\mathrm{d}x\,\mathrm{d}y + \int_{-\infty}^{\infty} \Lambda_\varepsilon^r \xi \Lambda_\varepsilon^r u|_{y=1}\,\mathrm{d}x$$

because Λ_ε^r commutes with partial derivatives and is symmetric with respect to the L^2 inner product. This equation can be rewritten as

$$\begin{aligned}\int_\Sigma P\nabla\Lambda_\varepsilon^r u\cdot\nabla\Lambda_\varepsilon^r u\,\mathrm{d}x\,\mathrm{d}y = -\int_\Sigma [\Lambda_\varepsilon^r, Q]\nabla u\cdot\nabla\Lambda_\varepsilon^r u\,\mathrm{d}x\,\mathrm{d}y + \int_\Sigma \Lambda_\varepsilon^r G\cdot\nabla\Lambda_\varepsilon^r u\,\mathrm{d}x\,\mathrm{d}y \\ - \int_{-\infty}^{\infty}\Lambda_\varepsilon^r\zeta(\Lambda_\varepsilon^r u|_{y=1})_x\,\mathrm{d}x\end{aligned}$$

and it follows from the coercivity of P and the continuity of the trace map $H_\star^1(\Sigma) \to H_\star^{1/2}(\mathbb{R})$ that

$$\begin{aligned}\|\Lambda_\varepsilon^r\nabla u\|_{L^2(\Sigma)} &\leqslant C(\|[\Lambda_\varepsilon^r, Q]\nabla u\|_{L^2(\Sigma)} + \|\Lambda_\varepsilon^r G\|_{L^2(\Sigma)} + \|\Lambda_\varepsilon^r\Lambda^{1/2}\zeta\|_{L^2(\mathbb{R})}) \\ &\leqslant C(\|[\Lambda_\varepsilon^r, Q]\nabla u\|_{L^2(\Sigma)} + \|G\|_{t+1,1} + \|\zeta\|_{t+3/2}).\end{aligned}$$

The next step is to estimate the commutator $[\Lambda_\varepsilon^r, Q]$. For $r \leqslant s+\frac{1}{2}$ we choose $\tilde\Delta \in (0, \min(s,1))$ and estimate

$$\begin{aligned}\|[\Lambda_\varepsilon^r, Q]\nabla u\|_{L^2(\Sigma)} &\leqslant C\|Q\|_{s+1/2,\infty}\|\nabla u\|_{r-\tilde\Delta,0} \\ &\leqslant C\|Q\|_{s+1,1}\|\nabla u\|_{r-\tilde\Delta,0}\end{aligned}$$

using lemma 2.24 (with $r_0 = s+\frac{1}{2}-\tilde\Delta$, $\Delta = \tilde\Delta$). For $r \in (s+\frac{1}{2}, s+1]$, on the other hand, we choose $\tilde\Delta \in (0, \min(s,\frac{1}{2}))$ and estimate

$$\begin{aligned}\|[\Lambda_\varepsilon^r, Q]\nabla u\|_{L^2(\Sigma)} &\leqslant C\|Q\|_{s+1,0}\|\nabla u\|_{r-\tilde\Delta-1/2,\infty} \\ &\leqslant C\|Q\|_{s+1,0}\|\nabla u\|_{r-\tilde\Delta,1}\end{aligned}$$

using lemma 2.24 (with $r_0 = s+\frac{1}{2}-\tilde\Delta$ and $\Delta = \tilde\Delta+\frac{1}{2}$) and

$$\|\nabla u\|_{r-\tilde\Delta,1} \leqslant C(\|G\|_{t+1,1} + \|\nabla u\|_{r-\tilde\Delta,0})$$

using lemma 2.23.

Combining the above estimates yields

$$\|\Lambda_\varepsilon^r\nabla u\|_{L^2(\Sigma)} \leqslant C(\|\nabla u\|_{r-\tilde\Delta,0} + \|G\|_{t+1,1} + \|\zeta\|_{t+3/2}),$$

where $\tilde\Delta \in (0, \min(s,\frac{1}{2}))$, and letting $\varepsilon \to 0$ and using the resulting estimate iteratively, we find that

$$\|\nabla u\|_{t+1,0} \leqslant C(\|G\|_{t+1,1} + \|\zeta\|_{t+3/2} + \|u\|_{H_\star^1(\Sigma)}),$$

from which the result follows by lemmas 2.18 and 2.23. □

The following result shows that lemma 2.25 is applicable to the boundary-value problems (2.14)–(2.16) and (2.17)–(2.19).

Lemma 2.26. *The mapping $W^{s+3/2} \to (H^{s+1,2})^{2\times 2}$ given by $\eta \mapsto Q(\eta)$ is analytic.*

Remark 2.27. Observe that

$$\begin{aligned} Q_x(\eta) &= S_0(\eta) + R_0(\eta)L_0^\delta\eta'' + R_1(\eta)L_1^\delta\eta'', \\ Q_y(\eta) &= T_0(\eta) + R_0(\eta)L_1^\delta\eta'' + R_1(\eta)L_2^\delta\eta'', \end{aligned}$$

where $L_j^\delta(\cdot) = \mathcal{F}^{-1}[(\mathrm{i}\delta)^j\chi^{(j)}((1-y)\delta k)\mathcal{F}[\cdot]]$, $j = 0, 1, 2$, are bounded bilinear functions $L^2(\mathbb{R}) \to L^\infty H^0$ and

$$S_0 \colon \eta \to \begin{pmatrix} \eta_x^\delta & 0 \\ 0 & -\dfrac{\eta_x^\delta}{1+f_y^\delta} - \dfrac{(-f_y^\delta + (f_x^\delta)^2)\eta_x^\delta}{(1+f_y^\delta)^2} \end{pmatrix},$$

$$T_0 \colon \eta \to \begin{pmatrix} 2L_1^\delta\eta' & -\eta_x^\delta \\ -\eta_x^\delta & -\dfrac{2L_1^\delta\eta'}{1+f_y^\delta} + \dfrac{2f_x^\delta\eta_x^\delta}{1+f_y^\delta} - \dfrac{2(-f_y^\delta + (f_x^\delta)^2)L_1^\delta\eta'}{(1+f_y^\delta)^2} \end{pmatrix},$$

$$R_0 \colon \eta \to \begin{pmatrix} 0 & -y \\ -y & \dfrac{2yf_x^\delta}{1+f_y^\delta} \end{pmatrix},$$

$$R_1 \colon \eta \to \begin{pmatrix} y & 0 \\ 0 & -\dfrac{y}{1+f_y^\delta} - \dfrac{y(-f_y^\delta + (f_x^\delta)^2)}{(1+f_y^\delta)^2} \end{pmatrix}$$

are analytic functions $W \to (L^\infty(\bar{\Sigma}))^{2\times 2}$.

The regularity assertion in theorem 2.22 now follows from the next result and the continuity of the trace operator $H^{s+1,1} \to H^{s+1/2}(\mathbb{R})$.

Theorem 2.28. *The mapping $W^{s+3/2} \to \mathcal{L}(H^{s+3/2}(\mathbb{R}), (H^{s+1,1})^2)$ given by $\eta \mapsto (\zeta \mapsto \nabla u)$, where $u \in H^1_\star(\Sigma)$ is the unique weak solution of (2.7)–(2.9) with $\xi = \zeta'$, is analytic.*

Proof. Repeating the proof of theorem 2.19, replacing lemma 2.18 by lemma 2.25, lemma 2.16 by lemma 2.26 and inequality (2.20) by

$$\|G^n\|_{s+1,1} \leqslant \sum_{k=1}^n \|Q^k\|_{s+1,1}\|\nabla u^{n-k}\|_{s+1,1}$$

($H^{s+1,1}$ is a Banach algebra), we obtain the representation

$$\nabla u(x,y) = \sum_{n=0}^\infty \nabla u^n(x,y), \qquad \nabla u^n = m_2^n(\{\tilde{\eta}\}^{(n)}),$$

where $m_2^n \in \mathcal{L}_{\mathrm{s}}^n(H^{s+3/2}(\mathbb{R}), (H^{s+1,1})^2)$ is linear in ζ and satisfies

$$|||m_2^n||| \leqslant C_1 B^n\|\zeta\|_{s+3/2}$$

for some constant $B > 0$. $\square$

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We conclude this section with a useful supplementary estimate for $\|K^n(\tilde{\eta})\|$.

PROPOSITION 2.29. *There exists a constant $B>0$ such that*

$$\|K^n(\tilde{\eta})\zeta\|_0 \leqslant C_1 B^n(\|\tilde{\eta}\|_{1,\infty} + \|\tilde{\eta}'' + k_0^2\tilde{\eta}\|_0)^n \|\zeta\|_{3/2}, \quad n \in \mathbb{N}_0.$$

Proof. It suffices to establish the estimate

$$\|\nabla u^n\|_1 \leqslant C_1 B^n(\|\tilde{\eta}\|_{1,\infty} + \|\tilde{\eta}'' + k_0^2\tilde{\eta}\|_0)^n \|\zeta\|_{3/2}, \quad n \in \mathbb{N}_0;$$

for $n=0$ this result follows from lemma 2.25 (with $t=0$ and $s=\frac{1}{2}$).

Proceeding inductively, suppose that the estimate for $\|\nabla u^k\|_1$ is true for all $k<n$ and recall from the proof of theorem 2.19 that

$$\|Q^k\|_\infty \leqslant C_2 r^{-k}\|\tilde{\eta}\|_{1,\infty}^k, \qquad \|G^n\|_0 \leqslant C_1 C_2 B^n \|\zeta\|_{3/2}\|\tilde{\eta}\|_{1,\infty}^n \sum_{k=1}^n (Br)^{-k}.$$

Writing

$$\begin{aligned} Q_x^k &= S_0^k + R_0^k L_0^\delta \eta_0'' + R_0^{k-1} L_0^\delta \tilde{\eta}'' + R_1^k L_0^\delta \eta_0'' + R_1^{k-1} L_0^\delta \tilde{\eta}'' \\ &= S_0^k + \sum_{j=0}^1 (-k_0^2 R_j^{k-1} L_j^\delta \tilde{\eta} + R_j^k L_j^\delta \eta_0'' + R_j^{k-1} L_j^\delta(\tilde{\eta}'' + k_0^2\tilde{\eta})), \end{aligned}$$

where

$$\|S_0^k\|_\infty \leqslant C_2 r^{-k}\|\tilde{\eta}\|_{1,\infty}^k, \qquad \|R_j^k\|_\infty \leqslant C_2 r^{-k}\|\tilde{\eta}\|_{1,\infty}^k, \quad j=0,1$$

(see remark 2.27), we find that

$$\begin{aligned} G_x^n &= -\sum_{k=1}^n (Q_x^k \nabla u^{n-k} + Q^k \nabla u_x^{n-k}) \\ &= \sum_{k=1}^n \bigg(S_0^k \nabla u^{n-k} + \sum_{j=0}^1 (-k_0^2 R_j^{k-1} L_j^\delta \tilde{\eta} + R_j^k L_j^\delta \eta_0'' + R_j^{k-1} L_j^\delta (\tilde{\eta}'' + k_0^2\tilde{\eta}))\nabla u^{n-k} \\ &\qquad\qquad + Q^k \nabla u_x^{n-k} \bigg). \end{aligned}$$

It follows that

$$\begin{aligned} \|G_x^n\|_0 \leqslant{}& \sum_{k=1}^n ((\|S_0^k\|_\infty + k_0^2(\|R_0^{k-1}\|_\infty + \|R_1^{k-1}\|_\infty)\|\tilde{\eta}\|_\infty)\|\nabla u^{n-k}\|_0 \\ &\quad + (\|R_0^k\|_\infty \|L_0^\delta\| + \|R_1^k\|_\infty \|\|L_1^\delta\|)\|\eta_0''\|_0 \|\nabla u^{n-k}\|_1 \\ &\quad + (\|R_0^{k-1}\|_\infty \|L_0^\delta\| + \|R_1^{k-1}\|_\infty \|L_1^\delta\|)\|\tilde{\eta}'' + k_0^2\tilde{\eta}\|_0 \|\nabla u^{n-k}\|_1 \\ &\quad + \|Q^k\|_\infty \|\nabla u_x^{n-k}\|_0) \\ \leqslant{}& C_1 C_2 B^n (1 + 2k_0^2 r + (\|L_0^\delta\| + \|L_1^\delta\|)(\|\eta_0''\|_0 + r) + 1) \\ &\quad \times \|\zeta\|_{3/2}(\|\tilde{\eta}\|_{1,\infty} + \|\tilde{\eta}'' + k_0^2\tilde{\eta}\|_0)^n \sum_{k=1}^n (Br)^{-k}, \end{aligned}$$

in which proposition 2.5 has been used. A similar calculation yields the same estimate for $\|G_y^n\|_0$.

Combining the estimates for $\|G^n\|_0$, $\|G_x^n\|_0$ and $\|G_y^n\|_0$ and applying lemma 2.25 (with $t = 0$ and $s = \frac{1}{2}$), one finds that

$$\|\nabla u^n\|_1 \leqslant \sqrt{3} C_1 C_2 C_3 B^n (1 + 2k_0^2 r + (\|L_0^\delta\| + \|L_1^\delta\|)(\|\eta_0''\|_0 + r) + 1) \\ \times \|\zeta\|_{3/2} (\|\tilde{\eta}\|_{1,\infty} + \|\tilde{\eta}'' + k_0^2 \tilde{\eta}\|_0)^n \sum_{k=1}^{n} (Br)^{-k},$$

so that

$$\|\nabla u^n\|_1 \leqslant C_1 B^n (\|\tilde{\eta}\|_{1,\infty} + \|\tilde{\eta}'' + k_0^2 \tilde{\eta}\|_0)^n \|\zeta\|_{3/2}$$

for sufficiently large values of B (independently of n). □

2.2. Variational functionals

In this section we study the functional

$$\mathcal{T}(\eta) = \int_{-\infty}^{\infty} f_1(\eta) K(\eta) f_2(\eta) \, \mathrm{d}x, \tag{2.22}$$

where $f_1, f_2 \colon \mathbb{R} \to \mathbb{R}$ are polynomials with $f_1(0) = f_2(0) = 0$, and apply our results to the functionals $\mathcal{G}$, $\mathcal{K}$ and $\mathcal{L}$.

2.2.1. *Analyticity of the functionals*

In this section we again suppose that $s > 0$. The first result follows from theorem 2.21(i).

Lemma 2.30. *Equation (2.22) defines a functional $\mathcal{T} \colon W^{s+3/2} \to \mathbb{R}$ that is analytic and satisfies $\mathcal{T}(0) = 0$.*

We now turn to the construction of the gradient $\mathcal{T}'(\eta)$ in $L^2(\mathbb{R})$, the main step of which is accomplished by the following lemma.

Lemma 2.31. *Define $\mathcal{H} \colon W^{s+3/2} \to \mathcal{L}_s^2(H^{s+3/2}(\mathbb{R}), \mathbb{R})$ by the formula*

$$\mathcal{H}(\eta)(\zeta_1, \zeta_2) = \langle \zeta_1, K(\eta)\zeta_2 \rangle_0.$$

The gradient $\mathcal{H}'(\eta)(\zeta_1, \zeta_2)$ in $L^2(\mathbb{R})$ exists for each $\eta \in W^{s+3/2}$ and $\zeta_1, \zeta_2 \in H^{s+3/2}(\mathbb{R})$ and is given by the formula

$$\mathcal{H}'(\eta)(\zeta_1, \zeta_2) = -u_{1x} u_{2x} + \frac{1 + \eta'^2}{(1 + \eta)^2} u_{1y} u_{2y} \bigg|_{y=1},$$

where u_j is the weak solution of (2.7)–(2.9) with $\xi = \zeta_j'$, $j = 1, 2$. This formula defines an analytic function $\mathcal{H}' \colon W^{s+3/2} \to \mathcal{L}_{\mathrm{s}}^2(H^{s+3/2}(\mathbb{R}), H^{s+1/2}(\mathbb{R}))$.

Proof. It follows from the formula

$$\mathcal{H}(\eta) = \int_\Sigma (I + Q(\eta)) \nabla u_1 \cdot \nabla u_2 \, \mathrm{d}x \, \mathrm{d}y$$

that

$$\mathrm{d}\mathcal{H}[\eta](\omega) = \int_\Sigma \mathrm{d}Q[\eta](\omega)\nabla u_1 \cdot \nabla u_2 \,\mathrm{d}x\,\mathrm{d}y + \int_\Sigma (I + Q(\eta))\nabla w_1 \cdot \nabla u_2 \,\mathrm{d}x\,\mathrm{d}y + \int_\Sigma (I + Q(\eta))\nabla u_1 \cdot \nabla w_2 \,\mathrm{d}x\,\mathrm{d}y, \tag{2.23}$$

where $w_j = \mathrm{d}u_j(\eta)[\omega]$, $j = 1, 2$. Recall that

$$\int_\Sigma (I + Q(\eta))\nabla u_j \cdot \nabla v \,\mathrm{d}x\,\mathrm{d}y = \int_{-\infty}^{\infty} \zeta_j' v|_{y=1} \,\mathrm{d}x, \quad j = 1, 2,$$

for every $v \in H^1_\star(\Sigma)$ (see definition 2.17 with $\xi = \zeta_j'$ and $G = 0$), so that

$$\int_\Sigma (\mathrm{d}Q[\eta](\omega)\nabla u_j \cdot \nabla v + (I + Q(\eta))\nabla w_j \cdot \nabla v) \,\mathrm{d}x\,\mathrm{d}y = 0, \quad j = 1, 2, \tag{2.24}$$

for every $v \in H^1_\star(\Sigma)$. Subtracting (2.24) with $j = 1$, $v = u_2$ and $j = 2$, $v = u_1$ from (2.23) yields

$$\mathrm{d}\mathcal{H}[\eta](\omega) = -\int_\Sigma \mathrm{d}Q[\eta](\omega)\nabla u_1 \cdot \nabla u_2 \,\mathrm{d}x\,\mathrm{d}y.$$

Finally, write $h^\delta(x, y) = y\omega^\delta(x, y)$, where

$$\omega^\delta(x, y) = \mathcal{F}^{-1}[\chi(\delta(y - 1)|k|)\hat{\omega}(k)](x),$$

so that $h^\delta = \mathrm{d}f^\delta[\eta](\omega)$, and observe that

$$\begin{aligned}
&\int_{-\infty}^{\infty} \left(-h^\delta\left(u_{1x} - \frac{f_x^\delta y u_{1y}}{1 + f_y^\delta}\right)\left(u_{2x} - \frac{f_x^\delta y u_{2y}}{1 + f_y^\delta}\right) + \frac{h^\delta u_{1y} u_{2y}}{(1 + f_y^\delta)^2}\right)\bigg|_{y=1} \mathrm{d}x \\
&= \frac{1}{2}\int_\Sigma \frac{\mathrm{d}}{\mathrm{d}y}\left(-h^\delta\left(u_{1x} - \frac{f_x^\delta y u_{1y}}{1 + f_y^\delta}\right)\left(u_{2x} - \frac{f_x^\delta y u_{2y}}{1 + f_y^\delta}\right) + \frac{h^\delta u_{1y} u_{2y}}{(1 + f_y^\delta)^2}\right) \mathrm{d}x\,\mathrm{d}y \\
&= \int_\Sigma \bigg(-h_x^\delta u_{1x} u_{2x} + h_x^\delta u_{1x} u_{2y} + h_x^\delta u_{1y} u_{2x} \\
&\qquad\qquad + \frac{h_y^\delta u_{1y} u_{2y}}{(1 + f_y^\delta)^2} + \frac{2(f_x^\delta)^2 h_y^\delta u_{1y} u_{2y}}{(1 + f_y^\delta)^2} - \frac{2 f_x^\delta h_x^\delta u_{1y} u_{2y}}{1 + f_y^\delta}\bigg) \mathrm{d}x\,\mathrm{d}y \\
&\quad + \int_\Sigma \frac{h^\delta u_{1y}}{1 + f_y^\delta}\left(((1 + f_y^\delta)u_{2x} - f_x^\delta u_{2y})_x + \left(-f_x^\delta u_{2x} + \frac{1 + (f_x^\delta)^2}{1 + f_y^\delta} u_{2y}\right)_y\right) \mathrm{d}x\,\mathrm{d}y \\
&\quad + \int_\Sigma \frac{h^\delta u_{2y}}{1 + f_y^\delta}\left(((1 + f_y^\delta)u_{1x} - f_x^\delta u_{1y})_x + \left(-f_x^\delta u_{1x} + \frac{1 + (f_x^\delta)^2}{1 + f_y^\delta} u_{1y}\right)_y\right) \mathrm{d}x\,\mathrm{d}y \\
&\quad + \int_{-\infty}^{\infty} \left(\frac{h^\delta f_x^\delta u_{1y}}{1 + f_y^\delta}\left(u_{2x} - \frac{f_x^\delta y u_{2y}}{1 + f_y^\delta}\right) + \frac{h^\delta f_x^\delta u_{2y}}{1 + f_y^\delta}\left(u_{1x} - \frac{f_x^\delta y u_{1y}}{1 + f_y^\delta}\right)\right)\bigg|_{y=1} \mathrm{d}x
\end{aligned}$$

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$$
\begin{aligned}
&= -\int_\Sigma \Bigg(\mathrm{d}Q[\eta](\omega)\nabla u_1 \cdot \nabla u_2 \\
&\qquad\qquad + \frac{h^\delta u_{1y}}{1+f_y^\delta} \underbrace{\nabla\cdot((I+Q(\eta))\nabla u_1)}_{=0} + \frac{h^\delta u_{2y}}{1+f_y^\delta} \underbrace{\nabla\cdot((I+Q(\eta))\nabla u_2)}_{=0} \Bigg)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \int_{-\infty}^{\infty} \left(\frac{h^\delta f_x^\delta u_{1y}}{1+f_y^\delta}\left(u_{2x} - \frac{f_x^\delta y u_{2y}}{1+f_y^\delta}\right) + \frac{h^\delta f_x^\delta u_{2y}}{1+f_y^\delta}\left(u_{1x} - \frac{f_x^\delta y u_{1y}}{1+f_y^\delta}\right)\right)\Bigg|_{y=1} \mathrm{d}x,
\end{aligned}
$$

in which the third line follows from the second by differentiating the term in braces with respect to y (note that $h^\delta|_{y=0} = 0$) and integrating by parts. One concludes that

$$
\mathrm{d}\mathcal{H}[\eta](\omega) = \int_{-\infty}^{\infty} \left(-u_{1x}u_{2x} + \frac{1+(f_x^\delta)^2}{(1+f_y^\delta)^2} u_{1y}u_{2y} \right) h^\delta \Bigg|_{y=1} \mathrm{d}x,
$$

and the stated formula follows from this result and the facts that $f^\delta|_{y=1} = \eta$ and $h^\delta|_{y=1} = \omega$.

The hypotheses of the lemma imply that $\nabla u_j \in H^{s+1,1}$ and $\nabla u_j|_{u=1} \in H^{s+1/2}(\mathbb{R})$, $j = 1, 2$. This observation ensures that the above algebraic manipulations are valid and that $\mathrm{d}\mathcal{H}[\eta]$ belongs to $H^{s+1/2}(\mathbb{R})$ because $H^{s+1,1}$ and $H^{s+1/2}(\mathbb{R})$ are Banach algebras. □

Corollary 2.32. *The gradient $\mathcal{T}'(\eta)$ in $L^2(\mathbb{R})$ exists for each $\eta \in W^{s+3/2}$ and is given by the formula*

$$
\mathcal{T}'(\eta) = \mathcal{H}'(\eta)(f_1(\eta), f_2(\eta)) + f_1'(\eta)K(\eta)f_2(\eta) + f_2'(\eta)K(\eta)f_1(\eta).
$$

This formula defines an analytic function $\mathcal{T}'\colon W^{s+3/2} \to H^{s+1/2}(\mathbb{R})$ that satisfies $\mathcal{T}'(0) = 0$.

Theorem 2.33.

(i) *Equations (1.10)–(1.12) define analytic functionals $\mathcal{G}, \mathcal{K}, \mathcal{L}\colon W^{s+3/2} \to \mathbb{R}$ that satisfy $\mathcal{G}(0), \mathcal{K}(0), \mathcal{L}(0) = 0$.*

(ii) *Equation (1.9) defines an analytic functional $\mathcal{J}_\mu\colon W^{s+3/2} \setminus \{0\} \to \mathbb{R}$.*

(iii) *The gradients $\mathcal{G}'(\eta)$ and $\mathcal{L}'(\eta)$ in $L^2(\mathbb{R})$ exist for each $\eta \in W^{s+3/2}$ and are given by the equations*

$$
\mathcal{G}'(\eta) = \tfrac{1}{4}\omega\mathcal{H}'(\eta)(\eta^2, \eta) + \tfrac{1}{4}\omega K(\eta)\eta^2 + \tfrac{1}{2}\omega\eta K(\eta)\eta - \tfrac{1}{2}\omega\eta, \tag{2.25}
$$

$$
\mathcal{L}'(\eta) = \tfrac{1}{2}\mathcal{H}'(\eta)(\eta, \eta) + K(\eta)\eta. \tag{2.26}
$$

These equations define analytic functions $\mathcal{G}', \mathcal{L}'\colon W^{s+3/2} \to H^{s+1/2}(\mathbb{R})$ that satisfy $\mathcal{G}'(0) = 0$ and $\mathcal{L}'(0) = 0$.

(iv) *The gradient $\mathcal{K}'(\eta)$ in $L^2(\mathbb{R})$ exists for each $\eta \in W^2$ and is given by*

$$
\mathcal{K}'(\eta) = \eta - \beta\left(\frac{\eta'}{\sqrt{1+\eta'^2}}\right)' - \frac{\omega^2}{8}\mathcal{H}'(\eta)(\eta^2, \eta^2) - \frac{\omega^2}{2}\eta^2 K(\eta)\eta + \frac{\omega^2}{3}\eta^2. \tag{2.27}
$$

This equation defines an analytic function $\mathcal{K}'\colon W^2 \to L^2(\mathbb{R})$ that satisfies $\mathcal{K}'(0) = 0$.

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(v) *The gradient* $\mathcal{J}'_\mu(\eta)$ *in* $L^2(\mathbb{R})$ *exists for each* $\eta \in W^2 \setminus \{0\}$ *and defines an analytic function* $\mathcal{J}'_\mu \colon W^2 \setminus \{0\} \to L^2(\mathbb{R})$.

Corollary 2.34. *Choose* $M > 0$ *so that* $\bar{B}_M(0) \subseteq H^2(\mathbb{R})$ *lies in* $W^{s+3/2}$. *Equations (1.10)–(1.12) define analytic functionals* $\mathcal{G}, \mathcal{K}, \mathcal{L} \colon U \to \mathbb{R}$, *while (2.25)–(2.27) define analytic functions* $\mathcal{G}', \mathcal{K}', \mathcal{L}' \colon U \to L^2(\mathbb{R})$, *where* $U = B_M(0)$.

Finally, we state some further useful estimates for the operators $\mathcal{G}$, $\mathcal{K}$ and $\mathcal{L}$. Here and in the remainder of this paper, the constant M is chosen small enough for the validity of our calculations.

Proposition 2.35. *The estimates*

$$|\mathcal{G}(\eta)| \leqslant c\|\eta\|_{1/2}^2, \qquad \mathcal{K}(\eta) \geqslant c\|\eta\|_1^2, \qquad c\|\eta\|_{1/2}^2 \leqslant \mathcal{L}(\eta) \leqslant c\|\eta\|_{1/2}^2$$

hold for each $\eta \in U$.

Proof. The estimate for $\mathcal{G}$ follows from the calculation

$$|\mathcal{G}(\eta)| \leqslant c(\|\eta\|_0^2 \|K(\eta)\eta\|_0 + \|\eta\|_0^2) \leqslant c\|\eta\|_{1/2}^2,$$

while that for $\mathcal{L}$ is a direct consequence of theorem 2.21(ii). Turning to the estimate for $\mathcal{K}$, observe that

$$\mathcal{K}(\eta) = \underbrace{\int_{-\infty}^{\infty} \left\{ \frac{\beta \eta'^2}{1 + \sqrt{1 + \eta'^2}} + \frac{\eta^2}{2} \right\} \mathrm{d}x}_{\geqslant c\|\eta\|_1^2} - \frac{\omega^2}{8} \int_{-\infty}^{\infty} \eta^2 K(\eta) \eta^2 \, \mathrm{d}x + \frac{\omega^2}{6} \int_{-\infty}^{\infty} \eta^3 \, \mathrm{d}x$$

and

$$\left| \int_{-\infty}^{\infty} \eta^3 \, \mathrm{d}x \right| \leqslant c\|\eta\|_1^3, \qquad \left| \int_{-\infty}^{\infty} \eta^2 K(\eta) \eta^2 \, \mathrm{d}x \right| \leqslant c\|\eta^2\|_{1/2}^2 \leqslant c\|\eta\|_1^4$$

for each $\eta \in U$, so that $\mathcal{K}(\eta) \geqslant c\|\eta\|_1^2$. □

2.2.2. *Pseudo-local properties of the operator* $\mathcal{T}$

In this section we consider sequences $\{\eta_m^{(1)}\}$, $\{\eta_m^{(2)}\} \subset U$ with the properties that

$$\operatorname{supp} \eta_m^{(1)} \subset [-R_m, R_m], \qquad \operatorname{supp} \eta_m^{(2)} \subset \mathbb{R} \setminus (-S_m, S_m)$$

and

$$\sup_{m \in \mathbb{N}} \|\eta_m^{(1)} + \eta_m^{(1)}\|_2 < M,$$

where $\{R_m\}$, $\{S_m\}$ are sequences of positive real numbers with $R_m, S_m \to \infty$, $R_m/S_m \to 0$ as $m \to \infty$. We establish the following 'pseudo-local' property of the operator $\mathcal{T}$.

Theorem 2.36. *The operator* $\mathcal{T}$ *satisfies*

$$\lim_{m \to \infty} (\mathcal{T}(\eta_m^{(1)} + \eta_m^{(2)}) - \mathcal{T}(\eta_m^{(1)}) - \mathcal{T}(\eta_m^{(2)})) = 0,$$
$$\lim_{m \to \infty} \|\mathcal{T}'(\eta_m^{(1)} + \eta_m^{(2)}) - \mathcal{T}'(\eta_m^{(1)}) - \mathcal{T}'(\eta_m^{(2)})\|_0 = 0,$$
$$\lim_{m \to \infty} \langle \mathcal{T}'(\eta_m^{(2)}), \eta_m^{(1)} \rangle_0 = 0.$$

In particular, this result applies to $\mathcal{G}$, $\mathcal{K}$ *and* $\mathcal{L}$.

We begin the proof of theorem 2.36 by re-examining the general boundary-value problem (2.10)–(2.12).

Lemma 2.37. *Suppose that* $\{R_m\}$, $\{S_m\}$ *and* $\{U_m\}$ *are sequences of positive real numbers and*

$$\{Q_m\} \subseteq (L^\infty(\bar{\Sigma}))^{2\times 2}, \qquad \{G_m^{(1)}\}, \{G_m^{(2)}\} \subseteq L^2(\Sigma), \qquad \{\zeta_m^{(1)}\}, \{\zeta_m^{(2)}\} \subseteq H^{1/2}(\mathbb{R})$$

are bounded sequences with the properties that

(i) $S_m - U_m, U_m - R_m \to \infty$ *as* $m \to \infty$*;*

(ii) $\operatorname{supp} \zeta_m^{(1)} \subset [-R_m, R_m]$ *and* $\operatorname{supp} \zeta_m^{(2)} \subset \mathbb{R} \setminus (-S_m, S_m)$*;*

(iii) $\|G_m^{(1)}\|_{L^2(|x|>R_m)}, \|G_m^{(2)}\|_{L^2(|x|<S_m)} \to 0$ *as* $m \to \infty$*;*

(iv) *there exists a constant* $p_0 > 0$ *such that*

$$(I + Q_m)(x, y)\nu \cdot \nu \geqslant p_0 |\nu|^2$$

for all $(x, y) \in \bar{\Sigma}$*, all* $m \in \mathbb{N}$ *and all* $\nu \in \mathbb{R}^2$*.*

The unique weak solutions $u_m^{(j)} \in H_\star^1(\Sigma)$ *of the boundary-value problems*

$$\nabla \cdot ((I + Q_m)\nabla u_m^{(j)}) = \nabla \cdot G_m^{(j)}, \qquad 0 < y < 1, \tag{2.28}$$

$$(I + Q_m)\nabla u_m^{(j)} \cdot (0, 1) = \zeta_{m,x}^{(j)} + G_m^{(j)} \cdot (0, 1), \quad y = 1, \tag{2.29}$$

$$(I + Q_m)\nabla u_m^{(j)} \cdot (0, -1) = G_m^{(j)} \cdot (0, -1), \qquad y = 0, \tag{2.30}$$

$j = 1, 2$*, satisfy the estimates*

$$\lim_{m\to\infty} \|\nabla u_m^{(1)}\|_{L^2(|x|>U_m)} = 0, \qquad \lim_{m\to\infty} \|\nabla u_m^{(2)}\|_{L^2(|x|<U_m)} = 0.$$

Proof. Write $\zeta_m^{(2)} = \zeta_{m,+}^{(2)} + \zeta_{m,-}^{(2)}$, where

$$\operatorname{supp} \zeta_{m,+}^{(2)} \subseteq [S_m, \infty), \qquad \operatorname{supp} \zeta_{m,-}^{(2)} \subseteq (-\infty, -S_m],$$

and let $u_{m,+}^{(2)}, u_{m,-}^{(2)}$ be the weak solutions of the boundary-value problem (2.28)–(2.29) with $\zeta_m^{(2)}$, $G_m^{(2)}$ replaced by

$$\zeta_{m,+}^{(2)},\ G_{m,+}^{(2)} := G_m^{(2)} \chi_{\{x>0\}}, \qquad \zeta_{m,-}^{(2)},\ G_{m,-}^{(2)} := G_m^{(2)} \chi_{\{x<0\}},$$

respectively, so that $u_m^{(2)} = u_{m,+}^{(2)} + u_{m,-}^{(2)}$.

Choose $T > 0$ and take m large enough so that $T + 1 < S_m$. Define $\phi \in C^\infty(\mathbb{R})$ by

$$\phi_T(x) = \begin{cases} 1, & x \leqslant T, \\ \chi(2(x - T)), & x > T, \end{cases}$$

and set

$$w_m(x, y) = \phi_T^2(x)(u_{m,+}^{(2)}(x, y) - M_T),$$

where

$$M_T = \int_{T \leqslant x \leqslant T+1} u_{m,+}^{(2)}(x, y) \, \mathrm{d}x \, \mathrm{d}y,$$

so that $\operatorname{supp} w_m \subseteq (-\infty, T+1] \times [0,1]$ and the mean value of $u^{(2)}_{m,+}(x,y) - M_T$ over $(T, T+1) \times (0,1)$ is zero. Using definition 2.17, we find that

$$\int_\Sigma (I+Q_m)\nabla u^{(2)}_{m,+} \cdot \nabla w_m \, \mathrm{d}x \, \mathrm{d}y = \int_\Sigma G^{(2)}_{m,+} \cdot \nabla w_m \, \mathrm{d}x \, \mathrm{d}y + \underbrace{\int_{-\infty}^{\infty} \partial_x \zeta^{(2)}_{m,+} w_m|_{y=1} \, \mathrm{d}x}_{=0},$$

from which it follows that

$$\begin{aligned}
&\int_\Sigma (I+Q_m)\phi_T^2 \nabla u^{(2)}_{m,+} \cdot \nabla u^{(2)}_{m,+} \, \mathrm{d}x \, \mathrm{d}y \\
&\quad \leqslant c\biggl(\biggl(\int_\Sigma \phi_T^2 |\nabla u^{(2)}_{m,+}|^2 \, \mathrm{d}x \, \mathrm{d}y\biggr)^{1/2} \biggl(\int_{T \leqslant x \leqslant T+1} |u^{(2)}_{m,+} - M_T|^2 \, \mathrm{d}x \, \mathrm{d}y\biggr)^{1/2} \\
&\qquad + \biggl(\int_{x \leqslant T+1} |G^{(2)}_{m,+}|^2 \, \mathrm{d}x \, \mathrm{d}y\biggr)^{1/2} \biggl(\int_{T \leqslant x \leqslant T+1} |u^{(2)}_{m,+} - M_T|^2 \, \mathrm{d}x \, \mathrm{d}y\biggr)^{1/2} \\
&\qquad + \biggl(\int_{x \leqslant T+1} |G^{(2)}_{m,+}|^2 \, \mathrm{d}x \, \mathrm{d}y\biggr)^{1/2} \biggl(\int_\Sigma \phi_T^2 |\nabla u^{(2)}_{m,+}|^2 \, \mathrm{d}x \, \mathrm{d}y\biggr)^{1/2}\biggr),
\end{aligned}$$

and hence that

$$\int_\Sigma \phi_T^2 |\nabla u^{(2)}_{m,+}|^2 \, \mathrm{d}x \, \mathrm{d}y \leqslant c\biggl(\int_{T \leqslant x \leqslant T+1} |\nabla u^{(2)}_{m,+}|^2 \, \mathrm{d}x \, \mathrm{d}y + \int_{x \leqslant T+1} |G^{(2)}_{m,+}|^2 \, \mathrm{d}x \, \mathrm{d}y\biggr),$$

where the Poincaré inequality

$$\int_{T \leqslant x \leqslant T+1} |u^{(2)}_{m,+} - M_T|^2 \, \mathrm{d}x \, \mathrm{d}y \leqslant c \int_{T \leqslant x \leqslant T+1} |\nabla u^{(2)}_{m,+}|^2 \, \mathrm{d}x \, \mathrm{d}y$$

has been used.

The above inequality implies that

$$\Phi(T) \leqslant c_\star(\Phi(T+1) - \Phi(T) + \Psi(T+1))$$

for some $c_\star > 0$, where

$$\Phi(T) = \int_{x \leqslant T} |\nabla u^{(2)}_{m,+}|^2 \, \mathrm{d}x \, \mathrm{d}y, \qquad \Psi(T) = \int_{x \leqslant T} |G^{(2)}_{m,+}|^2 \, \mathrm{d}x \, \mathrm{d}y,$$

so that

$$\Phi(T) \leqslant d_\star(\Phi(T+1) + \Psi(T+1)),$$

where $d_\star = c_\star/(c_\star + 1) \in (0,1)$, and using this inequality recursively, one finds that

$$\Phi(T) \leqslant d_\star^{[r]} \Phi(T+r) + \frac{d_\star}{1-d_\star} \Psi(T+r), \quad r \geqslant 1.$$

In particular, this result asserts that

$$\Phi(U_m) \leqslant d_\star^{S_m - U_m - 1} \Phi(S_m) + \frac{d_\star}{1-d_\star} \Psi(S_m)$$

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and, because

$$\varPhi(S_m) = \int_{x<S_m} |\nabla u_{m,+}^{(2)}|^2 \,\mathrm{d}x\,\mathrm{d}y \leqslant \int_{\varSigma} |\nabla u_{m,+}^{(2)}|^2 \,\mathrm{d}x\,\mathrm{d}y \leqslant \|\zeta_m^{(2)}\|_{1/2} = O(1)$$

and

$$\varPsi(S_m) = \int_{x<S_m} |G_{m,+}^{(2)}|^2 \,\mathrm{d}x\,\mathrm{d}y \leqslant \int_{|x|<S_m} |G_m^{(2)}|^2 \,\mathrm{d}x\,\mathrm{d}y = o(1)$$

as $m \to \infty$, we conclude that

$$\varPhi(U_m) = \int_{x<U_m} |\nabla u_{m,+}^{(2)}|^2 \,\mathrm{d}x\,\mathrm{d}y = o(1)$$

as $m \to \infty$.

A similar argument shows that

$$\int_{x>-U_m} |\nabla u_{m,-}^{(2)}|^2 \,\mathrm{d}x\,\mathrm{d}y = o(1)$$

as $m \to \infty$, so that

$$\begin{aligned}\int_{|x|<U_m} |\nabla u_m^{(2)}|^2 &\leqslant \int_{|x|<U_m} |\nabla u_{m,+}^{(2)}|^2 \,\mathrm{d}x\,\mathrm{d}y + \int_{|x|<U_m} |\nabla u_{m,-}^{(2)}|^2 \,\mathrm{d}x\,\mathrm{d}y \\ &\leqslant \int_{x<U_m} |\nabla u_{m,+}^{(2)}|^2 \,\mathrm{d}x\,\mathrm{d}y + \int_{x>-U_m} |\nabla u_{m,-}^{(2)}|^2 \,\mathrm{d}x\,\mathrm{d}y \\ &= o(1)\end{aligned}$$

as $m \to \infty$.

The complementary estimate

$$\int_{|x|>U_m} |\nabla u_m^{(1)}|^2 = o(1)$$

as $m \to \infty$ is obtained in a similar fashion. □

The next step is to apply lemma 2.37 to the boundary-value problem (2.7)–(2.9).

LEMMA 2.38. *Let $u(\eta)$ be the solution to (2.7)–(2.9) with $\xi = \partial_x f(\eta)$, $\eta \in U$, where f is a real polynomial. The estimates*

$$\lim_{m\to\infty} \|\nabla u(\eta_m^{(1)})\|_{H^1(|x|>T_m)} = 0, \qquad \lim_{m\to\infty} \|\nabla u(\eta_m^{(2)})\|_{H^1(|x|<T_m)} = 0$$

hold for each sequence $\{T_m\}$ of positive real numbers with $S_m - T_m, T_m - R_m \to \infty$ as $m \to \infty$.

Proof. Choose sequences $\{\tilde{R}_m\}$, $\{\tilde{S}_m\}$ of positive real numbers with $S_m - \tilde{S}_m, \tilde{S}_m - T_m \to \infty$ and $T_m - \tilde{R}_m, \tilde{R}_m - R_m \to \infty$ as $m \to \infty$. The quantities $u_m^{(j)} = u(\eta_m^{(j)})$, $j = 1, 2$, satisfy the boundary-value problems

$$\begin{aligned} \nabla\cdot((I + Q_m^{(j)})\nabla u_m^{(j)}) &= 0, && 0 < y < 1, \\ (I + Q_m^{(j)})\nabla u_m^{(j)}\cdot(0,1) &= f(\eta_m^{(j)})_x, && y = 1, \\ (I + Q_m^{(j)})\nabla u_m^{(j)}\cdot(0,-1) &= 0, && y = 0, \end{aligned}$$

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where $Q_m^{(j)} = Q(\eta_m^{(j)})$, and lemma 2.37 asserts that

$$\lim_{m\to\infty} \|\nabla u_m^{(1)}\|_{L^2(|x|>\tilde{R}_m)} = 0, \qquad \lim_{m\to\infty} \|\nabla u_m^{(2)}\|_{L^2(|x|<\tilde{S}_m)} = 0.$$

The derivatives $u_{mx}^{(j)}$, $j = 1, 2$ are weak solutions of the boundary-value problems

$$\begin{aligned} \nabla\cdot((I+Q_m^{(j)})\nabla u_{mx}^{(j)}) &= \nabla\cdot G_m^{(j)}, && 0<y<1,\\ (I+Q_m^{(j)})\nabla u_{mx}^{(j)}\cdot(0,1) &= f(\eta_m^{(j)})_{xx} + G_m^{(j)}\cdot(0,1), && y=1,\\ (I+Q_m^{(j)})\nabla u_{mx}^{(j)}\cdot(0,-1) &= G_m^{(j)}\cdot(0,-1), && y=0, \end{aligned}$$

where $G_m^{(j)} = -Q_{mx}^{(j)}\nabla u_m^{(j)}$. Using remark 2.27 and writing $S_{0m}^{(j)} = S_0(\eta_m^{(j)})$, $R_{0m}^{(j)} = R_0(\eta_m^{(j)})$, $R_{1m}^{(j)} = R_1(\eta_m^{(j)})$, one finds that

$$\begin{aligned} \|Q_{mx}^{(1)}\nabla u_m^{(1)}\|_{L^2(|x|>\tilde{R}_m)} &\leqslant \|S_{0m}^{(1)}\|_\infty \|\nabla u_m^{(1)}\|_{L^2(|x|>\tilde{R}_m)}\\ &\qquad + c(\|R_{0m}^{(1)}\|_\infty\|L_0^\delta\| + \|R_{1m}^{(1)}\|_\infty\|L_1^\delta\|)\\ &\qquad\quad \times \|(\eta_m^{(1)})''\|_0 \|\nabla u_m^{(1)}\|_{L^2(|x|>\tilde{R}_m)}^{1/2} \|\nabla u_m^{(1)}\|_{H^1(|x|>\tilde{R}_m)}^{1/2}\\ &= o(1) \end{aligned} \tag{2.31}$$

as $m\to\infty$. (Lemma 2.25 asserts that $\{\nabla u_m^{(j)}\} \subseteq H^{3/2,1}$ and hence $\{\nabla u_m^{(j)}\} \subseteq H^1(\Sigma)$ is bounded; it follows that $\|\nabla u_m^{(1)}\|_{H^1(|x|>\tilde{R}_m)} = O(1)$ as $m\to\infty$.) A similar calculation shows that

$$\|Q_{mx}^{(2)}\nabla u_m^{(2)}\|_{L^2(|x|<\tilde{S}_m)} = o(1) \quad \text{as } m\to\infty,$$

and lemma 2.37 yields the estimates

$$\lim_{m\to\infty} \|\nabla u_{mx}^{(1)}\|_{L^2(|x|>T_m)} = 0, \qquad \lim_{m\to\infty} \|\nabla u_{mx}^{(2)}\|_{L^2(|x|<T_m)} = 0.$$

The calculation

$$u_{myy}^{(j)} = -\frac{1}{1+q_{m22}^{(j)}}(\partial_x[(1+q_{m11}^{(j)})u_{mx}^{(j)} + q_{m12}^{(j)}u_{my}^{(j)}] + \partial_y(q_{m12}^{(j)}u_{mx}^{(j)}) - q_{m22y}^{(j)}u_{my}^{(j)})$$

and estimates

$$\|q_{mij}^{(1)}\nabla u_{mx}^{(1)}\|_{L^2(|x|>T_m)} \leqslant \|q_{mij}^{(1)}\|_\infty \|\nabla u_{mx}^{(1)}\|_{L^2(|x|>T_m)} = o(1),$$

$$\left\| \begin{Bmatrix} \partial_x \\ \partial_y \end{Bmatrix} q_{mij}^{(1)}\nabla u_m^{(1)} \right\|_{L^2(|x|>T_m)} = o(1)$$

as $m\to\infty$ (cf. (2.31)) show that

$$\lim_{m\to\infty} \|u_{myy}^{(1)}\|_{L^2(|x|>T_m)} = 0$$

(recall that $\|(1+q_{m22}^{(j)})^{-1}\|_\infty$ is bounded); the complementary limit

$$\lim_{m\to\infty} \|u_{myy}^{(2)}\|_{L^2(|x|<T_m)} = 0$$

is obtained in a similar fashion. □

Lemma 2.40 states another useful application of lemma 2.37 to the boundary-value problem (2.7)–(2.9); the following proposition is used in its proof.

Proposition 2.39. *Choose $N \in \mathbb{N}$. The estimates*

$$|(Q(\eta_m^{(1)} + \eta_m^{(2)}) - Q(\eta_m^{(2)}))(x, y)| \leqslant c \operatorname{dist}(x, [-R_m, R_m])^{-N}$$

and

$$|(Q(\eta_m^{(1)} + \eta_m^{(2)}) - Q(\eta_m^{(1)}))(x, y)| \leqslant c \operatorname{dist}(x, \mathbb{R} \setminus (-S_m, S_m))^{-N}$$

hold for all $(x, y) \in \bar{\Sigma}$, where $|\cdot|$ denotes the 2×2 matrix maximum norm, and remain valid when Q is replaced by Q_x or Q_y.

Proof. Observe that

$$\eta^\delta(x, y') = \frac{1}{1-y} \int_{\operatorname{supp}\eta} K\left(\frac{x-s}{1-y}\right) \eta(s) \, \mathrm{d}s,$$

where $K = (2\pi)^{-1/2}\delta^{-1}\mathcal{F}^{-1}[\chi] \in \mathcal{S}(\mathbb{R})$. The above equation shows that $\eta^\delta \in C^\infty(\bar{\Sigma} \setminus \operatorname{supp}\eta \times \{1\})$ with

$$|\partial_x^j \partial_y^k \eta^\delta(x, y)| \leqslant c \operatorname{dist}(x, \operatorname{supp}\eta)^{-N} \|\eta\|_\infty$$

for each $N \in \mathbb{N}$.

Note that

$$\begin{aligned} |(Q(\eta_1 + \eta_2) - Q(\eta_2))(x, y)| &= \left| \begin{pmatrix} f_{1y}^\delta & -f_{1x}^\delta \\ -f_{1x}^\delta & \dfrac{-f_{3y}^\delta + (f_{3x}^\delta)^2}{1 + f_{3y}^\delta} - \dfrac{-f_{2y}^\delta + (f_{2x}^\delta)^2}{1 + f_{2y}^\delta} \end{pmatrix} (x, y) \right| \\ &\leqslant c |(f_{1x}^\delta, f_{1y}^\delta)(x, y)| \end{aligned}$$

for all η_1, η_2 and $\eta_3 := \eta_1 + \eta_2 \in U$. It follows that

$$\begin{aligned} |(Q(\eta_m^{(1)} + \eta_m^{(2)}) - Q(\eta_m^{(2)}))(x, y)| &\leqslant |((\eta_m^{(1)})^\delta(x, y), (\eta_{mx}^{(1)})^\delta(x, y), (\eta_{my}^{(1)})^\delta(x, y))| \\ &\leqslant c \operatorname{dist}(x, [-R_m, R_m])^{-N}. \end{aligned}$$

The same argument yields the estimate for $Q(\eta_m^{(1)} + \eta_m^{(2)}) - Q(\eta_m^{(1)})$ and the corresponding results for Q_x and Q_y. □

Lemma 2.40. *Let $u(\eta)$ be the solution to (2.7)–(2.9) with $\xi = \partial_x f(\eta)$, $\eta \in U$, where f is a real polynomial. The estimates*

$$\begin{aligned} \lim_{m \to \infty} \|\nabla u(\eta_m^{(1)} + \eta_m^{(2)}) - \nabla u(\eta_m^{(1)})\|_{H^1(|x| < T_m)} &= 0, \\ \lim_{m \to \infty} \|\nabla u(\eta_m^{(1)} + \eta_m^{(2)}) - \nabla u(\eta_m^{(2)})\|_{H^1(|x| > T_m)} &= 0 \end{aligned}$$

hold for each sequence $\{T_m\}$ of positive real numbers with $S_m - T_m, T_m - R_m \to \infty$ as $m \to \infty$.

Proof. Choose sequences $\{\tilde{R}_m\}$, $\{\tilde{S}_m\}$ of positive real numbers with $S_m - \tilde{S}_m, \tilde{S}_m - T_m \to \infty$ and $T_m - \tilde{R}_m, \tilde{R}_m - R_m \to \infty$ as $m \to \infty$. The quantities $w_m^{(1)} =$

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$u(\eta_m^{(1)} + \eta_m^{(2)}) - u(\eta_m^{(2)})$ and $w_m^{(2)} = u(\eta_m^{(1)} + \eta_m^{(2)}) - u(\eta_m^{(1)})$ satisfy the boundary-value problems

$$\begin{aligned} \nabla \cdot ((I + Q_m)\nabla w_m^{(j)}) &= \nabla \cdot G_m^{(j)}, && 0 < y < 1, \\ (I + Q_m)\nabla w_m^{(j)} \cdot (0, 1) &= f(\eta_m^{(j)})_x + G_m^{(j)} \cdot (0, 1), && y = 1, \\ (I + Q_m)\nabla w_m^{(j)} \cdot (0, -1) &= G_m^{(j)} \cdot (0, -1), && y = 0, \end{aligned}$$

where $Q_m = Q(\eta_m^{(1)} + \eta_m^{(2)})$ and

$$G_m^{(1)} = (Q_m^{(2)} - Q_m)\nabla u_m^{(2)}, \qquad G_m^{(2)} = (Q_m^{(1)} - Q_m)\nabla u_m^{(1)}.$$

Using the estimate

$$|(Q_m^{(2)} - Q_m)(x, y)| \leqslant c\,\mathrm{dist}(x, [-R_m, R_m])^{-N}$$

(see proposition 2.39), one finds that

$$\|G_m^{(1)}\|^2_{L^2(|x|>\tilde{R}_m)} \leqslant c(\tilde{R}_m - R_m)^{-N}\|\nabla u_m^{(2)}\|_0^2 \leqslant c(\tilde{R}_m - R_m)^{-N}\|f(\eta_m^{(2)})\|_{1/2}^2 = o(1)$$

as $m \to \infty$ and a similar argument shows that $\|G_m^{(2)}\|^2_{L^2(|x|<\tilde{S}_m)} = o(1)$ as $m \to \infty$. It follows from lemma 2.37 that

$$\lim_{m\to\infty} \|w_m^{(1)}\|_{L^2(|x|>T_m)} = 0, \qquad \lim_{m\to\infty} \|w_m^{(2)}\|_{L^2(|x|<T_m)} = 0.$$

The derivatives $w_{mx}^{(j)}$, $j = 1, 2$, are weak solutions of the boundary-value problems

$$\begin{aligned} \nabla \cdot ((I + Q_m^{(j)})\nabla w_{mx}^{(j)}) &= \nabla \cdot H_m^{(j)}, && 0 < y < 1, \\ (I + Q_m^{(j)})\nabla w_{mx}^{(j)} \cdot (0, 1) &= \partial_x^2 f(\eta_m^{(j)}) + H_m^{(j)} \cdot (0, 1), && y = 1, \\ (I + Q_m^{(j)})\nabla w_{mx}^{(j)} \cdot (0, -1) &= H_m^{(j)} \cdot (0, -1), && y = 0, \end{aligned}$$

where

$$\begin{aligned} H_m^{(1)} &= -Q_{mx}\nabla w_m^{(1)} + (Q_m^{(2)} - Q_m)\nabla u_{mx}^{(2)} + (Q_{mx}^{(2)} - Q_{mx})\nabla u_m^{(2)}, \\ H_m^{(2)} &= -Q_{mx}\nabla w_m^{(2)} + (Q_m^{(1)} - Q_m)\nabla u_{mx}^{(1)} + (Q_{mx}^{(1)} - Q_{mx})\nabla u_m^{(1)}. \end{aligned}$$

Treating $\|Q_{mx}\nabla w_m^{(1)}\|_{L^2(|x|>\tilde{R}_m)}$ using the method given in the proof of lemma 2.38 (see estimate (2.31)) and treating

$$\|(Q_m^{(2)} - Q_m)\nabla u_{mx}^{(2)}\|_{L^2(|x|>\tilde{R}_m)}, \qquad \|(Q_{mx}^{(2)} - Q_{mx})\nabla u_m^{(2)}\|_{L^2(|x|>\tilde{R}_m)}$$

using the method given above, one finds that $\|H_m^{(1)}\|_{L^2(|x|>\tilde{R}_m)} = o(1)$ as $m \to \infty$. A similar argument yields $\|H_m^{(2)}\|_{L^2(|x|<\tilde{S}_m)} = o(1)$ as $m \to \infty$ and it follows from lemma 2.37 that

$$\lim_{m\to\infty} \|\nabla u_{mx}^{(1)}\|_{L^2(|x|>T_m)} = 0, \qquad \lim_{m\to\infty} \|\nabla u_{mx}^{(2)}\|_{L^2(|x|<T_m)} = 0.$$

Finally, observe that

$$\begin{aligned} w_{myy}^{(1)} = -\frac{1}{1 + q_{m22}^{(1)}}(\partial_x[(1 + q_{m11}^{(1)})w_{mx}^{(1)} + q_{m12}^{(1)}w_{my}^{(1)}] + \partial_y(q_{m12}^{(1)}w_{mx}^{(1)}) \\ - q_{m22y}^{(1)}w_{my}^{(1)} + \nabla(Q_m^{(2)} - Q_m) \cdot \nabla u_m^{(1)} + (Q_m^{(2)} - Q_m)\Delta u_m^{(1)}). \end{aligned}$$

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The argument given in the proof of lemma 2.38 shows that

$$\|\partial_x[(1+q_{m11}^{(1)})w_{mx}^{(1)}+q_{m12}^{(1)}w_{my}^{(1)}]+\partial_y(q_{m12}^{(1)}w_{mx}^{(1)})-q_{m22y}^{(1)}w_{my}^{(1)}\|_{L^2(|x|>T_m)}=o(1),$$

and the method given above shows that

$$\|\nabla(Q_m^{(2)}-Q_m)\cdot\nabla u_m^{(1)}\|_{L^2(|x|>T_m)},\|(Q_m^{(2)}-Q_m)\Delta u_m^{(1)}\|_{L^2(|x|>T_m)}=o(1)$$

as $m\to\infty$. One concludes that

$$\lim_{m\to\infty}\|w_{myy}^{(1)}\|_{L^2(|x|>T_m)}=0,$$

and the complementary limit

$$\lim_{m\to\infty}\|w_{myy}^{(2)}\|_{L^2(|x|<T_m)}=0$$

is obtained in a similar fashion. □

Corollary 2.41. *The estimate*

$$\lim_{m\to\infty}\|\nabla u(\eta_m^{(1)}+\eta_m^{(2)})-\nabla u(\eta_m^{(1)})-\nabla u(\eta_m^{(2)})\|_1=0$$

holds under the hypotheses of lemmas 2.38 and 2.40.

The proof of theorem 2.36 is completed by applying the next lemma to the equation for $\mathcal{T}'$ given in corollary 2.32.

Lemma 2.42.

(i) *The estimates*

$$\lim_{m\to\infty}\|f_1(\eta_m^{(1)}+\eta_m^{(2)})K(\eta_m^{(1)}+\eta_m^{(2)})f_2(\eta_m^{(1)}+\eta_m^{(2)})\\ -f_1(\eta_m^{(1)})K(\eta_m^{(1)})f_2(\eta_m^{(1)})-f_1(\eta_m^{(2)})K(\eta_m^{(2)})f_2(\eta_m^{(2)})\|_0=0$$

and

$$\lim_{m\to\infty}\|f_1(\eta_m^{(1)}+\eta_m^{(2)})K(\eta_m^{(1)}+\eta_m^{(2)})f_2(\eta_m^{(1)}+\eta_m^{(2)})\\ -f_1(\eta_m^{(1)})K(\eta_m^{(1)})f_2(\eta_m^{(1)})-f_1(\eta_m^{(2)})K(\eta_m^{(2)})f_2(\eta_m^{(2)})\|_{L^1(\mathbb{R})}=0$$

hold for all real polynomials f_1, f_2.

(ii) *The estimate*

$$\lim_{m\to\infty}\|\mathcal{H}'(\eta_m^{(1)}+\eta_m^{(2)})(f_1(\eta_m^{(1)}+\eta_m^{(2)}),f_2(\eta_m^{(1)}+\eta_m^{(2)}))\\ -\mathcal{H}'(\eta_m^{(1)})(f_1(\eta_m^{(1)}),f_2(\eta_m^{(1)}))-\mathcal{H}'(\eta_m^{(2)})(f_1(\eta_m^{(2)}),f_2(\eta_m^{(2)}))\|_0=0$$

holds for all real polynomials f_1, f_2.

(iii) *The estimate*

$$\lim_{m\to\infty}\langle\mathcal{H}'(\eta_m^{(1)})(f_1(\eta_m^{(1)}),f_2(\eta_m^{(1)})),\eta_m^{(2)}\rangle_0=0$$

holds for all real polynomials f_1, f_2.

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Proof. (i) Observe that

$$\begin{aligned}
&f_1(\eta_m^{(1)}+\eta_m^{(2)})K(\eta_m^{(1)}+\eta_m^{(2)})f_2(\eta_m^{(1)}+\eta_m^{(2)})\\
&\quad - f_1(\eta_m^{(1)})K(\eta_m^{(1)})f_2(\eta_m^{(1)}) - f_1(\eta_m^{(2)})K(\eta_m^{(2)})f_2(\eta_m^{(2)})\\
&\qquad = f_1(\eta_m^{(1)})(u_x(\eta_m^{(1)}+\eta_m^{(2)}) - u_x(\eta_m^{(1)})) + f_2(\eta_m^{(2)})(u_x(\eta_m^{(1)}+\eta_m^{(2)}) - u_x(\eta_m^{(2)})).
\end{aligned}$$

The $L^1(\mathbb{R})$- and $L^2(\mathbb{R})$-norms of this quantity can both be estimated by

$$\begin{aligned}
&\|f_1(\eta_m^{(1)})\|_1 \|u_x(\eta_m^{(1)}+\eta_m^{(2)}) - u_x(\eta_m^{(1)})|_{y=1}\|_{L^2(|x|<R_m)}\\
&\quad + \|f_2(\eta_m^{(1)})\|_1 \|u_x(\eta_m^{(1)}+\eta_m^{(2)}) - u_x(\eta_m^{(2)})|_{y=1}\|_{L^2(|x|>S_m)}\\
&\qquad \leqslant \underbrace{\|f_1(\eta_m^{(1)})\|_1}_{O(1)} \underbrace{\|\nabla u(\eta_m^{(1)}+\eta_m^{(2)}) - \nabla u(\eta_m^{(1)})\|_{H^1(|x|<T_m)}}_{o(1)}\\
&\qquad\quad + \underbrace{\|f_2(\eta_m^{(1)})\|_1}_{O(1)} \underbrace{\|\nabla u(\eta_m^{(1)}+\eta_m^{(2)}) - \nabla u(\eta_m^{(2)})\|_{H^1(|x|>T_m)}}_{o(1)}\\
&\qquad = o(1)
\end{aligned}$$

(use the Cauchy–Schwarz inequality or the maximum norm for the polynomials).

(ii) Observe that

$$\begin{aligned}
&\mathcal{H}'(\eta_m^{(1)}+\eta_m^{(2)})(f_1(\eta_m^{(1)}+\eta_m^{(2)}), f_2(\eta_m^{(1)}+\eta_m^{(2)}))\\
&\quad - \mathcal{H}'(\eta_m^{(1)})(f_1(\eta_m^{(1)}), f_2(\eta_m^{(1)})) - \mathcal{H}'(\eta_m^{(2)})(f_1(\eta_m^{(2)}), f_2(\eta_m^{(2)}))\\
&\qquad = -u_x(\eta_m^{(1)}+\eta_m^{(2)})v_x(\eta_m^{(1)}+\eta_m^{(2)}) + u_x(\eta_m^{(1)})v_x(\eta_m^{(1)}) + u_x(\eta_m^{(2)})v_x(\eta_m^{(2)})\\
&\qquad\quad + u_y(\eta_m^{(1)}+\eta_m^{(2)})v_y(\eta_m^{(1)}+\eta_m^{(2)}) - u_y(\eta_m^{(1)})v_y(\eta_m^{(1)}) - u_y(\eta_m^{(2)})v_y(\eta_m^{(2)})\\
&\qquad\quad + h(\eta_m^{(1)}+\eta_m^{(2)})u_y(\eta_m^{(1)}+\eta_m^{(2)})v_y(\eta_m^{(1)}+\eta_m^{(2)})\\
&\qquad\quad - h(\eta_m^{(1)})u_y(\eta_m^{(1)})v_y(\eta_m^{(1)}) - h(\eta_m^{(2)})u_y(\eta_m^{(2)})v_y(\eta_m^{(2)})|_{y=1},
\end{aligned}$$

where

$$h(\eta) = \frac{\eta'^2 - \eta^2 - 2\eta}{(1+\eta)^2}$$

and $u(\eta)$, $v(\eta)$ are the solutions to (2.7)–(2.9) with $\xi = \partial_x f_1(\eta)$, $\eta \in U$, and $\xi = \partial_x f_2(\eta)$, $\eta \in U$, respectively.

The estimates

$$\begin{aligned}
&\|u_x(\eta_m^{(1)}+\eta_m^{(2)})v_x(\eta_m^{(1)}+\eta_m^{(2)}) - (u_x(\eta_m^{(1)}) + u_x(\eta_m^{(2)}))(v_x(\eta_m^{(1)}) + v_x(\eta_m^{(2)}))|_{y=1}\|_0\\
&\quad \leqslant \underbrace{\|v_x(\eta_m^{(1)}+\eta_m^{(2)})|_{y=1}\|_1}_{=O(1)} \underbrace{\|u_x(\eta_m^{(1)}+\eta_m^{(2)}) - u_x(\eta_m^{(1)}) - u_x(\eta_m^{(2)})|_{y=1}\|_0}_{=o(1)}\\
&\qquad + \underbrace{\|u_x(\eta_m^{(1)}) + u_x(\eta_m^{(2)})|_{y=1}\|_1}_{=O(1)} \underbrace{\|v_x(\eta_m^{(1)}+\eta_m^{(2)}) - v_x(\eta_m^{(1)}) - v_x(\eta_m^{(2)})|_{y=1}\|_0}_{=o(1)}\\
&\quad = o(1)
\end{aligned}$$

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and

$$\begin{aligned}
&\|(u_x(\eta_m^{(1)}) + u_x(\eta_m^{(2)}))(v_x(\eta_m^{(1)}) + v_x(\eta_m^{(2)})) \\
&\qquad - u_x(\eta_m^{(1)})v_x(\eta_m^{(1)}) - u_x(\eta_m^{(2)})v_x(\eta_m^{(2)})|_{y=1}\|_0 \\
&\qquad\qquad \leqslant \|u_x(\eta_m^{(1)})v_x(\eta_m^{(2)})|_{y=1}\|_0 + \|u_x(\eta_m^{(2)})v_x(\eta_m^{(1)})|_{y=1}\|_0 \\
&\qquad\qquad \leqslant c(\underbrace{\|u_x(\eta_n^{(1)})|_{y=1}\|_{L^2(|x|>T_m)}}_{=o(1)} \underbrace{\|v_x(\eta_n^{(2)})|_{y=1}\|_1}_{=O(1)} \\
&\qquad\qquad\qquad + \underbrace{\|u_x(\eta_n^{(1)})|_{y=1}\|_1}_{=O(1)} \underbrace{\|v_x(\eta_n^{(2)})|_{y=1}\|_{L^2(|x|<T_m)}}_{=o(1)} \\
&\qquad\qquad\qquad + \underbrace{\|u_x(\eta_n^{(2)})|_{y=1}\|_{L^2(|x|<T_m)}}_{=o(1)} \underbrace{\|v_x(\eta_n^{(1)})|_{y=1}\|_1}_{=O(1)} \\
&\qquad\qquad\qquad + \underbrace{\|u_x(\eta_n^{(2)})|_{y=1}\|_1}_{=O(1)} \underbrace{\|v_x(\eta_n^{(1)})|_{y=1}\|_{L^2(|x|>T_m)}}_{=o(1)}) \\
&\qquad\qquad = o(1)
\end{aligned}$$

imply that

$$\begin{aligned}
&\|(u_x(\eta_m^{(1)}) + u_x(\eta_m^{(2)}))(v_x(\eta_m^{(1)}) + v_x(\eta_m^{(2)})) \\
&\qquad\qquad - u_x(\eta_m^{(1)})v_x(\eta_m^{(1)}) - u_x(\eta_m^{(2)})v_x(\eta_m^{(2)})|_{y=1}\|_0 = o(1)
\end{aligned}$$

as $m \to \infty$; here we have used the estimate

$$\|u_x(\eta)|_{y=1}\|_1 \leqslant c\|\nabla u\|_{3/2,1} \leqslant c\|f_1(\eta)\|_2, \quad \eta \in U,$$

and its counterpart for v. The same argument shows that

$$\begin{aligned}
&\|(u_y(\eta_m^{(1)}) + u_y(\eta_m^{(2)}))(v_y(\eta_m^{(1)}) + v_y(\eta_m^{(2)})) \\
&\qquad\qquad - u_y(\eta_m^{(1)})v_y(\eta_m^{(1)}) - u_y(\eta_m^{(2)})v_y(\eta_m^{(2)})|_{y=1}\|_0 = o(1)
\end{aligned}$$

as $m \to \infty$.

Because $h(\eta_m^{(1)} + \eta_m^{(2)}) = h(\eta_m^{(1)}) + h(\eta_m^{(2)})$ and

$$\begin{aligned}
&\|(u_y(\eta_m^{(1)}) + u_y(\eta_m^{(2)}))(v_y(\eta_m^{(1)}) + v_y(\eta_m^{(2)})) \\
&\qquad\qquad - u_y(\eta_m^{(1)})v_y(\eta_m^{(1)}) - u_y(\eta_m^{(2)})v_y(\eta_m^{(2)})|_{y=1}\|_0 = o(1)
\end{aligned}$$

as $m \to \infty$ (see above), repeating the proof of part (i) yields the estimate

$$\begin{aligned}
&\|h(\eta_m^{(1)} + \eta_m^{(2)})u_y(\eta_m^{(1)} + \eta_m^{(2)})v_y(\eta_m^{(1)} + \eta_m^{(2)}) \\
&\qquad - h(\eta_m^{(1)})u_y(\eta_m^{(1)})v_y(\eta_m^{(1)}) - h(\eta_m^{(2)})u_y(\eta_m^{(2)})v_y(\eta_m^{(2)})|_{y=1}\|_0 = o(1)
\end{aligned}$$

as $m \to \infty$.

(iii) The methods used in part (ii) show that

$$\|\mathcal{H}'(\eta_m^{(1)})(f_1(\eta_m^{(1)}), f_2(\eta_m^{(1)}))\|_{L^2(|x|>T_m)} = o(1),$$

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so that

$$
\begin{aligned}
&|\langle \mathcal{H}'(\eta_m^{(1)})(f_1(\eta_m^{(1)}), f_2(\eta_m^{(1)})), \eta_m^{(2)}\rangle_0| \\
&\qquad \leqslant \underbrace{\|\mathcal{H}'(\eta_m^{(1)})(f_1(\eta_m^{(1)}), f_2(\eta_m^{(1)}))\|_{L^2(|x|>S_m)}}_{=O(1)} \underbrace{\|\eta_m^{(2)}\|_0}_{=o(1)} \\
&\qquad \to 0
\end{aligned}
$$

as $m \to \infty$. □

3. Minimizing sequences

The goal of this section is the proof of the following theorem, the existence of the sequence advertised in which is a key ingredient in the proof that the infimum of $\mathcal{J}_\mu$ over $U \setminus \{0\}$ is a strictly subadditive function of μ. The subadditivity property of c_μ is in turn used to establish the convergence (up to subsequences and translations) of *any* minimizing sequence for $\mathcal{J}_\mu$ over $U \setminus \{0\}$ that does not approach the boundary of U.

Theorem 3.1. *There exists a minimizing sequence $\{\tilde{\eta}_m\}$ for $\mathcal{J}_\mu$ over $U \setminus \{0\}$ with the properties that $\|\tilde{\eta}_m\|_2^2 \leqslant c\mu$ for each $m \in \mathbb{N}$ and $\lim_{m\to\infty} \|\mathcal{J}'_\mu(\tilde{\eta}_m)\|_0 = 0$.*

3.1. The penalized minimization problem

We begin by studying the functional $\mathcal{J}_{\rho,\mu}\colon H^2(\mathbb{R}) \to \mathbb{R} \cup \{\infty\}$ defined by

$$
\mathcal{J}_{\rho,\mu}(\eta) = \begin{cases} \mathcal{K}(\eta) + \dfrac{(\mu + \mathcal{G}(\eta))^2}{\mathcal{L}(\eta)} + \rho(\|\eta\|_2^2), & \eta \in U \setminus \{0\}, \\ \infty, & \eta \notin U \setminus \{0\}, \end{cases}
$$

in which $\rho\colon [0, M^2) \to \mathbb{R}$ is a smooth increasing 'penalization' function such that $\rho(t) = 0$ for $0 \leqslant t \leqslant \tilde{M}^2$ and $\rho(t) \to \infty$ as $t \uparrow M^2$. We allow negative values of the small parameter, so that $0 < |\mu| < \mu_0$ (see the comments below lemma 3.8) and the number $\tilde{M} \in (0, M)$ is chosen so that

$$
\tilde{M}^2 > (c^\star + D\nu_0 + D\nu_0^-)|\mu|;
$$

the following analysis is valid for every such choice of $\tilde{M}$, which, in particular, may be chosen arbitrarily close to M. In this inequality ν_0 and ν_0^- are the speeds of linear waves with frequency k_0 riding shear flows with vorticities ω and $-\omega$ and $c^\star$, D are constants identified in lemmas 3.2(i) and 3.3. In § 3.2 we give a detailed description of the qualitative properties of an arbitrary minimizing sequence $\{\eta_m\}$ for $\mathcal{J}_{\rho,\mu}$; the penalization function ensures that $\{\eta_m\}$ does not approach the boundary of the set $U \setminus \{0\}$, in which $\mathcal{J}_\mu$ is defined.

We first give some useful *a priori* estimates. Lemma 3.2(i) shows in particular that

$$
c_{\rho,\mu} := \inf \mathcal{J}_{\rho,\mu} < 2\nu_0^\mu|\mu| - c|\mu|^{r^\star}, \qquad c_\mu := \inf_{\eta \in U\setminus\{0\}} \mathcal{J}_\mu(\eta) < 2\nu_0^\mu|\mu| - c|\mu|^{r^\star},
$$

where ν_0^μ is the speed of linear waves with frequency k_0 riding a shear flow with vorticity $(\operatorname{sgn}\mu)\omega$ (which depends only upon the sign of μ), while lemma 3.3, whose proof is a straightforward modification of the argument presented by Buffoni *et al*. [9, propositions 2.34 and 3.2], gives estimates on the size of critical points of $\mathcal{J}_\mu$ and a class of related functionals.

LEMMA 3.2.

(i) *There exists* $\eta_\mu^\star \in U \setminus \{0\}$ *with compact support and a positive constant* $c^\star$ *such that* $\|\eta_\mu^\star\|_2 \leqslant c^\star|\mu|^{1/2}$, $\rho(\|\eta_\mu^\star\|_2^2) = 0$ *and*

$$\mathcal{J}_{\rho,\mu}(\eta_\mu^\star) = \mathcal{J}_\mu(\eta_\mu^\star) < 2\nu_0^\mu|\mu| - c|\mu|^{r^\star}, \quad r^\star = \begin{cases} \frac{5}{3}, & \beta > \beta_{\mathrm{c}}, \\ 3, & \beta < \beta_{\mathrm{c}}. \end{cases}$$

(ii) *The inequality*

$$\mathcal{K}_2(\eta) + \frac{(\mu + \mathcal{G}_2(\eta))^2}{\mathcal{L}_2(\eta)} \geqslant 2\nu_0^\mu|\mu|$$

holds for each $\eta \in H^2(\mathbb{R}) \setminus \{0\}$.

Proof. First suppose that $\mu > 0$. The proof of part (i) is recorded in Appendix A, while part (ii) follows from the calculation

$$\begin{aligned}
\mathcal{K}_2(\eta) &+ \frac{(\mu + \mathcal{G}_2(\eta))^2}{\mathcal{L}_2(\eta)} \\
&= \mathcal{K}_2(\eta) + 2\nu_0\mathcal{G}_2(\eta) - \nu_0^2\mathcal{L}_2(\eta) + \frac{(\mu + \mathcal{G}_2(\eta) - \nu_0\mathcal{L}_2(\eta))^2}{\mathcal{L}_2(\eta)} + 2\nu_0\mu \\
&= \frac{1}{2}\int_{-\infty}^{\infty} g(k)|\hat{\eta}|^2 + \frac{(\mu + \mathcal{G}_2(\eta) - \nu_0\mathcal{L}_2(\eta))^2}{\mathcal{L}_2(\eta)} + 2\nu_0\mu \\
&\geqslant 2\nu_0\mu.
\end{aligned}$$

For $\mu < 0$ we observe that $\mathcal{J}_\mu(\eta)$, $\mathcal{J}_{\rho,\mu}(\eta)$ and $\mathcal{K}_2(\eta) + (\mu + \mathcal{G}_2(\eta))^2/\mathcal{L}_2(\eta)$ are invariant under the transformation $(\mu, \omega) \mapsto (-\mu, -\omega)$. □

LEMMA 3.3. *Suppose that* γ_1 *and* γ_2 *belong to a bounded set of real numbers. Any critical point* η *of the functional* $\tilde{\mathcal{J}}_\gamma \colon U \to \mathbb{R}$ *defined by*

$$\tilde{\mathcal{J}}_\gamma(\eta) = \mathcal{K}(\eta) - \gamma_1\mathcal{G}(\eta) - \gamma_2\mathcal{L}(\eta) + \gamma_3\|\eta\|_2^2, \quad \gamma_3 \geqslant 0,$$

satisfies the estimate

$$\|\eta\|_2^2 \leqslant D\mathcal{K}(\eta),$$

where D *is a positive constant that does not depend upon* γ_1, γ_2 *or* γ_3.

COROLLARY 3.4. *Any critical point* η *of* $\mathcal{J}_{\rho,\mu}$ *with* $\mathcal{J}_{\rho,\mu}(\eta) < 2\nu_0^\mu|\mu|$ *satisfies*

$$\|\eta\|_2^2 \leqslant 2D\nu_0^\mu|\mu|, \quad \rho(\|\eta\|_2^2) = 0.$$

Proof. Notice that any critical point η of $\mathcal{J}_{\rho,\mu}$ is also a critical point of the functional $\tilde{\mathcal{J}}_\gamma$, where

$$\gamma_1 = -\frac{2(\mu + \mathcal{G}(\eta))}{\mathcal{L}(\eta)}, \qquad \gamma_2 = \frac{(\mu + \mathcal{G}(\eta))^2}{\mathcal{L}(\eta)^2}, \qquad \gamma_3 = 2\rho'(\|\eta\|_2^2).$$

Furthermore, any function $\eta \in U$ such that

$$\frac{(\mu + \mathcal{G}(\eta))^2}{\mathcal{L}(\eta)} \leqslant 2\nu_0^\mu |\mu|$$

satisfies

$$\frac{\mu^2}{\mathcal{L}(\eta)} \leqslant 2\nu_0^\mu |\mu| - \frac{2\mu\mathcal{G}(\eta)}{\mathcal{L}(\eta)} - \frac{\mathcal{G}(\eta)^2}{\mathcal{L}(\eta)} \leqslant 2\nu_0^\mu |\mu| + \frac{2|\mu||\mathcal{G}(\eta)|}{\mathcal{L}(\eta)} \leqslant c|\mu|$$

(see proposition 2.35), so that

$$\frac{|\mu|}{\mathcal{L}(\eta)} \leqslant c. \tag{3.1}$$

Observing that

$$\frac{(\mu + \mathcal{G}(\eta))^2}{\mathcal{L}(\eta)} \leqslant \mathcal{J}_{\rho,\mu}(\eta) \leqslant 2\nu_0^\mu |\mu|,$$

we find from proposition 2.35 and inequality (3.1) that γ_1 and γ_2 are bounded. The previous lemma shows that $\|\eta\|_2^2 \leqslant D\mathcal{K}(\eta) \leqslant D\mathcal{J}_{\rho,\mu}(\eta) < 2D\nu_0^\mu|\mu|$, and hence $\rho(\|\eta\|_2^2) = 0$ because of the choice of $\tilde{M}$. □

Finally, we establish some basic properties of a minimizing sequence $\{\eta_m\}$ for $\mathcal{J}_{\rho,\mu}$. Without loss of generality we may assume that

$$\sup_{m\in\mathbb{N}} \|\eta_m\|_2 < M$$

($\|\eta_m\|_2 \to M$ would imply that $\mathcal{J}_{\rho,\mu}(\eta_m) \to \infty$) and it follows that $\{\eta_m\}$ admits a subsequence such that $\lim_{m\to\infty} \|\eta_m\|_2$ exists and is positive ($\eta_m \to 0$ in $H^2(\mathbb{R})$ would also imply that $\mathcal{J}_{\rho,\mu}(\eta_m) \to \infty$). The following lemma records further useful properties of $\{\eta_m\}$.

Lemma 3.5. *Every minimizing sequence $\{\eta_m\}$ for $\mathcal{J}_{\rho,\mu}$ has the properties that*

$$\mathcal{J}_{\rho,\mu}(\eta_m) < 2\nu_0^\mu|\mu| - c|\mu|^{r^\star}, \qquad \mathcal{L}(\eta_m) \geqslant c|\mu|, \qquad \mathcal{L}_2(\eta_m) \geqslant c|\mu|,$$
$$\mathcal{M}_{\rho,\mu}(\eta_m) \leqslant -c|\mu|^{r^\star}, \qquad \|\eta_m\|_{1,\infty} \geqslant c|\mu|^{r^\star}$$

for each $m \in \mathbb{N}$, where

$$\mathcal{M}_{\rho,\mu}(\eta) = \mathcal{J}_{\rho,\mu}(\eta) - \mathcal{K}_2(\eta) - \frac{(\mu + \mathcal{G}_2(\eta))^2}{\mathcal{L}_2(\eta)}.$$

Proof. The first and second estimates are obtained from lemma 3.2(i) and the remark leading to (3.1), while the third is a consequence of the calculation

$$c\|\eta\|_{1/2}^2 \leqslant \begin{Bmatrix} \mathcal{L}_2(\eta) \\ \mathcal{L}(\eta) \end{Bmatrix} \leqslant c\|\eta\|_{1/2}^2, \quad \eta \in U. \tag{3.2}$$

Turning to the fourth estimate, observe that

$$\mathcal{M}_{\rho,\mu}(\eta_m) \leqslant \mathcal{J}_{\rho,\mu}(\eta_m) - 2\nu_0^\mu|\mu| \leqslant -c|\mu|^{r^\star}$$

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because

$$\mathcal{K}_2(\eta) + \frac{(\mu + \mathcal{G}_2(\eta))^2}{\mathcal{L}_2(\eta)} \geqslant 2\nu_0^\mu |\mu|$$

(see lemma 3.2(ii)).

Finally, it follows from the calculation

$$\begin{aligned}
&\mathcal{M}_{\rho,\mu}(\eta_m) - \rho(\|\eta_m\|_2^2) \\
&\quad = \mathcal{K}_{\mathrm{nl}}(\eta_m) - \frac{\mu^2 \mathcal{L}_{\mathrm{nl}}(\eta_m)}{\mathcal{L}(\eta_m)\mathcal{L}_2(\eta_m)} - \frac{2\mu\mathcal{G}(\eta_m)\mathcal{L}_{\mathrm{nl}}(\eta_m)}{\mathcal{L}(\eta_m)\mathcal{L}_2(\eta_m)} + \frac{2\mu\mathcal{G}_{\mathrm{nl}}(\eta_m)}{\mathcal{L}(\eta_m)} \\
&\qquad - \frac{\mathcal{G}_2(\eta_m)\mathcal{L}_{\mathrm{nl}}(\eta_m)}{\mathcal{L}(\eta_m)\mathcal{L}_2(\eta_m)} + \frac{(\mathcal{G}(\eta_m) + \mathcal{G}_2(\eta_m))\mathcal{G}_{\mathrm{nl}}(\eta_m)}{\mathcal{L}(\eta_m)},
\end{aligned}$$

the inequalities

$$\begin{aligned}
|\mathcal{G}_2(\eta_m)|, |\mathcal{G}(\eta_m)| &\leqslant c\|\eta_m\|_{1/2}^2, \\
|\mathcal{G}_{\mathrm{nl}}(\eta_m)|, |\mathcal{K}_{\mathrm{nl}}(\eta_m)| &\leqslant c\|\eta_m\|_{1,\infty}, \\
|\mathcal{L}_{\mathrm{nl}}(\eta_m)| &\leqslant c\|\eta_m\|_{1,\infty}\|\eta_m\|_{1/2}^2
\end{aligned}$$

and (3.2) that

$$|\mathcal{M}_{\rho,\mu}(\eta_m) - \rho(\|\eta_m\|_2^2)| \leqslant c\|\eta_m\|_{1,\infty}.$$

The fifth estimate is obtained from this result and the fact that

$$\mathcal{M}_{\rho,\mu}(\eta_m) - \rho(\|\eta_m\|_2^2) \leqslant -c|\mu|^{r^\star}.$$

□

Remark 3.6. Replacing $\mathcal{J}_{\rho,\mu}(\eta)$ by $\mathcal{J}_\mu(\eta)$ and $\mathcal{M}_{\rho,\mu}(\eta)$ by

$$\mathcal{M}_\mu(\eta) := \mathcal{J}_\mu(\eta) - \mathcal{K}_2(\eta) - \frac{(\mu + \mathcal{G}_2(\eta))^2}{\mathcal{L}_2(\eta)}$$

in its statement, one finds that lemma 3.5 is also valid for a minimizing sequence $\{\eta_m\}$ for $\mathcal{J}_\mu$ over $U \setminus \{0\}$.

3.2. Minimizing sequences for the penalized problem

3.2.1. *Application of the concentration-compactness principle*

The next step is to perform a more detailed analysis of the behaviour of a minimizing sequence $\{\eta_m\}$ for $\mathcal{J}_{\rho,\mu}$ by applying the concentration-compactness principle (see Lions [21, 22]); theorem 3.7 states this result in a form suitable for the present situation.

Theorem 3.7. *Any sequence $\{u_m\} \subset L^1(\mathbb{R})$ of non-negative functions with the property that*

$$\lim_{m\to\infty} \int_{-\infty}^{\infty} u_m(x)\,\mathrm{d}x = \ell > 0$$

admits a subsequence for which precisely one of the following phenomena occurs.

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- *Vanishing: for each $r > 0$ one has that*

$$\lim_{m\to\infty}\left(\sup_{\tilde{x}\in\mathbb{R}}\int_{\tilde{x}-r}^{\tilde{x}+r} u_m(x)\,\mathrm{d}x\right) = 0.$$

- *Concentration: there is a sequence $\{x_m\} \subset \mathbb{R}$ with the property that for each $\varepsilon > 0$ there exists a positive real number R with*

$$\int_{-R}^{R} u_m(x + x_m)\,\mathrm{d}x \geqslant \ell - \varepsilon$$

  for each $m \in \mathbb{N}$.

- *Dichotomy: there are sequences $\{x_m\} \subset \mathbb{R}$, $\{M_m^{(1)}\}, \{M_m^{(2)}\} \subset \mathbb{R}$ and a real number $\kappa \in (0, \ell)$ with the properties that*

$$M_m^{(1)}, M_m^{(2)} \to \infty, \qquad M_m^{(1)}/M_m^{(2)} \to 0,$$
$$\int_{-M_m^{(1)}}^{M_m^{(1)}} u_m(x + x_m)\,\mathrm{d}x \to \kappa, \qquad \int_{-M_m^{(2)}}^{M_m^{(2)}} u_m(x + x_m)\,\mathrm{d}x \to \kappa$$

  as $m \to \infty$. Furthermore,

$$\lim_{m\to\infty}\left(\sup_{\tilde{x}\in\mathbb{R}}\int_{\tilde{x}-r}^{\tilde{x}+r} u_m(x)\,\mathrm{d}x\right) \leqslant \kappa$$

  for each $r > 0$, and for each $\varepsilon > 0$ there is a positive real number R such that

$$\int_{-R}^{R} u_m(x + x_m)\,\mathrm{d}x \geqslant \kappa - \varepsilon$$

  for each $m \in \mathbb{N}$.

Standard interpolation inequalities show that the norms $\|\cdot\|_r$ are metrically equivalent on U for $r \in [0, 2)$; we therefore study the convergence properties of $\{\eta_m\}$ in $H^r(\mathbb{R})$ for $r \in [0, 2)$ by focusing on the concrete choice $r = 1$. One may assume that $\|\eta_m\|_1 \to \ell$ as $m \to \infty$, where $\ell > 0$ because $\eta_m \to 0$ in $H^r(\mathbb{R})$ for $r > \frac{3}{2}$ would imply that $\mathcal{J}_{\rho,\mu}(\eta_m) \to \infty$. This observation suggests applying theorem 3.7 to the sequence $\{u_m\}$ defined by

$$u_m = \eta_m'^2 + \eta_m^2,$$

so that $\|u_m\|_{L^1(\mathbb{R})} = \|\eta_m\|_1^2$. The following result deals with 'vanishing' and 'concentration' (see Buffoni *et al*. [9, lemmas 3.7 and 3.9]).

Lemma 3.8.

(i) *The sequence $\{u_m\}$ does not have the 'vanishing' property.*

(ii) *Suppose $\{u_m\}$ has the 'concentration' property. The sequence $\{\eta_m(\cdot + x_m)\}$ admits a subsequence, abbreviated, with a slight abuse of notation, to $\{\eta_m\}$, which satisfies*

$$\lim_{m\to\infty}\|\eta_m\|_2 \leqslant \tilde{M}$$

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and converges in $H^r(\mathbb{R})$ *for* $r \in [0,2)$ *to* $\eta^{(1)}$*. The function* $\eta^{(1)}$ *satisfies the estimate*

$$\|\eta^{(1)}\|_2^2 \leqslant D\mathcal{K}(\eta^{(1)}) < 2D\nu_0^{\mu}|\mu|,$$

minimizes $\mathcal{J}_{\rho,\mu}$ *and minimizes* $\mathcal{J}_\mu$ *over* $\tilde{U} \setminus \{0\}$*, where*

$$\tilde{U} = \{\eta \in H^2(\mathbb{R}) \colon \|\eta\|_2 < \tilde{M}\}.$$

We now present the more involved discussion of the remaining case ('dichotomy'), again abbreviating the subsequence of $\{\eta_m(\cdot + x_m)\}$ identified by theorem 3.7 to $\{\eta_m\}$. The analysis is similar to that given by Buffoni *et al.* [9] in their study of three-dimensional irrotational solitary waves, the main difference being that negative values of μ are also considered, so that μ is replaced by $|\mu|$ in estimates (this change is necessary since the numbers $\mu^{(1)}$ and $\mu^{(2)}$ appearing in part (iv) of the following lemma, which are later used iteratively, may be negative). We therefore omit proofs that are straightforward modifications of those given by Buffoni *et al.*; note, however, that references in their paper to Appendix D (in particular theorem D.6) for 'pseudo-local' properties of operators should be replaced by references to § 2.2.2 (in particular theorem 2.36) here.

Define sequences $\{\eta_m^{(1)}\}$, $\{\eta_m^{(2)}\}$ by the formulas

$$\eta_m^{(1)}(x) = \eta_m(x)\chi\left(\frac{x}{M_m^{(1)}}\right), \qquad \eta_m^{(2)}(x) = \eta_m(x)\left(1 - \chi\left(\frac{x}{M_m^{(2)}}\right)\right),$$

so that

$$\operatorname{supp} \eta_m^{(1)} \subset [-2M_m^{(1)}, 2M_m^{(1)}], \qquad \operatorname{supp} \eta_m^{(2)} \subset \mathbb{R} \setminus (-M_m^{(2)}, M_m^{(2)}).$$

Lemma 3.9.

(i) *The sequences* $\{\eta_m\}$, $\{\eta_m^{(1)}\}$ *and* $\{\eta_m^{(2)}\}$ *have the limiting behaviour*

$$\|\eta_m^{(1)}\|_2^2 \to \kappa, \qquad \|\eta_m^{(2)}\|_2^2 \to \ell - \kappa, \qquad \|\eta_m - \eta_m^{(1)} - \eta_m^{(2)}\|_2 \to 0$$

as $m \to \infty$ *and satisfy the bounds*

$$\sup_{m\in\mathbb{N}} \|\eta_m^{(1)}\|_2 < M, \qquad \sup_{m\in\mathbb{N}} \|\eta_m^{(2)}\|_2 < M, \qquad \sup_{m\in\mathbb{N}} \|\eta_m^{(1)} + \eta_m^{(2)}\|_2 < M.$$

(ii) *The limits* $\lim_{m\to\infty} \mathcal{L}(\eta_m^{(1)})$ *and* $\lim_{m\to\infty} \mathcal{L}(\eta_m^{(2)})$ *are positive.*

(iii) *The functionals* $\mathcal{G}$, $\mathcal{K}$ *and* $\mathcal{L}$ *satisfy*

$$\begin{Bmatrix} \mathcal{G} \\ \mathcal{K} \\ \mathcal{L} \end{Bmatrix}(\eta_m) - \begin{Bmatrix} \mathcal{G} \\ \mathcal{K} \\ \mathcal{L} \end{Bmatrix}(\eta_m^{(1)}) - \begin{Bmatrix} \mathcal{G} \\ \mathcal{K} \\ \mathcal{L} \end{Bmatrix}(\eta_m^{(2)}) \to 0,$$

$$\left\| \begin{Bmatrix} \mathcal{G}' \\ \mathcal{K}' \\ \mathcal{L}' \end{Bmatrix}(\eta_m) - \begin{Bmatrix} \mathcal{G}' \\ \mathcal{K}' \\ \mathcal{L}' \end{Bmatrix}(\eta_m^{(1)}) - \begin{Bmatrix} \mathcal{G}' \\ \mathcal{K}' \\ \mathcal{L}' \end{Bmatrix}(\eta_m^{(2)}) \right\|_0 \to 0$$

as $m \to \infty$.

(iv) *The sequences* $\{\eta_m\}$, $\{\eta_m^{(1)}\}$ *and* $\{\eta_m^{(2)}\}$ *satisfy*

$$\lim_{m\to\infty} \mathcal{J}_\mu(\eta_m) = \lim_{m\to\infty} \mathcal{J}_{\mu^{(1)}}(\eta_m^{(1)}) + \lim_{m\to\infty} \mathcal{J}_{\mu^{(2)}}(\eta_m^{(2)}),$$
$$\lim_{m\to\infty} \mathcal{J}'_\mu(\eta_m) = \lim_{m\to\infty} \mathcal{J}'_{\mu^{(1)}}(\eta_m^{(1)}) + \lim_{m\to\infty} \mathcal{J}'_{\mu^{(2)}}(\eta_m^{(2)}),$$

where

$$\mu^{(1)} = \alpha^{(1)}\Big(\mu + \lim_{m\to\infty} \mathcal{G}(\eta_m)\Big) - \lim_{m\to\infty} \mathcal{G}(\eta_m^{(1)}),$$
$$\mu^{(2)} = \alpha^{(2)}\Big(\mu + \lim_{m\to\infty} \mathcal{G}(\eta_m)\Big) - \lim_{m\to\infty} \mathcal{G}(\eta_m^{(2)}),$$

and the positive numbers $\alpha^{(1)}$, $\alpha^{(2)}$ *are defined by*

$$\alpha^{(1)} = \frac{\lim_{m\to\infty} \mathcal{L}(\eta_m^{(1)})}{\lim_{m\to\infty} \mathcal{L}(\eta_m)}, \qquad \alpha^{(2)} = \frac{\lim_{m\to\infty} \mathcal{L}(\eta_m^{(2)})}{\lim_{m\to\infty} \mathcal{L}(\eta_m)}.$$

(v) *The sequence* $\{\eta_m^{(1)}\}$ *converges weakly in* $H^2(\mathbb{R})$, *and strongly in* $H^r(\mathbb{R})$ *for* $r \in [0, 2)$, *to a function* $\eta^{(1)} \in H^2(\mathbb{R})$ *with* $\|\eta^{(1)}\|_2^2 \leqslant D\mathcal{K}(\eta^{(1)})$ *and* $\|\eta^{(1)}\|_1 \geqslant c|\mu|^{2r^\star}$.

(vi) *The sequence* $\{\eta_m^{(2)}\}$ *is a minimizing sequence for the functional*

$$\mathcal{J}_{\rho_2,\mu^{(2)}} \colon H^2(\mathbb{R}) \to \mathbb{R} \cup \{\infty\}$$

defined by

$$\mathcal{J}_{\rho_2,\mu^{(2)}}(\eta) = \begin{cases} \mathcal{K}(\eta) + \dfrac{(\mu^{(2)} + \mathcal{G}(\eta))^2}{\mathcal{L}(\eta)} + \rho_2(\|\eta\|_2^2), & \eta \in U_2 \setminus \{0\}, \\ \infty, & \eta \notin U_2 \setminus \{0\}, \end{cases}$$

where

$$U_2 = \{\eta \in H^2(\mathbb{R}) \colon \|\eta\|_2^2 \leqslant M^2 - \|\eta^{(1)}\|_2^2\}, \qquad \rho_2(\|\eta\|_2^2) = \rho(\|\eta^{(1)}\|_2^2 + \|\eta\|_2^2).$$

(vii) *The sequences* $\{\eta_m\}$ *and* $\{\eta_m^{(2)}\}$ *satisfy*

$$\lim_{m\to\infty} \rho(\|\eta_m\|_2^2) = \lim_{m\to\infty} \rho_2(\|\eta_m^{(2)}\|_2^2),$$
$$\lim_{m\to\infty} \mathcal{J}_{\rho,\mu}(\eta_m) = \mathcal{J}_{\mu^{(1)}}(\eta^{(1)}) + \lim_{m\to\infty} \mathcal{J}_{\rho_2,\mu^{(2)}}(\eta_m^{(2)})$$

and

$$\|\eta^{(1)}\|_2^2 + \lim_{m\to\infty} \|\eta_m^{(2)}\|_2^2 \leqslant \lim_{m\to\infty} \|\eta_m\|_2^2$$

with equality if $\lim_{m\to\infty} \rho(\|\eta_m\|_2^2) > 0$.

Proof. For part (i), see Buffoni *et al*. [9, lemma 3.10(i) and (ii)].

Turning to part (ii), observe that $\mathcal{L}(\eta_m^{(1)}) \to 0$ as $m \to \infty$ implies that $\|\eta_m^{(1)}\|_{1/2} \to 0$, and hence $\|\eta_m^{(1)}\|_1 \to 0$ as $m \to \infty$, which contradicts part (i). The same argument shows us that $\mathcal{L}(\eta_m^{(2)}) \not\to 0$ as $m \to \infty$. Because the derivative of $\mathcal{G}$ is bounded on U, we find that

$$|\mathcal{G}(\eta_m) - \mathcal{G}(\eta_m^{(1)} + \eta_m^{(2)})| \leqslant c\|\eta_m - \eta_m^{(1)} - \eta_m^{(2)}\|_2 \to 0$$

(see part (i)), and therefore that

$$\begin{aligned}\mathcal{G}(\eta_m) - \mathcal{G}(\eta_m^{(1)}) - \mathcal{G}(\eta_m^{(2)}) \\ = \underbrace{\mathcal{G}(\eta_m) - \mathcal{G}(\eta_m^{(1)} + \eta_m^{(2)})}_{=o(1)} + \underbrace{\mathcal{G}(\eta_m^{(1)} + \eta_m^{(2)}) - \mathcal{G}(\eta_m^{(1)}) - \mathcal{G}(\eta_m^{(2)})}_{=o(1)}\end{aligned}$$

as $m \to \infty$, in which theorem 2.36 has been used. The same argument applies to $\mathcal{K}$ and $\mathcal{L}$ and establishes part (iii).

Part (iv) follows from part (iii) by a direct calculation (cf. Buffoni *et al*. [9, corollary 3.11]); for parts (v), (vi) and (vii) see Buffoni *et al*. [9, lemmas 3.12, 3.15(i) and 3.15(ii)]. □

3.2.2. *Iteration*

The next step is to apply the concentration-compactness principle to the sequence $\{u_{2,m}\}$ given by

$$u_{2,m} = \eta_{2,m}'^2 + \eta_{2,m}^2,$$

where $\eta_{2,m} = \eta_m^{(2)}$, and repeat the above analysis. We proceed iteratively in this fashion, writing $\{\eta_m\}$, μ and U in iterative formulas as $\{\eta_{1,m}\}$, μ_1 and U_1, respectively. The following lemma describes the result of one step in this procedure (see Buffoni *et al*. [9, § 3.3]).

Lemma 3.10. *Suppose that there exist functions $\eta^{(1)}, \dots, \eta^{(k)} \in H^2(\mathbb{R})$ and a sequence $\{\eta_{k+1,m}\} \subset H^2(\mathbb{R})$ with the following properties.*

(i) *The sequence $\{\eta_{k+1,m}\}$ is a minimizing sequence for*

$$\mathcal{J}_{\rho_{k+1},\mu_{k+1}} \colon H^2(\mathbb{R}) \to \mathbb{R} \cup \{\infty\}$$

defined by

$$\mathcal{J}_{\rho_{k+1},\mu_{k+1}}(\eta) = \begin{cases} \mathcal{K}(\eta) + \dfrac{(\mu_{k+1} + \mathcal{G}(\eta))^2}{\mathcal{L}(\eta)} + \rho_{k+1}(\|\eta\|_2^2), & \eta \in U_{k+1} \setminus \{0\}, \\ \infty, & \eta \notin U_{k+1} \setminus \{0\}, \end{cases}$$

where

$$U_{k+1} = \left\{\eta \in H^2(\mathbb{R}) \colon \|\eta\|_2^2 \leqslant M^2 - \sum_{j=1}^{k} \|\eta^{(j)}\|_2^2\right\}$$

and

$$\rho_{k+1}(\|\eta\|_2^2) = \rho\left(\sum_{j=1}^{k} \|\eta^{(j)}\|_2^2 + \|\eta\|_2^2\right),$$

$$\mu_{k+1} = \frac{\lim_{m\to\infty} \mathcal{L}(\eta_{k+1,m})}{\lim_{m\to\infty} \mathcal{L}(\eta_m)}\left(\mu + \lim_{m\to\infty} \mathcal{G}(\eta_m)\right) - \lim_{m\to\infty} \mathcal{G}(\eta_{k+1,m}).$$

(ii) *The functions $\eta^{(1)}, \dots, \eta^{(k)}$ satisfy*

$$0 < \|\eta^{(j)}\|_2^2 \leqslant D\mathcal{K}(\eta^{(j)}), \quad j = 1, \dots, k,$$

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and

$$c_{\rho,\mu} = \sum_{j=1}^{k} \mathcal{J}_{\mu_j^{(1)}}(\eta^{(j)}) + c_{\rho_{k+1},\mu_{k+1}},$$

where

$$\mu_j^{(1)} = \frac{\mathcal{L}(\eta^{(j)})}{\lim_{m\to\infty}\mathcal{L}(\eta_m)}\Big(\mu + \lim_{m\to\infty}\mathcal{G}(\eta_m)\Big) - \lim_{m\to\infty}\mathcal{G}(\eta^{(j)}), \quad j = 1,\dots,k,$$

and $c_{\rho_{k+1},\mu_{k+1}} = \inf \mathcal{J}_{\rho_{k+1},\mu_{k+1}}$.

(iii) *The sequences* $\{\eta_m\}$, $\{\eta_{k+1,m}\}$ *and functions* $\eta^{(1)},\dots,\eta^{(k)}$ *satisfy*

$$\sum_{j=1}^{k} \left\{\begin{matrix}\mathcal{G}\\ \mathcal{K}\\ \mathcal{L}\end{matrix}\right\}(\eta^{(j)}) + \lim_{m\to\infty}\left\{\begin{matrix}\mathcal{G}\\ \mathcal{K}\\ \mathcal{L}\end{matrix}\right\}(\eta_{k+1,m}) = \lim_{m\to\infty}\left\{\begin{matrix}\mathcal{G}\\ \mathcal{K}\\ \mathcal{L}\end{matrix}\right\}(\eta_m),$$
$$\lim_{m\to\infty}\rho(\|\eta_m\|_2^2) = \lim_{m\to\infty}\rho_{k+1}(\|\eta_{k+1,m}\|_2^2)$$

and

$$\sum_{j=1}^{k}\|\eta^{(j)}\|_2^2 + \lim_{m\to\infty}\|\eta_{k+1,m}\|_2^2 \leqslant \lim_{m\to\infty}\|\eta_m\|_2^2$$

with equality if $\lim_{m\to\infty}\rho(\|\eta_m\|_2^2) > 0$.

Precisely one of the following phenomena occurs.

(1) *There exists a sequence* $\{x_{k+1,m}\} \subset \mathbb{R}$ *and a subsequence of* $\{\eta_{k+1,m}(\cdot + x_{k+1,m})\}$ *that satisfies*

$$\lim_{m\to\infty}\|\eta_{k+1,m}(\cdot + x_{k+1,m})\|_2^2 \leqslant \tilde{M}^2 - \sum_{j=1}^{k}\|\eta^{(j)}\|_2^2$$

and converges in $H^r(\mathbb{R})$ *for* $r \in [0,2)$. *The limiting function* $\eta^{(k+1)}$ *satisfies*

$$\sum_{j=1}^{k+1}\left\{\begin{matrix}\mathcal{G}\\ \mathcal{K}\\ \mathcal{L}\end{matrix}\right\}(\eta^{(j)}) = \lim_{m\to\infty}\left\{\begin{matrix}\mathcal{G}\\ \mathcal{K}\\ \mathcal{L}\end{matrix}\right\}(\eta_m),$$
$$0 < \|\eta^{(k+1)}\|_2^2 \leqslant D\mathcal{K}(\eta^{(k+1)}), \qquad c_{\rho,\mu} = \sum_{j=1}^{k+1}\mathcal{J}_{\mu_j^{(1)}}(\eta^{(j)})$$

with $\mu_{k+1}^{(1)} = \mu_{k+1}$, *minimizes* $\mathcal{J}_{\rho_{k+1},\mu_{k+1}}$ *and minimizes* $\mathcal{J}_{\mu_{k+1}^{(1)}}$ *over* $\tilde{U}_{k+1}\backslash\{0\}$, *where*

$$\tilde{U}_{k+1} = \left\{\eta \in H^2(\mathbb{R})\colon \|\eta\|_2^2 \leqslant \tilde{M}^2 - \sum_{j=1}^{k}\|\eta^{(j)}\|_2^2\right\}.$$

The iteration terminates with this step.

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(2) *There exist sequences* $\{\eta_{k+1,m}^{(1)}\}$, $\{\eta_{k+1,m}^{(2)}\}$ *with the following properties.*

(i) *The sequence* $\{\eta_{k+1,m}^{(1)}\}$ *converges in* $H^r(\mathbb{R}^2)$ *for* $r \in [0,2)$ *to a function* $\eta^{(k+1)}$ *that satisfies the estimates*

$$0 < \|\eta^{(k+1)}\|_2^2 \leqslant D\mathcal{K}(\eta^{(k+1)}), \qquad \|\eta^{(k+1)}\|_2 \geqslant c|\mu|_{k+1}^{2r^\star}.$$

(ii) *The sequence* $\{\eta_{k+1,m}^{(2)}\}$ *is a minimizing sequence for*

$$\mathcal{J}_{\rho_{k+2},\mu_{k+1}^{(2)}} : H^2(\mathbb{R}) \to \mathbb{R} \cup \{\infty\}$$

defined by

$$\mathcal{J}_{\rho_{k+2},\mu_{k+1}^{(2)}}(\eta) = \begin{cases} \mathcal{K}(\eta) \\ \quad + \dfrac{(\mu_{k+1}^{(2)} + \mathcal{G}(\eta))^2}{\mathcal{L}(\eta)} + \rho_{k+2}(\|\eta\|_2^2), & \eta \in U_{k+2} \setminus \{0\}, \\ \infty, & \eta \notin U_{k+2} \setminus \{0\}, \end{cases}$$

where

$$U_{k+2} = \left\{\eta \in H^2(\mathbb{R}) \colon \|\eta\|_2^2 \leqslant M^2 - \sum_{j=1}^{k+1} \|\eta^{(j)}\|_2^2\right\}$$

and

$$\rho_{k+2}(\|\eta\|_2^2) = \rho\left(\sum_{j=1}^{k+1} \|\eta^{(j)}\|_2^2 + \|\eta\|_2^2\right),$$

$$\mu_{k+1}^{(2)} = \frac{\lim_{m\to\infty} \mathcal{L}(\eta_{k+1,m}^{(2)})}{\lim_{m\to\infty} \mathcal{L}(\eta_m)}\left(\mu + \lim_{m\to\infty} \mathcal{G}(\eta_m)\right) - \lim_{m\to\infty} \mathcal{G}(\eta_{k+1,m}^{(2)});$$

furthermore

$$c_{\rho,\mu} = \sum_{j=1}^{k+1} \mathcal{J}_{\mu_j^{(1)}}(\eta^{(j)}) + c_{\rho_{k+2},\mu_{k+1}^{(2)}},$$

where

$$\mu_{k+1}^{(1)} = \mu \frac{\mathcal{L}(\eta^{(k+1)})}{\lim_{m\to\infty} \mathcal{L}(\eta_m)}, \qquad c_{\rho_{k+2},\mu_{k+1}^{(2)}} = \inf \mathcal{J}_{\rho_{k+2},\mu_{k+1}^{(2)}}.$$

(iii) *The sequences* $\{\eta_m\}$, $\{\eta_{k+1,m}^{(2)}\}$ *and functions* $\eta^{(1)}, \ldots, \eta^{(k+1)}$ *satisfy*

$$\sum_{j=1}^{k} \begin{Bmatrix} \mathcal{G} \\ \mathcal{K} \\ \mathcal{L} \end{Bmatrix} (\eta^{(j+1)}) + \lim_{m\to\infty} \begin{Bmatrix} \mathcal{G} \\ \mathcal{K} \\ \mathcal{L} \end{Bmatrix} (\eta_{k+1,m}^{(2)}) = \lim_{m\to\infty} \begin{Bmatrix} \mathcal{G} \\ \mathcal{K} \\ \mathcal{L} \end{Bmatrix} (\eta_m),$$

$$\lim_{m\to\infty} \rho(\|\eta_m\|_2^2) = \lim_{m\to\infty} \rho_{k+2}(\|\eta_{k+1,m}^{(2)}\|_2^2)$$

and

$$\sum_{j=1}^{k+1} \|\eta^{(j)}\|_2^2 + \lim_{m\to\infty} \|\eta_{k+1,m}^{(2)}\|_2^2 \leqslant \lim_{m\to\infty} \|\eta_m\|_2^2$$

with equality if $\lim_{m\to\infty} \rho(\|\eta_m\|_2^2) > 0$.

The iteration continues to the next step with $\eta_{k+2,m} = \eta_{k+1,m}^{(2)}$, $m \in \mathbb{N}$.

The above construction does not assume that the iteration terminates (that is 'concentration' occurs after a finite number of iterations). If it does not terminate, we let $k \to \infty$ in lemma 3.10 and find that $\|\eta^{(k)}\|_2 \to 0$ (because

$$\sum_{j=1}^{k} \|\eta^{(j)}\|_2^2 \leqslant D \sum_{j=1}^{k} \mathcal{K}(\eta^{(j)}) \leqslant D \sum_{j=1}^{k} \mathcal{J}_{\mu_j^{(1)}}(\eta^{(j)}) < Dc_{\rho,\mu} < 2D\nu_0^{\mu}|\mu|$$

for each $k \in \mathbb{N}$, so that the series $\sum_{j=1}^{\infty} \|\eta^{(j)}\|_2^2$ converges), $\mu_k \to 0$ (because $\|\eta^{(k)}\|_2^2 \geqslant c|\mu_k|^{2r^\star}$), $c_{\rho_k,\mu_k} \to 0$ (because $c_{\rho_k,\mu_k} < 2\nu_0^{\mu_k}|\mu_k|$) and

$$c_{\rho,\mu} = \sum_{j=1}^{\infty} \mathcal{J}_{\mu_j^{(1)}}(\eta^{(j)}).$$

For completeness we record the following corollary of lemma 3.10, which is not used in the remainder of the paper (cf. Buffoni *et al*. [9, corollary 3.17]).

Corollary 3.11. *Every minimizing sequence $\{\eta_m\}$ for $\mathcal{J}_{\rho,\mu}$ satisfies*

$$\lim_{m\to\infty} \|\eta_m\|_2 \leqslant \tilde{M}.$$

3.3. Construction of the special minimizing sequence

The sequence $\{\tilde{\eta}_m\}$ advertised in theorem 3.1 is constructed by gluing together the functions $\eta^{(j)}$ identified in § 3.2.2 with increasingly large distances between them (the index j is taken between 1 and k, where $k = \infty$ if the iteration does not terminate). The minimal distance between the functions is chosen so that the interaction between the 'tails' of the individual functions is negligible and $\|\tilde{\eta}_m\|_2^2$ is approximately $\sum_{j=1}^{k} \|\eta^{(j)}\|_2^2 = O(\mu)$ (we return to the original physical setting in which μ is positive). The algorithm is stated precisely in part (ii) of the following proposition (which follows immediately from part (i)); for the proof of part (i), see Buffoni *et al*. [9, proposition 3.20].

Proposition 3.12.

(i) *There exists a constant $C > 0$ such that*

$$\left\| \sum_{j=1}^{k} \tau_{S_j} \eta^{(j)} \right\|_2^2 \leqslant 2C^2 D\nu_0 \mu,$$

where $(\tau_X \eta^{(j)})(x) := \eta^{(j)}(x + X)$, for all choices of $\{S_j\}_{j=1}^{k}$. Moreover, in the case $k = \infty$ the series converges uniformly over all such sequences.

(ii) *The sequence $\{\tilde{\eta}_m\}$ defined by the following algorithm satisfies*

$$\|\tilde{\eta}_m\|_3^2 \leqslant 2C^2 D\nu_0 \mu.$$

(1) *Choose $R_j > 1$ large enough so that*

$$\|\eta^{(j)}\|_{H^2(|x|>R_j)} < \frac{\mu}{2^j}.$$

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(2) *Write* $S_1 = 0$ *and choose* $S_j > S_{j-1} + 2R_j + 2R_{j-1}$ *for* $j = 2, \dots, k$.

(3) *Define*

$$\tilde{\eta}_m = \sum_{j=1}^{k} \tau_{S_j+(j-1)m}\eta^{(j)}, \quad m \in \mathbb{N}.$$

Observe that a local translation-invariant analytic operator $\mathcal{T}\colon U \to \mathbb{R}$ has the property that

$$\lim_{m\to\infty} \mathcal{T}(\tilde{\eta}_m) = \sum_{j=1}^{k} \mathcal{T}(\eta^{(j)}).$$

Part (i) of the next lemma states that the functionals $\mathcal{G}$, $\mathcal{K}$ and $\mathcal{L}$ behave in the same fashion (with corresponding estimates for their L^2-gradients); it is deduced from theorem 2.36 using the method given by Buffoni *et al*. [9, lemma 3.22]. Part (ii) follows from part (i) by a straightforward calculation that shows that

$$\lim_{m\to\infty} \mathcal{J}_\mu(\tilde{\eta}_m) = \sum_{j=1}^{k} \mathcal{J}_{\mu_j^{(1)}}(\eta^{(j)}), \qquad \lim_{m\to\infty} \left\| \mathcal{J}'_\mu(\tilde{\eta}_m) - \sum_{j=1}^{k} \mathcal{J}'_{\mu_j^{(1)}}(\eta^{(j)}) \right\|_0 = 0$$

(cf. Buffoni *et al*. [9, corollary 3.23]).

Lemma 3.13.

(i) *The sequence* $\{\tilde{\eta}_m\}$ *and functions* $\{\eta^{(i)}\}_{i=1}^{m}$ *satisfy*

$$\lim_{m\to\infty} \begin{Bmatrix} \mathcal{G} \\ \mathcal{K} \\ \mathcal{L} \end{Bmatrix} (\tilde{\eta}_m) = \sum_{i=1}^{k} \begin{Bmatrix} \mathcal{G} \\ \mathcal{K} \\ \mathcal{L} \end{Bmatrix} (\eta^{(i)}),$$

$$\lim_{m\to\infty} \left\| \begin{Bmatrix} \mathcal{G}' \\ \mathcal{K}' \\ \mathcal{L}' \end{Bmatrix} (\tilde{\eta}_m) - \sum_{i=1}^{k} \begin{Bmatrix} \mathcal{G}' \\ \mathcal{K}' \\ \mathcal{L}' \end{Bmatrix} (\eta^{(i)}) \right\|_0 = 0.$$

(ii) *The sequence* $\{\tilde{\eta}_m\}$ *has the properties that*

$$\lim_{m\to\infty} \mathcal{J}_\mu(\tilde{\eta}_m) = c_{\rho,\mu}, \qquad \lim_{m\to\infty} \|\mathcal{J}'_\mu(\tilde{\eta}_m)\|_0 = 0.$$

The proof of theorem 3.1 is completed by the following proposition.

Proposition 3.14. *The sequence* $\{\tilde{\eta}_m\}$ *is a minimizing sequence for* $\mathcal{J}_\mu$ *over* $U \setminus \{0\}$.

Proof. Let us first note that $\{\tilde{\eta}_m\}$ is a minimizing sequence for $\mathcal{J}_\mu$ over $\tilde{U} \setminus \{0\}$ since the existence of a minimizing sequence $\{v_m\}$ for $\mathcal{J}_\mu$ over $\tilde{U} \setminus \{0\}$ with $\lim_{m\to\infty} \mathcal{J}_\mu(v_m) < \lim_{m\to\infty} \mathcal{J}_\mu(\tilde{\eta}_m)$ would lead to the contradiction

$$\lim_{m\to\infty} \mathcal{J}_{\rho,\mu}(v_m) = \lim_{m\to\infty} \mathcal{J}_\mu(v_m) < \lim_{m\to\infty} \mathcal{J}_\mu(\tilde{\eta}_m) = \lim_{m\to\infty} \mathcal{J}_{\rho,\mu}(\tilde{\eta}_m) = c_{\rho,\mu}.$$

It follows from this fact and the estimate $\|\tilde{\eta}_m\|_2^2 \leqslant 2C^2 D\nu_0\mu$ that

$$\inf\{\mathcal{J}_\mu(\eta)\colon \|\eta\|_2 \in (0, \tilde{M})\} = \inf\{\mathcal{J}_\mu(\eta)\colon \|\eta\|_2 \in (0, \sqrt{2C^2 D\nu_0\mu})\}$$

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for all $\tilde{M} \in (\sqrt{2C^2 D\nu_0\mu}, M)$. The right-hand side of this equation does not depend upon $\tilde{M}$; letting $\tilde{M} \to M$ on the left-hand side, one therefore finds that

$$\begin{aligned}\inf\{\mathcal{J}_\mu(\eta)\colon \|\eta\|_2 \in (0, M)\} &= \inf\{\mathcal{J}_\mu(\eta)\colon \|\eta\|_2 \in (0, \sqrt{2C^2 D\nu_0\mu})\}\\ &= \lim_{m\to\infty} \mathcal{J}_\mu(\tilde{\eta}_m).\end{aligned}$$

□

4. Strict subadditivity

The goal of this section is to establish that c_μ is *strictly subadditive*, that is,

$$c_{\mu_1+\mu_2} < c_{\mu_1} + c_{\mu_2}, \quad 0 < |\mu_1|, |\mu_2|, \mu_1 + \mu_2 < \mu_0, \tag{4.1}$$

where negative values of the small parameter are again allowed. This fact is deduced from the facts that c_μ is an increasing *strictly subhomogeneous* function of $\mu > 0$, that is,

$$c_{a\mu} < ac_\mu, \quad a > 1. \tag{4.2}$$

The strict subhomogeneity property of c_μ is established by considering a 'near minimizer' of $\mathcal{J}_\mu$ over $U \setminus \{0\}$, that is, a function in $U \setminus \{0\}$ with

$$\|\tilde{\eta}\|_2^2 \leqslant c\mu, \qquad \mathcal{J}_\mu(\tilde{\eta}) < 2\nu_0\mu - c\mu^{r^\star}, \qquad \|\mathcal{J}'_\mu(\tilde{\eta})\|_0 \leqslant \mu^N,$$

and hence $\mathcal{L}(\tilde{\eta}), \mathcal{L}_2(\tilde{\eta}) > c\mu$ (see the remark above (3.1) and inequality (3.2)), and identifying the dominant term in the 'nonlinear' part $\mathcal{M}_\mu(\tilde{\eta})$ of $\mathcal{J}_\mu(\tilde{\eta})$. In §§ 4.2 and 4.3 we show that

$$0 > \mathcal{M}_\mu(\tilde{\eta}) = \begin{cases} c\displaystyle\int_{-\infty}^{\infty} \tilde{\eta}_1^3 \,\mathrm{d}x + o(\mu^{5/3}), & \beta > \beta_{\mathrm{c}}, \\ -c\displaystyle\int_{-\infty}^{\infty} \tilde{\eta}_1^4 \,\mathrm{d}x + o(\mu^3), & \beta < \beta_{\mathrm{c}}, \end{cases} \tag{4.3}$$

where η_1 is obtained from $\eta \in H^2(\mathbb{R})$ by multiplying its Fourier transform by the characteristic function of the set $S = [-k_0 - \delta_0, -k_0 + \delta_0] \cup [k_0 - \delta_0, k_0 + \delta_0]$ with $\delta_0 > 0$ if $\beta > \beta_{\mathrm{c}}$ and $\delta_0 \in (0, k_0/3)$ if $\beta < \beta_{\mathrm{c}}$; inequality (4.2) is readily verified by approximating $\mathcal{M}(\tilde{\eta}_m)$ by the homogeneous term identified in (4.3). The details of this procedure are given in § 4.4.

Straightforward estimates of the kind

$$\mathcal{G}_j(\tilde{\eta}_m), \mathcal{K}_j(\tilde{\eta}_m), \mathcal{L}_j(\tilde{\eta}_m) = O(\|\tilde{\eta}_m\|_2^j) = O(\mu^{j/2})$$

do not suffice to establish (4.3). According to the calculations presented in Appendix A, the function $\eta_\mu^\star$, which is constructed using the Korteweg–de Vries scaling for $\beta > \beta_{\mathrm{c}}$ and the nonlinear Schrödinger scaling for $\beta < \beta_{\mathrm{c}}$, satisfies the estimate (4.3) (with $\tilde{\eta}$ replaced by $\eta_\mu^\star$). The choice of $\eta_\mu^\star$ is of course motivated by the expectation that a minimizer, and hence any near minimizer, should have the Korteweg–de Vries or nonlinear Schrödinger length-scales. Our strategy is therefore to show that $\tilde{\eta}_1$ is $O(\mu^{1/2})$ with respect to a weighted norm. To this end we consider the norm

$$|\!|\!|\eta|\!|\!|_\alpha^2 := \int_{-\infty}^{\infty} (1 + \mu^{-4\alpha}(|k| - k_0)^4)|\hat{\eta}(k)|^2 \,\mathrm{d}k$$

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and choose $\alpha > 0$ as large as possible so that $|||\tilde{\eta}_1|||_\alpha$ is $O(\mu^{1/2})$; this more detailed description of the behaviour of $\tilde{\eta}$ allows one to obtain better estimates for $\mathcal{G}_j(\tilde{\eta})$, $\mathcal{K}_j(\tilde{\eta})$ and $\mathcal{L}_j(\tilde{\eta})$, and thus establish (4.3) (see §§ 4.2 and 4.3 for $\beta > \beta_{\mathrm{c}}$ and $\beta < \beta_{\mathrm{c}}$, respectively).

4.1. Preliminaries

In this section we establish some basic facts that are used in §§ 4.2–4.4.

4.1.1. *Splitting of* η

In view of the expected frequency distribution of $\tilde{\eta}$, we split each $\eta \in U$ into the sum of a function η_1 with spectrum near $k = \pm k_0$ and a function η_2 whose spectrum is bounded away from these points. To this end we write the equation

$$\begin{aligned}
\mathcal{J}_\mu'(\eta) &= \mathcal{K}_2'(\eta) + \mathcal{K}_{\mathrm{nl}}'(\eta) + 2\left(\frac{\mu + \mathcal{G}(\eta)}{\mathcal{L}(\eta)}\right)\mathcal{G}_2'(\eta) + 2\left(\frac{\mu + \mathcal{G}(\eta)}{\mathcal{L}(\eta)}\right)\mathcal{G}_{\mathrm{nl}}'(\eta) \\
&\qquad - \left(\frac{\mu + \mathcal{G}(\eta)}{\mathcal{L}(\eta)}\right)^2\mathcal{L}_2'(\eta) - \left(\frac{\mu + \mathcal{G}(\eta)}{\mathcal{L}(\eta)}\right)^2\mathcal{L}_{\mathrm{nl}}'(\eta) \\
&= \mathcal{K}_2'(\eta) + 2\nu_0\mathcal{G}_2'(\eta) - \nu_0^2\mathcal{L}_2'(\eta) \\
&\qquad + \mathcal{K}_{\mathrm{nl}}'(\eta) + 2\left(\frac{\mu + \mathcal{G}(\eta)}{\mathcal{L}(\eta)} - \nu_0\right)\mathcal{G}_2'(\eta) + 2\left(\frac{\mu + \mathcal{G}(\eta)}{\mathcal{L}(\eta)}\right)\mathcal{G}_{\mathrm{nl}}'(\eta) \\
&\qquad - \left(\frac{\mu + \mathcal{G}(\eta)}{\mathcal{L}(\eta)} + \nu_0\right)\left(\frac{\mu + \mathcal{G}(\eta)}{\mathcal{L}(\eta)} - \nu_0\right)\mathcal{L}_2'(\eta) - \left(\frac{\mu + \mathcal{G}(\eta)}{\mathcal{L}(\eta)}\right)^2\mathcal{L}_{\mathrm{nl}}'(\eta)
\end{aligned}$$

in the form

$$\begin{aligned}
g(k)\hat{\eta} = \mathcal{F}\bigg[&\mathcal{J}_\mu'(\eta) - \mathcal{K}_{\mathrm{nl}}'(\eta) - 2\left(\frac{\mu + \mathcal{G}(\eta)}{\mathcal{L}(\eta)} - \nu_0\right)\mathcal{G}_2'(\eta) - 2\left(\frac{\mu + \mathcal{G}(\eta)}{\mathcal{L}(\eta)}\right)\mathcal{G}_{\mathrm{nl}}'(\eta) \\
&+ \left(\frac{\mu + \mathcal{G}(\eta)}{\mathcal{L}(\eta)} + \nu_0\right)\left(\frac{\mu + \mathcal{G}(\eta)}{\mathcal{L}(\eta)} - \nu_0\right)\mathcal{L}_2'(\eta) + \left(\frac{\mu + \mathcal{G}(\eta)}{\mathcal{L}(\eta)}\right)^2\mathcal{L}_{\mathrm{nl}}'(\eta)\bigg]
\end{aligned}$$

and decompose it into two coupled equations by defining $\eta_2 \in H^2(\mathbb{R})$ by the formula

$$\begin{aligned}
\eta_2 = \mathcal{F}^{-1}\bigg[\frac{1 - \chi_S(k)}{g(k)}\mathcal{F}\bigg[&\mathcal{J}'(\eta) - \mathcal{K}_{\mathrm{nl}}'(\eta) \\
&- 2\left(\frac{\mu + \mathcal{G}(\eta)}{\mathcal{L}(\eta)} - \nu_0\right)\mathcal{G}_2'(\eta) - 2\left(\frac{\mu + \mathcal{G}(\eta)}{\mathcal{L}(\eta)}\right)\mathcal{G}_{\mathrm{nl}}'(\eta) \\
&+ \left(\frac{\mu + \mathcal{G}(\eta)}{\mathcal{L}(\eta)} + \nu_0\right)\left(\frac{\mu + \mathcal{G}(\eta)}{\mathcal{L}(\eta)} - \nu_0\right)\mathcal{L}_2'(\eta) \\
&+ \left(\frac{\mu + \mathcal{G}(\eta)}{\mathcal{L}(\eta)}\right)^2\mathcal{L}_{\mathrm{nl}}'(\eta)\bigg]\bigg]
\end{aligned}$$

and $\eta_1 \in H^2(\mathbb{R})$ by $\eta_1 = \eta - \eta_2$, so that $\hat{\eta}_1$ has support in S; here we have used the fact that

$$f \mapsto \mathcal{F}^{-1}\left[\frac{1 - \chi_S(k)}{g(k)}\hat{f}(k)\right]$$

is a bounded linear operator $L^2(\mathbb{R}) \to H^2(\mathbb{R})$.

4.1.2. *Estimates for* $|||\cdot|||_\alpha$

PROPOSITION 4.1.

(i) *The estimates* $\|\eta\|_{1,\infty} \leqslant c\mu^{\alpha/2}|||\eta|||_\alpha$, $\|K^0\eta\|_\infty \leqslant c\mu^{\alpha/2}|||\eta|||_\alpha$ *hold for each* $\eta \in H^2(\mathbb{R})$.

(ii) *The estimates*

$$\|\eta'' + k_0^2\eta\|_0 \leqslant c\mu^\alpha|||\eta|||_\alpha, \quad k_0 \neq 0,$$

and

$$\|(K^0\eta)^{(n)}\|_\infty \leqslant \mu^{\alpha/2}|||\eta|||_\alpha, \quad n \in \mathbb{N}_0,$$

hold for each $\eta \in H^2(\mathbb{R})$ *with* $\operatorname{supp}\hat{\eta} \subseteq S$.

Proof. (i) Observe that

$$\|\eta^{(j)}\|_\infty^2 \leqslant c\||k|^j\hat{\eta}\|_{L^1(\mathbb{R})}, \quad j = 0, 1, \tag{4.4}$$

$$\begin{aligned}\|K^0\eta\|_\infty &\leqslant \|(K^0 - 1)\eta\|_\infty + \|\eta\|_\infty \\ &\leqslant c(\|(|k|\coth|k| - 1)\hat{\eta}\|_{L^1(\mathbb{R})} + \|\eta\|_\infty) \\ &\leqslant c(\||k|\hat{\eta}\|_{L^1(\mathbb{R})} + \|\hat{\eta}\|_{L^1(\mathbb{R})})\end{aligned} \tag{4.5}$$

and

$$\begin{aligned}&\||k|^j\hat{\eta}\|_{L^1(\mathbb{R})}^2 \\ &\quad\leqslant \left(\int_{-\infty}^{\infty} \frac{k^{2j}}{1 + \mu^{-4\alpha}(k - k_0)^4}\,\mathrm{d}k\right)\int_0^\infty (1 + \mu^{-4\alpha}(k - k_0)^4)|\hat{\eta}(k)|^2\,\mathrm{d}k \\ &\qquad + \left(\int_{-\infty}^{\infty} \frac{k^{2j}}{1 + \mu^{-4\alpha}(k + k_0)^4}\,\mathrm{d}k\right)\int_{-\infty}^0 (1 + \mu^{-4\alpha}(k + k_0)^4)|\hat{\eta}(k)|^2\,\mathrm{d}k \\ &\quad\leqslant c\mu^\alpha|||\eta|||^2, \quad j = 0, 1.\end{aligned}$$

(ii) The first result follows from the calculation

$$\begin{aligned}\|\eta'' + k_0^2\eta\|_0^2 &= \|(k^2 - k_0^2)\hat{\eta}\|_0^2 \\ &\leqslant c\bigg(\int_{k_0-\delta_0}^{k_0+\delta_0} |k - k_0|^2|\hat{\eta}(k)|^2\,\mathrm{d}k + \int_{-k_0-\delta_0}^{-k_0+\delta_0} |k + k_0|^2|\hat{\eta}(k)|^2\,\mathrm{d}k\bigg) \\ &\leqslant c\bigg(\int_{k_0-\delta_0}^{k_0+\delta_0} (\mu^{2\alpha} + \mu^{-2\alpha}|k - k_0|^4)|\hat{\eta}(k)|^2\,\mathrm{d}k \\ &\qquad + \int_{-k_0-\delta_0}^{-k_0+\delta_0} (\mu^{2\alpha} + \mu^{-2\alpha}|k + k_0|^4)|\hat{\eta}(k)|^2\,\mathrm{d}k\bigg) \\ &\leqslant c\mu^{2\alpha}\bigg(\int_{k_0-\delta_0}^{k_0+\delta_0} (1 + \mu^{-4\alpha}|k - k_0|^4)|\hat{\eta}(k)|^2\,\mathrm{d}k \\ &\qquad + \int_{-k_0-\delta_0}^{-k_0+\delta_0} (1 + \mu^{-4\alpha}|k + k_0|^4)|\hat{\eta}(k)|^2\,\mathrm{d}k\bigg) \\ &= c\mu^{2\alpha}|||\eta|||_\alpha^2,\end{aligned}$$

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while the second is established by repeating the proof of the second inequality in part (i) and estimating $|k| \leqslant k_0 + \delta_0$. □

4.1.3. *Estimates for the wave speed*

The following proposition is used in particular to bound the deviation of the quantity $(\mu + \mathcal{G}(\tilde{\eta}))/\mathcal{L}(\tilde{\eta})$ (the speed of the corresponding travelling wave when $\tilde{\eta}$ is a minimizer of $\mathcal{J}_\mu$ over $U \setminus \{0\}$) from the linear wave speed ν_0.

Proposition 4.2. *The function $\tilde{\eta}$ satisfies the inequalities*

$$\mathcal{R}_1(\tilde{\eta}) \leqslant \frac{\mu + \mathcal{G}(\tilde{\eta})}{\mathcal{L}(\tilde{\eta})} - \nu_0 \leqslant \mathcal{R}_2(\tilde{\eta})$$

and

$$\mathcal{R}_1(\tilde{\eta}) - \tilde{\mathcal{M}}_\mu(\tilde{\eta}) \leqslant \frac{\mu + \mathcal{G}_2(\tilde{\eta})}{\mathcal{L}_2(\tilde{\eta})} - \nu_0 \leqslant \mathcal{R}_2(\tilde{\eta}) - \tilde{\mathcal{M}}_\mu(\tilde{\eta}),$$

where

$$\mathcal{R}_1(\tilde{\eta}) = -\frac{\langle \mathcal{J}_\mu'(\tilde{\eta}), \tilde{\eta}\rangle}{4\mu} + \frac{1}{4\mu}(\langle \mathcal{M}_\mu'(\tilde{\eta}), \tilde{\eta}\rangle + 4\mu\tilde{\mathcal{M}}_\mu(\tilde{\eta})),$$
$$\mathcal{R}_2(\tilde{\eta}) = -\frac{\langle \mathcal{J}_\mu'(\tilde{\eta}), \tilde{\eta}\rangle}{4\mu} + \frac{1}{4\mu}(\langle \mathcal{M}_\mu'(\tilde{\eta}), \tilde{\eta}\rangle + 4\mu\tilde{\mathcal{M}}_\mu(\tilde{\eta})) - \frac{\mathcal{M}_\mu(\tilde{\eta})}{2\mu}$$

and

$$\tilde{\mathcal{M}}_\mu(\tilde{\eta}) = \frac{\mu + \mathcal{G}(\tilde{\eta})}{\mathcal{L}(\tilde{\eta})} - \frac{\mu + \mathcal{G}_2(\tilde{\eta})}{\mathcal{L}_2(\tilde{\eta})}.$$

Proof. Taking the scalar product of the equation

$$\mathcal{J}_\mu'(\tilde{\eta}) = \mathcal{K}_2'(\tilde{\eta}) - \left(\frac{\mu + \mathcal{G}_2(\tilde{\eta})}{\mathcal{L}_2(\tilde{\eta})}\right)^2 \mathcal{L}_2'(\tilde{\eta}) + 2\left(\frac{\mu + \mathcal{G}_2(\tilde{\eta})}{\mathcal{L}_2(\tilde{\eta})}\right)\mathcal{G}_2'(\tilde{\eta}) + \mathcal{M}_\mu'(\tilde{\eta})$$

with $\tilde{\eta}$ yields the identity

$$\frac{\mu + \mathcal{G}(\tilde{\eta})}{\mathcal{L}(\tilde{\eta})} = -\frac{\langle \mathcal{J}_\mu'(\tilde{\eta}), \tilde{\eta}\rangle}{4\mu} + \frac{1}{2\mu}\left(\mathcal{K}_2(\tilde{\eta}) + \frac{(\mu + \mathcal{G}_2(\tilde{\eta}))^2}{\mathcal{L}_2(\tilde{\eta})}\right) + \frac{1}{4\mu}(\langle \mathcal{M}_\mu'(\tilde{\eta}), \tilde{\eta}\rangle + 4\mu\tilde{\mathcal{M}}_\mu(\tilde{\eta})).$$

The first inequality is derived by estimating the quantity in brackets from above and below by means of the estimate

$$2\nu_0\mu \leqslant \mathcal{K}_2(\tilde{\eta}) + \frac{(\mu + \mathcal{G}_2(\tilde{\eta}))^2}{\mathcal{L}_2(\tilde{\eta})} = \mathcal{J}_\mu(\tilde{\eta}) - \mathcal{M}_\mu(\tilde{\eta}) < 2\nu_0\mu - \mathcal{M}_\mu(\tilde{\eta})$$

and the second inequality follows directly from the first. □

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4.1.4. *Estimates for the functionals $\mathcal{G}$, $\mathcal{K}$ and $\mathcal{L}$*

Turning to the functionals $\mathcal{G}$, $\mathcal{K}$ and $\mathcal{L}\colon U \to \mathbb{R}$, denote their non-quadratic parts by $\mathcal{G}_{\mathrm{nl}}$, $\mathcal{K}_{\mathrm{nl}}$, $\mathcal{L}_{\mathrm{nl}}$ and write

$$\mathcal{G}_{\mathrm{nl}}(\eta) = \sum_{k=3}^{4} \mathcal{G}_k(\eta) + \mathcal{G}_{\mathrm{r}}(\eta),$$
$$\mathcal{K}_{\mathrm{nl}}(\eta) = \sum_{k=3}^{4} \mathcal{K}_k(\eta) + \mathcal{K}_{\mathrm{r}}(\eta),$$
$$\mathcal{L}_{\mathrm{nl}}(\eta) = \sum_{k=3}^{4} \mathcal{L}_k(\eta) + \mathcal{L}_{\mathrm{r}}(\eta),$$

so that

$$\mathcal{G}_{\mathrm{r}}(\eta) = \frac{\omega}{4}\int_{-\infty}^{\infty} \eta^2 (K(\eta) - K^0 - K^1(\eta))\eta \,\mathrm{d}x, \tag{4.6}$$

$$\mathcal{K}_{\mathrm{r}}(\eta) = \beta \int_{-\infty}^{\infty} \left(\sqrt{1+\eta'^2} - 1 - \frac{\eta'^2}{2} + \frac{\eta'^4}{8}\right) \mathrm{d}x - \frac{\omega^2}{2}\int_{-\infty}^{\infty} \frac{\eta^2}{2}(K(\eta) - K^0)\frac{\eta^2}{2}\,\mathrm{d}x, \tag{4.7}$$

$$\mathcal{L}_{\mathrm{r}}(\eta) = \frac{1}{2}\int_{-\infty}^{\infty} \eta(K(\eta) - K^0 - K^1(\eta) - K^2(\eta))\eta \,\mathrm{d}x. \tag{4.8}$$

We now record useful explicit formulas for the cubic and quartic parts of the functionals in terms of the Fourier-multiplier operator K^0 and give order-of-magnitude estimates for their cubic, quartic and higher-order parts.

PROPOSITION 4.3. *The formulas*

$$\mathcal{G}_3(\eta) = \frac{\omega}{4}\int_{-\infty}^{\infty} \eta^2 K^0 \eta \,\mathrm{d}x, \qquad \mathcal{K}_3(\eta) = \frac{\omega^2}{6}\int_{-\infty}^{\infty} \eta^3 \,\mathrm{d}x,$$
$$\mathcal{L}_3(\eta) = \frac{1}{2}\int_{-\infty}^{\infty} (-(K^0\eta)^2\eta + \eta'^2\eta)\,\mathrm{d}x$$

and

$$\mathcal{G}_4(\eta) = \frac{\omega}{2}\eta^2\eta'^2\,\mathrm{d}x - \frac{\omega}{4}\int_{-\infty}^{\infty} \eta^2 K^0(\eta K^0 \eta)\,\mathrm{d}x,$$
$$\mathcal{K}_4(\eta) = -\frac{\beta}{8}\int_{-\infty}^{\infty} \eta'^4\,\mathrm{d}x - \frac{\omega^2}{8}\int_{-\infty}^{\infty} \eta^2 K^0 \eta^2\,\mathrm{d}x,$$
$$\mathcal{L}_4(\eta) = \frac{1}{2}\int_{-\infty}^{\infty} (K^0(\eta K^0\eta)\eta K^0\eta + (K^0\eta)\eta^2\eta'')\,\mathrm{d}x$$

hold for each $\eta \in U$.

Proof. The formulas for $\mathcal{G}_3$ and $\mathcal{K}_3$, $\mathcal{K}_4$ follow directly from (1.10) and (1.11).

Equations (1.12) and (2.26) imply that

$$\mathcal{L}_3(\eta) = \frac{1}{2}\int_{-\infty}^{\infty} \eta K_1(\eta)\eta\,\mathrm{d}x, \qquad \mathcal{L}_3'(\eta) = \tfrac{1}{2}\mathcal{H}_1'(\eta)(\eta,\eta) + K_1(\eta)\eta,$$

while lemma 2.31 shows that

$$\mathcal{H}_1'(\eta)(\zeta_1,\zeta_2) = -u_{1x}^0 u_{2x}^0 + u_{1y}^0 u_{2y}^0|_{y=1} = -(K^0\zeta_1)(K^0\zeta_2) + \zeta_1'\zeta_2',$$

where u_j is the weak solution of (2.7)–(2.9) with $\xi = \zeta_j'$, $j = 1, 2$, so that

$$\mathcal{L}_3'(\eta) = -\tfrac{1}{2}(K^0\eta)^2 + \tfrac{1}{2}\eta'^2 + K_1(\eta)\eta. \tag{4.9}$$

Taking the inner product of this equation with η, we therefore find that

$$3\mathcal{L}_3(\eta) = \frac{1}{2}\int_{-\infty}^{\infty} (-(K^0\eta)^2\eta + \eta'^2\eta)\,\mathrm{d}x + 2\mathcal{L}_3(\eta),$$

which yields the given formula for $\mathcal{L}_3(\eta)$.

Similarly, (1.10) and (2.25) imply that

$$\mathcal{G}_4(\eta) = \frac{\omega}{4}\int_{-\infty}^{\infty} \eta^2 K_1(\eta)\eta\,\mathrm{d}x$$

and

$$\begin{aligned}\mathcal{G}_4'(\eta) &= \tfrac{1}{4}\omega\mathcal{H}_1'(\eta)(\eta^2,\eta) + \tfrac{1}{4}\omega K_1(\eta)\eta^2 + \tfrac{1}{2}\omega\eta K_1(\eta)\eta \\ &= -\tfrac{1}{4}\omega(K^0\eta^2)K^0\eta + \tfrac{1}{4}\omega(\eta^2)'\eta' + \tfrac{1}{4}\omega K_1(\eta)\eta^2 + \tfrac{1}{2}\omega\eta K_1(\eta)\eta.\end{aligned}$$

The formula for $\mathcal{G}_4(\eta)$ follows by taking the inner product of the latter equation with η.

Finally, (1.12) and (2.26) imply that

$$\mathcal{L}_4(\eta) = \frac{1}{2}\int_{-\infty}^{\infty} \eta K_2(\eta)\eta\,\mathrm{d}x, \qquad \mathcal{L}_4'(\eta) = \tfrac{1}{2}\mathcal{H}_2'(\eta)(\eta,\eta) + K_2(\eta)\eta$$

and lemma 2.31 shows that

$$\mathcal{H}_2'(\eta)(\zeta_1,\zeta_2) = -u_{1x}^0 u_{2x}^1 - u_{2x}^0 u_{1x}^1 + u_{1y}^0 u_{2y}^1 + u_{2y}^0 u_{1y}^1 - 2\eta u_{1y}^0 u_{2y}^0|_{y=1}.$$

Using (2.18), we find that

$$u_y^1|_{y=1} = G^1\cdot(0,1)|_{y=1} = -(Q^1\nabla u^0)\cdot(0,1)|_{y=1} = \eta u_y^0 + \eta' u_x^0|_{y=1} = \eta\zeta' - \eta' K^0\zeta,$$

where u is the weak solution of (2.7)–(2.9) with $\xi = \zeta'$, so that

$$\mathcal{H}_2'(\eta)(\eta,\eta) = -2\eta'^2 K^0\eta - 2K^0\eta K^1(\eta)\eta.$$

Equating (4.9) and

$$\mathcal{L}_3'(\eta) = -K^0(\eta K^0\eta) - \tfrac{1}{2}(K^0\eta)^2 - \tfrac{1}{2}\eta'^2 - \eta''\eta,$$

which follows from the formula for $\mathcal{L}_3(\eta)$, we find that

$$K^1(\eta)\eta = -K^0(\eta K^0\eta) - (\eta'\eta)'$$

so that

$$\mathcal{L}_4'(\eta) = -\eta'^2 K^0\eta + K^0\eta K^0(\eta K^0\eta) + K^0\eta(\eta'\eta)' + K_2(\eta)\eta.$$

The formula for $\mathcal{L}_4(\eta)$ is obtained by taking the inner product of this expression with η. □

PROPOSITION 4.4. *The estimates*

$$\left.\begin{cases}|\mathcal{G}_3(\eta)|\\|\mathcal{K}_3(\eta)|\\|\mathcal{L}_3(\eta)|\end{cases}\right\} \leqslant c\|\eta\|_2^2(\|\eta\|_{1,\infty}+\|\eta''+k_0^2\eta\|_0),$$

$$\left.\begin{cases}|\mathcal{G}_4(\eta)|\\|\mathcal{K}_4(\eta)|\\|\mathcal{L}_4(\eta)|\end{cases}\right\} \leqslant c\|\eta\|_2^2(\|\eta\|_{1,\infty}+\|\eta''+k_0^2\eta\|_0)^2,$$

$$\left.\begin{cases}|\mathcal{G}_\mathrm{r}(\eta)|\\|\mathcal{K}_\mathrm{r}(\eta)|\\|\mathcal{L}_\mathrm{r}(\eta)|\end{cases}\right\} \leqslant c\|\eta\|_2^3(\|\eta\|_{1,\infty}+\|\eta''+k_0^2\eta\|_0)^2$$

hold for each $\eta \in U$.

Proof. These results are obtained by estimating the right-hand sides of the formulas given in proposition 4.3 and (4.6)–(4.8) using proposition 2.29. □

PROPOSITION 4.5. *The estimates*

$$\left.\begin{cases}\|\mathcal{G}_3'(\eta)\|_0\\\|\mathcal{K}_3'(\eta)\|_0\\\|\mathcal{L}_3'(\eta)\|_0\end{cases}\right\} \leqslant c\|\eta\|_2(\|\eta\|_{1,\infty}+\|\eta''+k_0^2\eta\|_0+\|K^0\eta\|_\infty),$$

$$\left.\begin{cases}\|\mathcal{G}_4'(\eta)\|_0\\\|\mathcal{K}_4'(\eta)\|_0\\\|\mathcal{L}_4'(\eta)\|_0\end{cases}\right\} \leqslant c\|\eta\|_2(\|\eta\|_{1,\infty}+\|\eta''+k_0^2\eta\|_0+\|K^0\eta\|_\infty)^2,$$

$$\left.\begin{cases}\|\mathcal{G}_\mathrm{r}'(\eta)\|_0\\\|\mathcal{K}_\mathrm{r}'(\eta)\|_0\\\|\mathcal{L}_\mathrm{r}'(\eta)\|_0\end{cases}\right\} \leqslant c\|\eta\|_2^2(\|\eta\|_{1,\infty}+\|\eta''+k_0^2\eta\|_0)^2$$

hold for each $\eta \in U$.

Proof. We estimate the right-hand sides of the formulas

$$\mathcal{G}_3'(\eta) = \tfrac{1}{4}\omega K^0\eta^2 + \tfrac{1}{2}\omega\eta K^0\eta, \tag{4.10}$$

$$\mathcal{K}_3'(\eta) = \tfrac{1}{2}\omega^2\eta^2, \tag{4.11}$$

$$\mathcal{L}_3'(\eta) = -K^0(\eta K^0\eta) - \tfrac{1}{2}(K^0\eta)^2 - \tfrac{1}{2}\eta'^2 - \eta''\eta, \tag{4.12}$$

$$\mathcal{G}_4'(\eta) = -\tfrac{1}{4}\omega(K^0\eta^2)K^0\eta - \tfrac{1}{4}\omega K^0(\eta K^0\eta^2) - \omega\eta\eta'^2 - \omega\eta^2\eta'' - \tfrac{1}{2}\omega\eta K^0(\eta K^0\eta),$$

$$\mathcal{K}_4'(\eta) = \tfrac{3}{2}\beta\eta'^2\eta'' - \tfrac{1}{4}\omega^2\eta^2K^0\eta^2,$$

$$\mathcal{L}_4'(\eta) = -2\eta'^2K^0\eta - 2K^0\eta K^1(\eta)\eta + K_2(\eta)\eta$$

and

$$\begin{aligned}\mathcal{G}_\mathrm{r}'(\eta) = {}& \tfrac{1}{4}\omega(\mathcal{H}'(\eta)-\mathcal{H}_1'(\eta))(\eta^2,\eta) + \tfrac{1}{4}\omega(K(\eta)-K^0-K^1(\eta))\eta^2\\ &+ \tfrac{1}{2}\omega\eta(K(\eta)-K^0-K^1(\eta))\eta,\end{aligned}$$

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$$\mathcal{K}_{\mathrm{r}}'(\eta) = \beta\left(1 - \tfrac{3}{2}\eta'^2 - \frac{1}{(1+\eta'^2)^{3/2}}\right)\eta'' - \tfrac{1}{8}\omega^2\mathcal{H}'(\eta)(\eta^2,\eta^2) - \tfrac{1}{2}\omega^2\eta^2(K(\eta)-K^0)\eta,$$
$$\mathcal{L}_{\mathrm{r}}'(\eta) = \tfrac{1}{2}(\mathcal{H}'(\eta) - \mathcal{H}_1'(\eta) - \mathcal{H}_2'(\eta))(\eta,\eta) + (K(\eta) - K^0 - K^1(\eta) - K^2(\eta))\eta$$

using proposition 2.29 and the estimate

$$\|\mathcal{H}_{j+1}'(\eta)(\zeta_1,\zeta_2)\|_0 \leqslant CB^j(\|\eta\|_{1,\infty} + \|\eta'' + k_0^2\eta\|_0)^j\|\zeta_1\|_{3/2}\|\zeta_2\|_{3/2}, \quad j \in \mathbb{N}_0.$$

$\square$

It is also helpful to write

$$\mathcal{K}_3'(\eta) = m_1(\eta,\eta), \qquad \mathcal{G}_3'(\eta) = m_2(\eta,\eta), \qquad \mathcal{L}_3'(\eta) = m_3(\eta,\eta),$$

where $m_j \in \mathcal{L}_{\mathrm{s}}^2(H^2(\mathbb{R}), L^2(\mathbb{R}))$, $j = 1,2,3$, are defined by

$$\begin{aligned} m_1(u_1,u_2) &= \tfrac{1}{2}\omega^2 u_1 u_2, \\ m_2(u_1,u_2) &= \tfrac{1}{4}\omega K^0(u_1u_2) + \tfrac{1}{4}\omega u_1 K^0 u_2 + \tfrac{1}{4}\omega u_2 K^0 u_1, \\ m_3(u_1,u_2) &= -\tfrac{1}{2}K^0(u_1K^0u_2) - \tfrac{1}{2}K^0(u_2K^0u_1) \\ &\qquad - \tfrac{1}{2}K^0u_1K^0u_2 - \tfrac{1}{2}u_{1x}u_{2x} - \tfrac{1}{2}u_{1xx}u_2 - \tfrac{1}{2}u_1u_{2xx} \end{aligned}$$

and, similarly,

$$\mathcal{K}_3(\eta) = n_1(\eta,\eta,\eta), \qquad \mathcal{G}_3(\eta) = n_2(\eta,\eta,\eta), \qquad \mathcal{L}_3(\eta) = n_3(\eta,\eta,\eta),$$

where $n_j \in \mathcal{L}_{\mathrm{s}}^3(H^2(\mathbb{R}), \mathbb{R})$, $j = 1,2,3$, are defined by

$$\begin{aligned} n_1(u_1,u_2,u_3) &= \frac{1}{6}\omega^2\int_{-\infty}^{\infty} u_1u_2u_3\,\mathrm{d}x, \\ n_2(u_1,u_2,u_3) &= \frac{1}{12}\omega\int_{-\infty}^{\infty}\mathcal{P}[u_1u_2K^0u_3]\,\mathrm{d}x, \\ n_3(u_1,u_2,u_3) &= \frac{1}{6}\int_{-\infty}^{\infty}\mathcal{P}[u_1'u_2'u_3]\,\mathrm{d}x - \frac{1}{6}\int_{-\infty}^{\infty}\mathcal{P}[(K^0u_1)(K^0u_2)u_3]\,\mathrm{d}x \end{aligned}$$

and the symbol $\mathcal{P}[\cdot]$ denotes the sum of all distinct expressions resulting from permutations of the variables appearing in its argument.

Proposition 4.6. *The estimates*

$$\|m_j(\eta_1,u_2)\|_0 \leqslant c(\|\eta_1\|_{1,\infty} + \|\eta_1'' + k_0^2\eta_1\|_0 + \|K^0\eta_1\|_{1,\infty})\|u_2\|_2, \quad j = 1,2,3,$$

and

$$|n_j(\eta_1,u_2,u_3)| \leqslant c(\|\eta_1\|_{1,\infty} + \|\eta_1'' + k_0^2\eta_1\|_0 + \|K^0\eta_1\|_{1,\infty})\|u_2\|_2\|u_3\|_2, \quad j = 1,2,3,$$

hold for each $\eta \in U$ and u_2, $u_3 \in H^2(\mathbb{R})$.

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4.1.5. *Formulae for the functionals $\mathcal{M}_\mu$ and $\tilde{\mathcal{M}}_\mu$*

LEMMA 4.7. *The estimates*

$$\begin{aligned}
\mathcal{M}_\mu(\eta) &= \mathcal{K}_3(\eta) + 2\nu_0\mathcal{G}_3(\eta) - \nu_0^2\mathcal{L}_3(\eta) + \mathcal{K}_4(\eta) + 2\nu_0\mathcal{G}_4(\eta) - \nu_0^2\mathcal{L}_4(\eta) \\
&\quad + 2\biggl(\frac{\mu + \mathcal{G}_2(\eta)}{\mathcal{L}_2(\eta)} - \nu_0\biggr)(\mathcal{G}_3(\eta) + \mathcal{G}_4(\eta)) \\
&\quad - \biggl(\frac{\mu + \mathcal{G}_2(\eta)}{\mathcal{L}_2(\eta)} - \nu_0\biggr)\biggl(\frac{\mu + \mathcal{G}_2(\eta)}{\mathcal{L}_2(\eta)} + \nu_0\biggr)(\mathcal{L}_3(\eta) + \mathcal{L}_4(\eta)) \\
&\quad + \frac{1}{\mathcal{L}_2(\eta)}\biggl(\mathcal{G}_3(\eta) - \biggl(\frac{\mu + \mathcal{G}_2(\eta)}{\mathcal{L}_2(\eta)}\biggr)\mathcal{L}_3(\eta)\biggr)^2 \\
&\quad + O(\mu^{3/2}(\|\eta\|_{1,\infty} + \|\eta'' + k_0^2\eta\|_0)^2),
\end{aligned}$$

$$\begin{aligned}
&\langle \mathcal{M}_\mu(\eta), \eta\rangle + 4\mu\tilde{\mathcal{M}}_\mu(\eta) \\
&\quad = 3(\mathcal{K}_3(\eta) + 2\nu_0\mathcal{G}_3(\eta) - \nu_0^2\mathcal{L}_3(\eta)) + 4(\mathcal{K}_3(\eta) + 2\nu_0\mathcal{G}_3(\eta) - \nu_0^2\mathcal{L}_3(\eta)) \\
&\qquad + 2\biggl(\frac{\mu + \mathcal{G}_2(\eta)}{\mathcal{L}_2(\eta)} - \nu_0\biggr)(3\mathcal{G}_3(\eta) + 4\mathcal{G}_4(\eta)) \\
&\qquad - \biggl(\frac{\mu + \mathcal{G}_2(\eta)}{\mathcal{L}_2(\eta)} - \nu_0\biggr)\biggl(\frac{\mu + \mathcal{G}_2(\eta)}{\mathcal{L}_2(\eta)} + \nu_0\biggr)(3\mathcal{L}_3(\eta) + 4\mathcal{L}_4(\eta)) \\
&\qquad + \frac{4}{\mathcal{L}_2(\eta)}\biggl(\mathcal{G}_3(\eta) - \biggl(\frac{\mu + \mathcal{G}_2(\eta)}{\mathcal{L}_2(\eta)}\biggr)\mathcal{L}_3(\eta)\biggr)^2 \\
&\qquad + O(\mu^{3/2}(\|\eta\|_{1,\infty} + \|\eta'' + k_0^2\eta\|_0)^2)
\end{aligned}$$

and

$$\begin{aligned}
\tilde{\mathcal{M}}_\mu(\eta) &= \mu^{-1}(\mathcal{G}_3(\eta) + \mathcal{G}_4(\eta)) + \mu^{-1}\biggl(\frac{\mu + \mathcal{G}_2(\eta)}{\mathcal{L}_2(\eta)}\biggr)(\mathcal{L}_3(\eta) + \mathcal{L}_4(\eta)) \\
&\quad + O(\mu^{1/2}(\|\eta\|_{1,\infty} + \|\eta'' + k_0^2\eta\|_0)^2)
\end{aligned}$$

hold for each $\eta \in U$ with $\|\eta\|_2 \leqslant c\mu^{1/2}$ and $\mathcal{L}_2(\eta) > c\mu$.

Proof. Using the formulas

$$\mathcal{M}_\mu(\eta) = \mathcal{K}_{\mathrm{nl}}(\eta) + \frac{(\mu + \mathcal{G}(\eta))^2}{\mathcal{L}(\eta)} - \frac{(\mu + \mathcal{G}_2(\eta))^2}{\mathcal{L}_2(\eta)}$$

and

$$\frac{1}{\mathcal{L}(\eta)} = \frac{1}{\mathcal{L}_2(\eta)}\biggl(1 - \frac{\mathcal{L}_{\mathrm{nl}}(\eta)}{\mathcal{L}(\eta)}\biggr),$$

one finds that

$$\begin{aligned}
\mathcal{M}_\mu(\eta) &= \mathcal{K}_{\mathrm{nl}}(\eta) + 2\biggl(\frac{\mu + \mathcal{G}_2(\eta)}{\mathcal{L}_2(\eta)}\biggr)\mathcal{G}_{\mathrm{nl}}(\eta) - \biggl(\frac{\mu + \mathcal{G}_2(\eta)}{\mathcal{L}_2(\eta)}\biggr)^2\mathcal{L}_{\mathrm{nl}}(\eta) \\
&\quad + \frac{\mathcal{G}_{\mathrm{nl}}(\eta)^2}{\mathcal{L}(\eta)} - 2\biggl(\frac{\mu + \mathcal{G}_2(\eta)}{\mathcal{L}_2(\eta)}\biggr)\frac{\mathcal{G}_{\mathrm{nl}}(\eta)\mathcal{L}_{\mathrm{nl}}(\eta)}{\mathcal{L}(\eta)} + \biggl(\frac{\mu + \mathcal{G}_2(\eta)}{\mathcal{L}_2(\eta)}\biggr)^2\frac{\mathcal{L}_{\mathrm{nl}}(\eta)^2}{\mathcal{L}(\eta)}.
\end{aligned}$$

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We estimate the first line by substituting

$$\begin{Bmatrix} \mathcal{G}_{\rm nl}(\eta) \\ \mathcal{K}_{\rm nl}(\eta) \\ \mathcal{L}_{\rm nl}(\eta) \end{Bmatrix} = \begin{Bmatrix} \mathcal{G}_3(\eta)+\mathcal{G}_4(\eta) \\ \mathcal{K}_3(\eta)+\mathcal{K}_4(\eta) \\ \mathcal{L}_3(\eta)+\mathcal{L}_4(\eta) \end{Bmatrix} + O(\mu^{3/2}(\|\eta\|_{1,\infty}+\|\eta''+k_0^2\eta\|_0)^2)$$

(see proposition 4.4) and

$$\frac{\mu+\mathcal{G}_2(\eta)}{\mathcal{L}_2(\eta)} = O(1).$$

Writing

$$\mathcal{G}_{\rm nl}(\eta) = \mathcal{G}_3(\eta) + O(\mu(\|\eta\|_{1,\infty}+\|\eta''+k_0^2\eta\|_0)^2)$$

(see proposition 4.4) and estimating

$$\mathcal{G}_3(\eta) = O(\|\eta\|_\infty\|\eta\|_2^2) = O(\mu\|\eta\|_\infty)$$

(using the formula for $\mathcal{G}_3(\eta)$ given in proposition 4.3) yields

$$\mathcal{G}_{\rm nl}(\eta)^2 = \mathcal{G}_3(\eta)^2 + O(\mu^2(\|\eta\|_{1,\infty}+\|\eta''+k_0^2\eta\|_0)^3)$$

and

$$\frac{\mathcal{L}_{\rm nl}(\eta)\mathcal{G}_3(\eta)^2}{\mathcal{L}_2(\eta)\mathcal{L}(\eta)} = O(\mu^2(\|\eta\|_{1,\infty}+\|\eta''+k_0^2\eta\|_0)^3)$$

(recall that $\mathcal{L}(\eta) \geqslant c\mathcal{L}_2(\eta)$ for $\eta\in U$), so that

$$\frac{\mathcal{G}_{\rm nl}(\eta)^2}{\mathcal{L}(\eta)} = \frac{\mathcal{G}_3(\eta)^2}{\mathcal{L}_2(\eta)} + O(\mu^{3/2}(\|\eta\|_{1,\infty}+\|\eta''+k_0^2\eta\|_0)^2);$$

the remaining terms on the second line are estimated in the same fashion.

Altogether we find that

$$\begin{aligned}\mathcal{M}_\mu(\eta) = \mathcal{K}_3(\eta) &+ 2\left(\frac{\mu+\mathcal{G}_2(\eta)}{\mathcal{L}_2(\eta)}\right)\mathcal{G}_3(\eta) - \left(\frac{\mu+\mathcal{G}_2(\eta)}{\mathcal{L}_2(\eta)}\right)^2\mathcal{L}_3(\eta) \\ &+ \mathcal{K}_4(\eta) + 2\left(\frac{\mu+\mathcal{G}_2(\eta)}{\mathcal{L}_2(\eta)}\right)\mathcal{G}_4(\eta) - \left(\frac{\mu+\mathcal{G}_2(\eta)}{\mathcal{L}_2(\eta)}\right)^2\mathcal{L}_4(\eta) \\ &+ \frac{1}{\mathcal{L}_2(\eta)}\left(\mathcal{G}_3(\eta) - \mathcal{L}_3(\eta)\left(\frac{\mu+\mathcal{G}_2(\eta)}{\mathcal{L}_2(\eta)}\right)\right)^2 \\ &+ O(\mu^{3/2}(\|\eta\|_{1,\infty}+\|\eta''+k_0^2\eta\|_0)^2),\end{aligned}$$

from which the stated formula for $\mathcal{M}_\mu(\eta)$ follows by an algebraic manipulation.

The other estimates are derived by similar calculations. □

4.2. The case $\beta > \beta_c$

We begin by estimating the wave speed.

Proposition 4.8. *The function $\tilde{\eta}$ satisfies the estimates*

$$\begin{Bmatrix} \left|\dfrac{\mu+\mathcal{G}(\tilde\eta)}{\mathcal{L}(\tilde\eta)} - \nu_0\right| \\ \left|\dfrac{\mu+\mathcal{G}_2(\tilde\eta)}{\mathcal{L}_2(\tilde\eta)} - \nu_0\right| \end{Bmatrix} \leqslant c(\|\tilde\eta\|_{1,\infty} + \|\tilde\eta''\|_0 + \mu^{N-1/2}).$$

Proof. Proposition 4.4 implies that

$$\left.\begin{cases} |\mathcal{G}_j(\tilde{\eta})| \\ |\mathcal{K}_j(\tilde{\eta})| \\ |\mathcal{L}_j(\tilde{\eta})| \end{cases}\right\} \leqslant c\mu(\|\tilde{\eta}\|_{1,\infty} + \|\tilde{\eta}''\|_0), \quad j = 3, 4,$$

and lemma 4.7 shows that

$$\begin{gathered} |\mathcal{M}_\mu(\tilde{\eta})|, |\langle \mathcal{M}_\mu'(\tilde{\eta}), \tilde{\eta}\rangle + 4\mu\tilde{\mathcal{M}}_\mu(\tilde{\eta})| \leqslant c\mu(\|\tilde{\eta}\|_{1,\infty} + \|\tilde{\eta}''\|_0), \\ |\tilde{\mathcal{M}}_\mu(\tilde{\eta})| \leqslant c(\|\tilde{\eta}\|_{1,\infty} + \|\tilde{\eta}''\|_0). \end{gathered}$$

The results are obtained by combining these estimates with proposition 4.2. □

Corollary 4.9. *The quantity*

$$\begin{aligned} \mathcal{S}(\tilde{\eta}) = \mathcal{J}_\mu'(\tilde{\eta}) - \mathcal{K}_{\mathrm{nl}}'(\tilde{\eta}) - 2\left(\frac{\mu + \mathcal{G}(\tilde{\eta})}{\mathcal{L}(\tilde{\eta})} - \nu_0\right)\mathcal{G}_2'(\tilde{\eta}) - 2\left(\frac{\mu + \mathcal{G}(\tilde{\eta})}{\mathcal{L}(\tilde{\eta})}\right)\mathcal{G}_{\mathrm{nl}}'(\tilde{\eta}) \\ + \left(\frac{\mu + \mathcal{G}(\tilde{\eta})}{\mathcal{L}(\tilde{\eta})} + \nu_0\right)\left(\frac{\mu + \mathcal{G}(\tilde{\eta})}{\mathcal{L}(\tilde{\eta})} - \nu_0\right)\mathcal{L}_2'(\tilde{\eta}) + \left(\frac{\mu + \mathcal{G}(\tilde{\eta})}{\mathcal{L}(\tilde{\eta})}\right)^2\mathcal{L}_{\mathrm{nl}}'(\tilde{\eta}) \end{aligned}$$

satisfies

$$\|\mathcal{S}(\tilde{\eta})\|_0 \leqslant c(\mu^{1/2}(\|\tilde{\eta}\|_{1,\infty} + \|\tilde{\eta}''\|_0 + \|K^0\tilde{\eta}\|_\infty) + \mu^N).$$

The next step is an estimate for $|||\tilde{\eta}_1|||_\alpha$ and $\|\tilde{\eta}_2\|_2$.

Lemma 4.10. *The function $\tilde{\eta}$ satisfies $|||\tilde{\eta}_1|||_\alpha^2 \leqslant c\mu$ and $\|\tilde{\eta}_2\|_2^2 \leqslant c\mu^{2+\alpha}$ for $\alpha < \frac{1}{3}$.*

Proof. Using the equations

$$g(k)\hat{\tilde{\eta}}_1 = \mathcal{F}[\mathcal{S}(\eta)], \qquad \tilde{\eta}_2 = \mathcal{F}^{-1}\left[\frac{1 - \chi_S(k)}{g(k)}\mathcal{F}[\mathcal{S}(\tilde{\eta})]\right],$$

we find from corollary 4.9 that

$$\|\tilde{\eta}_2\|_2 \leqslant c(\mu^{1/2}(\|\tilde{\eta}_1\|_{1,\infty} + \|\tilde{\eta}_1''\|_0 + \|K^0\tilde{\eta}_1\|_\infty) + \mu^{1/2}\|\tilde{\eta}_2\|_2 + \mu^N),$$

and therefore

$$\|\tilde{\eta}_2\|_2 \leqslant c(\mu^{1/2}(\|\tilde{\eta}_1\|_{1,\infty} + \|\tilde{\eta}_1''\|_0 + \|K^0\tilde{\eta}_1\|_\infty) + \mu^N), \tag{4.13}$$

and

$$\int_{-\infty}^{\infty} g(k)^2|\hat{\tilde{\eta}}_1(k)|^2\,\mathrm{d}k \leqslant c(\mu(\|\tilde{\eta}_1\|_{1,\infty} + \|\tilde{\eta}_1''\|_0 + \|K^0\tilde{\eta}_1\|_\infty)^2 + \mu\|\tilde{\eta}_2\|_2^2 + \mu^{2N})$$

(see proposition 4.1). Multiplying the above inequality by $\mu^{-4\alpha}$, using (4.13) and adding $\|\tilde{\eta}_1\|_0^2 \leqslant \|\tilde{\eta}\|_0^2 \leqslant c\mu$, one finds that

$$\begin{aligned} |||\tilde{\eta}|||_\alpha^2 &\leqslant c(\mu^{1-4\alpha}(\|\tilde{\eta}_1\|_{1,\infty} + \|\tilde{\eta}_1''\|_0 + \|K^0\tilde{\eta}_1\|_\infty)^2 + \mu) \\ &\leqslant c(\mu^{1-3\alpha}|||\tilde{\eta}|||_\alpha^2 + \mu), \end{aligned} \tag{4.14}$$

so that $|||\tilde{\eta}|||_\alpha^2 \leqslant c\mu$ for $\alpha < \frac{1}{3}$. The estimate for $\tilde{\eta}_2$ follows from inequality (4.13). □

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It remains to identify the dominant terms in the formulas for

$$\mathcal{M}_\mu(\tilde{\eta}) \quad \text{and} \quad \langle \mathcal{M}'_\mu(\tilde{\eta}), \tilde{\eta}\rangle + 4\mu\tilde{\mathcal{M}}_\mu(\tilde{\eta})$$

given in lemma 4.7; this task is accomplished by combining the estimates in propositions 4.11 and 4.12 and lemma 4.13.

Proposition 4.11. *The function $\tilde{\eta}$ satisfies the estimate*

$$\begin{Bmatrix} \mathcal{G}_3(\tilde{\eta}) \\ \mathcal{K}_3(\tilde{\eta}) \\ \mathcal{L}_3(\tilde{\eta}) \end{Bmatrix} = \begin{Bmatrix} \mathcal{G}_3(\tilde{\eta}_1) \\ \mathcal{K}_3(\tilde{\eta}_1) \\ \mathcal{L}_3(\tilde{\eta}_1) \end{Bmatrix} + o(\mu^{5/3}).$$

Proof. Using proposition 4.6, we find that

$$\begin{aligned} \left| n_j\left(\tilde{\eta}_1, \begin{Bmatrix} \tilde{\eta}_1 \\ \tilde{\eta}_2 \end{Bmatrix}, \tilde{\eta}_2\right)\right| &\leqslant c\mu^{\alpha/2} |||\tilde{\eta}_1|||_\alpha \begin{Bmatrix} \|\tilde{\eta}_1\|_2 \\ \|\tilde{\eta}_2\|_2 \end{Bmatrix} \|\tilde{\eta}_2\|_2 \\ &\leqslant c\mu^{2+\alpha} \\ &= o(\mu^{5/3}), \end{aligned}$$

while

$$|n_j(\tilde{\eta}_2, \tilde{\eta}_2, \tilde{\eta}_2)| \leqslant c\|\tilde{\eta}_2\|_2^3 \leqslant c\mu^{3+3\alpha/2} = o(\mu^{5/3});$$

it follows that

$$n_j(\tilde{\eta}_1 + \tilde{\eta}_2, \tilde{\eta}_1 + \tilde{\eta}_2, \tilde{\eta}_1 + \tilde{\eta}_2) - n_j(\tilde{\eta}_1, \tilde{\eta}_1, \tilde{\eta}_1) = o(\mu^{5/3})$$

for $j = 1, 2, 3$. □

Proposition 4.12. *The function $\tilde{\eta}$ satisfies the estimate*

$$\mathcal{K}_3(\tilde{\eta}_1) + 2\nu_0\mathcal{G}_3(\tilde{\eta}_1) - \nu_0^2\mathcal{L}_3(\tilde{\eta}_1) = \tfrac{1}{2}(\tfrac{1}{3}\omega^2 + 1)\int_{-\infty}^{\infty} \tilde{\eta}_1^3 \, \mathrm{d}x + o(\mu^{5/3}).$$

Proof. Note that

$$\begin{aligned} \mathcal{G}_3(\tilde{\eta}_1) &= \tfrac{1}{4}\omega \int_{-\infty}^{\infty} \tilde{\eta}_1^3 \, \mathrm{d}x + \tfrac{1}{4}\omega \int_{-\infty}^{\infty} \tilde{\eta}_1^2 (K^0\tilde{\eta}_1 - \tilde{\eta}_1) \, \mathrm{d}x, \\ \mathcal{K}_3(\tilde{\eta}_1) &= \tfrac{1}{6}\omega^2 \int_{-\infty}^{\infty} \tilde{\eta}_1^3 \, \mathrm{d}x, \\ \mathcal{L}_3(\tilde{\eta}_1) &= -\frac{1}{2}\int_{-\infty}^{\infty} \tilde{\eta}_1^3 \, \mathrm{d}x - \int_{-\infty}^{\infty} (K^0\tilde{\eta}_1 - \tilde{\eta}_1)\tilde{\eta}_1^2 \, \mathrm{d}x \\ &\quad - \frac{1}{2}\int_{-\infty}^{\infty} (K^0\tilde{\eta}_1 - \tilde{\eta}_1)^2 \tilde{\eta}_1 \, \mathrm{d}x + \frac{1}{2}\int_{-\infty}^{\infty} \tilde{\eta}_1'^2 \tilde{\eta}_1 \, \mathrm{d}x \end{aligned}$$

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(see proposition 4.3) and estimate

$$\begin{aligned}
\left|\int_{-\infty}^{\infty} \tilde{\eta}_1'^2 \tilde{\eta}_1 \,\mathrm{d}x\right| &\leqslant \|\tilde{\eta}_1\|_\infty \|\tilde{\eta}_1'\|_0^2 \\
&\leqslant c\mu^{5\alpha/2} |||\tilde{\eta}_1|||_\alpha^3 \\
&\leqslant c\mu^{3/2+5\alpha/2} \\
&= o(\mu^{5/3}), \\
\left|\int_{-\infty}^{\infty} \tilde{\eta}_1^2 (K^0\tilde{\eta}_1 - \tilde{\eta}_1) \,\mathrm{d}x\right| &\leqslant \|\tilde{\eta}_1\|_\infty \|\tilde{\eta}_1\|_0 \|K^0\tilde{\eta}_1 - \tilde{\eta}_1\|_0 \\
&\leqslant c\mu^{1/2+5\alpha/2} |||\tilde{\eta}_1|||_\alpha^2 \\
&\leqslant c\mu^{3/2+5\alpha/2} \\
&= o(\mu^{5/3}), \\
\left|\int_{-\infty}^{\infty} \tilde{\eta}_1 (K^0\tilde{\eta}_1 - \tilde{\eta}_1)^2 \,\mathrm{d}x\right| &\leqslant \|\tilde{\eta}_1\|_\infty \|K^0\tilde{\eta}_1 - \tilde{\eta}_1\|_0^2 \\
&\leqslant c\mu^{9\alpha/2} |||\tilde{\eta}_1|||_\alpha^3 \\
&\leqslant c\mu^{3/2+9\alpha/2} \\
&= o(\mu^{5/3}),
\end{aligned}$$

in which the calculation

$$\begin{aligned}
\|K^0\eta - \eta\|_0^2 &= \int_{-\infty}^{\infty} (|k| \coth|k| - 1)^2 |\hat{\eta}(k)|^2 \,\mathrm{d}k \\
&\leqslant c \int_{-\infty}^{\infty} k^4 |\hat{\eta}(k)|^2 \,\mathrm{d}k \\
&= c\|\eta''\|_0^2 \leqslant c\mu^{4\alpha} |||\eta|||_\alpha^2
\end{aligned}$$

for $\eta \in H^2(\mathbb{R})$ has been used. One concludes that

$$\mathcal{K}_3(\tilde{\eta}_1) + 2\nu_0 \mathcal{G}_3(\tilde{\eta}_1) - \nu_0^2 \mathcal{L}_3(\tilde{\eta}_1) = \tfrac{1}{2}(\tfrac{1}{3}\omega^2 + \underbrace{\omega\nu_0 + \nu_0^2}_{=1}) \int_{-\infty}^{\infty} \tilde{\eta}_1^3 \,\mathrm{d}x + o(\mu^{5/3}).$$

□

LEMMA 4.13. *The estimates*

$$\mathcal{M}_{a^2\mu}(a\tilde{\eta}) = a^3(\mathcal{K}_3(\tilde{\eta}) + 2\nu_0\mathcal{G}_3(\tilde{\eta}) - \nu_0^2\mathcal{L}_3(\tilde{\eta})) + a^3 o(\mu^{5/3}),$$
$$\langle \mathcal{M}_{a^2\mu}'(a\tilde{\eta}), a\tilde{\eta}\rangle + 4a^2\mu\tilde{\mathcal{M}}_{a^2\mu}(a\tilde{\eta}) = 3a^3(\mathcal{K}_3(\tilde{\eta}) + 2\nu_0\mathcal{G}_3(\tilde{\eta}) - \nu_0^2\mathcal{L}_3(\tilde{\eta})) + a^3 o(\mu^{5/3})$$

hold uniformly over $a \in [1, 2]$.

Proof. Using lemma 4.7, the estimates given in proposition 4.4 and

$$\frac{\mu + \mathcal{G}_2(\eta)}{\mathcal{L}_2(\eta)} = O(1),$$

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we find that

$$\begin{aligned}\mathcal{M}_{a^2\mu}(a\tilde{\eta}) = a^3\bigg[&\mathcal{K}_3(\tilde{\eta}) + 2\nu_0\mathcal{G}_3(\tilde{\eta}) - \nu_0^2\mathcal{L}_3(\tilde{\eta}) + 2\bigg(\frac{\mu+\mathcal{G}_2(\tilde{\eta})}{\mathcal{L}_2(\tilde{\eta})} - \nu_0\bigg)\mathcal{G}_3(\tilde{\eta}) \\ &- \bigg(\frac{\mu+\mathcal{G}_2(\tilde{\eta})}{\mathcal{L}_2(\tilde{\eta})} - \nu_0\bigg)\bigg(\frac{\mu+\mathcal{G}_2(\tilde{\eta})}{\mathcal{L}_2(\tilde{\eta})} + \nu_0\bigg)\mathcal{L}_3(\tilde{\eta})\bigg] \\ &+ O(a^4\mu^{3/2}(\|\tilde{\eta}\|_{1,\infty} + \|\tilde{\eta}''\|_0))\end{aligned}$$

uniformly over $a \in [1,2]$. The first result follows by estimating

$$\|\tilde{\eta}\|_{1,\infty} + \|\tilde{\eta}''\|_0 \leqslant c(\mu^{\alpha/2}|||\tilde{\eta}_1|||_\alpha + \|\tilde{\eta}_2\|_2) \leqslant c\mu^{1/2+\alpha/2},$$

$$\frac{\mu+\mathcal{G}_2(\tilde{\eta})}{\mathcal{L}_2(\tilde{\eta})} - \nu_0 = O(\mu^{1/2+\alpha/2}), \qquad \begin{Bmatrix}\mathcal{G}_3(\tilde{\eta}) \\ \mathcal{L}_3(\tilde{\eta})\end{Bmatrix} = O(\mu^{3/2})$$

and $a^4 \leqslant 2a^3$. The second result is derived in a similar fashion. □

Corollary 4.14. *The estimates*

$$\mathcal{M}_{a^2\mu}(a\tilde{\eta}) = \tfrac{1}{2}a^3(\tfrac{1}{3}\omega^2+1)\int_{-\infty}^{\infty}\tilde{\eta}_1^3\,\mathrm{d}x + a^3 o(\mu^{5/3}),$$

$$\langle\mathcal{M}'_{a^2\mu}(a\tilde{\eta}), a\tilde{\eta}\rangle + 4a^2\mu\tilde{\mathcal{M}}_{a^2\mu}(a\tilde{\eta}) = \tfrac{3}{2}a^3(\tfrac{1}{3}\omega^2+1)\int_{-\infty}^{\infty}\tilde{\eta}_1^3\,\mathrm{d}x + a^3 o(\mu^{5/3})$$

hold uniformly over $a \in [1,2]$ *and*

$$\int_{-\infty}^{\infty}\tilde{\eta}_1^3\,\mathrm{d}x \leqslant -c\mu^{5/3}.$$

Proof. The estimates follow from propositions 4.11 and 4.12 and lemma 4.13, while the inequality for $\tilde{\eta}$ is a consequence of the first estimate (with $a = 1$) and the fact that $\mathcal{M}_\mu(\tilde{\eta}) \leqslant -c\mu^{5/3}$. □

4.3. The case $\boldsymbol{\beta < \beta_c}$

4.3.1. *Estimates for near minimizers*

We begin with an observation that shows that the equation for η_1 may be written as

$$g(k)\hat{\eta}_1 = \chi_S(k)\mathcal{F}[\mathcal{S}(\eta)], \tag{4.15}$$

where

$$\begin{aligned}\mathcal{S}(\eta) = \mathcal{J}'_\mu(\eta) - \mathcal{K}'_{\mathrm{nl}}(\eta) + \mathcal{K}_3(\eta_1) &- 2\bigg(\frac{\mu+\mathcal{G}(\eta)}{\mathcal{L}(\eta)} - \nu_0\bigg)\mathcal{G}'_2(\eta) \\ &- 2\bigg(\frac{\mu+\mathcal{G}(\eta)}{\mathcal{L}(\eta)}\bigg)(\mathcal{G}'_{\mathrm{nl}}(\eta) - \mathcal{G}'_3(\eta)) \\ &+ \bigg(\frac{\mu+\mathcal{G}(\eta)}{\mathcal{L}(\eta)} + \nu_0\bigg)\bigg(\frac{\mu+\mathcal{G}(\eta)}{\mathcal{L}(\eta)} - \nu_0\bigg)\mathcal{L}'_2(\eta) \\ &+ \bigg(\frac{\mu+\mathcal{G}(\eta)}{\mathcal{L}(\eta)}\bigg)^2(\mathcal{L}'_{\mathrm{nl}}(\eta) - \mathcal{L}'_3(\eta)).\end{aligned}$$

Proposition 4.15. *The identity*

$$\chi_S \mathcal{F}\left[\begin{Bmatrix} \mathcal{G}_3'(\eta_1) \\ \mathcal{K}_3'(\eta_1) \\ \mathcal{L}_3'(\eta_1) \end{Bmatrix}\right] = 0$$

holds for each $\eta \in U$.

Proof. Using (4.10)–(4.12) we find that the supports of $\mathcal{G}_3'(\eta_1)$, $\mathcal{K}_3'(\eta_1)$ and $\mathcal{L}_3'(\eta_1)$ lie in the set $[-2k_0 - 2\delta_0, -2k_0 + 2\delta_0] \cup [-2\delta_0, 2\delta_0] \cup [2k_0 - 2\delta_0, 2k_0 + 2\delta_0]$. □

In keeping with (4.15), we write the equation for η_2 in the form

$$\underbrace{\eta_2 + H(\eta)}_{:=\eta_3} = \mathcal{F}^{-1}\left[\frac{1-\chi_S(k)}{g(k)}\mathcal{F}[\mathcal{S}(\eta)]\right],$$

where

$$H(\eta) = \mathcal{F}^{-1}\left[\frac{1}{g(k)}\mathcal{F}\left[\mathcal{K}_3'(\eta_1) + 2\left(\frac{\mu + \mathcal{G}(\eta)}{\mathcal{L}(\eta)}\right)\mathcal{G}_3'(\eta_1) - \left(\frac{\mu + \mathcal{G}(\eta)}{\mathcal{L}(\eta)}\right)^2 \mathcal{L}_3'(\eta_1)\right]\right]; \tag{4.16}$$

the decomposition $\eta = \eta_1 - H(\eta) + \eta_3$ forms the basis of the calculations presented below. An estimate on the size of $H(\eta)$ is obtained from (4.16) and proposition 4.6.

Proposition 4.16. *The estimate*

$$\|H(\eta)\|_2 \leqslant c(\|\eta_1\|_{1,\infty} + \|\eta_1'' + k_0^2\eta_1\|_0 + \|K^0\eta_1\|_{1,\infty} + \|\eta_3\|_2)\|\eta_1\|_2$$

holds for each $\eta \in U$.

The above results may be used to derive estimates for the gradients of the cubic parts of the functionals that are used in the analysis below.

Proposition 4.17. *The function* $\tilde{\eta}$ *satisfies the estimates*

$$\begin{Bmatrix} \|\mathcal{G}_3'(\tilde{\eta}) - \mathcal{G}_3'(\tilde{\eta}_1)\|_0 \\ \|\mathcal{K}_3'(\tilde{\eta}) - \mathcal{K}_3'(\tilde{\eta}_1)\|_0 \\ \|\mathcal{L}_3'(\tilde{\eta}) - \mathcal{L}_3'(\tilde{\eta}_1)\|_0 \end{Bmatrix} \leqslant c\mu^{1/2}((\|\tilde{\eta}_1\|_{1,\infty} + \|\tilde{\eta}_1'' + k_0^2\tilde{\eta}_1\|_0 + \|K^0\tilde{\eta}_1\|_{1,\infty})^2 + \|\tilde{\eta}_3\|_2).$$

Proof. Observe that

$$\begin{aligned}\mathcal{G}_3'(\eta) - \mathcal{G}_3'(\eta_1) &= m_2(H(\eta), H(\eta)) + m_2(\eta_3, \eta_3) \\ &\quad - 2m_2(\eta_1, H(\eta)) - 2m_2(\eta_3, H(\eta)) + 2m_2(\eta_1, \eta_3)\end{aligned}$$

and estimate the right-hand side of this equation using propositions 4.6 and 4.16. The same method yields the results for $\mathcal{K}_3'$ and $\mathcal{L}_3'$. □

Estimates for $\mathcal{G}_3(\tilde{\eta})$, $\mathcal{K}_3(\tilde{\eta})$ and $\mathcal{L}_3(\tilde{\eta})$ are obtained in a similar fashion.

Proposition 4.18. *The function* $\tilde{\eta}$ *satisfies the estimates*

$$\begin{Bmatrix} |\mathcal{G}_3(\tilde{\eta})| \\ |\mathcal{K}_3(\tilde{\eta})| \\ |\mathcal{L}_3(\tilde{\eta})| \end{Bmatrix} \leqslant c(\mu(\|\tilde{\eta}_1\|_{1,\infty} + \|\tilde{\eta}_1'' + k_0^2\tilde{\eta}_1\|_0 + \|K^0\tilde{\eta}_1\|_{1,\infty}) + \mu\|\tilde{\eta}_3\|_2).$$

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Proof. Observe that

$$\begin{aligned}\mathcal{G}_3(\eta_1) = \tfrac{1}{3}\langle \mathcal{G}_3'(\eta_1), \eta_1\rangle &= \frac{1}{3}\int_{-\infty}^{\infty} \mathcal{F}[\mathcal{G}_3'(\eta_1)]\overline{\hat{\eta}_1}\,\mathrm{d}k \\ &= \frac{1}{3}\int_{-\infty}^{\infty} \underbrace{\chi_S(k)\mathcal{F}[\mathcal{G}_3'(\eta_1)]}_{=0}\overline{\hat{\eta}_1}\,\mathrm{d}k \\ &= 0\end{aligned}$$

(since $\hat{\eta}_1 = \chi_S(k)\hat{\eta}_1$), so that

$$\begin{aligned}\mathcal{G}_3(\eta) &= \mathcal{G}_3(\eta) - \mathcal{G}_3(\eta_1) \\ &= -n_2(H(\eta), H(\eta), H(\eta)) + n_2(\eta_3, \eta_3, \eta_3) - 6n_2(\eta_1, H(\eta), \eta_3) \\ &\quad - 3n_2(\eta_1, \eta_1, H(\eta)) + 3n_2(\eta_1, \eta_1, \eta_3) + 3n_2(H(\eta), H(\eta), \eta_3) \\ &\quad + 3n_2(H(\eta), H(\eta), \eta_1) + 3n_2(\eta_3, \eta_3, \eta_1) - 3n_2(\eta_3, \eta_3, H(\eta))\end{aligned}$$

and estimate the right-hand side of this equation using propositions 4.6 and 4.16. The same method yields the results for $\mathcal{K}_3$ and $\mathcal{L}_3$. □

Estimating the right-hand sides of the inequalities

$$\begin{aligned}\|\mathcal{G}_{\mathrm{nl}}'(\tilde{\eta}) - \mathcal{G}_3'(\tilde{\eta}_1)\|_0 &\leqslant \|\mathcal{G}_{\mathrm{r}}'(\tilde{\eta})\|_0 + \|\mathcal{G}_4'(\tilde{\eta})\|_0 + \|\mathcal{G}_3'(\tilde{\eta}) - \mathcal{G}_3'(\tilde{\eta}_1)\|_0, \\ |\mathcal{G}_{\mathrm{nl}}(\tilde{\eta})| &\leqslant |\mathcal{G}_{\mathrm{r}}(\tilde{\eta})| + |\mathcal{G}_4(\tilde{\eta})| + |\mathcal{G}_3(\tilde{\eta})|\end{aligned}$$

(together with the corresponding inequalities for $\mathcal{K}$ and $\mathcal{L}$) using propositions 4.4 and 4.5, the calculation

$$\begin{aligned}&\|\eta\|_{1,\infty} + \|\eta'' + k_0^2\eta\|_0 + \|K^0\eta\|_\infty \\ &\quad\leqslant c(\|\eta_1\|_{1,\infty} + \|\eta_1'' + k_0^2\eta_1\|_0 + \|K^0\eta_1\|_\infty + \|H(\eta)\|_2 + \|\eta_3\|_2) \\ &\quad\leqslant c(\|\eta_1\|_{1,\infty} + \|\eta_1'' + k_0^2\eta_1\|_0 + \|K^0\eta_1\|_{1,\infty} + \|\eta_3\|_2). \qquad (4.17)\end{aligned}$$

and propositions 4.17 and 4.18 yields the following estimates for the 'nonlinear' parts of the functionals.

LEMMA 4.19. *The function $\tilde{\eta}$ satisfies the estimates*

$$\begin{aligned}&\left\{\begin{array}{c}\|\mathcal{G}_{\mathrm{nl}}'(\tilde{\eta}) - \mathcal{G}_3'(\tilde{\eta}_1)\|_0 \\ \|\mathcal{K}_{\mathrm{nl}}'(\tilde{\eta}) - \mathcal{K}_3'(\tilde{\eta}_1)\|_0 \\ \|\mathcal{L}_{\mathrm{nl}}'(\tilde{\eta}) - \mathcal{L}_3'(\tilde{\eta}_1)\|_0\end{array}\right\} \\ &\qquad\qquad \leqslant c(\mu^{1/2}(\|\tilde{\eta}_1\|_{1,\infty} + \|\tilde{\eta}_1'' + k_0^2\tilde{\eta}_1\|_0 + \|K^0\tilde{\eta}_1\|_{1,\infty})^2 + \mu^{1/2}\|\tilde{\eta}_3\|_2), \\ &\left\{\begin{array}{c}|\mathcal{G}_{\mathrm{nl}}(\tilde{\eta})| \\ |\mathcal{K}_{\mathrm{nl}}(\tilde{\eta})| \\ |\mathcal{L}_{\mathrm{nl}}(\tilde{\eta})|\end{array}\right\} \leqslant c(\mu(\|\tilde{\eta}_1\|_{1,\infty} + \|\tilde{\eta}_1'' + k_0^2\tilde{\eta}_1\|_0 + \|K^0\tilde{\eta}_1\|_{1,\infty})^2 + \mu\|\tilde{\eta}_3\|_2).\end{aligned}$$

We now have all the ingredients necessary to estimate the wave speed and the quantity $|||\tilde{\eta}_1|||_\alpha$.

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PROPOSITION 4.20. *The function $\tilde{\eta}$ satisfies the estimates*

$$\left\{\begin{array}{l}\left|\dfrac{\mu+\mathcal{G}(\tilde{\eta})}{\mathcal{L}(\tilde{\eta})}-\nu_0\right| \\ \left|\dfrac{\mu+\mathcal{G}_2(\tilde{\eta})}{\mathcal{L}_2(\tilde{\eta})}-\nu_0\right|\end{array}\right\} \leqslant c((\|\tilde{\eta}_1\|_{1,\infty}+\|\tilde{\eta}_1''+k_0^2\tilde{\eta}_1\|_0+\|K^0\tilde{\eta}_1\|_{1,\infty})^2+\|\tilde{\eta}_3\|_2+\mu^{N-1/2}).$$

Proof. Combining lemma 4.7, inequality (4.17) and lemma 4.19, one finds that

$$\begin{aligned}&|\mathcal{M}(\tilde{\eta})|, |\langle\mathcal{M}'(\tilde{\eta}),\tilde{\eta}\rangle+4\mu\tilde{\mathcal{M}}_\mu(\tilde{\eta})| \\ &\qquad\qquad \leqslant c(\mu(\|\tilde{\eta}_1\|_{1,\infty}+\|\tilde{\eta}_1''+k_0^2\tilde{\eta}_1\|_0)^2+\|K^0\tilde{\eta}_1\|_{1,\infty})^2+\mu\|\tilde{\eta}_3\|_2), \\ &|\tilde{\mathcal{M}}_\mu(\tilde{\eta})| \leqslant c((\|\tilde{\eta}_1\|_{1,\infty}+\|\tilde{\eta}_1''+k_0^2\tilde{\eta}_1\|_0)^2+\|K^0\tilde{\eta}_1\|_{1,\infty})^2+\|\tilde{\eta}_3\|_2),\end{aligned}$$

from which the given estimates follow by proposition 4.2. □

LEMMA 4.21. *The function $\tilde{\eta}$ satisfies $|||\tilde{\eta}_1|||_\alpha^2 \leqslant c\mu$, $\|\tilde{\eta}_3\|_2^2 \leqslant c\mu^{3+2\alpha}$ and $\|H(\tilde{\eta})\|_2^2 \leqslant c\mu^{2+\alpha}$ for $\alpha<1$.*

Proof. Lemma 4.19 and proposition 4.20 assert that

$$\|\mathcal{S}(\tilde{\eta})\|_0 \leqslant c(\mu^{1/2}(\|\tilde{\eta}_1\|_{1,\infty}+\|\tilde{\eta}_1''+k_0^2\tilde{\eta}_1\|_0+\|K^0\tilde{\eta}_1\|_{1,\infty})^2+\mu^{1/2}\|\tilde{\eta}_3\|_2+\mu^N),$$

whereby

$$\|\tilde{\eta}_3\|_2 \leqslant c(\mu^{1/2}(\|\tilde{\eta}_1\|_{1,\infty}+\|\tilde{\eta}_1''+k_0^2\tilde{\eta}_1\|_0+\|K^0\tilde{\eta}_1\|_{1,\infty})^2+\mu^{1/2}\|\tilde{\eta}_3\|_2+\mu^N),$$

and therefore

$$\|\tilde{\eta}_3\|_2 \leqslant c(\mu^{1/2}(\|\tilde{\eta}_1\|_{1,\infty}+\|\tilde{\eta}_1''+k_0^2\tilde{\eta}_1\|_0+\|K^0\tilde{\eta}_1\|_{1,\infty})^2+\mu^N) \tag{4.18}$$

and

$$\begin{aligned}\int_{-\infty}^{\infty} g(k)^2|\hat{\tilde{\eta}}_1|^2\,\mathrm{d}k &\leqslant c(\mu(\|\tilde{\eta}_1\|_{1,\infty}+\|\tilde{\eta}_1''+k_0^2\tilde{\eta}_1\|_0+\|K^0\tilde{\eta}_1\|_{1,\infty})^4+\mu\|\tilde{\eta}_3\|_2^2+\mu^{2N}) \\ &\leqslant c(\mu(\|\tilde{\eta}_1\|_{1,\infty}+\|\tilde{\eta}_1''+k_0^2\tilde{\eta}_1\|_0+\|K^0\tilde{\eta}_1\|_{1,\infty})^4+\mu^{2N}).\end{aligned}$$

Multiplying the above inequality by $\mu^{-4\alpha}$ and adding $\|\tilde{\eta}_1\|_0^2 \leqslant \|\tilde{\eta}\|_0^2 \leqslant c\mu$, one finds that

$$\begin{aligned}|||\tilde{\eta}_1|||_\alpha^2 &\leqslant c(\mu^{1-4\alpha}(\|\tilde{\eta}_1\|_{1,\infty}+\|\tilde{\eta}_1''+k_0^2\tilde{\eta}_1\|_0+\|K^0\tilde{\eta}_1\|_{1,\infty})^4+\mu) \\ &\leqslant c(\mu^{1-2\alpha}|||\tilde{\eta}_1|||_\alpha^4+\mu),\end{aligned} \tag{4.19}$$

where proposition 4.1 and the fact that $g(k) \geqslant c(|k|-k_0)^2$ for $k \in S$ have also been used.

The estimate for $\tilde{\eta}_1$ follows from the previous inequality using the argument given by Groves and Wahlén [17, p. 401], while those for $\tilde{\eta}_3$ and $H(\tilde{\eta})$ are derived by estimating $|||\tilde{\eta}_1|||_\alpha^2 \leqslant c\mu$ in (4.18) and proposition 4.16. □

4.3.2. *Estimates for the variational functional*

The next step is to identify the dominant terms in the formulas for $\mathcal{M}_\mu(\tilde{\eta})$ and $\langle \mathcal{M}'_\mu(\tilde{\eta}), \tilde{\eta}\rangle + 4\mu\tilde{\mathcal{M}}_\mu(\tilde{\eta})$ given in lemma 4.7. We begin by examining the quantities $\mathcal{G}_4(\tilde{\eta})$, $\mathcal{K}_4(\tilde{\eta})$ and $\mathcal{L}_4(\tilde{\eta})$.

PROPOSITION 4.22. *The function $\tilde{\eta}$ satisfies the estimates*

$$\begin{Bmatrix} \mathcal{G}_4(\tilde{\eta}) \\ \mathcal{K}_4(\tilde{\eta}) \\ \mathcal{L}_4(\tilde{\eta}) \end{Bmatrix} = \begin{Bmatrix} \mathcal{G}_4(\tilde{\eta}_1) \\ \mathcal{K}_4(\tilde{\eta}_1) \\ \mathcal{L}_4(\tilde{\eta}_1) \end{Bmatrix} + o(\mu^3).$$

Proof. Write

$$\mathcal{K}_4(\eta) = p_1(\eta,\eta,\eta,\eta), \qquad \mathcal{G}_4(\eta) = p_2(\eta,\eta,\eta,\eta), \qquad \mathcal{L}_4(\eta) = p_3(\eta,\eta,\eta,\eta),$$

where $p_j \in \mathcal{L}^4_{\mathrm{s}}(H^2(\mathbb{R}),\mathbb{R})$, $j=1,2,3$, are defined by

$$\begin{aligned}
p_1(u_1,u_2,u_3,u_4) &= -\frac{1}{8}\int_{-\infty}^{\infty} u_1'u_2'u_3'u_4'\,\mathrm{d}x - \tfrac{1}{48}\omega^2\int_{-\infty}^{\infty}\mathcal{P}[u_1u_2K^0(u_3u_4)]\,\mathrm{d}x,\\
p_2(u_1,u_2,u_3,u_4) &= \tfrac{1}{12}\omega\int_{-\infty}^{\infty}\mathcal{P}[u_1u_2u_3'u_4']\,\mathrm{d}x - \tfrac{1}{48}\omega\int_{-\infty}^{\infty}\mathcal{P}[u_1u_2K^0(u_3K^0u_4)]\,\mathrm{d}x,\\
p_3(u_1,u_2,u_3,u_4) &= \frac{1}{24}\int_{-\infty}^{\infty}\mathcal{P}[u_1u_2(K^0u_3)u_4'']\,\mathrm{d}x\\
&\qquad + \frac{1}{48}\int_{-\infty}^{\infty}\mathcal{P}[K^0(u_1K^0u_2)u_3K^0u_4]\,\mathrm{d}x,
\end{aligned}$$

and estimate each term in the expansion of

$$p_j(\tilde{\eta}_1 - H(\tilde{\eta}) + \tilde{\eta}_3, \tilde{\eta}_1 - H(\tilde{\eta}) + \tilde{\eta}_3, \tilde{\eta}_1 - H(\tilde{\eta}) + \tilde{\eta}_3, \tilde{\eta}_1 - H(\tilde{\eta}) + \tilde{\eta}_3) - p_j(\tilde{\eta}_1,\tilde{\eta}_1,\tilde{\eta}_1,\tilde{\eta}_1)$$

for $j=1,2,3$. Terms with zero, one or two occurrences of $\tilde{\eta}_1$ are estimated by

$$\left| p_j\left(\begin{Bmatrix} \tilde{\eta}_1 \\ H(\tilde{\eta}) \\ \tilde{\eta}_3 \end{Bmatrix}^{(2)}, \begin{Bmatrix} H(\tilde{\eta}) \\ \tilde{\eta}_3 \end{Bmatrix}^{(2)} \right) \right| \leqslant c \begin{Bmatrix} \|\tilde{\eta}_1\|_2 \\ \|H(\tilde{\eta})\|_2 \\ \|\tilde{\eta}_3\|_2 \end{Bmatrix}^2 \begin{Bmatrix} \|H(\tilde{\eta})\|_2 \\ \|\tilde{\eta}_3\|_2 \end{Bmatrix}^2 \leqslant c\mu\mu^{2+\alpha} = o(\mu^3),$$

while terms with three occurrences of $\tilde{\eta}_1$ are estimated by

$$\begin{aligned}
\left| p_j\left(\{\tilde{\eta}_1\}^{(3)}, \begin{Bmatrix} H(\tilde{\eta}) \\ \tilde{\eta}_3 \end{Bmatrix} \right) \right| &\leqslant c \begin{Bmatrix} \|\tilde{\eta}_1\|_\infty \\ \|K^0\tilde{\eta}_1\|_{1,\infty} \\ \|\tilde{\eta}_1''\|_0 \end{Bmatrix} \|\tilde{\eta}_1\|_2^2 \begin{Bmatrix} \|H(\tilde{\eta})\|_2 \\ \|\tilde{\eta}_3\|_2 \end{Bmatrix}\\
&\leqslant c\mu^{\alpha/2}|||\tilde{\eta}_1|||_\alpha \mu\mu^{1+\alpha/2}\\
&\leqslant c\mu^{5/2+\alpha}\\
&= o(\mu^3).
\end{aligned}$$

□

To identify the dominant terms in $\mathcal{G}_4(\tilde{\eta}_1)$, $\mathcal{K}_4(\tilde{\eta}_1)$ and $\mathcal{L}_4(\tilde{\eta}_1)$ we use the following result, which shows how Fourier-multiplier operators acting upon the function η_1,

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whose spectrum is concentrated near $k = \pm k_0$, may be approximated by multiplication by constants.

LEMMA 4.23. *For each $\eta \in H^2(\mathbb{R})$ with $\|\eta\|_2 \leqslant c\mu^{1/2}$ the quantities*

$$\eta_1^+ := \mathcal{F}^{-1}[\chi_{[0,\infty)}\hat{\eta}_1], \qquad \eta_1^- := \mathcal{F}^{-1}[\chi_{(-\infty,0]}\hat{\eta}_1] = \overline{\eta_1^+}$$

satisfy the estimates

(i) $\eta_1^{\pm\prime} = \pm \mathrm{i}k_0\eta_1^\pm + \underline{O}(\mu^{1/2+\alpha})$,

(ii) $K^0(\eta_1^\pm) = f(k_0)\eta_1^\pm + \underline{O}(\mu^{1/2+\alpha})$,

(iii) $((\eta_1^\pm)^2)' = \pm 2k_0\mathrm{i}(\eta_1^\pm)^2 + \underline{O}(\mu^{1+3\alpha/2})$,

(iv) $(\eta_1^+\eta_1^-)' = \underline{O}(\mu^{1+3\alpha/2})$,

(v) $K^0((\eta_1^\pm)^2) = f(2k_0)(\eta_1^\pm)^2 + \underline{O}(\mu^{1+3\alpha/2})$,

(vi) $K^0(\eta_1^+\eta_1^-) = \eta_1^+\eta_1^- + \underline{O}(\mu^{1+3\alpha/2})$,

(vii) $\mathcal{F}^{-1}[g(k)^{-1}\mathcal{F}[(\eta_1^\pm)^2]] = g(2k_0)(\eta_1^\pm)^2 + \underline{O}(\mu^{1+3\alpha/2})$,

(viii) $\mathcal{F}^{-1}[g(k)^{-1}\mathcal{F}[\eta_1^+\eta_1^-]] = g(0)^{-1}\eta_1^+\eta_1^- + \underline{O}(\mu^{1+3\alpha/2})$.

Here the symbol $\underline{O}(\mu^\gamma)$ denotes a quantity whose Fourier transform has compact support and whose $L^2(\mathbb{R})$-norm (and hence $H^s(\mathbb{R})$-norm for $s \geqslant 0$) is $O(\mu^\gamma)$.

Proof. Estimates (i) and (ii) follow from the calculations

$$\|(\mathrm{i}k \mp \mathrm{i}k_0)\hat{\eta}_1^\pm\|_0^2 = \|(|k| - k_0)\hat{\eta}_1\|_0^2, \qquad \|(K^0 - f(k_0))(\eta_1^\pm)\|_0^2 \leqslant c\|(|k| - k_0)\hat{\eta}_1\|_0^2$$

(because $f(k) = f(k_0) + O(|k| - k_0)$ for $k \in S$) and

$$\|(|k| - k_0)\hat{\eta}_1\|_0^2 \leqslant \frac{1}{2}\int_{-\infty}^{\infty}(\mu^{2\alpha} + \mu^{-2\alpha}(|k| - k_0)^4)|\hat{\eta}_1|^2\,\mathrm{d}k \leqslant c\mu^{2\alpha}|\!|\!|\eta_1|\!|\!|_\alpha^2 \leqslant c\mu^{1+2\alpha},$$

while (iii) and (iv) are obtained from the observations

$$\begin{aligned}\|(\partial_x \mp 2\mathrm{i}k_0)(\eta_1^\pm)^2\|_0 &= \|2((\partial_x \mp k_0\mathrm{i})\eta_1^\pm)\eta_1^\pm\|_0 \\ &\leqslant 2\|(\partial_x \mp \mathrm{i}k_0)\eta_1^\pm\|_0\|\eta_1^\pm\|_\infty \\ &\leqslant c\mu^{1/2+3\alpha/2}|\!|\!|\eta_1^\pm|\!|\!|_\alpha \\ &\leqslant c\mu^{1+3\alpha/2}\end{aligned}$$

and

$$\begin{aligned}\|(\eta_1^+\eta_1^-)'\|_0 &= \|((\partial_x - \mathrm{i}k_0)\eta_1^+)\eta_1^- + \eta_1^+((\partial_x + \mathrm{i}k_0)\eta_1^-)\|_0 \\ &\leqslant \|(\partial_x - \mathrm{i}k_0)\eta_1^+\|_0\|\eta_1^-\|_\infty + \|\eta_1^+\|_\infty\|(\partial_x + \mathrm{i}k_0)\eta_1^-\|_0 \\ &\leqslant c\mu^{1+3\alpha/2},\end{aligned}$$

in which proposition 4.1 has been used. Estimates (v) and (vi) are deduced from (iii) and (iv), respectively, by means of the inequalities

$$\|(K^0 - f(2k_0))(\eta_1^\pm)^2\|_0^2 \leqslant c\|(|k| - 2k_0)\mathcal{F}[(\eta_1^\pm)^2]\|_0^2 = \|(\mathrm{i}k \mp \mathrm{i}k_0)\mathcal{F}[(\eta_1^\pm)^2]\|_0^2$$

(because $f(k) = f(2k_0) + O(|k| - 2k_0)$ for $k \in 2S$) and

$$\|(K^0 - \underbrace{f(0)}_{=1})\eta_1^+\eta_1^-\|_0^2 \leqslant c\||k|\mathcal{F}[\eta_1^+\eta_1^-]\|_0^2 = \|\mathrm{i}k\mathcal{F}[\eta_1^+\eta_1^-]\|_0^2$$

(because $f(k) = f(0) + O(|k|)$ for $k \in [-2\delta_0, 2\delta_0]$), and (vii) and (viii) are deduced from (iii) and (iv) in the same fashion. □

Proposition 4.24. *The function $\tilde{\eta}_1$ satisfies the estimates*

$$\mathcal{K}_4(\tilde{\eta}_1) = A_4^1 \int_{-\infty}^{\infty} \tilde{\eta}_1^4 \,\mathrm{d}x + o(\mu^3), \qquad A_4^1 = -\tfrac{1}{8}\beta\omega k_0^4 - \tfrac{1}{24}\omega^2(f(2k_0) + 2),$$

$$\mathcal{G}_4(\tilde{\eta}_1) = A_4^2 \int_{-\infty}^{\infty} \tilde{\eta}_1^4 \,\mathrm{d}x + o(\mu^3), \qquad A_4^2 = \tfrac{1}{6}\omega k_0^2 - \tfrac{1}{12}\omega f(k_0)(f(2k_0) + 2),$$

$$\mathcal{L}_4(\tilde{\eta}_1) = A_4^3 \int_{-\infty}^{\infty} \tilde{\eta}_1^4 \,\mathrm{d}x + o(\mu^3), \qquad A_4^3 = \tfrac{1}{6}f(k_0)^2(f(2k_0) + 2) - \tfrac{1}{2}k_0^2 f(k_0).$$

Proof. Using the formulas given in lemma 4.23, we find that

$$\begin{aligned}\int_{-\infty}^{\infty} \tilde{\eta}_1^2\tilde{\eta}_1'^2 \,\mathrm{d}x &= \int_{-\infty}^{\infty} ((\tilde{\eta}_1^+)^2((\tilde{\eta}_1^-)')^2 + (\tilde{\eta}_1^-)^2((\tilde{\eta}_1^+)')^2 + 4\tilde{\eta}_1^+\tilde{\eta}_1^-(\tilde{\eta}_1^+)'(\tilde{\eta}_1^-)')\,\mathrm{d}x \\ &= 2k_0^2 \int_{-\infty}^{\infty} (\tilde{\eta}_1^+)^2(\tilde{\eta}_1^-)^2 \,\mathrm{d}x + o(\mu^3),\end{aligned}$$

and similarly

$$\begin{aligned}\int_{-\infty}^{\infty} K^0(\tilde{\eta}_1^2)\tilde{\eta}_1 K^0\tilde{\eta}_1 \,\mathrm{d}x &= (2f(2k_0)f(k_0) + 4f(k_0)) \\ &\qquad \times \int_{-\infty}^{\infty} (\tilde{\eta}_1^+)^2(\tilde{\eta}_1^-)^2 \,\mathrm{d}x + o(\mu^3), \\ \int_{-\infty}^{\infty} (\tilde{\eta}_1')^4 \,\mathrm{d}x &= 6k_0^4 \int_{-\infty}^{\infty} (\tilde{\eta}_1^+)^2(\tilde{\eta}_1^-)^2 \,\mathrm{d}x + o(\mu^3), \\ \int_{-\infty}^{\infty} \tilde{\eta}_1^2 K^0(\tilde{\eta}_1^2) \,\mathrm{d}x &= (2f(2k_0) + 4) \int_{-\infty}^{\infty} (\tilde{\eta}_1^+)^2(\tilde{\eta}_1^-)^2 \,\mathrm{d}x + o(\mu^3), \\ \int_{-\infty}^{\infty} K^0(\tilde{\eta}_1 K^0\tilde{\eta}_1)\tilde{\eta}_1 K^0\tilde{\eta}_1 \,\mathrm{d}x &= (2f(2k_0)f(k_0)^2 + 4f(k_0)^2) \\ &\qquad \times \int_{-\infty}^{\infty} (\tilde{\eta}_1^+)^2(\tilde{\eta}_1^-)^2 \,\mathrm{d}x + o(\mu^3), \\ \int_{-\infty}^{\infty} (K^0\tilde{\eta}_1)\tilde{\eta}_1^2\tilde{\eta}_1'' \,\mathrm{d}x &= -6k_0^2 f(k_0) \int_{-\infty}^{\infty} (\tilde{\eta}_1^+)^2(\tilde{\eta}_1^-)^2 \,\mathrm{d}x + o(\mu^3).\end{aligned}$$

The result is obtained by substituting the above expressions into the explicit formulas for $\mathcal{K}_4$, $\mathcal{G}_4$ and $\mathcal{L}_4$ given in proposition 4.3. □

Corollary 4.25. *The function $\tilde{\eta}$ satisfies the estimate*

$$\mathcal{K}_4(\tilde{\eta}) + 2\nu_0\mathcal{G}_4(\tilde{\eta}) - \nu_0^2\mathcal{L}_4(\tilde{\eta}) = A_4 \int_{-\infty}^{\infty} \tilde{\eta}_1^4 \,\mathrm{d}x + o(\mu^3),$$

where $A_4 = A_4^1 + 2\nu_0 A_4^2 - \nu_0^2 A_4^3$.

We now turn to the corresponding result for $\mathcal{G}_3(\tilde{\eta})$, $\mathcal{K}_3(\tilde{\eta})$ and $\mathcal{L}_3(\tilde{\eta})$.

PROPOSITION 4.26. *The function $\tilde{\eta}$ satisfies the estimate*

$$\begin{Bmatrix} \mathcal{G}_3(\tilde{\eta}) \\ \mathcal{K}_3(\tilde{\eta}) \\ \mathcal{L}_3(\tilde{\eta}) \end{Bmatrix} = -\int_{-\infty}^{\infty} \begin{Bmatrix} \mathcal{G}_3'(\tilde{\eta}_1) \\ \mathcal{K}_3'(\tilde{\eta}_1) \\ \mathcal{L}_3'(\tilde{\eta}_1) \end{Bmatrix} H(\tilde{\eta})\,\mathrm{d}x + o(\mu^3).$$

Proof. Each term in the expansion of

$$n_2(\tilde{\eta}_1 - H(\tilde{\eta}) + \tilde{\eta}_3, \tilde{\eta}_1 - H(\tilde{\eta}) + \tilde{\eta}_3, \tilde{\eta}_1 - H(\tilde{\eta}) + \tilde{\eta}_3)$$

with zero or one occurrence of $\tilde{\eta}_1$ can be estimated by

$$\left| n_2\left(\begin{Bmatrix} \tilde{\eta}_1 \\ H(\tilde{\eta}) \\ \tilde{\eta}_3 \end{Bmatrix}, \begin{Bmatrix} H(\tilde{\eta}) \\ \tilde{\eta}_3 \end{Bmatrix}^{(2)} \right) \right| \leqslant c \begin{Bmatrix} \|\tilde{\eta}_1\|_2 \\ \|H(\tilde{\eta})\|_2 \\ \|\tilde{\eta}_3\|_2 \end{Bmatrix} \begin{Bmatrix} \|H(\tilde{\eta})\|_2 \\ \|\tilde{\eta}_3\|_2 \end{Bmatrix}^2 \leqslant c\mu^{1/2}\mu^{2+\alpha} = o(\mu^3),$$

while

$$|n_2(\tilde{\eta}_1, \tilde{\eta}_1, \tilde{\eta}_3)| \leqslant c\|\tilde{\eta}\|_2^2 \|\tilde{\eta}_3\|_2 \leqslant c\mu\mu^{3/2+\alpha} = o(\mu^3)$$

and

$$n_2(\tilde{\eta}_1, \tilde{\eta}_1, \tilde{\eta}_1) = \mathcal{G}_3(\tilde{\eta}_1) = 0.$$

It follows that

$$\begin{aligned} \mathcal{G}_3(\tilde{\eta}) &= -3n_2(\tilde{\eta}_1, \tilde{\eta}_1, H(\tilde{\eta})) + o(\mu^3) \\ &= -\mathrm{d}\mathcal{G}_3[\tilde{\eta}_1](H(\tilde{\eta})) + o(\mu^3) \\ &= -\int_{-\infty}^{\infty} \mathcal{G}_3'(\tilde{\eta}_1) H(\tilde{\eta})\,\mathrm{d}x + o(\mu^3). \end{aligned}$$

The same argument yields the results for $\mathcal{K}_3(\tilde{\eta})$ and $\mathcal{L}_3(\tilde{\eta})$. □

PROPOSITION 4.27. *The function $\tilde{\eta}$ satisfies the estimate*

$$H(\tilde{\eta}) = \mathcal{F}^{-1}\left[\frac{1}{g(k)} \mathcal{F}[\mathcal{K}_3'(\tilde{\eta}_1) + 2\nu_0 \mathcal{G}_3'(\tilde{\eta}_1) - \nu_0^2 \mathcal{L}_3'(\tilde{\eta}_1)]\right] + \underline{o}(\mu^3).$$

Proof. Noting that

$$\left| \frac{\mu + \mathcal{G}(\tilde{\eta})}{\mathcal{L}(\eta)} - \nu_0 \right| \leqslant c(\mu^\alpha |\!|\!|\tilde{\eta}_1|\!|\!|_\alpha^2 + \|\tilde{\eta}_3\|_2 + \mu^{N-1/2}) = O(\mu^{1+\alpha})$$

(see corollary 4.20) and

$$\begin{Bmatrix} \|\mathcal{G}_3'(\tilde{\eta}_1)\|_0 \\ \|\mathcal{K}_3'(\tilde{\eta}_1)\|_0 \\ \|\mathcal{L}_3'(\tilde{\eta}_1)\|_0 \end{Bmatrix} \leqslant c\mu^{\alpha/2} |\!|\!|\tilde{\eta}_1|\!|\!|_\alpha \|\tilde{\eta}_1\|_2 = O(\mu^{1+\alpha/2})$$

(see proposition 4.5), one finds that

$$H(\tilde{\eta}) = \mathcal{F}^{-1}\left[\frac{1}{g(k)} \mathcal{F}[\mathcal{K}_3'(\tilde{\eta}_1) + 2\nu_0 \mathcal{G}_3'(\tilde{\eta}_1) - \nu_0^2 \mathcal{L}_3'(\tilde{\eta}_1)]\right] + \underbrace{O(\mu^{1+\alpha})\underline{O}(\mu^{1+\alpha/2})}_{=\underline{o}(\mu^3)}.$$

□

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Combining propositions 4.26 and 4.27, one finds that

$$\begin{aligned}\mathcal{K}_3(\tilde{\eta}) &+ 2\nu_0\mathcal{G}_3(\tilde{\eta}) - \nu_0^2\mathcal{L}_3(\tilde{\eta})\\ &= -\int_{-\infty}^{\infty}(\mathcal{K}_3'(\tilde{\eta}_1) + 2\nu_0\mathcal{G}_3'(\tilde{\eta}_1) - \nu_0^2\mathcal{L}_3'(\tilde{\eta}_1))\\ &\qquad\times\mathcal{F}^{-1}\left[\frac{1}{g(k)}\mathcal{F}[\mathcal{K}_3'(\tilde{\eta}_1) + 2\nu_0\mathcal{G}_3'(\tilde{\eta}_1) - \nu_0^2\mathcal{L}_3'(\tilde{\eta}_1)]\right]\mathrm{d}x + o(\mu^3), \quad (4.20)\end{aligned}$$

which we write as

$$\begin{aligned}\mathcal{K}_3(\tilde{\eta}) &+ 2\nu_0\mathcal{G}_3(\tilde{\eta}) - \nu_0^2\mathcal{L}_3(\tilde{\eta})\\ &= -2\int_{-\infty}^{\infty}M(\tilde{\eta}_1^+,\tilde{\eta}_1^+)\mathcal{F}^{-1}[g(k)^{-1}M(\tilde{\eta}_1^-,\tilde{\eta}_1^-)]\,\mathrm{d}x\\ &\quad -4\int_{-\infty}^{\infty}M(\tilde{\eta}_1^+,\tilde{\eta}_1^-)\mathcal{F}^{-1}[g(k)^{-1}M(\tilde{\eta}_1^+,\tilde{\eta}_1^-)]\,\mathrm{d}x + o(\mu^3), \qquad (4.21)\end{aligned}$$

where

$$M = m_1 + 2\nu_0 m_2 - \nu_0^2 m_3,$$

in order to determine the dominant term on its right-hand side.

Proposition 4.28. *The function $\tilde{\eta}$ satisfies*

$$\mathcal{K}_3(\tilde{\eta}) + 2\nu_0\mathcal{G}_3(\tilde{\eta}) - \nu_0^2\mathcal{L}_3(\tilde{\eta}) = A_3\int_{-\infty}^{\infty}\tilde{\eta}_1^4\,\mathrm{d}x + o(\mu^3),$$

where

$$\begin{aligned}A_3 &= -\tfrac{1}{3}g(2k_0)^{-1}(A_3^1)^2 - \tfrac{2}{3}g(0)^{-1}(A_3^2)^2,\\ A_3^1 &= \tfrac{1}{2}\omega\nu_0 f(2k_0) + \omega\nu_0 f(k_0) + \tfrac{1}{2}\omega^2 + \nu_0^2 f(2k_0)f(k_0) + \tfrac{1}{2}\nu_0^2 f(k_0)^2 - \tfrac{3}{2}k_0^2\nu_0^2,\\ A_3^2 &= \tfrac{1}{2}\omega\nu_0 + \omega\nu_0 f(k_0) + \tfrac{1}{2}\omega^2 + \nu_0^2 f(k_0) + \tfrac{1}{2}\nu_0^2 f(k_0)^2 - \tfrac{1}{2}\nu_0^2 k_0^2.\end{aligned}$$

Proof. Lemma 4.23 implies that

$$M(\tilde{\eta}_1^+,\tilde{\eta}_1^+) = A_3^1(\tilde{\eta}_1^+)^2 + \underline{O}(\mu^{1+\alpha}),$$

so that

$$\mathcal{F}^{-1}[g(k)^{-1}M(\tilde{\eta}_1^-,\tilde{\eta}_1^-)] = \mathcal{F}^{-1}[g(k)^{-1}\overline{M(\tilde{\eta}_1^+,\tilde{\eta}_1^+)}] = g(2k_0)^{-1}A_3^1(\tilde{\eta}_1^-)^2 + \underline{O}(\mu^{1+\alpha}),$$

and

$$M(\tilde{\eta}_1^+,\tilde{\eta}_1^-) = A_3^2\tilde{\eta}_1^+\tilde{\eta}_1^- + \underline{O}(\mu^{1+\alpha}),$$

so that

$$\mathcal{F}^{-1}[g(k)^{-1}M(\tilde{\eta}_1^+,\tilde{\eta}_1^-)] = g(0)^{-1}A_3^2\tilde{\eta}_1^+\tilde{\eta}_1^- + \underline{O}(\mu^{1+\alpha});$$

the result follows from these calculations and (4.21). □

The requisite estimates for $\mathcal{M}_\mu(\tilde{\eta})$ and $\langle\mathcal{M}_\mu'(\tilde{\eta}),\tilde{\eta}\rangle + 4\mu\tilde{\mathcal{M}}_\mu(\tilde{\eta})$ may now be derived from corollary 4.25 and proposition 4.28.

LEMMA 4.29. *The estimates*

$$\begin{aligned}\mathcal{M}_{a^2\mu}(a\tilde{\eta}) &= a^3(\mathcal{K}_3(\tilde{\eta}) + 2\nu_0\mathcal{G}_3(\tilde{\eta}) - \nu_0^2\mathcal{L}_3(\tilde{\eta})) \\ &\qquad + a^4(\mathcal{K}_4(\tilde{\eta}) + 2\nu_0\mathcal{G}_4(\tilde{\eta}) - \nu_0^2\mathcal{L}_4(\tilde{\eta})) + a^3 o(\mu^3), \\ \langle \mathcal{M}'_{a^2\mu}(a\tilde{\eta}), a\tilde{\eta}\rangle &+ 4a^2\mu\tilde{\mathcal{M}}_{a^2\mu}(a\tilde{\eta}) \\ &= 3a^3(\mathcal{K}_3(\tilde{\eta}) + 2\nu_0\mathcal{G}_3(\tilde{\eta}) - \nu_0^2\mathcal{L}_3(\tilde{\eta})) \\ &\qquad + 4a^4(\mathcal{K}_4(\tilde{\eta}) + 2\nu_0\mathcal{G}_4(\tilde{\eta}) - \nu_0^2\mathcal{L}_4(\tilde{\eta})) + a^3 o(\mu^3)\end{aligned}$$

hold uniformly over $a \in [1, 2]$.

Proof. Lemma 4.7 asserts that

$$\begin{aligned}\mathcal{M}_{a^2\mu}(a\tilde{\eta}) &= a^3(\mathcal{K}_3(\tilde{\eta}) + 2\nu_0\mathcal{G}_3(\tilde{\eta}) - \nu_0^2\mathcal{L}_3(\tilde{\eta})) + a^4(\mathcal{K}_4(\tilde{\eta}) + 2\nu_0\mathcal{G}_4(\tilde{\eta}) - \nu_0^2\mathcal{L}_4(\tilde{\eta})) \\ &\quad + 2\left(\frac{\mu + \mathcal{G}_2(\tilde{\eta})}{\mathcal{L}_2(\tilde{\eta})} - \nu_0\right)(a^3\mathcal{G}_3(\tilde{\eta}) + a^4\mathcal{G}_4(\tilde{\eta})) \\ &\quad - \left(\frac{\mu + \mathcal{G}_2(\tilde{\eta})}{\mathcal{L}_2(\tilde{\eta})} - \nu_0\right)\left(\frac{\mu + \mathcal{G}_2(\tilde{\eta})}{\mathcal{L}_2(\tilde{\eta})} + \nu_0\right)(a^3\mathcal{L}_3(\tilde{\eta}) + a^4\mathcal{L}_4(\tilde{\eta})) \\ &\quad + \frac{a^4}{\mathcal{L}_2(\tilde{\eta})}\left(\mathcal{G}_3(\tilde{\eta}) - \left(\frac{\mu + \mathcal{G}_2(\tilde{\eta})}{\mathcal{L}_2(\tilde{\eta})}\right)\mathcal{L}_3(\tilde{\eta})\right)^2 \\ &\quad + O(a^5\mu^{3/2}(\|\tilde{\eta}\|_{1,\infty} + \|\tilde{\eta}'' + k_0^2\tilde{\eta}\|_0)^2)\end{aligned}$$

uniformly over $a \in [1, 2]$.

The first result follows by estimating

$$\begin{Bmatrix}\mathcal{G}_3(\tilde{\eta}) \\ \mathcal{L}_3(\tilde{\eta})\end{Bmatrix} = O(\mu^{3/2}), \qquad \begin{Bmatrix}\mathcal{G}_4(\tilde{\eta}) \\ \mathcal{L}_4(\tilde{\eta})\end{Bmatrix} = O(\mu^2),$$
$$\|\tilde{\eta}\|_{1,\infty} + \|\tilde{\eta}'' + k_0^2\tilde{\eta}\|_0 \leqslant c(\mu^{\alpha/2}|||\tilde{\eta}|||_\alpha + \|\tilde{\eta}_3\|_2) \leqslant c\mu^{1/2+\alpha/2}$$

(see (4.17)),

$$\left|\frac{\mu + \mathcal{G}_2(\tilde{\eta})}{\mathcal{L}_2(\tilde{\eta})} - \nu_0\right| \leqslant c(\mu^\alpha|||\tilde{\eta}_1|||_\alpha^2 + \|\eta_3\|_2 + \mu^{N-1/2}) \leqslant c\mu^{1+\alpha}$$

and noting that

$$\begin{aligned}\mathcal{G}_3(\tilde{\eta}) &- \left(\frac{\mu + \mathcal{G}_2(\tilde{\eta})}{\mathcal{L}_2(\tilde{\eta})}\right)\mathcal{L}_3(\tilde{\eta}) \\ &= \mathcal{G}_3(\tilde{\eta}) - \nu_0\mathcal{L}_3(\tilde{\eta}) + o(\mu^3) \\ &= -\int_{-\infty}^{\infty}(\mathcal{G}_3'(\tilde{\eta}_1) - \nu_0\mathcal{L}_3'(\tilde{\eta}_1)) \\ &\qquad\qquad \times \mathcal{F}^{-1}\left[\frac{1}{g(k)}\mathcal{F}[\mathcal{K}_3'(\tilde{\eta}_1) + 2\nu_0\mathcal{G}_3'(\tilde{\eta}_1) - \nu_0^2\mathcal{L}_3'(\tilde{\eta}_1)]\right]\mathrm{d}x + o(\mu^3)\end{aligned}$$

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$$
\begin{aligned}
&= -\int_{-\infty}^{\infty} (\tilde{M}(\tilde{\eta}_1^+, \tilde{\eta}_1^+)\mathcal{F}^{-1}[g(k)^{-1}M(\tilde{\eta}_1^-, \tilde{\eta}_1^-)] \\
&\qquad\qquad + \tilde{M}(\tilde{\eta}_1^-, \tilde{\eta}_1^-)\mathcal{F}^{-1}[g(k)^{-1}M(\tilde{\eta}_1^+, \tilde{\eta}_1^+)])\,\mathrm{d}x \\
&\qquad - 4\int_{-\infty}^{\infty} \tilde{M}(\tilde{\eta}_1^+, \tilde{\eta}_1^-)\mathcal{F}^{-1}[g(k)^{-1}M(\tilde{\eta}_1^+, \tilde{\eta}_1^-)]\,\mathrm{d}x + o(\mu^3) \\
&= \gamma \int_{-\infty}^{\infty} \tilde{\eta}_1^4\,\mathrm{d}x + o(\mu^3) \\
&= O(\mu^{2+\alpha}) + o(\mu^3),
\end{aligned}
$$

where $\tilde{M} = m_2 - \nu_0 m_3$ and γ is a (possibly negative) constant. Here, the third line follows from the second by propositions 4.26 and 4.27 and the fifth from the fourth by repeating the proof of proposition 4.28.

The second result is derived in a similar fashion. □

Corollary 4.30. *The estimates*

$$
\mathcal{M}_{a^2\mu}(a\tilde{\eta}) = (a^3 A_3 + a^4 A_4)\int_{-\infty}^{\infty} \tilde{\eta}_1^4\,\mathrm{d}x + a^3 o(\mu^3),
$$

$$
\langle \mathcal{M}'_{a^2\mu}(a\tilde{\eta}), a\tilde{\eta}\rangle + 4a^2\mu\tilde{\mathcal{M}}_{a^2\mu}(a\tilde{\eta}) = (3a^3 A_3 + 4a^4 A_4)\int_{-\infty}^{\infty} \tilde{\eta}_1^4\,\mathrm{d}x + a^3 o(\mu^3)
$$

hold uniformly over $a \in [1, 2]$ *and*

$$
\int_{-\infty}^{\infty} \tilde{\eta}_1^4\,\mathrm{d}x \geqslant c\mu^3.
$$

Proof. The estimates follow by combining corollary 4.25, proposition 4.28 and lemma 4.29, while the inequality for $\tilde{\eta}_1$ is a consequence of the first estimate (with $a = 1$) and the fact that $\mathcal{M}_\mu(\tilde{\eta}) \leqslant -c\mu^3$. □

4.4. Derivation of the strict subadditivity property

In this section we derive the strict subadditivity property (4.1). We begin by showing that c_μ is a strictly subhomogeneous increasing function of $\mu > 0$. The first of these properties is a corollary of the next proposition.

Proposition 4.31. *There exist* $a_0 \in (1, 2]$ *and* $q > 2$ *with the property that the function* $a \mapsto a^{-q}\mathcal{M}_{a^2\mu}(a\tilde{\eta})$, $a \in [1, a_0]$, *is decreasing and strictly negative.*

Proof. This result follows from the calculations

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}a}(a^{-5/2}\mathcal{M}_{a^2\mu}(a\tilde{\eta})) &= a^{-7/2}(-\tfrac{5}{2}\mathcal{M}_{a^2\mu}(a\tilde{\eta}) + \langle \mathcal{M}'_{a^2\mu}(a\tilde{\eta}), a\tilde{\eta}\rangle_0 + 4a^2\mu\tilde{\mathcal{M}}_{a^2\mu}(a\tilde{\eta})) \\
&= \tfrac{1}{4}a^{-7/2}\left(\tfrac{1}{4}a^3(\tfrac{1}{3}\omega^3 + 1)\int_{-\infty}^{\infty} \tilde{\eta}_1^3\,\mathrm{d}x + a^3 o(\mu^{5/3})\right) \\
&= a^{-1/2}\left(\tfrac{1}{4}(\tfrac{1}{3}\omega^3 + 1)\int_{-\infty}^{\infty} \tilde{\eta}_1^3\,\mathrm{d}x + o(\mu^{5/3})\right) \\
&\leqslant -c\mu^{5/3} \\
&< 0, \quad a \in (1, 2),
\end{aligned}
$$

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for $\beta > \beta_{\rm c}$ (see corollary 4.14) and

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}a}(a^{-q}\mathcal{M}_{a^2\mu}(a\tilde{\eta}))\\
&\quad = a^{-(q+1)}(-q\mathcal{M}_{a^2\mu}(a\tilde{\eta}) + \langle\tilde{\mathcal{M}}'_{a^2\mu}(a\tilde{\eta}), a\tilde{\eta}\rangle_0 + 4a^2\mu\tilde{\mathcal{M}}_{a^2\mu}(a\tilde{\eta}))\\
&\quad = a^{-(q+1)}\bigg((-q(a^3A_3 + a^4A_4) + 3a^3A_3 + 4a^4A_4)\int_{-\infty}^{\infty}\tilde{\eta}_1^4\,\mathrm{d}x + a^3o(\mu^3)\bigg)\\
&\quad = a^{2-q}\bigg(((3-q)A_3 + a(4-q)A_4)\int_{-\infty}^{\infty}\tilde{\eta}_1^4\,\mathrm{d}x + o(\mu^3)\bigg)\\
&\quad \leqslant -c\mu^3\\
&\quad < 0, \quad a \in (1, a_0),\ q \in (2, q_0),
\end{aligned}$$

for $\beta < \beta_{\rm c}$ (see corollary 4.30); here $a_0 > 1$ and $q_0 > 2$ are chosen so that $(3 - q)A_3 + a(4-q)A_4$, which is negative for $a = 1$ and $q = 2$ (see Appendix B), is also negative for $a \in (1, a_0]$ and $q \in (2, q_0]$. □

Corollary 4.32. *The number c_μ is a strictly subhomogeneous function of $\mu > 0$.*

Proof. The previous lemma implies that

$$\mathcal{M}_{a\mu}(a^{1/2}\tilde{\eta}_m) \leqslant a^{1/2}q\mathcal{M}_\mu(\tilde{\eta}_m) < 0, \quad a \in [1, a_0^2],$$

from which it follows that

$$\begin{aligned}
c_{a\mu} &\leqslant \mathcal{J}_{a\mu}(a^{1/2}\tilde{\eta}_m)\\
&= \mathcal{K}_2(a^{1/2}\tilde{\eta}_m) + \frac{(a\mu + \mathcal{G}_2(a^{1/2}\tilde{\eta}_m))^2}{\mathcal{L}_2(a^{1/2}\tilde{\eta}_m)} + \mathcal{M}(a^{1/2}\tilde{\eta}_m)\\
&\leqslant a\bigg(\mathcal{K}_2(\tilde{\eta}_m) + \frac{(\mu + \mathcal{G}(\tilde{\eta}_m))^2}{\mathcal{L}(\tilde{\eta}_m)}\bigg) + a^{1/2}q\mathcal{M}_\mu(\tilde{\eta}_m)\\
&= a\bigg(\mathcal{K}_2(\tilde{\eta}_m) + \frac{\mu^2}{\mathcal{L}(\tilde{\eta}_m)} + \mathcal{M}_\mu(\tilde{\eta}_m)\bigg) + (a^{1/2}q - a)\mathcal{M}_\mu(\tilde{\eta}_m)\\
&\leqslant a\mathcal{J}(\tilde{\eta}_m) - c(a^{1/2}q - a)\mu^{r^\star}, \quad a \in [1, a_0^2].
\end{aligned}$$

In the limit $n \to \infty$ the above inequality yields

$$c_{a\mu} \leqslant ac_\mu - c(a^{1/2}q - a)\mu^{r^\star} < ac_\mu, \quad a \in (1, a_0^2].$$

For $a > a_0^2$ we choose $p \geqslant 2$ such that $a \in (1, a_0^{2p}]$ (and hence $a^{1/p} \in (1, a_0^2]$) and observe that

$$c_{a\mu} < a^{1/p}c_{a^{(p-1)/p}\mu} < a^{2/p}c_{a^{(p-2)/p}\mu} < \cdots < ac_\mu.$$

□

Lemma 4.33. *The number c_μ is an increasing function of $\mu > 0$.*

Proof. Using proposition 4.8 for $\beta > \beta_{\rm c}$ and proposition 4.20 for $\beta < \beta_{\rm c}$, one finds that

$$\mu + \mathcal{G}(\tilde{\eta}_m) = \nu_0\mathcal{L}(\tilde{\eta}_m) + O(\mu^{3/2}) \geqslant c\mu + O(\mu^{3/2})$$

so that

$$\mu + \mathcal{G}(\tilde{\eta}_m) \geqslant c_\star \mu$$

for some $c_\star \in (0,1)$. Let $d_\star = 1 - c_\star$ so that $d_\star \in (0,1)$.

First suppose that $\mu_1 \in [d_\star \mu_2, \mu_2]$. Let $\{\tilde{\eta}_m^2\}$ be the special minimizing sequence constructed in theorem 3.1 for $\mu = \mu_2$ and note that

$$\mu_1 + \mathcal{G}(\tilde{\eta}_m^2) = \mu_2 + \mathcal{G}(\tilde{\eta}_m^2) - (\mu_2 - \mu_1) \geqslant \mu_1 - d_\star \mu_2 \geqslant 0,$$

so that $\mathcal{J}_{\mu_1}(\tilde{\eta}_m^2) \leqslant \mathcal{J}_{\mu_2}(\tilde{\eta}_m^2)$. It follows that

$$c_{\mu_1} \leqslant \mathcal{J}_{\mu_1}(\tilde{\eta}_m^2) \leqslant \mathcal{J}_{\mu_2}(\tilde{\eta}_m^2) \to c_{\mu_2}$$

as $n \to \infty$, that is,

$$c_{\mu_1} \leqslant c_{\mu_2}.$$

For $\mu_1 < d_\star \mu_2$ we choose $p \geqslant 2$ such that $\mu_1 \in [d_\star^p \mu_2, \mu_2]$ (and hence $\mu_1 \in [d_\star d_\star^{p-1} \mu_2, d_\star^{p-1} \mu_2]$ and obviously $d_\star^{q+1} \mu_2 \in [d_\star d_\star^q \mu_2, d_\star^q \mu_2]$, $q = 0, \dots, p-2$) and observe that

$$c_{\mu_1} \leqslant c_{d_\star^{p-1}\mu_2} \leqslant c_{d_\star^{p-2}\mu_2} \leqslant \cdots \leqslant c_{\mu_2}.$$

□

Our final result is stated in the following theorem.

THEOREM 4.34. *The number c_μ has the strict subadditivity property*

$$c_{\mu_1+\mu_2} < c_{\mu_1} + c_{\mu_2}, \qquad 0 < |\mu_1|, |\mu_2|, \mu_1 + \mu_2 < \mu_0.$$

Proof. Using the strict subhomogeneity of $c(\mu)$ for $\mu > 0$, we find that

$$c_{\mu_1+\mu_2} < \frac{\mu_1 + \mu_2}{\mu_1} c_{\mu_1} = c_{\mu_1} + \frac{\mu_2}{\mu_1} c_{\mu_1} \leqslant c_{\mu_1} + c_{\mu_2}$$

for $0 < \mu_1 \leqslant \mu_2$, and for $\mu_1 < 0$, $\mu_2 > 0$ with $\mu_1 + \mu_2 > 0$ its monotonicity for $\mu > 0$ shows that

$$c_{\mu_1+\mu_2} \leqslant c_{\mu_2} < c_{\mu_1} + c_{\mu_2}.$$

□

5. Existence theory and consequences

5.1. Minimization

The following theorem, which is proved using the results of §§ 3 and 4, is our final result concerning the set of minimizers of $\mathcal{J}_\mu$ over $U \setminus \{0\}$.

THEOREM 5.1.

(i) *The set B_μ of minimizers of $\mathcal{J}_\mu$ over $U \setminus \{0\}$ is non-empty.*

(ii) *Suppose that $\{\eta_m\}$ is a minimizing sequence for $\mathcal{J}_\mu$ on $U \setminus \{0\}$ that satisfies*

$$\sup_{m \in \mathbb{N}} \|\eta_m\|_2 < M.$$

There exists a sequence $\{x_m\} \subset \mathbb{R}$ with the property that a subsequence of $\{\eta_m(x_m + \cdot)\}$ converges in $H^r(\mathbb{R})$, $r \in [0,2)$, to a function $\eta \in B_\mu$.

Proof. It suffices to prove part (ii), since an application of this result to the sequence $\{\tilde{\eta}_m\}$ constructed in theorem 3.1 yields part (i).

In order to establish part (ii) we choose $\tilde{M} \in (\sup_{m\in\mathbb{N}} \|\eta_m\|_2, M)$, so that $\{\eta_m\}$ is also a minimizing sequence for the functional $\mathcal{J}_{\rho,\mu}$ introduced in § 3.1; the existence of a minimizing sequence $\{v_m\}$ for $\mathcal{J}_{\rho,\mu}$ with $\lim_{m\to\infty} \mathcal{J}_{\rho,\mu}(v_m) < \lim_{m\to\infty} \mathcal{J}_{\rho,\mu}(\eta_m)$ would lead to the contradiction

$$\lim_{m\to\infty} \mathcal{J}_\mu(v_m) \leqslant \lim_{m\to\infty} \mathcal{J}_{\rho,\mu}(v_m) < \lim_{m\to\infty} \mathcal{J}_{\rho,\mu}(\eta_m) = \lim_{m\to\infty} \mathcal{J}_\mu(\eta_m) = c_\mu.$$

We may therefore study $\{\eta_m\}$ using the theory given in § 3.2, noting that the sequence $\{u_m\}$ with $u_m = (\eta_m')^2 + \eta_m^2$ does not have the 'dichotomy' property: the existence of two sequences $\{\eta_m^{(1)}\}$, $\{\eta_m^{(2)}\}$ with the features listed in lemma 3.9 is incompatible with the strict subadditivity property of c_μ (see theorem 4.34). Recall that the numbers $\mu^{(1)}$, $\mu^{(2)}$ sum to μ; this fact leads to the contradiction

$$\begin{aligned} c_\mu &< c_{\mu^{(1)}} + c_{\mu^{(2)}} \\ &\leqslant \lim_{m\to\infty} \mathcal{J}_{\mu^{(1)}}(\eta_m^{(1)}) + \lim_{m\to\infty} \mathcal{J}_{\mu^{(2)}}(\eta_m^{(2)}) \\ &= \lim_{m\to\infty} \mathcal{J}_\mu(\eta_m) \\ &= c_\mu. \end{aligned}$$

We conclude that $\{u_m\}$ has the 'concentration' property, and hence $\eta_m(\cdot + x_m) \to \eta^{(1)}$ as $n \to \infty$ in $H^r(\mathbb{R})$ for every $r \in [0, 2)$ (see lemma 3.8(ii)), whereby $\mathcal{J}_\mu(\eta) = \lim_{m\to\infty} \mathcal{J}_\mu(\eta_m(\cdot + x_m)) = c_\mu$ so that $\eta^{(1)}$ is a minimizer of $\mathcal{J}_\mu$ over $U \setminus \{0\}$. □

The next step is to relate the above result to our original problem of finding minimizers of $\mathcal{H}(\eta, \xi)$ subject to the constraint $\mathcal{I}(\eta, \xi) = 2\mu$, where $\mathcal{H}$ and $\mathcal{I}$ are defined in (1.6) and (1.7).

Theorem 5.2.

(i) *The set D_μ of minimizers of $\mathcal{H}$ on the set*

$$S_\mu = \{(\eta, \xi) \in U \times H_\star^{1/2}(\mathbb{R}) \colon \mathcal{I}(\eta, \xi) = 2\mu\}$$

is non-empty.

(ii) *Suppose that $\{(\eta_m, \xi_m)\} \subset S_\mu$ is a minimizing sequence for $\mathcal{H}$ with the property that $\sup_{m\in\mathbb{N}} \|\eta_m\|_2 < M$. There exists a sequence $\{x_m\} \subset \mathbb{R}$ with the property that a subsequence of $\{(\eta_m(x_m + \cdot), \xi_m(x_m + \cdot))\}$ converges in $H^r(\mathbb{R}) \times H_\star^{1/2}(\mathbb{R})$, $r \in [0, 2)$, to a function in D_μ.*

Proof. (i) We consider the minimization problem in two steps.

(1) *Fix $\eta \in U \setminus \{0\}$ and minimize $\mathcal{H}(\eta, \cdot)$ over $T_\mu = \{\xi \in H_\star^{1/2}(\mathbb{R}) \colon \mathcal{I}(\eta, \xi) = 2\mu\}$:* notice that $\mathcal{H}(\eta, \cdot)$ is weakly lower semi-continuous on $H_\star^{1/2}(\mathbb{R})$ (since $\xi \mapsto \langle G(\eta)\xi, \xi\rangle_0^{1/2}$ is equivalent to its usual norm), while $\mathcal{I}(\eta, \cdot)$ is weakly continuous on $H_\star^{1/2}(\mathbb{R})$; furthermore, $\mathcal{H}(\eta, \cdot)$ is convex and coercive. A familiar argument shows that $\mathcal{H}(\eta, \cdot)$ has a unique minimizer ξ_η over T_μ.

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(2) *Minimize* $\mathcal{H}(\eta, \xi_\eta)$ *over* $U \setminus \{0\}$*:* because ξ_η minimizes $\mathcal{H}(\eta, \cdot)$ over T_μ, there exists a Lagrange multiplier ν_η such that

$$G(\eta)\xi_\eta + \omega\eta\eta' = \nu_\eta \eta',$$

and a straightforward calculation shows that

$$\xi_\eta = G(\eta)^{-1}(\nu_\eta\eta' - \omega\eta\eta'), \quad \nu_\eta = \frac{\mu + \mathcal{G}(\eta)}{\mathcal{L}(\eta)}. \tag{5.1}$$

According to theorem 5.1(i), the set B_μ of minimizers of $\mathcal{J}_\mu(\eta) := \mathcal{H}(\eta, \xi_\eta)$ over $U \setminus \{0\}$ is not empty; it follows that D_μ is also not empty.

(ii) Let $\{(\eta_m, \xi_m)\} \subset U \times H_\star^{1/2}(\mathbb{R})$ be a minimizing sequence for $\mathcal{H}$ over S_μ with $\sup_{m\in\mathbb{N}} \|\eta_m\|_2 < M$. The inequality

$$\mathcal{H}(\eta_m, \xi_{\eta_m}) \leqslant \mathcal{H}(\eta_m, \xi_m)$$

shows that $\{(\eta_k, \xi_{\eta_k})\} \subset U \times H_\star^{1/2}(\mathbb{R})$ is also a minimizing sequence; it follows that $\{\eta_m\} \subset U \setminus \{0\}$ is a minimizing sequence for $\mathcal{J}_\mu$, which therefore converges (up to translations and subsequences) in $H^r(\mathbb{R})$, $r \in [0, 2)$, to a minimizer η of $\mathcal{J}_\mu$ over $U \setminus \{0\}$.

The relations (5.1) show that $\xi_{\eta_m} \to \xi_\eta$ in $H_\star^{1/2}(\mathbb{R})$ and, using this result and the calculation

$$\begin{aligned}
c\|\xi_m - \xi_{\eta_m}\|_{*,1/2}^2 &\leqslant \tfrac{1}{2}\langle G(\eta_m)(\xi_m - \xi_{\eta_m}), (\xi_m - \xi_{\eta_m})\rangle \\
&= 2\mathcal{H}(\eta_m, \xi_m) + 2\mathcal{H}(\eta_m, \xi_{\eta_m}) - 4\mathcal{H}(\eta_m, \tfrac{1}{2}(\xi_m + \xi_{\eta_m})) \\
&\leqslant 2\mathcal{H}(\eta_m, \xi_m) + 2\mathcal{H}(\eta_m, \xi_{\eta_m}) - 4c_\mu \\
&\to 2c_\mu + 2c_\mu - 4c_\mu \\
&= 0
\end{aligned}$$

as $n \to \infty$ (recall that $\mathcal{H}(\eta_m, \xi) \geqslant \mathcal{H}(\eta_m, \xi_{\eta_m}) = \mathcal{J}(\eta_m) \geqslant c_\mu$ for all $\xi \in H_\star^{1/2}(\mathbb{R})$), one finds that $\xi_m \to \xi_\eta$ in $H_\star^{1/2}(\mathbb{R})$ as $m \to \infty$. □

5.2. Convergence to solitary-wave solutions of model equations

5.2.1. *The case* $\beta > \beta_{\mathrm{c}}$

Suppose that η is a minimizer of $\mathcal{J}$ over $U \setminus \{0\}$, write $\eta = \eta_1 + \eta_2$ according to the decomposition introduced in § 4.1 and define $\phi_\eta \in H^2(\mathbb{R})$ by the formula

$$\eta_1(x) = \mu^{2/3}\phi_\eta(\mu^{1/3}x).$$

In this section we prove that $\operatorname{dist}(\phi_\eta, D_{\mathrm{KdV}}) \to 0$ as $\mu \downarrow 0$, uniformly over $\eta \in B_\mu$, where D_{KdV} is the set of solitary-wave solutions to the Korteweg–de Vries equation and 'dist' denotes the distance in $H^1(\mathbb{R})$.

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REMARK 5.3. Observe that

$$\begin{Bmatrix}\mathcal{K}_2(\eta)\\ \mathcal{G}_2(\eta)\\ \mathcal{L}_2(\eta)\end{Bmatrix} = \begin{Bmatrix}\mathcal{K}_2(\eta_1)\\ \mathcal{G}_2(\eta_1)\\ \mathcal{L}_2(\eta_1)\end{Bmatrix} + \underbrace{\begin{Bmatrix}\mathcal{K}_2(\eta_2)\\ \mathcal{G}_2(\eta_2)\\ \mathcal{L}_2(\eta_2)\end{Bmatrix}}_{\substack{=O(\|\eta\|_2^2)\\ =O(\mu^{2+\alpha})}}$$

because $\hat{\eta}_1$ and $\hat{\eta}_2$ have disjoint supports, and

$$\mathcal{G}_2(\eta_1) = -\frac{\mu\omega}{4}\int_{-\infty}^{\infty}\phi_\eta^2\,\mathrm{d}x, \qquad \mathcal{K}_2(\eta_1) = \frac{\mu}{2}\int_{-\infty}^{\infty}\phi_\eta^2\,\mathrm{d}x,$$

while the estimates

$$\begin{aligned}\int_{-\infty}^{\infty}(|k|\coth|k| - 1)|\hat{\eta}_1|^2\,\mathrm{d}k &\leqslant c\int_{-\infty}^{\infty}k^2|\hat{\eta}_1|^2\,\mathrm{d}k = c\|\eta'\|_0^2\\ &\leqslant c\mu^{2\alpha}|||\eta|||_\alpha^2 \leqslant c\mu^{1+2\alpha},\\ \int_{-\infty}^{\infty}(|k|\coth|k| - 1 - \tfrac{1}{3}k^2)|\hat{\eta}_1|^2\,\mathrm{d}k &\leqslant c\int_{-\infty}^{\infty}k^4|\hat{\eta}_1|^2\,\mathrm{d}k = c\|\eta''\|_0^2\\ &\leqslant c\mu^{4\alpha}|||\eta|||_\alpha^2 = c\mu^{1+4\alpha}\end{aligned}$$

show that

$$\mathcal{L}_2(\eta_1) = \frac{\mu}{2}\int_{-\infty}^{\infty}\phi_\eta^2\,\mathrm{d}x + O(\mu^{1+2\alpha})$$

and

$$\mathcal{L}_2(\eta_1) = \frac{\mu}{2}\int_{-\infty}^{\infty}\phi_\eta^2\,\mathrm{d}x - \frac{\beta}{3}\mu^{5/3}\int_{-\infty}^{\infty}(\phi_\eta')^2\,\mathrm{d}x + O(\mu^{1+4\alpha}).$$

Furthermore, corollary 4.14 implies that

$$\mathcal{M}_\mu(\eta) = \tfrac{1}{2}(\tfrac{1}{3}\omega^2 + 1)\mu^{5/3}\int_{-\infty}^{\infty}\phi_\eta^3\,\mathrm{d}x + o(\mu^{5/3}).$$

Our first result concerns the convergence of the $L^2(\mathbb{R})$-norm of minimizers of $\mathcal{J}_\mu$ over $U\setminus\{0\}$.

PROPOSITION 5.4. *The estimate* $\|\phi_\eta\|_0^2 = 4(\omega^2+4)^{-1/2} + O(\mu^{2\alpha})$ *holds for each* $\eta \in B_\mu$.

Proof. It follows from

$$\left|\frac{\mu + \mathcal{G}_2(\eta)}{\mathcal{L}_2(\eta)} - \nu_0\right| \leqslant c\mu^{\alpha/2+1/2}, \quad \mathcal{L}(\eta) \leqslant c\mu,$$

that

$$\nu_0\mathcal{L}_2(\eta) - \mathcal{G}_2(\eta) = \mu + O(\mu^{\alpha/2+3/2}),$$

and the result is obtained by combining this estimate with

$$\nu_0\mathcal{L}_2(\eta) - \mathcal{G}_2(\eta) = \tfrac{1}{4}\underbrace{(2\nu_0+\omega)}_{=\sqrt{\omega^2+4}}\mu\int_{-\infty}^{\infty}\phi_\eta^2\,\mathrm{d}x + O(\mu^{1+2\alpha}).$$

□

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The next step is to show that the Korteweg–de Vries energy $\mathcal{E}_{\mathrm{KdV}}(\phi_\eta)$ corresponding to a minimizer η of $\mathcal{J}_\mu$ over $U \setminus \{0\}$ approaches c_{KdV} in the limit $\mu \downarrow 0$.

Theorem 5.5.

(i) *The number c_μ satisfies $c_\mu = 2\nu_0\mu + c_{\mathrm{KdV}}\mu^{5/3} + o(\mu^{5/3})$.*

(ii) *Each $\eta \in B_\mu$ satisfies $\mathcal{E}_{\mathrm{KdV}}(\phi_\eta) \to c_{\mathrm{KdV}}$ as $\mu \downarrow 0$.*

Proof. Notice that

$$\begin{aligned} c_\mu &= \mathcal{J}_\mu(\eta) \\ &= \mathcal{K}_2(\eta) + \frac{(\mu + \mathcal{G}_2(\eta))^2}{\mathcal{L}_2(\eta)} + \mathcal{M}_\mu(\eta) \\ &= 2\nu_0\mu + \mathcal{K}_2(\eta) + 2\nu_0\mathcal{G}_2(\eta) - \nu_0^2\mathcal{L}_2(\eta) + \left(\frac{\mu + \mathcal{G}_2(\eta)}{\sqrt{\mathcal{L}_2(\eta)}} - \nu_0\sqrt{\mathcal{L}_2(\eta)}\right)^2 + \mathcal{M}_\mu(\eta) \\ &\geqslant 2\nu_0\mu + \mathcal{K}_2(\eta) + 2\nu_0\mathcal{G}_2(\eta) - \nu_0^2\mathcal{L}_2(\eta) + \mathcal{M}_\mu(\eta) \\ &= 2\nu_0\mu + \frac{\mu^{5/3}}{2}\int_{-\infty}^{\infty}\left(\left(\beta - \frac{\nu_0^2}{3}\right)(\phi_\eta')^2 + \left(\frac{\omega^2}{3} + 1\right)\phi_\eta^3\right)\mathrm{d}x + o(\mu^{5/3}) \\ &= 2\nu_0\mu + \mu^{5/3}\mathcal{E}_{\mathrm{KdV}}(\phi_\eta) + o(\mu^{5/3}), \end{aligned} \tag{5.2}$$

and combining this estimate with lemma A.1 yields

$$\mathcal{E}_{\mathrm{KdV}}(\phi_\eta) \leqslant c_{\mathrm{KdV}} + o(1).$$

A straightforward scaling argument shows that

$$\inf\{\mathcal{E}_{\mathrm{KdV}}(\phi)\colon \phi \in H^1(\mathbb{R}),\ \|\phi\|_0^2 = 4(\omega^2+4)^{-1/2}a\} = a^{5/3}c_{\mathrm{KdV}},$$

whence

$$\mathcal{E}_{\mathrm{KdV}}(\phi_\eta) \geqslant (1 + O(\mu^{2\alpha}))^{5/3}c_{\mathrm{KdV}} = c_{\mathrm{KdV}} + o(1)$$

because $\|\phi_\eta\|_0^2 = 4(\omega^2+4)^{-1/2} + O(\mu^{2\alpha})$ (see proposition 5.4), and it follows from (5.2) that

$$c_\mu \geqslant 2\nu_0\mu + \mu^{5/3}c_{\mathrm{KdV}} + o(\mu^{5/3}).$$

The complementary estimate

$$c_\mu \leqslant 2\nu_0\mu + \mu^{5/3}c_{\mathrm{KdV}} + o(\mu^{5/3})$$

is a consequence of lemma A.1. □

We now present our main convergence result.

Theorem 5.6. *The set B_μ of minimizers of $\mathcal{J}_\mu$ over $U \setminus \{0\}$ satisfies*

$$\sup_{\eta \in B_\mu}\inf_{x \in \mathbb{R}}\|\phi_\eta - \phi_{\mathrm{KdV}}(\cdot + x)\|_1 \to 0$$

as $\mu \downarrow 0$.

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Proof. Suppose that the limit is positive so that there exists $\varepsilon > 0$ and a sequence $\{\mu_m\}$ with $\mu_m \downarrow 0$ such that

$$\sup_{\eta \in C_{\mu_m}} \inf_{x\in\mathbb{R}} \|\phi_\eta - \phi_{\mathrm{KdV}}(\cdot + x)\|_1 \geqslant \varepsilon, \quad m \in \mathbb{N},$$

and hence a further sequence $\{\eta_m\} \subset U \setminus \{0\}$ with $\eta_m \in C_{\mu_m}$ and

$$\mathrm{dist}(\phi_{\eta_m}, D_{\mathrm{KdV}}) = \inf_{x\in\mathbb{R}} \|\phi_\eta - \phi_{\mathrm{KdV}}(\cdot + x)\|_1 \geqslant \tfrac{1}{2}\varepsilon, \quad m \in \mathbb{N}.$$

On the other hand, $\mathcal{E}_{\mathrm{KdV}}(\phi_{\eta_m}) \to c_{\mathrm{KdV}}$ and $\|\phi_{\eta_m}\|_0^2 \to 4(\omega^2+4)^{-1/2}$ as $n \to \infty$ (see proposition 5.4 and theorem 5.5(ii)); combining lemma 1.2(ii) with a straightforward scaling argument, we arrive at the contradiction of the existence of a sequence $\{x_m\} \subset \mathbb{R}$ such that a subsequence of $\{\phi_{\eta_m}(x_m + \cdot)\}$ converges in $H^1(\mathbb{R})$ to an element of D_{KdV}. □

Remark 5.7. The previous theorem implies that $\{\|\phi_\eta\|_1 \colon \eta \in B_\mu\}$ is bounded, so that

$$\begin{aligned}
\|\hat{\eta}_1\|_{L^1(\mathbb{R})}^2 &\leqslant \biggl(\int_{-\infty}^{\infty} \frac{1}{1+\mu^{-2/3}k^2}\,\mathrm{d}k\biggr)\biggl(\int_{-\infty}^{\infty} (1+\mu^{-2/3}k^2)|\hat{\eta}_1(k)|^2\,\mathrm{d}k\biggr) \\
&= \mu^{2/3}\biggl(\int_{-\infty}^{\infty} \frac{1}{1+\mu^{-2/3}k^2}\,\mathrm{d}k\biggr)\biggl(\int_{-\infty}^{\infty} (1+\mu^{-2/3}k^2)\biggl|\hat{\phi}_\eta\biggl(\frac{k}{\mu^{1/3}}\biggr)\biggr|^2\,\mathrm{d}k\biggr) \\
&= 2\pi\mu^{4/3}\|\phi_\eta\|_1^2 \\
&\leqslant c\mu^{4/3},
\end{aligned}$$

and hence $\|\eta_1\|_{1,\infty}$, $\|K^0\eta_1\|_\infty \leqslant c\mu^{2/3}$ (see (4.4) and (4.5)) and it follows from (4.13) and (4.14) that

$$|\!|\!|\eta_1|\!|\!|_{1/3}^2 \leqslant c\mu, \qquad \|\eta_2\|_2^2 \leqslant \mu^{7/3}.$$

For $\eta \in B_\mu$, lemma 4.10 therefore also holds with $\alpha = \frac{1}{3}$ (the result predicted in the Korteweg–de Vries scaling limit).

Our final result shows that the speed ν_μ of a solitary wave corresponding to $\eta \in B_\mu$, which is given by the formula

$$\nu_\mu = \frac{\mu + \mathcal{G}(\eta)}{\mathcal{L}(\eta)},$$

satisfies

$$\nu_\mu = \nu_0 + 2(\omega^2+4)^{-1/2}\nu_{\mathrm{KdV}}\mu^{2/3} + o(\mu^{2/3})$$

uniformly over $\eta \in B_\mu$.

Theorem 5.8. *The set B_μ of minimizers of $\mathcal{J}_\mu$ over $U \setminus \{0\}$ satisfies*

$$\sup_{\eta\in B_\mu}\biggl|\frac{\mu + \mathcal{G}(\eta)}{\mathcal{L}(\eta)} - (\nu_0 + 2(\omega^2+4)^{-1/2}\nu_{\mathrm{KdV}}\mu^{2/3})\biggr| = o(\mu^{2/3}).$$

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Proof. Using the identity

$$\frac{\mu+\mathcal{G}(\eta)}{\mathcal{L}(\eta)}=\frac{1}{2\mu}(c_\mu-\mathcal{M}_\mu(\eta))+\frac{1}{4\mu}(\langle\mathcal{M}_\mu'(\eta),\eta\rangle+4\mu\tilde{\mathcal{M}}_\mu(\eta))$$

(see the proof of proposition 4.2), we find that

$$\begin{aligned}\frac{\mu+\mathcal{G}(\eta)}{\mathcal{L}(\eta)}&=\nu_0+\tfrac{1}{2}c_{\mathrm{KdV}}\mu^{2/3}+\frac{1}{8\mu}\left(\frac{\omega^2}{3}+1\right)\int_{-\infty}^{\infty}\eta_1^3\,\mathrm{d}x+o(\mu^{2/3})\\&=\nu_0+\tfrac{1}{2}c_{\mathrm{KdV}}\mu^{2/3}+\frac{1}{8}\left(\frac{\omega^2}{3}+1\right)\mu^{2/3}\int_{-\infty}^{\infty}\phi_\eta^3\,\mathrm{d}x+o(\mu^{2/3})\\&=\nu_0+\tfrac{1}{2}\mathcal{E}_{\mathrm{KdV}}(\phi_{\mathrm{KdV}})\mu^{2/3}+\frac{1}{8}\left(\frac{\omega^2}{3}+1\right)\mu^{2/3}\int_{-\infty}^{\infty}\phi_{\mathrm{KdV}}^3\,\mathrm{d}x+o(\mu^{2/3})\\&=\nu_0+\tfrac{1}{4}\mu^{2/3}\underbrace{\int_{-\infty}^{\infty}\left(\left(\beta-\frac{\nu_0^2}{3}\right)(\phi_{\mathrm{KdV}}')^2+\frac{3}{2}\left(\frac{\omega^2}{3}+1\right)\phi_{\mathrm{KdV}}^3\right)\mathrm{d}x}_{=8(\omega^2+4)^{-1/2}\nu_{\mathrm{KdV}}}+o(\mu^{2/3})\\&=\nu_0+2(\omega^2+4)^{-1/2}\nu_{\mathrm{KdV}}\mu^{2/3}+o(\mu^{2/3}),\end{aligned}$$

in which theorem 5.5(i), corollary 4.14 and theorem 5.6 have been used. □

5.2.2. *The case $\beta<\beta_{\mathrm{c}}$*

Suppose that η is a minimizer of $\mathcal{J}_\mu$ over $U\setminus\{0\}$, write $\eta=\eta_1-H(\eta_1)+\eta_3$ and $\eta_1=\eta_1^++\eta_1^-$ according to the decompositions introduced in § 4.3, and define $\phi_\eta\in H^2(\mathbb{R})$ by the formula

$$\eta_1^+(x)=\tfrac{1}{2}\mu\phi_\eta(\mu x)\mathrm{e}^{\mathrm{i}k_0x}.$$

In this section we prove that $\mathrm{dist}(\phi_\eta,D_{\mathrm{NLS}})\to 0$ as $\mu\downarrow 0$, uniformly over $\eta\in B_\mu$, where D_{NLS} is the set of solitary-wave solutions to the nonlinear Schrödinger equation and 'dist' denotes the distance in $H^1(\mathbb{R})$.

Remark 5.9. Note that

$$\begin{Bmatrix}\mathcal{K}_2(\eta)\\\mathcal{G}_2(\eta)\\\mathcal{L}_2(\eta)\end{Bmatrix}=\begin{Bmatrix}\mathcal{K}_2(\eta_1)\\\mathcal{G}_2(\eta_1)\\\mathcal{L}_2(\eta_1)\end{Bmatrix}+\begin{Bmatrix}\mathcal{K}_2(-H(\eta)+\eta_3)\\\mathcal{G}_2(-H(\eta)+\eta_3)\\\mathcal{L}_2(-H(\eta)+\eta_3)\end{Bmatrix}\tag{5.3}$$

because $\hat{\eta}_1$ and $\mathcal{F}[-H(\eta)+\eta_3]$ have disjoint supports.

Our first result concerns the convergence of the $L^2(\mathbb{R})$-norm of minimizers of $\mathcal{J}_\mu$ over $U_2\setminus\{0\}$.

Proposition 5.10. *The estimate $\|\phi_\eta\|_0^2=(\frac{1}{4}\nu_0f(k_0)+\frac{1}{8}\omega)^{-1}+O(\mu^\alpha)$ holds for each $\eta\in B_\mu$.*

Proof. It follows from

$$\left|\frac{\mu+\mathcal{G}_2(\eta)}{\mathcal{L}_2(\eta)}-\nu_0\right|\leqslant c\mu^{1+\alpha},\quad \mathcal{L}_2(\eta)\leqslant c\mu,$$

that

$$\nu_0 \mathcal{L}_2(\eta) - \mathcal{G}_2(\eta) = \mu + O(\mu^{2+\alpha}). \tag{5.4}$$

On the other hand,

$$\begin{aligned} \nu_0 \mathcal{L}_2(\eta) - \mathcal{G}_2(\eta) &= \nu_0 \mathcal{L}_2(\eta_1) - \mathcal{G}_2(\eta_1) + O(\|H(\eta)\|_2^2 + \|\eta_3\|_2^2) \\ &= \nu_0 \mathcal{L}_2(\eta_1) - \mathcal{G}_2(\eta_1) + O(\mu^{2+\alpha}) \\ &= \nu_0 \int_{-\infty}^{\infty} \eta_1^+ K^0 \eta_1^- \, \mathrm{d}x + \frac{\omega}{2} \int_{-\infty}^{\infty} \eta_1^+ \eta_1^- \, \mathrm{d}x + O(\mu^{2+\alpha}) \\ &= \left(\nu_0 f(k_0) + \frac{\omega}{2} \right) \int_{-\infty}^{\infty} \eta_1^+ \eta_1^- \, \mathrm{d}x + O(\mu^{1+\alpha}) \\ &= \left(\frac{1}{4} \nu_0 f(k_0) + \frac{\omega}{8} \right) \mu \int_{-\infty}^{\infty} |\phi_\eta|^2 \, \mathrm{d}x + O(\mu^{1+\alpha}), \end{aligned}$$

and the result is obtained by combining this estimate with (5.4). □

The next step is to show that the nonlinear Schrödinger energy $\mathcal{E}_{\mathrm{NLS}}(\phi_\eta)$ corresponding to a minimizer η of $\mathcal{J}_\mu$ over $U \setminus \{0\}$ approaches c_{NLS} in the limit $\mu \downarrow 0$.

Theorem 5.11.

(i) *The number c_μ satisfies $c_\mu = 2\nu_0 \mu + c_{\mathrm{NLS}} \mu^3 + o(\mu^3)$.*

(ii) *Each $\eta \in B_\mu$ satisfies $\mathcal{E}_{\mathrm{NLS}}(\phi_\eta) \to c_{\mathrm{NLS}}$ as $\mu \downarrow 0$.*

Proof. Notice that

$$\begin{aligned} c_\mu &= \mathcal{J}_\mu(\eta) \\ &= \mathcal{K}_2(\eta) + \frac{(\mu + \mathcal{G}_2(\eta))^2}{\mathcal{L}_2(\eta)} + \mathcal{M}_\mu(\eta) \\ &= 2\nu_0 \mu + \mathcal{K}_2(\eta) + 2\nu_0 \mathcal{G}_2(\eta) - \nu_0^2 \mathcal{L}_2(\eta) + \left(\frac{\mu + \mathcal{G}_2(\eta)}{\sqrt{\mathcal{L}_2(\eta)}} - \nu_0 \sqrt{\mathcal{L}_2(\eta)} \right)^2 + \mathcal{M}_\mu(\eta) \\ &\geqslant 2\nu_0 \mu + \mathcal{K}_2(\eta) + 2\nu_0 \mathcal{G}_2(\eta) - \nu_0^2 \mathcal{L}_2(\eta) + \mathcal{M}_\mu(\eta), \end{aligned} \tag{5.5}$$

where

$$\begin{aligned} &\mathcal{K}_2(\eta) + 2\nu_0 \mathcal{G}_2(\eta) - \nu_0^2 \mathcal{L}_2(\eta) \\ &\qquad = (\mathcal{K}_2 + 2\nu_0 \mathcal{G}_2 - \nu_0^2 \mathcal{L}_2)(\eta_1) + (\mathcal{K}_2 + 2\nu_0 \mathcal{G}_2 - \nu_0^2 \mathcal{L}_2)(-H(\eta) + \eta_3). \end{aligned} \tag{5.6}$$

The second term on the right-hand side of (5.6) is estimated using the calculation

$$\begin{aligned} &(\mathcal{K}_2 + 2\nu_0 \mathcal{G}_2 - \nu_0^2 \mathcal{L}_2)(-H(\eta) + \eta_3) \\ &\qquad = (\mathcal{K}_2 + 2\nu_0 \mathcal{G}_2 - \nu_0^2 \mathcal{L}_2)(H(\eta)) + O(\|H(\eta)\|_2 \|\eta_3\|_2) + O(\|\eta_3\|_2^2) \\ &\qquad = \frac{1}{2} \int_{-\infty}^{\infty} g(k) |\mathcal{F}[H(\eta)]|^2 \, \mathrm{d}k + o(\mu^3) \\ &\qquad = \frac{1}{2} \int_{-\infty}^{\infty} g(k)^{-1} |\mathcal{F}[\mathcal{K}_3(\eta_1) + 2\nu_0 \mathcal{G}_3(\eta_1) - \nu_0^2 \mathcal{L}_3(\eta_1)]|^2 \, \mathrm{d}k + o(\mu^3) \end{aligned}$$

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$$\begin{aligned}
&= -\tfrac{1}{2}(\mathcal{K}_3(\eta) + 2\nu_0\mathcal{G}_3(\eta) - \nu_0^2\mathcal{L}_3(\eta)) + o(\mu^3)\\
&= -\tfrac{1}{2}A_3\int_{-\infty}^{\infty}\eta_1^4\,\mathrm{d}x + o(\mu^3)\\
&= -\tfrac{3}{16}A_3\mu^3\int_{-\infty}^{\infty}|\phi_\eta|^4\,\mathrm{d}x + o(\mu^3),
\end{aligned}$$

where we have used proposition 4.27, (4.20) and proposition 4.28. Turning to the first term on the right-hand side of (5.6), write

$$(\mathcal{K}_2 + 2\nu_0\mathcal{G}_2 - \nu_0^2\mathcal{L}_2)(\eta_1) = \frac{1}{2}\int_{-\infty}^{\infty} g(k)|\hat{\eta}_1|^2\,\mathrm{d}k = \int_{-\infty}^{\infty} g(k)|\hat{\eta}_1^+(k)|^2\,\mathrm{d}k$$

and note that

$$g(k) = \tfrac{1}{2}g''(k_0)(k-k_0)^2 + O(|k-k_0|^3), \quad k \in [k_0-\delta_0, k_0+\delta_0].$$

One finds that

$$\begin{aligned}
\int_{-\infty}^{\infty}(k-k_0)^2|\hat{\eta}_1^+(k)|^2\,\mathrm{d}k &= \int_{-\infty}^{\infty}k^2|\hat{\eta}_1^+(k+k_0)|^2\,\mathrm{d}k\\
&= \frac{\mu^2}{4}\int_{-\infty}^{\infty}\left|\frac{\mathrm{d}}{\mathrm{d}x}\phi_\eta(\mu x)\right|^2\mathrm{d}x = \frac{\mu^3}{4}\int_{-\infty}^{\infty}|\phi_\eta'|^2\,\mathrm{d}x
\end{aligned}$$

(because $\hat{\eta}_1^+(k+k_0) = \frac{1}{2}\mu\mathcal{F}[\phi_\eta(\mu x)]$) and

$$\int_{-\infty}^{\infty}(k-k_0)^3|\hat{\eta}_1^+(k)|^2\,\mathrm{d}k \leqslant c\mu^{3\alpha}|||\eta_1|||_\alpha^2 = O(\mu^{1+3\alpha}),$$

so that

$$\int_{-\infty}^{\infty}(g(k) - \tfrac{1}{2}(k-k_0)^2)|\hat{\eta}_1^+(k)|^2\,\mathrm{d}k = o(\mu^3).$$

Altogether these calculations show that

$$\begin{aligned}
&(\mathcal{K}_2 + 2\nu_0\mathcal{G}_2 - \nu_0^2\mathcal{L}_2)(\eta_1)\\
&\quad = \tfrac{1}{8}g''(k_0)\mu^3\int_{-\infty}^{\infty}|\phi_\eta'|^2\,\mathrm{d}x - \frac{3A_3}{16}\mu^3\int_{-\infty}^{\infty}|\phi_\eta|^4\,\mathrm{d}x + o(\mu^3).
\end{aligned} \tag{5.7}$$

Substituting (5.7) and

$$\mathcal{M}_\mu(\eta) = (A_3 + A_4)\int_{-\infty}^{\infty}\eta_1^4\,\mathrm{d}x + o(\mu^3) = \tfrac{3}{8}(A_3+A_4)\mu^3\int_{-\infty}^{\infty}|\phi_\eta|^4\,\mathrm{d}x + o(\mu^3)$$

(see corollary 4.30) into inequality (5.5) yields

$$\begin{aligned}
c_\mu &\geqslant 2\nu_0\mu + \tfrac{1}{8}g''(k_0)\mu^3\int_{-\infty}^{\infty}|\phi_\eta'|^2\,\mathrm{d}x + \tfrac{3}{8}(\tfrac{1}{2}A_3 + A_4)\mu^3\int_{-\infty}^{\infty}|\phi_\eta|^4\,\mathrm{d}x + o(\mu^3)\\
&= 2\nu_0\mu + \mu^3\mathcal{E}_{\mathrm{NLS}}(\phi_\eta) + o(\mu^3),
\end{aligned} \tag{5.8}$$

and combining this estimate with lemma A.2 yields

$$\mathcal{E}_{\mathrm{NLS}}(\phi_\eta) \leqslant c_{\mathrm{NLS}} + o(1).$$

A straightforward scaling argument shows that

$$\inf\{\mathcal{E}_{\mathrm{NLS}}(\phi)\colon \phi \in H^1(\mathbb{R}),\ \|\phi\|_0^2 = (\tfrac{1}{4}\nu_0 f(k_0) + \tfrac{1}{8}\omega)^{-1}a\} = a^3 c_{\mathrm{NLS}},$$

whence

$$\mathcal{E}_{\mathrm{NLS}}(\phi_\eta) \geqslant (1 + O(\mu^\alpha))^3 c_{\mathrm{NLS}} = c_{\mathrm{NLS}} + o(1)$$

because $\|\phi_\eta\|_0^2 = (\frac{1}{4}\nu_0 f(k_0) + \frac{1}{8}\omega)^{-1} + O(\mu^\alpha)$ (see proposition 5.10), and it follows from (5.8) that

$$c_\mu \geqslant 2\nu_0\mu + \mu^3 c_{\mathrm{NLS}} + o(\mu^3).$$

The complementary estimate

$$c_\mu \leqslant 2\nu_0\mu + \mu^3 c_{\mathrm{NLS}} + o(\mu^3)$$

is a consequence of lemma A.2. □

Our main convergence result is derived from theorem 5.11 in the same way as the corresponding result for $\beta > \beta_{\mathrm{c}}$ (see Appendix A.1).

THEOREM 5.12. *The set B_μ of minimizers of $\mathcal{J}_\mu$ over $U \setminus \{0\}$ satisfies*

$$\sup_{\eta \in B_\mu} \inf_{\substack{\omega \in [0,2\pi],\\ x \in \mathbb{R}}} \|\phi_\eta - \mathrm{e}^{\mathrm{i}\omega}\phi_{\mathrm{NLS}}(\cdot + x)\|_1 \to 0$$

as $\mu \downarrow 0$.

REMARK 5.13. The previous theorem implies that $\{\|\phi_\eta\|_1\colon \eta \in B_\mu\}$ is bounded, so that

$$\begin{aligned}
\|\hat{\eta}_1\|_{L^1(\mathbb{R})}^2 &\leqslant 2\biggl(\int_{k_0-\delta_0}^{k_0+\delta_0} \frac{1}{1 + \mu^{-2}(k - k_0)^2}\,\mathrm{d}k\biggr) \\
&\qquad \times \biggl(\int_{k_0-\delta_0}^{k_0+\delta_0} (1 + \mu^{-2}(k - k_0)^2)|\hat{\eta}_1(k)|^2\,\mathrm{d}k\biggr) \\
&\leqslant 2\biggl(\int_{-\infty}^{\infty} \frac{1}{1 + \mu^{-2}(k - k_0)^2}\,\mathrm{d}k\biggr) \\
&\qquad \times \biggl(\int_{-\infty}^{\infty} (1 + \mu^{-2}(k - k_0)^2)\biggl|\hat{\phi}_\eta\biggl(\frac{k - k_0}{\mu}\biggr)\biggr|^2\,\mathrm{d}k\biggr) \\
&= 2\pi\mu^2\|\phi_\eta\|_1^2 \\
&\leqslant c\mu^2,
\end{aligned}$$

and hence $\|\eta_1\|_{1,\infty}$, $\|K^0\eta_1\|_{1,\infty} \leqslant c\mu$ (see (4.4) and (4.5)) and it follows from proposition 4.16 and inequalities (4.18), (4.19) that

$$|||\eta_1|||_1^2 \leqslant c\mu, \qquad \|H(\eta_1)\|_2^2 \leqslant c\mu^3, \qquad \|u_3\|_2^2 \leqslant c\mu^5.$$

For $\eta \in B_\mu$, lemma 4.21 therefore also holds with $\alpha = 1$ (the result predicted in the nonlinear Schrödinger scaling limit).

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Our final result shows that the speed ν_μ of a solitary wave corresponding to $\eta \in B_\mu$, which is given by the formula

$$\nu_\mu + \frac{\mu + \mathcal{G}(\eta)}{\mathcal{L}(\eta)},$$

satisfies

$$\nu_\mu = \nu_0 + 4(\omega + 2\nu_0 f(k_0))^{-1}\nu_{\mathrm{NLS}}\mu^2 + o(\mu^2)$$

uniformly over $\eta \in B_\mu$.

THEOREM 5.14. *The set B_μ of minimizers of $\mathcal{J}_\mu$ over $U \setminus \{0\}$ satisfies*

$$\sup_{\eta \in B_\mu} \left| \frac{\mu + \mathcal{G}(\eta)}{\mathcal{L}(\eta)} - (\nu_0 + 4(\omega + 2\nu_0 f(k_0))^{-1}\nu_{\mathrm{NLS}}\mu^2) \right| = o(\mu^2).$$

Proof. Using the identity

$$\frac{\mu + \mathcal{G}(\eta)}{\mathcal{L}(\eta)} = \frac{1}{2\mu}(c_\mu - \mathcal{M}_\mu(\eta)) + \frac{1}{4\mu}(\langle \mathcal{M}_\mu'(\eta), \eta\rangle + 4\mu\tilde{\mathcal{M}}_\mu(\eta))$$

(see the proof of proposition 4.2), we find that

$$\begin{aligned}
\frac{\mu + \mathcal{G}(\eta)}{\mathcal{L}(\eta)} &= \nu_0 + \tfrac{1}{2}c_{\mathrm{NLS}}\mu^2 + \frac{1}{2\mu}\left(\frac{A_3}{2} + A_4\right)\int_{-\infty}^{\infty} \eta_1^4 \, \mathrm{d}x + o(\mu^2) \\
&= \nu_0 + \tfrac{1}{2}c_{\mathrm{NLS}}\mu^2 + \frac{3}{16}\left(\frac{A_3}{2} + A_4\right)\mu^2 \int_{-\infty}^{\infty} |\phi_\eta|^4 \, \mathrm{d}x + o(\mu^2) \\
&= \nu_0 + \tfrac{1}{2}\mathcal{E}_{\mathrm{NLS}}(\phi_{\mathrm{NLS}})\mu^2 + \frac{3}{16}\left(\frac{A_3}{2} + A_4\right)\mu^2 \int_{-\infty}^{\infty} |\phi_{\mathrm{NLS}}|^4 \, \mathrm{d}x + o(\mu^2) \\
&= \nu_0 + \tfrac{1}{4}\mu^2 \underbrace{\int_{-\infty}^{\infty} (\tfrac{1}{4}g''(k_0)|\phi_{\mathrm{NLS}}'|^2 + \tfrac{3}{2}(\tfrac{1}{2}A_3 + A_4)|\phi_{\mathrm{NLS}}|^4) \, \mathrm{d}x}_{=2\left(\frac{1}{4}\nu_0 f(k_0) + \frac{1}{8}\omega\right)^{-1}\nu_{\mathrm{NLS}}} + o(\mu^2) \\
&= \nu_0 + 4(\omega + 2\nu_0 f(k_0))^{-1}\nu_{\mathrm{NLS}}\mu^2 + o(\mu^2),
\end{aligned}$$

in which theorem 5.11(i), corollary 4.30 and theorem 5.12 have been used. □

Appendix A. Proof of lemma 3.2(i)

A.1. The case $\boldsymbol{\beta > \beta_c}$

LEMMA A.1. *Suppose that $\mu > 0$. There exists a continuous invertible mapping $\mu \to \alpha(\mu)$ such that*

$$\mathcal{J}_\mu(\eta^\star) = 2\nu_0\mu + c_{\mathrm{KdV}}\mu^{5/3} + o(\mu^{5/3}),$$

where

$$\eta^\star(x) = \alpha^2 \phi_{\mathrm{KdV}}(\alpha x).$$

Proof. Let us first note that

$$K^0\eta^\star - \eta^\star + \tfrac{1}{3}(\eta^\star)'' = \mathcal{F}^{-1}[\underbrace{(|k|\coth|k| - 1 - \tfrac{1}{3}|k|^2)}_{\leqslant c|k|^4}\hat{\eta}^\star] = \underline{O}(\alpha^{11/2}),$$

and hence

$$K^0\eta^\star - \eta^\star = \mathcal{F}^{-1}[(|k|\coth|k| - 1)\hat{\eta}^\star] = \underline{O}(\alpha^{7/2}).$$

Using these estimates and $\|\eta^\star\|_0 = O(\alpha^{3/2})$, one finds that

$$\begin{aligned}
\mathcal{K}_2(\eta^\star) &= \frac{1}{2}\alpha^3 \int_{-\infty}^{\infty} \phi_{\mathrm{KdV}}^2 \,\mathrm{d}x + \frac{1}{2}\alpha^5\beta \int_{-\infty}^{\infty} \phi_{\mathrm{KdV}}'^2 \,\mathrm{d}x,\\
\mathcal{G}_2(\eta^\star) &= -\frac{1}{4}\alpha^3\omega \int_{-\infty}^{\infty} \phi_{\mathrm{KdV}}^2 \,\mathrm{d}x,\\
\mathcal{L}_2(\eta^\star) &= \frac{1}{2}\int_{-\infty}^{\infty} \eta^\star K^0\eta^\star \,\mathrm{d}x\\
&= \frac{1}{2}\alpha^3 \int_{-\infty}^{\infty} \phi_{\mathrm{KdV}}^2 \,\mathrm{d}x + \frac{1}{6}\alpha^5 \int_{-\infty}^{\infty} \phi_{\mathrm{KdV}}'^2 \,\mathrm{d}x + O(\alpha^7),
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{K}_3(\eta^\star) &= \frac{1}{6}\alpha^5\omega^2 \int_{-\infty}^{\infty} \phi_{\mathrm{KdV}}^3 \,\mathrm{d}x,\\
\mathcal{G}_3(\eta^\star) &= \frac{1}{4}\omega \int_{-\infty}^{\infty} (\eta^\star)^2 K^0\eta^\star \,\mathrm{d}x\\
&= \frac{1}{4}\omega \int_{-\infty}^{\infty} (\eta^\star)^3 \,\mathrm{d}x + \frac{1}{4}\omega \int_{-\infty}^{\infty} (\eta^\star)^2 (K^0\eta^\star - \eta^\star) \,\mathrm{d}x\\
&= \frac{1}{4}\alpha^5\omega \int_{-\infty}^{\infty} \phi_{\mathrm{KdV}}^3 \,\mathrm{d}x + O(\alpha^7),\\
\mathcal{L}_3(\eta^\star) &= \frac{1}{2}\int_{-\infty}^{\infty} (-(K^0\eta^\star)^2\eta^\star + (\eta^{\star\prime})^2\eta^\star) \,\mathrm{d}x\\
&= -\frac{1}{2}\int_{-\infty}^{\infty} (\eta^\star)^3 \,\mathrm{d}x\\
&\qquad + \frac{1}{2}\int_{-\infty}^{\infty} (-2(K^0\eta^\star - \eta^\star)(\eta^\star)^2 - (K^0\eta^\star - \eta^\star)^2\eta^\star + (\eta^{\star\prime})^2\eta^\star) \,\mathrm{d}x\\
&= -\frac{1}{2}\alpha^5 \int_{-\infty}^{\infty} \phi_{\mathrm{KdV}}^3 \,\mathrm{d}x + O(\alpha^7),
\end{aligned}$$

in which the further estimate $\|\eta^\star\|_\infty = O(\alpha^2)$ has been used (see proposition 4.3 for the formulas for $\mathcal{G}_3$, $\mathcal{K}_3$ and $\mathcal{L}_3$). Finally, proposition 4.4 shows that $\mathcal{G}_4(\eta^\star)$, $\mathcal{K}_4(\eta^\star)$, $\mathcal{L}_4(\eta^\star)$ and $\mathcal{G}_{\mathrm{r}}(\eta^\star)$, $\mathcal{K}_{\mathrm{r}}(\eta^\star)$, $\mathcal{L}_{\mathrm{r}}(\eta^\star)$ are all $O(\alpha^7)$.

The above calculations show that

$$\begin{aligned}
&\mathcal{K}(\eta^\star) + 2\nu_0\mathcal{G}(\eta^\star) - \nu_0^2\mathcal{L}(\eta^\star)\\
&\qquad = \frac{\alpha^3}{2}\underbrace{(1-\omega\nu_0-\nu_0^2)}_{=0}\int_{-\infty}^{\infty}\phi_{\mathrm{KdV}}^2\,\mathrm{d}x + \frac{1}{2}\left(\beta-\frac{\nu_0^2}{3}\right)\alpha^5\int_{-\infty}^{\infty}\phi_{\mathrm{KdV}}'^2\,\mathrm{d}x\\
&\qquad\qquad + \frac{1}{2}\left(\frac{\omega^2}{3}+\underbrace{\omega\nu_0+\nu_0^2}_{=1}\right)\alpha^5\int_{-\infty}^{\infty}\phi_{\mathrm{KdV}}^3\,\mathrm{d}x + O(\alpha^7)\\
&\qquad = \alpha^5\mathcal{E}_{\mathrm{KdV}}(\phi_{\mathrm{KdV}}) + O(\alpha^7)\\
&\qquad = c_{\mathrm{KdV}}\alpha^5 + O(\alpha^7).
\end{aligned}$$

The mapping

$$\begin{aligned}
\alpha \mapsto{}& \nu_0\mathcal{L}(\eta^\star) - \mathcal{G}(\eta^\star)\\
&= \alpha^3\left(\frac{\nu_0}{2}+\frac{\omega}{4}\right)\int_{-\infty}^{\infty}\phi_{\mathrm{KdV}}^2\,\mathrm{d}x + O(\alpha^5)\\
&= \frac{\alpha^3}{4}\sqrt{\omega^2+4}\int_{-\infty}^{\infty}\phi_{\mathrm{KdV}}^2\,\mathrm{d}x + O(\alpha^5)
\end{aligned}$$

is continuous and strictly increasing and therefore has a continuous inverse $\mu \mapsto \alpha(\mu)$; furthermore, $\alpha(\mu) = \mu^{1/3} + o(\mu^{1/3})$ and

$$\mathcal{J}_\mu(\eta^\star) - 2\nu_0\mu = \mathcal{K}(\eta^\star) + 2\nu_0\mathcal{G}(\eta^\star) - \nu_0^2\mathcal{L}(\eta^\star) = c_{\mathrm{KdV}}\mu^{5/3} + o(\mu^{5/3}).$$

□

A.2. The case $\beta < \beta_c$

Lemma A.2. *Suppose that $\mu > 0$. There exists a continuous invertible mapping $\mu \to \alpha(\mu)$ such that*

$$\mathcal{J}_\mu(\eta^\star) = 2\nu_0\mu + c_{\mathrm{NLS}}\mu^3 + o(\mu^3),$$

where

$$\begin{aligned}
\eta^\star(x) &= \alpha\phi_{\mathrm{NLS}}(\alpha x)\cos k_0x\\
&\qquad - \frac{\alpha^2}{2}g(2k_0)^{-1}A_3^1\phi_{\mathrm{NLS}}(\alpha x)^2\cos 2k_0x - \frac{\alpha^2}{2}g(0)^{-1}A_3^2\phi_{\mathrm{NLS}}(\alpha x)^2.
\end{aligned}$$

Proof. We seek a test function $\eta^\star$ of the form

$$\eta^\star(x) = \alpha\phi_{\mathrm{NLS}}(\alpha x)\cos k_0x + \alpha^2\psi(\alpha x)\cos 2k_0x + \alpha^2\xi(\alpha x)$$

with $\psi, \xi \in \mathcal{S}(\mathbb{R})$.

Choose $n \in \mathbb{N}$ and $\chi \in C_0^\infty(\mathbb{R})$. Straightforward calculations yield the formulas

$$K^0(\chi(\alpha x)) = \chi(\alpha x) + S_1(x),$$

where

$$S_1(x) = \frac{1}{\alpha}\mathcal{F}^{-1}\left[(|k|\coth|k|-1)\hat{\chi}\left(\frac{k}{\alpha}\right)\right],$$

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and

$$K^0(\chi(\alpha x)\cos nk_0 x) = f(nk_0)\chi(\alpha x)\cos nk_0 x + \alpha f'(nk_0)\chi'(\alpha x)\sin nk_0 x - \tfrac{1}{2}\alpha^2 f''(nk_0)\chi''(\alpha x)\cos nk_0 x + S_2(x),$$

where

$$S_2(x) = \tfrac{1}{2}\mathcal{F}^{-1}\biggl[R_{nk_0}(k)(k-nk_0)^3\hat{\chi}\biggl(\frac{k-nk_0}{\alpha}\biggr)\biggr] + \tfrac{1}{2}\mathcal{F}^{-1}\biggl[R_{-nk_0}(k)(k+nk_0)^3\hat{\chi}\biggl(\frac{k+nk_0}{\alpha}\biggr)\biggr]$$

and $R_\omega(k) = \frac{1}{6}f'''(k_\omega)$ for some k_ω between k and ω; the remainder terms S_1 and S_2 satisfy the estimates $\|S_1^{(m)}\|_0 = O(\alpha^{m+3/2})$ and $\|S_2\|_\infty = O(\alpha^3)$, $\|S_2\|_1 = O(\alpha^{7/2})$. Furthermore, repeated integration by parts shows that

$$\int_{-\infty}^{\infty} \chi(\alpha x)\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(mx)\,\mathrm{d}x = O(\alpha^n)$$

for each $m \in \mathbb{N}$, so that

$$\int_{-\infty}^{\infty} \chi(\alpha x)\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(m_1 x)\cdots\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(m_\ell x)\,\mathrm{d}x = O(\alpha^n)$$

for all $m_1,\dots,m_\ell \in \mathbb{N}$ with $m_1 \pm \cdots \pm m_\ell \neq 0$.

Estimating using the above rules, one finds that

$$\begin{aligned}
\mathcal{K}_2(\eta^\star) &= \frac{\alpha}{4}(1+\beta k_0^2)\int_{-\infty}^{\infty}\phi_{\mathrm{NLS}}^2\,\mathrm{d}x + \frac{\alpha^3}{4}\beta\int_{-\infty}^{\infty}\phi_{\mathrm{NLS}}'^2\,\mathrm{d}x \\
&\quad + \frac{\alpha^3}{4}(1+4\beta k_0^2)\int_{-\infty}^{\infty}\psi^2\,\mathrm{d}x + \frac{\alpha^3}{2}\int_{-\infty}^{\infty}\xi^2\,\mathrm{d}x + O(\alpha^4),\\
\mathcal{G}_2(\eta^\star) &= -\frac{\alpha}{8}\omega\int_{-\infty}^{\infty}\phi_{\mathrm{NLS}}^2\,\mathrm{d}x - \frac{\alpha^3}{8}\omega\int_{-\infty}^{\infty}\psi^2\,\mathrm{d}x - \frac{\alpha^3}{4}\omega\int_{-\infty}^{\infty}\xi^2\,\mathrm{d}x + O(\alpha^4),\\
\mathcal{L}_2(\eta^\star) &= \frac{\alpha}{4}f(k_0)\int_{-\infty}^{\infty}\phi_{\mathrm{NLS}}^2\,\mathrm{d}x + \frac{\alpha^3}{8}f''(k_0)\int_{-\infty}^{\infty}\phi_{\mathrm{NLS}}'^2\,\mathrm{d}x \\
&\quad + \frac{\alpha^3}{4}f(2k_0)\int_{-\infty}^{\infty}\psi^2\,\mathrm{d}x + \frac{\alpha^3}{2}\int_{-\infty}^{\infty}\xi^2\,\mathrm{d}x + O(\alpha^4),\\
\mathcal{K}_3(\eta^\star) &= \frac{\alpha^3}{8}\omega^2\int_{-\infty}^{\infty}\phi_{\mathrm{NLS}}^2\psi\,\mathrm{d}x + \frac{\alpha^3}{4}\omega^2\int_{-\infty}^{\infty}\phi_{\mathrm{NLS}}^2\xi\,\mathrm{d}x + O(\alpha^4),\\
\mathcal{G}_3(\eta^\star) &= \frac{\alpha^3}{8}\biggl(f(k_0)+\frac{f(2k_0)}{2}\biggr)\omega\int_{-\infty}^{\infty}\phi_{\mathrm{NLS}}^2\psi\,\mathrm{d}x \\
&\quad + \frac{\alpha^3}{4}\biggl(f(k_0)+\frac{1}{2}\biggr)\omega\int_{-\infty}^{\infty}\phi_{\mathrm{NLS}}^2\xi\,\mathrm{d}x + O(\alpha^4),\\
\mathcal{L}_3(\eta^\star) &= \frac{\alpha^3}{4}\biggl(-f(k_0)f(2k_0) - \frac{f(k_0)^2}{2} + \frac{3k_0^2}{2}\biggr)\int_{-\infty}^{\infty}\phi_{\mathrm{NLS}}^2\psi\,\mathrm{d}x \\
&\quad + \frac{\alpha^3}{4}(-2f(k_0) - f(k_0)^2 + k_0^2)\int_{-\infty}^{\infty}\phi_{\mathrm{NLS}}^2\xi\,\mathrm{d}x + O(\alpha^4)
\end{aligned}$$

and

$$\mathcal{K}_4(\eta^\star) = -\frac{\alpha^3}{64}(3\beta k_0^4 + \omega^2(f(2k_0)+2))\int_{-\infty}^{\infty}\phi_{\mathrm{NLS}}^4\,\mathrm{d}x + O(\alpha^4),$$
$$\mathcal{G}_4(\eta^\star) = \frac{\alpha^3}{16}\left(k_0^2 - \frac{f(k_0)(f(2k_0)+2)}{2}\right)\omega\int_{-\infty}^{\infty}\phi_{\mathrm{NLS}}^4\,\mathrm{d}x + O(\alpha^4),$$
$$\mathcal{L}_4(\eta^\star) = \frac{\alpha^3}{16}(f(k_0)^2(f(2k_0)+2) - 3k_0^2 f(k_0))\int_{-\infty}^{\infty}\phi_{\mathrm{NLS}}^4\,\mathrm{d}x + O(\alpha^4)$$

(see proposition 4.3 for the formulas for $\mathcal{K}_3$, $\mathcal{G}_3$, $\mathcal{L}_3$ and $\mathcal{K}_4$, $\mathcal{G}_4$, $\mathcal{L}_4$). Finally, observe that

$$\begin{aligned}\eta^{\star\prime\prime}(x) + k_0^2\eta^\star(x) = \alpha^3\phi_{\mathrm{NLS}}''(\alpha x)\cos k_0 x - 2\alpha^2 k_0\phi_{\mathrm{NLS}}'(\alpha x)\sin k_0 x\\ + \alpha^4\psi''(\alpha x)\cos 2k_0 x - 4\alpha^3 k_0\psi'(\alpha x)\sin 2k_0 x\\ - 3k_0^2\alpha^2\psi(\alpha x)\cos 2k_0 x + \alpha^4\xi''(\alpha x),\end{aligned}$$

so that $\|\eta^{\star\prime\prime} + k_0^2\eta^\star\|_0 = O(\alpha^{3/2})$, and using the further estimates $\|\eta^\star\|_2 = O(\alpha^{1/2})$ and $\|\eta^\star\|_{1,\infty} = O(\alpha)$, one finds from proposition 4.4 that $\mathcal{K}_{\mathrm{r}}(\eta^\star)$, $\mathcal{G}_{\mathrm{r}}(\eta^\star)$, $\mathcal{L}_{\mathrm{r}}(\eta^\star)$ are all $O(\alpha^{7/2})$.

The above calculations show that

$$\begin{aligned}&\mathcal{K}(\eta^\star) + 2\nu_0\mathcal{G}(\eta^\star) - \nu_0^2\mathcal{L}(\eta^\star)\\
&\quad= \frac{\alpha^3}{8}(2\beta - \nu_0^2 f''(k_0))\int_{-\infty}^{\infty}\phi_{\mathrm{NLS}}'^2\,\mathrm{d}x + \frac{\alpha^3}{4}\int_{-\infty}^{\infty}(g(2k_0)\psi^2 + A_3^1\phi_{\mathrm{NLS}}^2\psi)\,\mathrm{d}x\\
&\qquad+ \frac{\alpha^3}{2}\int_{-\infty}^{\infty}(g(0)\xi^2 + A_3^2\phi_{\mathrm{NLS}}^2\xi)\,\mathrm{d}x + \frac{3\alpha^3}{8}A_4\int_{-\infty}^{\infty}\phi_{\mathrm{NLS}}^4\,\mathrm{d}x + O(\alpha^{7/2})\\
&\quad= \frac{\alpha^3}{8}(2\beta - \nu_0^2 f''(k_0))\int_{-\infty}^{\infty}\phi_{\mathrm{NLS}}'^2\,\mathrm{d}x + \frac{\alpha^3}{4}g(2k_0)\int_{-\infty}^{\infty}\left(\psi + \frac{g(2k_0)^{-1}}{2}A_3^1\phi_{\mathrm{NLS}}^2\right)^2\mathrm{d}x\\
&\qquad+ \frac{\alpha^3}{4}g(0)\int_{-\infty}^{\infty}\left(\xi + \frac{g(0)^{-1}}{2}A_3^2\phi_{\mathrm{NLS}}^2\right)^2\mathrm{d}x\\
&\qquad+ \alpha^3\left(\frac{3}{8}A_4 - \frac{g(2k_0)^{-1}}{16}(A_3^1)^2 - \frac{g(0)^{-1}}{8}(A_3^2)^2\right)\int_{-\infty}^{\infty}\phi_{\mathrm{NLS}}^4\,\mathrm{d}x + O(\alpha^{7/2}),\end{aligned}$$

in which the second line follows from the first by the definitions of A_3^1, A_3^2, A_4 and the third from the second by completing the square. The choice

$$\psi = -\frac{g(2k_0)^{-1}}{2}A_3^1\phi_{\mathrm{NLS}}^2, \qquad \xi = -\frac{g(0)^{-1}}{2}A_3^1\phi_{\mathrm{NLS}}^2$$

therefore minimizes the value of $\mathcal{K}(\eta^\star)+2\nu_0\mathcal{G}(\eta^\star)-\nu_0^2\mathcal{L}(\eta^\star)$ up to $O(\alpha^{7/2})$, whereby

$$\begin{aligned}\mathcal{K}(\eta^\star) + 2\nu_0\mathcal{G}(\eta^\star) - \nu_0^2\mathcal{L}(\eta^\star) &= \alpha^3\mathcal{E}_{\mathrm{NLS}}(\phi_{\mathrm{NLS}}) + O(\alpha^{7/2})\\ &= c_{\mathrm{NLS}}\alpha^3 + O(\alpha^{7/2}).\end{aligned}$$

The mapping

$$\begin{aligned}\alpha \mapsto\ & \nu_0 \mathcal{L}(\eta^\star) - \mathcal{G}(\eta^\star)\\ &= \alpha\left(\frac{\nu_0}{4} f(k_0) + \frac{\omega}{8}\right) \int_{-\infty}^{\infty} \phi_{\mathrm{NLS}}^2 \, \mathrm{d}x + O(\alpha^2)\end{aligned}$$

is continuous and strictly increasing and therefore has a continuous inverse $\mu \mapsto \alpha(\mu)$; furthermore, $\alpha(\mu) = \mu + o(\mu)$ and

$$\mathcal{J}_\mu(\eta^\star) - 2\nu_0\mu = \mathcal{K}(\eta^\star) + 2\nu_0 \mathcal{G}(\eta^\star) - \nu_0^2 \mathcal{L}(\eta^\star) = c_{\mathrm{NLS}}\mu^3 + o(\mu^3).$$

□

Appendix B. The sign of $A_3 + 2A_4$

The quantities β, ω, k_0 and ν_0 are related by the fact that $g(k) \geqslant 0$ with equality precisely when $k = \pm k_0$. It follows from the simultaneous equations $g(k_0) = 0$, $g'(k_0) = 0$ that

$$\beta = \frac{\nu_0^2 f'(k_0)}{2k_0}, \qquad \omega = \frac{1 + \beta k_0^2 - \nu_0^2 f(k_0)}{\nu_0},$$

and inserting these expressions for β and ω into the formulas for A_3 and A_4 (see corollary 4.25 and proposition 4.28), one finds that

$$\nu_0^6 (A_3 + 2A_4) = a_8 \nu_0^8 + a_6 \nu_0^6 + a_4 \nu_0^4 + a_2 \nu_0^2 + a_0, \tag{B 1}$$

in which

$$\begin{aligned}
a_0 &= -\tfrac{1}{12} h_2(k_0)^{-1}(1 + 2h_1(k_0)),\\
a_2 &= -\tfrac{1}{3} h_2(k_0)^{-1}(\tfrac{1}{2} f(2k_0) + \tfrac{1}{2} k_0 f'(k_0) + 2h_1(k_0)(\tfrac{1}{2} + \tfrac{1}{2} k_0 f'(k_0))),\\
a_4 &= -\tfrac{1}{3} h_2(k_0)^{-1}((\tfrac{1}{2} f(2k_0) + \tfrac{1}{2} k_0 f'(k_0))^2 + 2h_1(k_0)(\tfrac{1}{2} + \tfrac{1}{2} k_0 f'(k_0))^2)\\
&\quad - 2(\tfrac{1}{12} + \tfrac{1}{24} f(2k_0)),\\
a_6 &= -\tfrac{2}{3} h_2(k_0)^{-1}(\tfrac{1}{2} f(k_0) f(2k_0) - \tfrac{3}{2} k_0^2 + \tfrac{1}{4} k_0 f'(k_0) f(2k_0) + \tfrac{1}{8} f'(k_0)^2)\\
&\qquad \times (\tfrac{1}{2} f(2k_0) + \tfrac{1}{2} k_0 f'(k_0))\\
&\qquad - \tfrac{4}{3} h_2(k_0)^{-1} h_1(k_0)(\tfrac{1}{4} k_0 f'(k_0) + \tfrac{1}{2} f(k_0) - \tfrac{1}{2} k_0^2 + \tfrac{1}{8} k_0^2 f'(k_0)^2)\\
&\qquad \times (\tfrac{1}{2} + \tfrac{1}{2} k_0 f'(k_0))\\
&\qquad + 2(-\tfrac{1}{24} k_0 f'(k_0) f(2k_0) + \tfrac{1}{3} k_0^2 - \tfrac{1}{12} k_0 f'(k_0) - \tfrac{1}{6} f(k_0) - \tfrac{1}{12} f(k_0) f(2k_0)),\\
a_8 &= -\tfrac{1}{3} h_2(k_0)^{-1}(\tfrac{1}{2} f(k_0) f(2k_0) - \tfrac{3}{2} k_0^2 + \tfrac{1}{4} f'(k_0) f(2k_0) + \tfrac{1}{8} f'(k_0)^2)^2\\
&\qquad - \tfrac{2}{3} h_2(k_0)^{-1} h_1(k_0)(\tfrac{1}{4} k_0 f'(k_0) + \tfrac{1}{2} f(k_0) - \tfrac{1}{2} k_0^2 + \tfrac{1}{8} k_0^2 f'(k_0)^2)^2\\
&\qquad - 2(\tfrac{1}{16} k_0^3 f'(k_0) + \tfrac{1}{6} f(k_0)^2 (f(k_0) + 2) - \tfrac{1}{2} k_0^2 f(k_0)\\
&\qquad\qquad - 2(\tfrac{1}{2} k_0 f'(k_0) - f(k_0))(\tfrac{1}{6} k_0^2 - \tfrac{1}{12} f(k_0)(f(2k_0) + 2))\\
&\qquad\qquad + \tfrac{1}{24}(\tfrac{1}{2} k_0 f'(k_0) - f(k_0))^2 (f(2k_0) + 2))
\end{aligned}$$

and

$$h_1(k_0) = \frac{-2f(2k_0) + 2f(k_0) + 3k_0 f'(k_0)}{-2 - k_0 f'(k_0) + 2f(k_0)}, \qquad h_2(k_0) = \tfrac{3}{2} k_0 f'(k_0) + f(k_0) - f(2k_0).$$

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The right-hand side of (B 1) defines a polynomial function of ν_0 with coefficients that depend upon k_0, and the following argument shows that it is negative for all positive values of ν_0.

First note that a_0, a_2 and a_4 are negative because

$$h_1(k_0) = g(0)^{-1} g(2k_0)^{-1} > 0, \qquad h_2(k_0) = \frac{g(2k_0)}{\nu_0^2} > 0.$$

A lengthy calculation shows that

$$a_8 = -\frac{k_0^3}{\sinh^6 k_0} \left(\sum_{j=0}^{\infty} \frac{a_{8,2j+1}}{(2j+1)!} k_0^{2j+1} \right)^{-1} \sum_{j=0}^{\infty} \frac{a_{8,2j}}{(2j)!} k_0^{2j},$$

in which explicit formulas for the coefficients $a_{8,j}$ are computed from the above expression for a_8. Elementary estimates are used to establish that $a_{8,j} > 0$, so that a_8 is also negative. The argument is completed by demonstrating that $4a_4a_8 - a_6^2$ is positive. For this purpose we use the calculation

$$4a_4a_8 - a_6^2 = \frac{k_0^4}{\sinh^8 k_0} \left(\sum_{j=0}^{\infty} \frac{b_j}{(2j)!} k_0^{2j} \right)^{-1} \sum_{j=0}^{\infty} \frac{c_j}{(2j)!} k_0^{2j}$$

with explicit formulas for the coefficients b_j and c_j, which are also found to be positive.

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