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A NOTE ON GENERATING SYSTEMS COMPOSED OF BINARY AND TERNARY RANDOM VARIABLES

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We present some tips for generating multiple sequences of two- and three-valued random variables.

1. INTRODUCTION

Suppose $X_j, j = 1,\ldots,n$, are independent Bernoulli random variables with respective means $p_j, j = 1,\ldots,n$, and that we are interested in using simulation to estimate $E[h(X_1,\ldots,X_n)]$ by generating m independent values of h. The usual simulation approach is to

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generate $X_{1,1}, X_{1,2}, \ldots, X_{1,n}$ and evaluate $h(X_{1,1}, X_{1,2}, \ldots, X_{1,n})$,
generate $X_{2,1}, X_{2,2}, \ldots, X_{2,n}$ and evaluate $h(X_{2,1}, X_{2,2}, \ldots, X_{2,n})$,
generate $X_{3,1}, X_{3,2}, \ldots, X_{3,n}$ and evaluate $h(X_{3,1}, X_{3,2}, \ldots, X_{3,n})$,

and so on (where $X_{i,j}$ is Bernoulli with parameter p_j).

However, a better approach is to first generate the sequence of values $X_{i,1}$, $i = 1, \ldots, m$, then the sequence of values $X_{i,2}, i = 1, \ldots, m$, and so on. This is true because we can generate a sequence of independent and identically distributed (i.i.d.) Bernoulli random variables by generating the times at which the least likely value (0 or 1) occurs; that is, suppose we want to generate $Y_1, \ldots, Y_m$, independent Bernoulli random variables with mean p. Let $a = \min(p, 1-p)$ and let v equal 1 if $a = p$ and let v equal 0 otherwise. Now, imagining that the random variables $Y_1, \ldots, Y_m$ are the results of m successive trials, we can generate the trial numbers at which we obtain the value v by generating independent geometric random variables having parameter a. To generate such a geometric random variable G, generate a random number U and set

$$G = \min(k : U > (1-a)^k) = [\ln(U)/\ln(1-a)] + 1,$$

where $[x]$ is the largest integer less than or equal to x. Continue to generate such geometrics until their sum is at least m. Suppose $G_1, \ldots, G_r$ are generated. If $\sum_{i=1}^{r} G_i = m$, then trial numbers $\sum_{i=1}^{j} G_i, j = 1, \ldots, r$ result in outcome v and the others in outcome $1 - v$; if $\sum_{i=1}^{r} G_i > m$, then trial numbers $\sum_{i=1}^{j} G_i$, $j = 1, \ldots, r-1$, result in outcome v and the others in outcome $1 - v$.

With R equal to the number of geometrics, we need generate

$$\begin{aligned} P\{R = r\} &= P\left\{\sum_{i=1}^{r-1} G_i < m \le \sum_{i=1}^{r} G_i\right\} \\ &= P\left\{\sum_{i=1}^{r-1} G_i \le m - 1 < \sum_{i=1}^{r} G_i\right\} \\ &= P\{N(m-1) = r-1\}, \end{aligned}$$

where $N(k)$ is the number of renewals by time k of the Bernoulli renewal process whose interarrival times are the G_i. Consequently,

$$\begin{aligned} E[R] &= 1 + \sum_{r=1}^{\infty} (r-1) P\{R = r\} \\ &= 1 + \sum_{r=1}^{\infty} (r-1) P\{N(m-1) = r-1\} \\ &= 1 + E[N(m-1)] \\ &= 1 + (m-1)a; \end{aligned}$$

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that is, the expected number of geometrics that are needed to generate the results of the m trials is $(m-1)a+1$. (Intuitively, the preceding is true because we generate all of the first $m-1$ trial numbers that result in successes, as well as the result of trial m.)

Remark: One downside of our suggestion to generate the Bernoulli array in a vertical rather than a horizontal fashion is the additional memory requirements. (If generated in a horizontal fashion, then once a row is generated and h evaluated the Bernoulli row data can be discarded.) However, a variant of our idea can still be utilized even if one wants to generate the complete row vector. For instance, suppose we want to generate $X_1,\ldots,X_n$, where X_i is Bernoulli with parameter p_i. Let $a_i = \min(p_i, 1-p_i)$ and let v_i equal 1 if $a_i = p_i$ and v_i equal 0 otherwise. Start by generating the first X_i, $i = 1,\ldots,n$, that results in its probability a_i value; that is, generate a random number U. If $U \le \prod_{i=1}^{n}(1-a_i)$, set

$$X_j = 1 - v_j, \qquad j = 1,\ldots,n,$$

and stop. Otherwise, let

$$F = \min\left(k : U > \prod_{i=1}^{k}(1-a_i)\right)$$

and set

$$X_j = 1 - v_j, \qquad j = 1,\ldots,F-1, \qquad X_F = v_F.$$

If $F = n$, stop; if $F < n$, use the same procedure to generate the values of $X_{F+1},\ldots,X_n$.

Remark: The results reported in [1] were what motivated us to consider more efficient ways to generate the data needed to estimate $E[h(X_1,\ldots,X_n)]$.

2. THE BINOMIAL APPROACH

Presented is a competing approach when you want m i.i.d. Bernoulli random variables having parameter p. Generate N, a binomial (m,p) random variable; N will equal the number of the random variables equal to 1. Now, let $M = \min(N, m-N)$ and generate the first M positions of a random permutation of $1,\ldots,m$ (this requires M random numbers and is easily done; see [2]). If $M = N$, these are the trial numbers whose value is 1; if $M \neq N$, these are the trial numbers whose value is zero. This method requires, on average, $1 + E[M] < 1 + ma$ random numbers, where $a = \min(p, 1-p)$ and where the 1 refers to the generation of the binomial. The binomial method has the advantage over the geometric in that it is easier to generate the first r positions of a random permutation than it is to generate r geometric random variables. On the other hand, it requires generating a binomial random variable (which can be efficiently done by the use of the inverse transform method starting at $[mp]$ and going either up or down (see [2])).

We now obtain an approximation to $E[M]$.

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LEMMA 1: *Let Z be a standard normal random variable with distribution function Φ. Then*

$$E[|Z-c|] = |c|(2\Phi(|c|)-1) + \sqrt{2/\pi}\, e^{-c^2/2}.$$

PROOF: For $c > 0$,

$$\begin{aligned} E[|Z-c|] &= \int_{-\infty}^{c} (c-x)\Phi'(x)\,dx + \int_{c}^{\infty} (x-c)\Phi'(x)\,dx \\ &= c\Phi(c) - \int_{-\infty}^{0} x\Phi'(x)\,dx - \int_{0}^{c} x\Phi'(x)\,dx + \int_{c}^{\infty} (x-c)\Phi'(x)\,dx \\ &= c\Phi(c) + \frac{e^{-c^2/2}}{\sqrt{2\pi}} + \frac{e^{-c^2/2}}{\sqrt{2\pi}} - c[1-\Phi(c)]. \end{aligned}$$

The argument for $c < 0$ is similar. ■

PROPOSITION 1: *If N is a binomial random variable with parameters m and p, then for m large,*

$$E[\min(N, m-N)] \approx m/2 - \sqrt{mp(1-p)}\,[|c|(2\Phi(|c|)-1) + \sqrt{2/\pi}\, e^{-c^2/2}],$$

where

$$c = \frac{m/2 - mp}{\sqrt{mp(1-p)}}.$$

PROOF:

$$\begin{aligned} E[\min(N, m-N)] &= m/2 + E[\min(N - m/2, m/2 - N)] \\ &= m/2 - E[|N - m/2|] \\ &= m/2 - \sqrt{mp(1-p)}E\left[\left|\frac{N-mp}{\sqrt{mp(1-p)}} - \frac{m/2-mp}{\sqrt{mp(1-p)}}\right|\right] \\ &\approx m/2 - \sqrt{mp(1-p)}E[|Z-c|], \end{aligned}$$

where Z is a standard normal. The proof now follows from Lemma 1. ■

3. GENERATING A SEQUENCE OF THREE-VALUED RANDOM VARIABLES

Suppose that we want to generate the results from m independent trials when each trial has three possible outcomes, with probabilities P_1, P_2, and P_3. Suppose that $P_{i_1} \le P_{i_2} \le P_{i_3}$. Two ways to generate the trial results are as follows:

1. Generate the trials that result in outcome i_1 by successively generating geometrics with parameter P_{i_1} as in the previous section. Suppose this results in trials $j_1, \ldots, j_r$ having value i_1. Then determine which of the other $m - r$ trials result in outcome i_2 by generating geometrics with parameter $P_{i_2}/(1-P_{i_1})$. For instance, suppose we want 10 trials, where $P_1 = 0.2$, $P_2 = 0.3$,

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and $P_3 = 0.5$. If the generated geometrics with parameter 0.2 are 4, 5, and 3 (we stopped when the sum of these geometrics was at least 10) and the generated geometrics with parameter 3/8 are 1, 3, and 4 (we stopped when the sum of these geometrics was at least 8), then the trial outcomes are

$$2, 3, 3, 1, 2, 3, 3, 3, 1, 2.$$

2. A second approach is to generate a binomial random variable N_{i_1} with parameters m and P_{i_1}, and then generate a binomial random variable N_{i_2} with parameters $m - N_{i_1}$, $P_{i_2}/(1 - P_{i_1})$. Let $N_{i_3} = m - N_{i_1} - N_{i_2}$. If $N_{i_r} \le N_{i_s} \le N_{i_t}$, generate the first $N_{i_r} + N_{i_s}$ positions of a random permutation of $1, \ldots, m$. The trial numbers resulting in outcome i_r are the first N_{i_r} values of this permutation, the trial numbers resulting in outcome i_s are the following N_{i_s} values of this permutation, and the other trials result in outcome i_t.

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