Proceedings of the Royal Society of Edinburgh, **134A**, 557–577, *2004*

On a variational characterization of the Fučík spectrum of the Laplacian and a superlinear Sturm–Liouville equation

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(MS received 31 March 2003; accepted 27 January 2004)

In the first part of this paper, a variational characterization of parts of the Fučík spectrum for the Laplacian in a bounded domain Ω is given. The proof uses a linking theorem on sets obtained through a suitable deformation of subspaces of $H^1(\Omega)$. In the second part, a nonlinear Sturm–Liouville equation with Neumann boundary conditions on an interval is considered, where the nonlinearity intersects all but a finite number of eigenvalues. It is proved that, under certain conditions, this equation is solvable for arbitrary forcing terms. The proof uses a comparison of the minimax levels of the functional associated to this equation with suitable values related to the Fučík spectrum.

1. Introduction

The purpose of this paper is twofold. First, we consider the so-called Fučík problem for the Laplacian, both with Dirichlet and Neumann boundary conditions,

$$\left.\begin{aligned} -\Delta u = \lambda^+ u^+ - \lambda^- u^- \quad &\text{in } \Omega, \\ \frac{\partial u}{\partial n} = 0 \quad \text{or} \quad u = 0 \quad &\text{in } \partial\Omega, \end{aligned}\right\} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ and $u^\pm(x) = \max\{0, \pm u(x)\}$.

The notion of Fučík spectrum was introduced in [4, 9]; it is defined as the set $\Sigma \subseteq \mathbb{R}^2$ of points (λ^+, λ^-) for which there exists a non-trivial solution of problem (1.1).

To know the Fučík spectrum is important in many applications; for example, in the study of problems with 'jumping nonlinearities', that is, nonlinearities which are asymptotically linear at both $+\infty$ and $-\infty$, but with different slopes. If the slopes correspond to a point (λ^+, λ^-) that is not in the Fučík spectrum, then it is possible to guarantee *a priori* estimates for the solutions and the Palais–Smale (PS) condition for the associated functional. Moreover, if the point (λ^+, λ^-) may be connected by a curve that does not intersect the Fučík spectrum to a point of the line $\{\lambda^+ = \lambda^-\}$ (not belonging to the Fučík spectrum), then it is possible to prove existence of solutions.

If one has also a variational characterization of this spectrum, then other interesting results can be obtained (cf. [2, 3, 5, 7]). However, these papers deal only with the first non-trivial curve of the Fučík spectrum or with the periodic case on an interval.

https://doi.org/10.1017/S0308210500003346 Published online by Cambridge University Press

In the following, we will call H the space $H^1(\Omega)$ when considering the Neumann problem and $H_0^1(\Omega)$ when considering the Dirichlet problem. We will denote by $0 \leqslant \lambda_1 < \lambda_2 \leqslant \lambda_3 \leqslant \cdots \leqslant \lambda_k \leqslant \cdots$ the eigenvalues of $-\Delta$ in H and by ϕ_k, $k = 1, 2, \dots$, the corresponding eigenfunctions, which will be taken orthogonal and normalized with $\|\phi_k\|_{L^2} = 1$.

First we give a variational characterization of parts of the Fučík spectrum for problem (1.1). In particular, we prove the following.

THEOREM 1.1. *Suppose that the point $(\alpha^+, \alpha^-) \in \mathbb{R}^2$ with $\alpha^+ \geqslant \alpha^-$ is Σ-connected to the diagonal between λ_k and λ_{k+1} in the sense of definition 2.1. Then we can find and characterize one intersection of the Fučík spectrum with the half-line*

$$\{(\alpha^+ + t, \alpha^- + rt),\ t > 0\},$$

for each value of $r \in (0, 1]$.

The cases $\alpha^+ \leqslant \alpha^-$ and $r \in [1, +\infty)$ can be done in a similar way.

The second main theme of the paper is the following Sturm–Liouville equation with Neumann boundary conditions,

$$\left.\begin{aligned} -u'' &= \lambda u + g(x, u) + h(x) \quad \text{in } (0, 1), \\ u'(0) &= u'(1) = 0, \end{aligned}\right\} \tag{1.2}$$

where $g \in \mathcal{C}^0([0, 1] \times \mathbb{R})$, with

$$\lim_{s \to -\infty} \frac{g(x, s)}{s} = 0, \qquad \lim_{s \to +\infty} \frac{g(x, s)}{s} = +\infty \tag{H1}$$

uniformly with respect to $x \in [0, 1]$, and $h \in L^2(0, 1)$.

We will compare it to the Fučík problem

$$\left.\begin{aligned} -u'' &= \lambda^+ u^+ - \lambda^- u^- \quad \text{in } (0, 1), \\ u'(0) &= u'(1) = 0, \end{aligned}\right\} \tag{1.3}$$

and, taking advantage of the fact that, in the one-dimensional case, the Fučík spectrum may be exactly calculated, we will prove existence results for problem (1.2). The proof uses the variational characterization above to make a comparison of these minimax levels with those of the functional associated to problem (1.2) in order to prove the existence of a linking structure for this last functional.

Some hypotheses on the growth at infinity of the nonlinearity g will be needed to obtain the PS condition for the functional associated to problem (1.2). Defining

$$G(x, s) = \int_0^s g(x, \xi)\, \mathrm{d}\xi,$$

we ask

$$\exists \theta \in (0, \tfrac{1}{2}),\ s_0 > 0 \quad \text{such that } 0 < G(x, s) \leqslant \theta s g(x, s) \quad \forall s > s_0, \tag{H2}$$

$$\exists s_1 > 0,\ C_0 > 0 \quad \text{such that } G(x, s) \leqslant \tfrac{1}{2} s g(x, s) + C_0 \quad \forall s < -s_1. \tag{H3}$$

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For certain 'resonant' values of λ, the following hypothesis will be needed:

$$\exists \rho_0 > 0,\ M_0 \in \mathbb{R} \quad \text{such that } G(x,s) + h(x)s \leqslant M_0 \quad \text{a.e. } x \in [0,1] \quad \forall s < -\rho_0. \tag{HR}$$

The exact statement of the results is as follows.

THEOREM 1.2. *Under hypotheses (H1), (H2) and (H3), if* $\lambda \in (\frac{1}{4}\lambda_k, \frac{1}{4}\lambda_{k+1})$ *for some* $k \geqslant 1$*, there exists a solution of problem (1.2) for all* $h \in L^2(0,1)$.

THEOREM 1.3. *Under hypotheses (H1), (H2), (H3) and (HR), if* $\lambda = \frac{1}{4}\lambda_{k+1}$ *for some* $k \geqslant 1$ *and* $h \in L^2(0,1)$*, there exists a solution of problem (1.2).*

REMARK 1.4. Hypotheses (H1)–(H3) are satisfied, for example, by the function $g(x,s) = \mathrm{e}^s$. In this case, in order to also satisfy (HR), we need $h(x) \geqslant 0$ a.e.

Another example of a nonlinearity that also satisfies (HR) and where there is some more freedom on h is when g behaves at $-\infty$ as $|s|^\delta$ with $\delta \in (0,1)$. Then h may be chosen arbitrarily in $L^\infty(0,1)$.

1.1. Related results

Theorem 1.2 extends the result obtained in [6], where the existence is proved for $\lambda \in (0, \frac{1}{4}\pi^2)$, that is, the case $k = 1$ of theorem 1.2.

Results similar to [6] (with slightly different hypotheses) can be found in [18].

Perera in [13] proved the existence of a solution for $\lambda \in (\frac{1}{4}\pi^2, \lambda^*)$, where λ^* is some value in $(\frac{1}{4}\pi^2, \frac{1}{2}\pi^2)$, and so theorem 1.2 extends this result as well.

We also mention that, for periodic boundary conditions, the equivalent of theorem 1.2 is proved in [7].

Theorem 1.3 deals with some kind of resonance (as will be clear from the proofs in the following). The case $\lambda = \frac{1}{4}\lambda_2$ was already discussed in [13], where the existence is proved under different hypotheses, while the case $\lambda = \frac{1}{4}\lambda_1$ (that is, $\lambda = 0$) is treated in [6].

For what concerns the variational characterization of the Fučík spectrum, we cite [3, 5], where the second curve in any spatial dimension is characterized (in two different ways), and [7], where the whole spectrum for periodic boundary conditions on an interval is characterized.

In the construction of the characterization of the Fučík spectrum, we will use a technique derived from one used in [8], which will be discussed in § 2.1.

For further references on problem (1.2), see [6].

1.2. The Fučík spectrum

The notion of the Fučík spectrum was introduced in [4, 9]. It is defined as the set $\Sigma \subseteq \mathbb{R}^2$ of points (λ^+, λ^-) for which there exists a non-trivial solution of problem (1.1).

In the case of problem (1.3), the spectrum can be completely calculated, with the corresponding non-trivial solutions. It is composed of curves (which we denote by Σ_k) in $\mathbb{R}^2$ arising from each point (λ_k, λ_k),

$$\left.\begin{aligned} &\Sigma_1 : \{\lambda^+ = \lambda_1\} \cup \{\lambda^- = \lambda_1\}, \\ &\Sigma_k : \frac{(k-1)\pi}{2\sqrt{\lambda^+}} + \frac{(k-1)\pi}{2\sqrt{\lambda^-}} = 1, \quad k = 2, 3, \dots \end{aligned}\right\} \tag{1.4}$$

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Note that each curve with $k \geqslant 2$ is monotone decreasing, has asymptotes at $\lambda^\pm = \frac{1}{4}\lambda_k$ and lies completely in the quadrant $\lambda^\pm > \frac{1}{4}\lambda_k$.

In the case of higher dimensions, less is known: Σ is always a closed set; the lines $\{\lambda^+ = \lambda_1\}$ and $\{\lambda^- = \lambda_1\}$ belong to Σ; Σ does not contain any other point with $\lambda^+ < \lambda_1$ or $\lambda^- < \lambda_1$. Moreover, we know (see, for example, [10, 11, 16]) that, in each square $(\lambda_{k-1}, \lambda_{k+m+1})^2$ where $\lambda_{k-1} < \lambda_k = \cdots = \lambda_{k+m} < \lambda_{k+m+1}$, from the point (λ_k, λ_k) arises a continuum composed of a lower and an upper curve, both decreasing (may be coincident); other points in $\Sigma \cap (\lambda_{k-1}, \lambda_{k+m+1})^2$ can only lie between the two curves (and hence, in the open squares $(\lambda_{k-1}, \lambda_k)^2$ and $(\lambda_{k+m}, \lambda_{k+m+1})^2$, there are never points of Σ). Something more can be said about the lower part of the continuum arising from (λ_2, λ_2) (see [5]).

In [1], it is proved, under a non-degeneracy condition (which was first introduced in [12, 14]), that the whole spectrum is composed by curves arising from a point (λ_k, λ_k), never intersecting and going to infinity. This non-degeneracy condition is discussed in [14], where it is proved that it holds for 'almost all' (in a suitable sense) domains. However, in the general case, it does not seem possible to arrive at the same conclusion.

For more references concerning the Fučík spectrum, see [17].

1.3. Idea and plan of the paper

If we consider a point $a \in (\lambda_k, \lambda_{k+1})$ and the functional $J_a : H \to \mathbb{R}$,

$$J_a(u) = \int_\Omega |\nabla u|^2 - a \int_\Omega u^2, \tag{1.5}$$

we have a natural splitting $H = V \oplus W$, where $V = \operatorname{span}\{\phi_1, \dots, \phi_k\}$.

Taking $\partial B^k_{L^2}$ to be the boundary of the unit ball in the L^2-norm in V, one knows that there exists $\mu > 0$ such that

$$J_a(u) \leqslant -\mu < 0 \qquad \text{for all } u \in \partial B^k_{L^2}, \tag{1.6}$$

$$J_a(u) \geqslant \mu \|u\|^2_H \geqslant 0 \quad \text{for all } u \in W, \tag{1.7}$$

and that the two sets link (for a definition of linking, see, for example, [15]).

The existence of this structure allows us to characterize the eigenvalue λ_{k+1} as

$$\lambda_{k+1} = a + \inf_{\gamma \in \Gamma} \sup_{u \in \gamma(B^k)} J_a(u), \tag{1.8}$$

where the family Γ is defined as

$$\Gamma = \{\gamma \in \mathcal{C}^0(B^k; \partial B_{L^2}) \text{ such that } \gamma|_{\partial B^k} \text{ is a homeomorphism onto } \partial B^k_{L^2}\}. \tag{1.9}$$

Here, B_{L^2} denotes the unit ball in the L^2-norm in H and

$$B^k = \left\{(x_1 \dots, x_k) \in \mathbb{R}^k \text{ such that } \sum_{i=1}^k x_i^2 \leqslant 1\right\}.$$

In this paper, we will build suitable sets to play the same role for the functional

$$J_\alpha(u) = \int_\Omega |\nabla u|^2 - \alpha^+ \int_\Omega (u^+)^2 - \alpha^- \int_\Omega (u^-)^2 \tag{1.10}$$

in order to characterize a point in the Fučík spectrum.

These sets will be obtained in § 2.1 as a deformation of the sets in (1.6) and (1.7), using a technique similar to the one described in [8].

Then the variational characterization will be carried out in § 2.2.

In § 3.1, a comparison of the obtained minimax levels with those of the functional associated to problem (1.2) will allow us to prove the existence of a linking structure for this last functional, and then to prove theorems 1.2 and 1.3.

Finally, in § 4, we give the complete proof of the PS condition for the functional associated to problem (1.2).

2. Variational characterization of the Fučík spectrum

2.1. Construction of the linking structure

Consider first the Dirichlet problem (thus, here, H will denote H_0^1 and we consider the norm $\|u\|_H^2 = \int_\Omega |\nabla u|^2$). Take a point (α^+, α^-), Σ-connected to the diagonal between λ_k and λ_{k+1}, in the sense of the following definition.

Definition 2.1. A point $(\alpha^+, \alpha^-) \notin \Sigma$ is Σ-connected to the diagonal between λ_k and λ_{k+1} if $\exists a \in (\lambda_k, \lambda_{k+1})$ and a $\mathcal{C}^1$ function $\alpha : [0, 1] \to \mathbb{R}^2$ such that the following hold.

(a) $\alpha(0) = (a, a)$, $\alpha(1) = (\alpha^+, \alpha^-)$.

(b) $\alpha([0, 1]) \cap \Sigma = \emptyset$.

Remark 2.2. Since Σ is closed and $\alpha([0, 1])$ is compact, this implies that the following property holds.

(b′) $\exists d > 0$ such that $\xi\alpha([0, 1]) \cap \Sigma = \emptyset$ for all $\xi \in [1 - d, 1 + d]$.

This property will be used in the following proofs.

Now consider

$$J_{\alpha(t)}(u) = \int_\Omega |\nabla u|^2 - \alpha^+(t) \int_\Omega (u^+)^2 - \alpha^-(t) \int_\Omega (u^-)^2, \tag{2.1}$$

where $\alpha(t) = (\alpha^+(t), \alpha^-(t))$. Then, splitting as before $H = V \oplus W$, we have

$$J_{\alpha(0)}(u) \leqslant -\mu\|u\|_H^2 \quad \text{for all } u \in V, \tag{2.2}$$

$$J_{\alpha(0)}(u) \geqslant \mu\|u\|_H^2 \quad \text{for all } u \in W, \tag{2.3}$$

for some $\mu > 0$.

Lemma 2.3 (from lemma 2.3 of [8]). *If (α^+, α^-) is as in definition 2.1, we can find $\eta \in (0, \mu)$ and $\delta_\eta > 0$ such that, $\forall t \in [0, 1]$, $u \in H$ with $\|u\|_H = 1$,*

$$\text{if } J_{\alpha(t)}(u) \in [-\eta, \eta], \text{ then } \|\nabla_u J_{\alpha(t)}(u)\|_H^2 - \langle \nabla_u J_{\alpha(t)}(u), u\rangle_H^2 \geqslant \delta_\eta.$$

Proof. Following [8], take $\eta = \min(d/(3(d + 1)), \mu)$, and suppose, by contradiction, the existence of a sequence $t_n \subseteq [0, 1]$ and $u_n \in H$, $\|u_n\|_H = 1$ such that

$$-\eta \leqslant J_{\alpha(t_n)}(u_n) \leqslant \eta \quad \text{and} \quad \|\nabla_u J_{\alpha(t_n)}(u_n)\|_H^2 - \langle \nabla_u J_{\alpha(t_n)}(u_n), u_n\rangle_H^2 \to 0 \tag{2.4}$$

as $n \to +\infty$.

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Define

$$j_n = \langle \nabla_u J_{\alpha(t_n)}(u_n), u_n \rangle_H = 2J_{\alpha(t_n)}(u_n) \in [-2\eta, 2\eta].$$

From Pythagoras' theorem, deduce that

$$\|\nabla_u J_{\alpha(t_n)}(u_n)\|_H^2 - \langle \nabla_u J_{\alpha(t_n)}(u_n), u_n \rangle_H^2 = \|\nabla_u J_{\alpha(t_n)}(u_n) - j_n u_n\|_H^2. \tag{2.5}$$

Then, evaluating the norm and considering the points in H as operators on H, one concludes that

$$(1 - j_n) \int_\Omega \nabla u_n \nabla v_n - \alpha^+(t_n) \int_\Omega u_n^+ v_n + \alpha^-(t_n) \int_\Omega u_n^- v_n \to 0 \tag{2.6}$$

for any bounded sequence $v_n \subseteq H$.

Up to a subsequence, we may say that $j_n \to j \in [-2\eta, 2\eta]$, $t_n \to t_0 \in [0,1]$ and $u_n \rightharpoonup u \in H$ (strongly in L^2). Taking the limit of (2.6) with $v_n = u_n$ gives

$$1 - j = \alpha^+(t_0) \int_\Omega (u^+)^2 + \alpha^-(t_0) \int_\Omega (u^-)^2, \tag{2.7}$$

where $j \leqslant 2\eta < 1$, and then u is not trivial.

From equation (2.6) with arbitrary test function, and using the weak convergence of u_n, we get that u is a solution of the Fučík problem with coefficients

$$\left(\frac{\alpha^+(t_0)}{1-j}, \frac{\alpha^-(t_0)}{1-j} \right),$$

but the choice of η and remark 2.2 imply that this is not possible, since

$$\frac{1}{1-j} \in [1-d, 1+d] \quad \forall j \in [-2\eta, 2\eta].$$

□

Then, as in [8], define a continuous flow $\sigma_t(u) : [0,1] \times H \to H$,

$$\left. \begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \sigma_t(u) &= M F_t(\sigma_t(u)), \\ \sigma_0(u) &= u, \end{aligned} \right\} \tag{2.8}$$

where the following conditions hold.

(1) M is a suitable positive constant, defined as $M = 2KS^2/\delta_\eta$, with

  (i) $K = \sup_{t \in [0,1]} (|\alpha^+(t)'| + |\alpha^-(t)'|)$;

  (ii) $S = \lambda_1^{-1/2} = \sup_{u \in H} \dfrac{\|u\|_{L^2}}{\|u\|_H}$.

(2) $F_t : H \to H$ is defined such that it is locally Lipschitz and

$$F_t(u) = \begin{cases} \nabla_u J_{\alpha(t)}(u) & \text{if } \dfrac{J_{\alpha(t)}(u)}{\|u\|_H^2} \geqslant \tfrac{1}{2}\eta, \\ -\nabla_u J_{\alpha(t)}(u) & \text{if } \dfrac{J_{\alpha(t)}(u)}{\|u\|_H^2} \leqslant -\tfrac{1}{2}\eta. \end{cases} \tag{2.9}$$

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Then $\sigma_t(u)$ has the following properties.

(1) $\sigma_t(0) = 0$ and $\sigma_t(u) \neq 0\ \forall u \neq 0$.

(2) $\sigma_t : H \to H$ is a homeomorphism $\forall t$.

Moreover, the following result holds.

LEMMA 2.4 (from lemma 2.6 of [8]). *Defining*

$$\Theta_t(u) = \frac{J_{\alpha(t)}(\sigma_t(u))}{\|\sigma_t(u)\|_H^2}$$

and fixing u*, we have that* $\Theta_t(u)$ *is increasing* (*respectively, decreasing*) *in the variable* t *in any interval* $[t_1, t_2]$ *such that*

$$\tfrac{1}{2}\eta \leqslant \Theta_t(u) \leqslant \eta \quad \forall t \in [t_1, t_2]$$

(*respectively,* $-\eta \leqslant \Theta_t(u) \leqslant -\frac{1}{2}\eta\ \forall t \in [t_1, t_2]$).

Proof. Consider first the case $\frac{1}{2}\eta \leqslant \Theta_t(u) \leqslant \eta$. Then the flow is defined by

$$\frac{\mathrm{d}}{\mathrm{d}t}\sigma_t(u) = M\nabla_u J_{\alpha(t)}(\sigma_t(u)) \tag{2.10}$$

for all $t \in [t_1, t_2]$.

Then we have (we will omit the dependence from u in the notation)

$$\begin{aligned}
\frac{\mathrm{d}\Theta_t}{\mathrm{d}t} &= \frac{1}{\|\sigma_t\|_H^2}\left[\frac{\partial J_{\alpha(t)}(\sigma_t)}{\partial t} + \left\langle \nabla_u J_{\alpha(t)}(\sigma_t), \frac{\mathrm{d}}{\mathrm{d}t}\sigma_t\right\rangle_H\right] + J_{\alpha(t)}(\sigma_t)\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\|\sigma_t\|_H^2}\right) \\
&= \frac{1}{\|\sigma_t\|_H^2}\bigg[-\alpha^+(t)'\int_\Omega (\sigma_t^+)^2 - \alpha^-(t)'\int_\Omega (\sigma_t^-)^2 \\
&\qquad\qquad + \langle \nabla_u J_{\alpha(t)}(\sigma_t), M\nabla_u J_{\alpha(t)}(\sigma_t)\rangle_H\bigg] \\
&\quad + \tfrac{1}{2}\langle \nabla_u J_{\alpha(t)}(\sigma_t), \sigma_t\rangle_H\left(-\frac{2}{\|\sigma_t\|_H^4}\left\langle \sigma_t, \frac{\mathrm{d}}{\mathrm{d}t}\sigma_t\right\rangle_H\right) \\
&\geqslant -KS^2 + M\left(\frac{\|\nabla_u J_{\alpha(t)}(\sigma_t)\|_H^2}{\|\sigma_t\|_H^2} - \frac{\langle \nabla_u J_{\alpha(t)}(\sigma_t), \sigma_t\rangle_H^2}{\|\sigma_t\|_H^4}\right) \\
&\geqslant -KS^2 + M\delta_\eta.
\end{aligned}$$

By the choice made above, $M > KS^2/\delta_\eta$ and then the proof of the first part is complete.

For the case $-\eta \leqslant \Theta_t(u) \leqslant -\frac{1}{2}\eta$, the proof follows the same ideas. □

Finally, denote $\sigma_1(u)$ with $\tau_{\alpha,\eta}(u)$ (to indicate its dependence on α and η) to obtain the following result.

LEMMA 2.5 (from equation (2.9) and lemma 2.7 of [8]). *We have*

$$J_\alpha(\tau_{\alpha,\eta}(u)) \leqslant -\eta\|\tau_{\alpha,\eta}(u)\|_H^2 \quad \textit{for all } u \in V, \tag{2.11}$$

$$J_\alpha(\tau_{\alpha,\eta}(u)) \geqslant \eta\|\tau_{\alpha,\eta}(u)\|_H^2 \quad \textit{for all } u \in W \tag{2.12}$$

and, $\forall R>0$, $\tau_{\alpha,\eta}(W)$ links with $R\tau_{\alpha,\eta}(\partial B_V^k)$, where B_V^k is the unit ball, in the H-norm, of V.

Proof. Equations (2.11) and (2.12) follow easily from lemma 2.4.

For the linking property, we need to prove that

$$\forall\gamma\in\Gamma=\{\gamma\in\mathcal{C}^0(R\tau_{\alpha,\eta}(B_V^k);H) \text{ and such that } \gamma(u)=u \text{ for } u\in R\tau_{\alpha,\eta}(\partial B_V^k)\},$$

there exists a point $\bar{u}\in\gamma(R\tau_{\alpha,\eta}(B_V^k))\cap\tau_{\alpha,\eta}(W)$.

We start by proving that

$$\xi\tau_{\alpha,\eta}(u)\neq\tau_{\alpha,\eta}(v) \tag{2.13}$$

for any $u\in\partial B_V^k$, $v\in W$ and $\xi>0$. Actually, if it were not so, from equations (2.11) and (2.12), we would get

$$\eta\|\tau_{\alpha,\eta}(v)\|_H^2\leqslant J_\alpha(\tau_{\alpha,\eta}(v))=J_\alpha(\xi\tau_{\alpha,\eta}(u))=\xi^2J_\alpha(\tau_{\alpha,\eta}(u))\leqslant-\eta\xi^2\|\tau_{\alpha,\eta}(u)\|_H^2,$$

which implies (using also the uniqueness of the Cauchy problem) that $u=v=0$; a contradiction, since $u\in\partial B_V^k$.

Now define P to be the orthogonal projection of H onto V and consider the map

$$H_t=P\circ\tau_{\alpha,\eta}^{-1}\circ(1+(R-1)t)\tau_{\alpha,\eta}.$$

Property (2.13) implies that $H_t\neq0$ on ∂B_V^k for any $t\in[0,1]$, and then

$$\deg(H_1,B_V^k,0)=\deg(H_0,B_V^k,0)=\deg(\mathrm{Id},B_V^k,0)=1.$$

Now, for any $\gamma\in\Gamma$,

$$\deg(P\circ\tau_{\alpha,\eta}^{-1}\circ\gamma\circ R\tau_{\alpha,\eta},B_V^k,0)=1,$$

since, on ∂B_V^k, the function is exactly H_1, and then there is a point $p\in B_V^k$ such that $\gamma(R\tau_{\alpha,\eta}(p))\in\tau_{\alpha,\eta}(W)$. □

For the Neumann problem, as shown in the proof of theorem 3.4 of [8] for the periodic case, one can get the same conclusions, working with the operator $-\Delta u+u$ to avoid the problems arising since the first eigenvalue is 0.

Finally, we prove one more property that we will need later.

Lemma 2.6. *If $u\in V$ or $u\in W$ and $\xi>0$, then $\tau_{\alpha,\eta}(\xi u)=\xi\tau_{\alpha,\eta}(u)$.*

Proof. From lemmas 2.4 and 2.5 and equations (2.8) and (2.9), we have that, in these two cases, the equation just contains the gradient of $J_{\alpha(t)}$.

If we take $u\in V$, then the flow is defined by

$$\left.\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\sigma_t(u)&=-M\nabla_uJ_{\alpha(t)}(\sigma_t(u)),\\ \sigma_0(u)&=u\in V.\end{aligned}\right\} \tag{2.14}$$

Consider then the change of variable $\sigma=k\pi$ with $k>0$. Equation (2.14) becomes

$$\left.\begin{aligned}k\frac{\mathrm{d}}{\mathrm{d}t}\pi_t(u)&=-M\nabla_uJ_{\alpha(t)}(k\pi_t(u)),\\ k\pi_0(u)&=u\in V,\end{aligned}\right\} \tag{2.15}$$

and, considering the linear positive homogeneity of $\nabla_u J_{\alpha(t)}$, it can be simplified to obtain

$$\left.\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\pi_t(u) &= -M\nabla_u J_{\alpha(t)}(\pi_t(u)),\\ \pi_0(u) &= \frac{u}{k} \in V, \end{aligned}\right\} \tag{2.16}$$

which is the same equation as (2.14), with a different initial condition. Then

$$\sigma_t(u) = k\pi_t(u) = k\sigma_t\left(\frac{u}{k}\right).$$

The case $u \in W$ is treated in the same way. □

2.2. Construction of the variational characterization

Now we use the results of § 2.1 to obtain a variational characterization of some parts of the Fučík spectrum (problem (1.1)).

The result is the one stated in theorem 1.1.

Note that, in the one-dimensional case, since the spectrum is known, (α^+, α^-) may be taken anywhere between the continuous curves arising from a point (λ_k, λ_k) and the ones arising from $(\lambda_{k+1}, \lambda_{k+1})$. In the multi-dimensional case, one has to be more careful, but Σ-connection may be assured at least for (α^+, α^-) in the square $(\lambda_{k-1}, \lambda_{k+m+1})^2$ (being $\lambda_{k-1} < \lambda_k = \cdots = \lambda_{k+m} < \lambda_{k+m+1}$) when it is not between (or on) the lower and the upper curve arising from (λ_k, λ_k).

We will now imitate the characterization of λ_{k+1} described in (1.8).

We fix a point (α^+, α^-), Σ-connected to the diagonal between λ_k and λ_{k+1} and with $\alpha^+ \geqslant \alpha^-$. Then we apply the results of § 2.1, obtaining the deformation $\tau_{\alpha,\eta}$, and choose $r \in (0, 1]$. We split again $H = V \oplus W$ with $V = \mathrm{span}\{\phi_1, \dots, \phi_k\}$ and we consider the following.

(i) The set

$$Q_r = \left\{u \in H \text{ such that } \int_\Omega (u^+)^2 + r(u^-)^2 = 1\right\}. \tag{2.17}$$

(ii) The radial projection on Q_r of the set obtained in § 2.1 by the deformation of ∂B_V^k, that is,

$$L_{\alpha,r} = P^r(\tau_{\alpha,\eta}(\partial B_V^k)), \tag{2.18}$$

where

$$P^r : u \to \frac{u}{\sqrt{\int_\Omega (u^+)^2 + r\int_\Omega (u^-)^2}}.$$

(iii) The class of maps

$$\Gamma_{\alpha,r} = \{\gamma \in \mathcal{C}^0(B^k; Q_r) \text{ such that } \gamma|_{\partial B^k} \text{ is a homeomorphism onto } L_{\alpha,r}\}, \tag{2.19}$$

where

$$B^k = \left\{(x_1 \dots, x_k) \in \mathbb{R}^k \text{ such that } \sum_{i=1}^k x_i^2 \leqslant 1\right\}.$$

(iv) The functional

$$J_\alpha(u) = \int_\Omega (\nabla u)^2 - \alpha^+ \int_\Omega (u^+)^2 - \alpha^- \int_\Omega (u^-)^2. \tag{2.20}$$

The idea now is to consider

$$d_{\alpha,r} = \inf_{\gamma \in \Gamma_{\alpha,r}} \sup_{u \in \gamma(B^k)} J_\alpha(u) \tag{2.21}$$

and to prove that this leads to a non-trivial solution of the Fučík problem (1.1), that is, to a point in Σ.

We first prove that the above definitions are well posed and derive some properties of the defined sets.

LEMMA 2.7. *For $u \in Q_r$, we have that $1 \leqslant \int_\Omega u^2 \leqslant 1/r$.*

Proof. We have

$$\begin{aligned} 1 &= \int_\Omega (u^+)^2 + r(u^-)^2 \\ &\leqslant \int_\Omega (u^+)^2 + (u^-)^2 = \int_\Omega u^2 \\ &\leqslant \frac{\int_\Omega (u^+)^2 + r(u^-)^2}{r} = \frac{1}{r}. \end{aligned}$$

□

LEMMA 2.8.

(i) *The set $L_{\alpha,r}$ is homeomorphic to ∂B^k.*

(ii) *$L_{\alpha,r} \subseteq \tau_{\alpha,\eta}(V)$.*

Proof. (i) Since ∂B^k_V is homeomorphic to ∂B^k and $\tau_{\alpha,\eta}$ is a homeomorphism, we just need to prove that P^r is a homeomorphism when restricted to $\tau_{\alpha,\eta}(\partial B^k_V)$.

$\tau_{\alpha,\eta}$ on ∂B^k_V has the property (see lemma 2.6) that $\tau_{\alpha,\eta}(\xi u) = \xi \tau_{\alpha,\eta}(u)$ $\forall \xi > 0$. Then P^r is one to one on $\tau_{\alpha,\eta}(\partial B^k_V)$ and so can be inverted.

Finally, P^r is continuous together with its inverse because, since ∂B^k_V is a compact set that does not contain the origin, $\int_\Omega (u^+)^2 + r\int_\Omega (u^-)^2$ is continuous, bounded and bounded away from zero on it.

(ii) The second point is a trivial consequence of lemma 2.6. □

LEMMA 2.9. *$\tau_{\alpha,\eta}(W)$ links with $L_{\alpha,r}$.*

Proof. From lemma 2.5, $\tau_{\alpha,\eta}(W)$ links with $\tau_{\alpha,\eta}(\partial B^k_V)$.

Then the claim could be false only if, for some $u \in L_{\alpha,r}$, $\xi > 0$ and $v \in \tau_{\alpha,\eta}(W)$, we had $\xi u = v$. But, by the homogeneity property of $\tau_{\alpha,\eta}$ in V and W (lemma 2.6), this would imply $\xi(\tau_{\alpha,\eta})^{-1}(u) = (\tau_{\alpha,\eta})^{-1}(v)$, and then $u = v = 0$, which is impossible since $u \in P^r(\tau_{\alpha,\eta}(\partial B^k_V))$. □

In the next three lemmas we verify the conditions for the 'linking theorem' which will be used to prove the criticality of $d_{\alpha,r}$.

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LEMMA 2.10. *The functional $J_\alpha(u)$ constrained to Q_r satisfies the PS condition.*

Proof. Consider the sequences $\{u_n\} \subseteq Q_r$, $\{\beta_n\} \subseteq \mathbb{R}$ (Lagrange multipliers) and $\varepsilon_n \to 0^+$ such that

$$\left|\int_\Omega (\nabla u_n)^2 - \alpha^+ \int_\Omega (u_n^+)^2 - \alpha^- \int_\Omega (u_n^-)^2\right| \leqslant C \tag{2.22}$$

$$\left|\int_\Omega \nabla u_n \nabla v - \alpha^+ \int_\Omega (u_n^+)v + \alpha^- \int_\Omega (u_n^-)v + \beta_n\left(\int_\Omega u_n^+ v - r u_n^- v\right)\right| \leqslant \varepsilon_n \|v\|_H \quad \forall v \in H. \tag{2.23}$$

Since $\{u_n\} \subseteq Q_r$, it is a bounded sequence in L^2, and then equation (2.22) implies that it is also a bounded sequence in H. Then there is a subsequence converging weakly in H and strongly in L^2 to some u.

The L^2 convergence implies that $u \in Q_r$.

Taking $v = u_n$, we get that

$$\beta_n + \left(\int_\Omega (\nabla u_n)^2 - \alpha^+ \int_\Omega (u_n^+)^2 - \alpha^- \int_\Omega (u_n^-)^2\right) \to 0. \tag{2.24}$$

Then, with $v = u_n - u$, we have

$$\begin{aligned}\int_\Omega \nabla u_n \nabla (u_n - u) - \alpha^+ \int_\Omega (u_n^+)(u_n - u) + \alpha^- \int_\Omega (u_n^-)(u_n - u)\\ - \left(\int_\Omega (\nabla u_n)^2 - \alpha^+ \int_\Omega (u_n^+)^2 - \alpha^- \int_\Omega (u_n^-)^2\right)\left(\int_\Omega (u_n^+ - r u_n^-)(u_n - u)\right) \to 0,\end{aligned}$$

where all terms except the first go to zero. Then we conclude that $\|\nabla u_n\|_{L^2} \to \|\nabla u\|_{L^2}$, and then $u_n \to u$ strongly in H. □

LEMMA 2.11. *We have* $\sup_{u\in\gamma(\partial B^k)} J_\alpha(u) \leqslant 0$ $\forall \gamma \in \Gamma_{\alpha,r}$.

Proof. By lemma 2.5, since $\gamma(\partial B^k) = L_{\alpha,r} \subseteq \tau_{\alpha,\eta}(V)$, then $J_\alpha(u) \leqslant -\eta\|u\|_H^2 < 0$. □

LEMMA 2.12. *We have* $+\infty > \sup_{u\in\gamma(B^k)} J_\alpha(u) \geqslant \eta > 0$ *for each* $\gamma \in \Gamma_{\alpha,r}$.

Proof. By lemma 2.9, there is always a point $u \in \gamma(B^k) \cap \tau_{\alpha,\eta}(W)$ and, by lemma 2.5, we have, in that point, that $J_\alpha(u) \geqslant \eta\|u\|_H^2$. Considering lemma 2.7 and the fact that $u \in Q_r$, this becomes $\geqslant \eta$.

Finally, it is less than $+\infty$, since each $\gamma(B^k)$ is a compact set. □

At this point, we can state the following standard 'linking theorem' (see, for example, [15]).

PROPOSITION 2.13. *The level $d_{\alpha,r} \geqslant \eta > 0$ is a critical value for $J_\alpha(u)$ constrained to Q_r.*

The importance of the criticality of the level $d_{\alpha,r}$ is clarified in the following proposition.

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Proposition 2.14. *The critical points associated to the critical value $d_{\alpha,r}$ are non-trivial solutions of the Fučík problem (1.1) with coefficients (λ^+, λ^-), where*

$$\lambda^+ - \alpha^+ = d_{\alpha,r} \quad \text{and} \quad \lambda^- - \alpha^- = rd_{\alpha,r}.$$

Proof. Criticality of u implies that there exists a Lagrange multiplier $\beta \in \mathbb{R}$ such that

$$\int_\Omega \nabla u \nabla v - \alpha^+ \int_\Omega (u^+)v + \alpha^- \int_\Omega (u^-)v + \beta\biggl(\int_\Omega u^+v - ru^-v\biggr) = 0 \quad \forall v \in H, \tag{2.25}$$

but testing against u, we get $\beta = -d_{\alpha,r}$, and so u solves

$$-\Delta u = \alpha^+u^+ - \alpha^-u^- + d_{\alpha,r}u^+ - d_{\alpha,r}ru^- = (\alpha^+ + d_{\alpha,r})u^+ - (\alpha^- + rd_{\alpha,r})u^- \tag{2.26}$$

in Ω, with the considered boundary conditions.

Finally, u is not trivial, since it is in Q_r. □

Propositions 2.13 and 2.14 imply that the point $(\alpha^+ + d_{\alpha,r}, \alpha^- + rd_{\alpha,r})$ belongs to the half-line $\{(\alpha^+ + t, \alpha^- + rt),\ t > 0\}$ (since $d_{\alpha,r} > 0$) and also to the Fučík spectrum. Thus theorem 1.1 is proved.

Remark 2.15. We did not prove that this solution corresponds to the first intersection (that is, the one with smallest t) of the half-line with Σ.

Thus, even in the one-dimensional case (that is, when the spectrum is known), we cannot assert that the point belongs to the continuum arising from $(\lambda_{k+1}, \lambda_{k+1})$. What we can say (since $d_{\alpha,r} > 0$) is just that it belongs to the continuum arising from (λ_h, λ_h) for some $h \geqslant k + 1$.

3. The superlinear problem

3.1. Proof of theorem 1.2

Consider now the superlinear problem (1.2). The idea here is to prove the existence of a non-constrained critical point of the functional

$$F(u) = \frac{1}{2}\int_0^1 (u')^2 - \frac{\lambda}{2}\int_0^1 u^2 - \int_0^1 G(x, u) - \int_0^1 hu, \tag{3.1}$$

which corresponds to a solution of the problem.

We will follow a strategy inspired by [7].

Note that $H^1(0, 1) \subseteq \mathcal{C}^0([0, 1])$, with compact inclusion, and recall that, in this case, the asymptotes of each Σ_k with $k \geqslant 2$ are at $\lambda^- = \frac{1}{4}\lambda_k$ and that Σ_k lies entirely in $\{\lambda^- > \frac{1}{4}\lambda_k\}$.

This structure of Σ implies that, for fixed $\lambda \in (\frac{1}{4}\lambda_k, \frac{1}{4}\lambda_{k+1})$, $k \geqslant 1$, it is always possible to find the following.

(i) A point (α^+, α^-), Σ-connected to the diagonal between λ_k and λ_{k+1} and such that $\alpha^- < \lambda$,

(ii) A $\delta > 0$ such that $\alpha^- < \lambda - \delta$ and $\lambda + \delta < \frac{1}{4}\lambda_{k+1}$.

Now, using the notation of § 2.2, we define, for $R > 0$, the family of maps

$$\Gamma^R_{\alpha,\bar{r}} = \{\gamma^* \in \mathcal{C}^0(B^k; H) \text{ such that } \gamma^*|_{\partial B^k} \text{ is a homeomorphism onto } RL_{\alpha,\bar{r}}\}. \tag{3.2}$$

We want to prove that, for a suitable $R > 0$, the level

$$f = \inf_{\gamma^* \in \Gamma^R_{\alpha,\bar{r}}} \sup_{u \in \gamma^*(B^k)} F(u) \tag{3.3}$$

is a critical value for the functional F.

REMARK 3.1. In the definition of $\Gamma^R_{\alpha,\bar{r}}$, the choice of $\bar{r} \in (0, 1]$ has no importance; it can be chosen arbitrarily.

Using the fact that $h \in L^2$ and hypothesis (H1), we can find constants C_1, C_2 and C_3 as follows.

(i) $C_1(\delta, h)$ such that

$$\left|\int_0^1 hu\right| \leqslant \tfrac{1}{4}\delta\|u\|^2_{L^2} + C_1(\delta, h). \tag{3.4}$$

(ii) $C_2(\delta, g)$ such that

$$\left|\int_0^1 G(x, -u^-)\right| \leqslant \tfrac{1}{4}\delta\|u\|^2_{L^2} + C_2(\delta, g). \tag{3.5}$$

(iii) $C_3(M, g)$ such that

$$\int_0^1 G(x, u^+) \geqslant \tfrac{1}{2}M\|u^+\|^2_{L^2} - C_3(M, g) \tag{3.6}$$

for any M.

To find a generalized mountain-pass structure, we first need the following result.

LEMMA 3.2. *$\forall C \in \mathbb{R}$, we can find $R > 0$ such that*

$$\sup_{u \in \gamma^*(\partial B^k)} F(u) < C \quad \forall \gamma^* \in \Gamma^R_{\alpha,\bar{r}}. \tag{3.7}$$

Proof. We evaluate, for $u \in L_{\alpha,\bar{r}}$ and $\rho > 0$,

$$\begin{aligned}
\frac{F(\rho u)}{\rho^2} &= \frac{1}{2}\int_0^1 (u')^2 - \tfrac{1}{2}\lambda\int_0^1 u^2 - \frac{\int_0^1 G(x, \rho u)}{\rho^2} - \frac{\int_0^1 h\rho u}{\rho^2} \\
&\leqslant \frac{1}{2}\int_0^1 (u')^2 - \tfrac{1}{2}\lambda\int_0^1 u^2 + \frac{|\int_0^1 G(x, -\rho u^-)|}{\rho^2} - \frac{\int_0^1 G(x, \rho u^+)}{\rho^2} + \frac{|\int_0^1 h\rho u|}{\rho^2} \\
&\leqslant \frac{1}{2}\int_0^1 (u')^2 - \tfrac{1}{2}\lambda\int_0^1 u^2 + \left(\tfrac{1}{4}\delta\int_0^1 u^2 + \frac{C_2(\delta, g)}{\rho^2}\right) \\
&\quad - \left(\tfrac{1}{2}M\int_0^1 (u^+)^2 - \frac{C_3(M, g)}{\rho^2}\right) + \left(\tfrac{1}{4}\delta\int_0^1 u^2 + \frac{C_1(\delta, h)}{\rho^2}\right)
\end{aligned}$$

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$$
\begin{aligned}
&\leqslant \frac{1}{2}\int_0^1 (u')^2 - \tfrac{1}{2}(\lambda-\delta)\int_0^1 u^2 \\
&\qquad - \tfrac{1}{2}M\int_0^1 (u^+)^2 + \frac{C_1(\delta,h)+C_2(\delta,g)+C_3(M,g)}{\rho^2} \\
&= \tfrac{1}{2}J_\alpha(u) - \tfrac{1}{2}(\lambda-\delta+M-\alpha^+)\int_0^1 (u^+)^2 \\
&\qquad - \tfrac{1}{2}(\lambda-\delta-\alpha^-)\int_0^1 (u^-)^2 + \frac{C_1+C_2+C_3(M,g)}{\rho^2}.
\end{aligned}
$$

Now, if we fix $M=\alpha^+-\alpha^-$ and consider that $J_\alpha(u)\leqslant 0$ and $\int_0^1 u^2 \geqslant 1$ on $L_{\alpha,\bar{r}}$, we get

$$
\frac{F(\rho u)}{\rho^2} \leqslant -\tfrac{1}{2}(\lambda-\delta-\alpha^-) + \frac{\tilde{C}(\delta,\alpha,g,h)}{\rho^2}, \tag{3.8}
$$

where the first part is negative by the choice made for δ and then we can find the required R, namely,

$$
R > \sqrt{\frac{2(\tilde{C}(\delta,\alpha,g,h)-C)}{\lambda-\delta-\alpha^-}}.
$$

□

We need also the following result.

LEMMA 3.3. *We have*

$$
\sup_{u\in\gamma^*(B^k)} F(u) \geqslant -C_1(\delta,h) - C_2(\delta,g) - 1 \quad \forall\gamma^*\in\Gamma^R_{\alpha,\bar{r}}. \tag{3.9}
$$

Proof. Fix a $\gamma^*\in\Gamma^R_{\alpha,\bar{r}}$.

Since $\gamma^*(B^k)$ is a compact set in a space of continuous functions, we can find

$$
b(\gamma^*) = \max\{|u(x)| : x\in[0,1],\ u\in\gamma^*(B^k)\}, \tag{3.10}
$$

and then there exists $\mu_{\gamma^*}>0$ such that

$$
\tfrac{1}{2}\mu_{\gamma^*}s^2 \geqslant G(x,s)-1 \quad \text{for all } s\in[0,b(\gamma^*)]. \tag{3.11}
$$

Then

$$
\begin{aligned}
&\int_0^1 G(x,u) + \int_0^1 hu \\
&\quad \leqslant \tfrac{1}{4}\delta\int_0^1 u^2 + C_1(\delta,h) + \tfrac{1}{4}\delta\int_0^1 u^2 + C_2(\delta,g) + \tfrac{1}{2}\mu_{\gamma^*}\int_0^1 (u^+)^2 + \int_0^1 1,
\end{aligned} \tag{3.12}
$$

and so

$$
\begin{aligned}
\sup_{u\in\gamma^*(B^k)} F(u) \geqslant \frac{1}{2}\sup_{u\in\gamma^*(B^k)}\biggl(\int_0^1 (u')^2 - (\lambda+\delta)\int_0^1 u^2 - \mu_{\gamma^*}\int_0^1 (u^+)^2\biggr)& \\
- C_1(\delta,h) - C_2(\delta,g) - 1.&
\end{aligned} \tag{3.13}
$$

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Now, if $0 \in \gamma^*(B^k)$, the sup on the right-hand side is clearly non-negative.

Otherwise, we can rearrange the terms in the sup on the right-hand side, adding and subtracting $\alpha^+ \int_0^1 (u^+)^2 + \alpha^- \int_0^1 (u^-)^2$, defining

$$r_{\gamma^*} = \frac{\lambda + \delta - \alpha^-}{\lambda + \delta + \mu_{\gamma^*} - \alpha^+}$$

and collecting $\int_0^1 (u^+)^2 + r_{\gamma^*} \int_0^1 (u^-)^2 > 0$. We thus obtain

$$\sup_{u \in \gamma^*(B^k)} \biggl[\biggl(\frac{J_\alpha(u)}{\int_0^1 (u^+)^2 + r_{\gamma^*} \int_0^1 (u^-)^2} - (\lambda + \delta + \mu_{\gamma^*} - \alpha^+)\biggr) \times \biggl(\int_0^1 (u^+)^2 + r_{\gamma^*} \int_0^1 (u^-)^2\biggr)\biggr]. \tag{3.14}$$

Now, if the sup of the first part is non-negative, then so is all the sup.

But

$$\sup_{u \in \gamma^*(B^k)} \frac{J_\alpha(u)}{\int_0^1 (u^+)^2 + r_{\gamma^*} \int_0^1 (u^-)^2}$$

is equivalent to $\sup_{u \in \gamma(B^k)} J_\alpha(u)$ for some $\gamma \in \varGamma_{\alpha, r_{\gamma^*}}$ (cf. (2.19) and (3.2), considering the definition 2.18). Then it is not lower than the value $d_{\alpha, r_{\gamma^*}}$ obtained in proposition 2.13. This means that, by proposition 2.14 and remark 2.15,

$$\sup_{u \in \gamma(B^k)} J_\alpha(u) \geqslant \lambda^+_{\gamma^*} - \alpha^+, \tag{3.15}$$

where $(\lambda^+_{\gamma^*}, \lambda^-_{\gamma^*}) \in \varSigma_h$ with $h \geqslant k+1$ and $(\lambda^-_{\gamma^*} - \alpha^-)/(\lambda^+_{\gamma^*} - \alpha^+) = r_{\gamma^*}$.

There remains the calculation

$$(\lambda^+_{\gamma^*} - \alpha^+) - (\lambda + \delta + \mu_{\gamma^*} - \alpha^+) = \frac{(\lambda^-_{\gamma^*} - \alpha^-) - (\lambda + \delta - \alpha^-)}{r_{\gamma^*}} = \frac{\lambda^-_{\gamma^*} - (\lambda + \delta)}{r_{\gamma^*}}, \tag{3.16}$$

which is positive for the choice made for δ, since the curves $\varSigma_h$, with $h \geqslant k+1$, have all points with $\lambda^- > \frac{1}{4}\lambda_{k+1}$.

To conclude, note that, in this way, we eliminated the dependence from γ^* (and from the values that depended upon it, r_{γ^*}, $\lambda^+_{\gamma^*}$ and $\lambda^-_{\gamma^*}$) in the estimates, and hence the lemma is proved. □

The PS condition for F was proved (using hypothesis (H2)) in [6] for $\lambda \in (0, \frac{1}{4}\pi^2)$, and in [7] (also using (H3)) for any $\lambda > 0$, in the case of periodic boundary conditions, but the proof can be extended to the Neumann case. The complete proof is given, for the sake of completeness, in § 4.

Using lemma 3.2 with $C < -C_1(\delta, h) - C_2(\delta, g) - 1$, lemma 3.3 and the PS condition, we have the conditions necessary to apply a linking theorem that proves the criticality of the level f defined in equation (3.3). Thus theorem 1.2 is proved.

3.2. Proof of theorem 1.3

For the values $\lambda = \frac{1}{4}\lambda_{k+1}$, one has a kind of resonance that creates difficulties for some of the estimates. Actually, the proof of lemma 3.2 can be done in the same

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way, choosing $\delta > 0$ such that $\alpha^- < \lambda - \delta$. But, for lemma 3.3, we cannot conclude with the same estimates, since no choice of $\delta > 0$ would allow us to infer that the expression in (3.16) is not negative.

Thus, in this case, we need to also impose the hypothesis (HR), and we proceed using the following estimates:

$$\int_{u<-\rho_0} G(x,u) + hu \leqslant M_0 \int_0^1 1,$$
$$\int_{u\in[-\rho_0,0]} G(x,u) + hu \leqslant \sup_{u\in[-\rho_0,0],x\in[0,1]} G(x,u) \int_0^1 1 + \rho_0 \int_0^1 |h| = C_4(h,g),$$
$$\int_{u>0} G(x,u) + hu \leqslant \tfrac{1}{2}\mu_{\gamma^*} \int_0^1 (u^+)^2 + \int_0^1 1 + \frac{1}{2} \int_0^1 (u^+)^2 + \frac{1}{2}\int_0^1 h^2.$$

Then we get, in place of (3.12), that

$$\int_0^1 G(x,u) + \int_0^1 hu \leqslant \tfrac{1}{2}(\mu_{\gamma^*}+1) \int_0^1 (u^+)^2 + M_0 + C_4(h,g) + 1 + \frac{1}{2}\int_0^1 h^2,$$

and we can estimate the sup, as in (3.13), as

$$\sup_{u\in\gamma^*(B^k)} F(u) \geqslant \frac{1}{2} \sup_{u\in\gamma^*(B^k)} \left(\int_0^1 (u')^2 - \lambda \int_0^1 u^2 - (\mu_{\gamma^*}+1)\int_0^1 (u^+)^2\right) - M_0 - 1 - \frac{1}{2}\int_0^1 h^2 - C_4(h,g). \quad (3.17)$$

We make the same calculations as we did before, but now with

$$r_{\gamma^*} = \frac{\lambda - \alpha^-}{\lambda + \mu_{\gamma^*} + 1 - \alpha^+},$$

to conclude that there is a point $(\lambda^+_{\gamma^*}, \lambda^-_{\gamma^*}) \in \Sigma_h$, with

$$h \geqslant k+1 \quad \text{and} \quad \frac{\lambda^-_{\gamma^*} - \alpha^-}{\lambda^+_{\gamma^*} - \alpha^+} = r_{\gamma^*},$$

such that the sup is not negative if the following expression is not negative too:

$$(\lambda^+_{\gamma^*} - \alpha^+) - (\lambda + \mu_{\gamma^*} + 1 - \alpha^+) = \frac{(\lambda^-_{\gamma^*} - \alpha^-) - (\lambda - \alpha^-)}{r_{\gamma^*}} = \frac{\lambda^-_{\gamma^*} - \lambda}{r_{\gamma^*}}. \quad (3.18)$$

But this is actually positive, since all points in Σ_h with $h \geqslant k+1$ have $\lambda^- > \lambda$.

4. Proof of the PS condition

Proposition 4.1. *Under hypotheses (H1), (H2) and (H3), the functional (3.1) satisfies the PS condition for any $\lambda > 0$.*

First note that, from hypothesis (H1), we can always construct the following estimates: for any $\varepsilon > 0$, $\bar{s} \in \mathbb{R}$ and $M \in R$, there exist C_M, $C_\varepsilon \in \mathbb{R}$ (also

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depending, of course, on $\bar{s}$) such that

$$g(x,s) \geqslant Ms - C_M \quad \text{for } s > \bar{s}, \tag{4.1}$$

$$|g(x,s)| \leqslant \varepsilon(-s) + C_\varepsilon \quad \text{for } s \leqslant \bar{s}. \tag{4.2}$$

Let $\{u_n\} \subseteq H^1(0,1)$ be a PS sequence, i.e. there exist $T > 0$ and $\varepsilon_n \to 0^+$ such that

$$|F(u_n)| = \left|\frac{1}{2}\int_0^1 |u_n'|^2 - \tfrac{1}{2}\lambda\int_0^1 u_n^2 - \int_0^1 G(x,u_n) - \int_0^1 hu_n\right| \leqslant T, \tag{4.3}$$

$$|\langle F'(u_n), v\rangle| = \left|\int_0^1 u_n'v' - \lambda\int_0^1 u_n v - \int_0^1 g(x,u_n)v - \int_0^1 hv\right| \leqslant \varepsilon_n\|v\|_{H^1} \quad \forall v \in H^1. \tag{4.4}$$

Step 1. Suppose u_n is not bounded. Then we can assume that $\|u_n\|_{H^1} \geqslant 1$, $\|u_n\|_{H^1} \to +\infty$ and define $z_n = u_n/\|u_n\|_{H^1}$, so that z_n is a bounded sequence in H^1 and we can select a subsequence such that $z_n \to z_0$ weakly in H^1 and strongly in $L^2(0,1)$ and $\mathcal{C}^0[0,1]$.

Step 2. We claim that $z_0 \leqslant 0$.

Proof of the claim. Consider $|\langle F'(u_n), z_0^+\rangle/\|u_n\|_{H^1}|$,

$$\left|\int_0^1 z_n'(z_0^+)' - \lambda\int_0^1 z_n z_0^+ - \int_0^1 \frac{g(x,u_n)z_0^+}{\|u_n\|_{H^1}} - \int_0^1 \frac{hz_0^+}{\|u_n\|_{H^1}}\right| \leqslant \frac{\varepsilon_n\|z_0^+\|_{H^1}}{\|u_n\|_{H^1}}, \tag{4.5}$$

from which

$$\int_0^1 \frac{g(x,u_n)z_0^+}{\|u_n\|_{H^1}} \leqslant \left|\int_0^1 z_n'(z_0^+)'\right| + \lambda\left|\int_0^1 z_n z_0^+\right| + \left|\int_0^1 \frac{hz_0^+}{\|u_n\|_{H^1}}\right| + \frac{\varepsilon_n\|z_0^+\|_{H^1}}{\|u_n\|_{H^1}}. \tag{4.6}$$

Now, for any $\bar{x}$ such that $z_0^+(\bar{x}) > 0$, we have that $u_n(\bar{x}) > 0$ for n large enough. We can then use estimate (4.1) to obtain

$$\frac{g(\bar{x},u_n)}{\|u_n\|_{H^1}} \geqslant Mz_n(\bar{x}) - \frac{C_M}{\|u_n\|_{H^1}}. \tag{4.7}$$

Taking $\liminf$, we get

$$\liminf_{n\to+\infty} \frac{g(\bar{x},u_n)}{\|u_n\|_{H^1}} \geqslant Mz_0(\bar{x}) \tag{4.8}$$

for any choice of M, and then

$$\lim_{n\to+\infty} \frac{g(\bar{x},u_n)}{\|u_n\|_{H^1}} = +\infty. \tag{4.9}$$

Joining equations (4.1) and (4.2) with $\bar{s} = 0$ and divided by $\|u_n\|_{H^1}$, we get

$$\frac{g(x,u_n)}{\|u_n\|_{H^1}} \geqslant \begin{cases} Mz_n - \dfrac{C_M}{\|u_n\|_{H^1}} & \text{if } z_n > 0, \\ -\varepsilon(-z_n) - \dfrac{C_\varepsilon}{\|u_n\|_{H^1}} & \text{if } z_n \leqslant 0, \end{cases}$$

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and so

$$\frac{g(x,u_n)}{\|u_n\|_{H^1}} \geqslant -\varepsilon|z_n| - \frac{C_{M,\varepsilon}}{\|u_n\|_{H^1}}. \tag{4.10}$$

Since z_n is uniformly bounded (by its $\mathcal{C}^0$ convergence) and $\|u_n\|_{H^1} \geqslant 1$, this implies that the functions $g(x,u_n)/\|u_n\|_{H^1}$ are bounded below uniformly, so that we can use Fatou's lemma and get, from (4.6) (supposing $z_0^+ \not\equiv 0$),

$$\begin{aligned}
+\infty &= \int_0^1 \lim_{n\to+\infty} \frac{g(x,u_n)z_0^+}{\|u_n\|_{H^1}} \\
&\leqslant \liminf_{n\to+\infty} \int_0^1 \frac{g(x,u_n)z_0^+}{\|u_n\|_{H^1}} \\
&\leqslant \liminf_{n\to+\infty}\biggl(\biggl|\int_0^1 z_n'(z_0^+)'\biggr| + \lambda\biggl|\int_0^1 z_n z_0^+\biggr| + \biggl|\int_0^1 \frac{hz_0^+}{\|u_n\|_{H^1}}\biggr| + \frac{\varepsilon_n\|z_0^+\|_{H^1}}{\|u_n\|_{H^1}}\biggr).
\end{aligned} \tag{4.11}$$

The right-hand side can be estimated, since the first two terms are bounded by $(1+\lambda)\|z_n\|_{H^1}\|z_0^+\|_{H^1} \leqslant 1+\lambda$ and the last two clearly go to zero. Then equation (4.11) gives rise to a contradiction unless $z_0 \leqslant 0$. □

Step 3. We claim that, using hypotheses (H2) and (H3), we obtain a constant A such that

$$\int_{u_n>s_0} g(x,u_n)u_n \leqslant A\|u_n\|_{H^1}, \tag{4.12}$$

at least for n big enough.

Proof of the claim. Consider first $|2F(u_n) - \langle F'(u_n), u_n\rangle|$,

$$\biggl|\int_0^1 -2G(x,u_n) + g(x,u_n)u_n + (1-2)\int_0^1 hu_n\biggr| \leqslant 2T + \varepsilon_n\|u_n\|_{H^1}, \tag{4.13}$$

from which

$$\begin{aligned}
\int_{u_n>s_0} & g(x,u_n)u_n - 2G(x,u_n) \\
&\leqslant \int_{u_n\leqslant s_0} 2G(x,u_n) - g(x,u_n)u_n + \biggl|\int_0^1 hu_n\biggr| + 2T + \varepsilon_n\|u_n\|_{H^1}.
\end{aligned} \tag{4.14}$$

The right-hand side may be estimated as follows.

(i) We have

$$\int_{-s_1\leqslant u_n\leqslant s_0} 2G(x,u_n) - g(x,u_n)u_n \leqslant \sup_{x\in[0,1],s\in[-s_1,s_0]} (2G(x,s) - g(x,s)s). \tag{4.15}$$

(ii) Using hypothesis (H3),

$$\int_{u_n\leqslant -s_1} 2G(x,u_n) - g(x,u_n)u_n \leqslant 2C_0. \tag{4.16}$$

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(iii) We have

$$\left|\int_0^1 hu_n\right| \leqslant \|h\|_{L^2}\|u_n\|_{L^2} \leqslant \|h\|_{L^2}\|u_n\|_{H^1}.$$

For the left-hand side, we use hypothesis (H2) to obtain

$$(1-2\theta)\int_{u_n>s_0} g(x,u_n)u_n \leqslant \int_{u_n>s_0} g(x,u_n)u_n - 2G(x,u_n), \tag{4.17}$$

and then, since $(1-2\theta)>0$, joining all estimates (4.14)–(4.17), we get

$$\int_{u_n>s_0} g(x,u_n)u_n \leqslant \tfrac{1}{2}A\|u_n\|_{H^1} + \tfrac{1}{2}A \leqslant A\|u_n\|_{H^1} \tag{4.18}$$

for some constant A. □

STEP 4. We claim that

$$\int_0^1 \frac{|g(x,u_n)|}{\|u_n\|_{H^1}} \to 0. \tag{4.19}$$

Proof of the claim. Fix $\varepsilon>0$ and k such that $A/k \leqslant \varepsilon$ and $k > s_0$.

Estimate (4.2) shows that

$$\int_{u_n\leqslant k} \frac{|g(x,u_n)|}{\|u_n\|_{H^1}} \leqslant \int_0^1 \frac{\varepsilon|u_n| + C_\varepsilon}{\|u_n\|_{H^1}} \leqslant \varepsilon C\frac{\|u_n\|_{L^2}}{\|u_n\|_{H^1}} + \frac{C_\varepsilon}{\|u_n\|_{H^1}}, \tag{4.20}$$

from which there exists $\bar{n}$ such that

$$\int_{u_n\leqslant k} \frac{|g(x,u_n)|}{\|u_n\|_{H^1}} \leqslant (C+1)\varepsilon \quad \text{for } n>\bar{n}. \tag{4.21}$$

Since $k>s_0$, using estimate (4.12), we have

$$\int_{u_n>k} \frac{g(x,u_n)}{\|u_n\|_{H^1}} \leqslant \int_{u_n>k} \frac{g(x,u_n)}{\|u_n\|_{H^1}}\frac{u_n}{k} \leqslant \int_{u_n>s_0} \frac{g(x,u_n)}{\|u_n\|_{H^1}}\frac{u_n}{k} \leqslant \frac{A}{k} \leqslant \varepsilon. \tag{4.22}$$

Then we conclude that, for $n>\bar{n}$,

$$\int_0^1 \frac{|g(x,u_n)|}{\|u_n\|_{H^1}} \leqslant (2+C)\varepsilon. \tag{4.23}$$

By the arbitrariness of ε, the claim is proved. □

STEP 5. We claim that $\lambda>0$ implies $z_0=0$.

Proof of the claim. For any $v\in H^1$, we consider $|\langle F'(u_n),v\rangle/\|u_n\|_{H^1}|$,

$$\left|\int_0^1 z_n'v' - \lambda\int_0^1 z_n v - \int_0^1 \frac{g(x,u_n)v}{\|u_n\|_{H^1}} - \int_0^1 \frac{hv}{\|u_n\|_{H^1}}\right| \leqslant \frac{\varepsilon_n\|v\|_{H^1}}{\|u_n\|_{H^1}}, \tag{4.24}$$

from which

$$\left|\int_0^1 z_n'v' - \lambda\int_0^1 z_n v\right| \leqslant \int_0^1 \frac{|g(x,u_n)|}{\|u_n\|_{H^1}}|v| + \left|\int_0^1 \frac{hv}{\|u_n\|_{H^1}}\right| + \frac{\varepsilon_n\|v\|_{H^1}}{\|u_n\|_{H^1}}. \tag{4.25}$$

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But now the right-hand side goes to zero by equation (4.19), and then we get, taking the limit and using the weak convergence of z_n, that

$$\int_0^1 z_0'v' - \lambda\int_0^1 z_0 v = 0 \quad \text{for any } v \in H^1. \tag{4.26}$$

Since all eigenfunctions of the Neumann problem with $\lambda > 0$ change sign (while $z_0 \leqslant 0$), this implies that $z_0 = 0$. □

Step 6. We claim that u_n is bounded.

Proof of the claim. Consider $|\langle F'(u_n), z_n\rangle/\|u_n\|_{H^1}|$,

$$\left|\int_0^1 (z_n')^2 - \lambda\int_0^1 z_n^2 - \int_0^1 \frac{g(x,u_n)z_n}{\|u_n\|_{H^1}} - \int_0^1 \frac{hz_n}{\|u_n\|_{H^1}}\right| \leqslant \frac{\varepsilon_n\|z_n\|_{H^1}}{\|u_n\|_{H^1}}, \tag{4.27}$$

from which

$$\int_0^1 (z_n')^2 \leqslant \lambda\int_0^1 z_n^2 + \int_0^1 \frac{|g(x,u_n)||z_n|}{\|u_n\|_{H^1}} + \int_0^1 \frac{hz_n}{\|u_n\|_{H^1}} + \frac{\varepsilon_n\|z_n\|_{H^1}}{\|u_n\|_{H^1}}. \tag{4.28}$$

Now, using (4.19) and the fact that $z_n \to 0$ in L^2, equation (4.28) becomes

$$\int_0^1 (z_n')^2 \to 0, \tag{4.29}$$

which gives a contradiction, since we would get

$$1 = \int_0^1 (z_n')^2 + \int_0^1 z_n^2 \to 0.$$

□

Step 7. The PS condition follows now with standard calculations from the boundedness of u_n.

Acknowledgments

The author expresses thanks for the hospitality of UNICAMP, Campinas, SP, Brasil.

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(*Issued 29 June 2004*)