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Fourier transforms and p-adic 'Weil II'

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Abstract

We give a purity theorem in the manner of Deligne's 'Weil II' theorem for rigid cohomology with coefficients in an overconvergent F-isocrystal; the proof mostly follows Laumon's Fourier-theoretic approach, transposed into the setting of arithmetic $\mathcal{D}$-modules. This yields in particular a complete, purely p-adic proof of the Weil conjectures when combined with recent results on p-adic differential equations by André, Christol, Crew, Kedlaya, Matsuda, Mebkhout and Tsuzuki.

1. Introduction

1.1 About this paper

The purpose of this paper is to establish a weak analogue, in Berthelot's rigid (p-adic) cohomology, of Deligne's 'Weil II' theorem [Del80, Théorème 3.3.1] on purity of higher direct images between two schemes of finite type over a finite field. The theorem we establish here (Theorem 5.3.2) asserts that given an overconvergent F-isocrystal on a variety over a finite field which is pure of some weight i (relative to some embedding of the relevant p-adic field into the complex numbers), the jth compactly supported rigid cohomology with coefficients in that isocrystal is mixed of weights no greater than $i + j$.

The basis for the argument is some recent progress in rigid cohomology, notably the finite dimensionality of rigid cohomology with coefficients in an overconvergent F-isocrystal, proved by the present author in [Ked05b]. Indeed, this paper may be most easily read as (and was, in fact, written as) a companion to [Ked05b], since the two papers involve a shared set of ideas, including construction of some higher direct images; here, such arguments are augmented by what amounts to an analysis of nearby cycles.

On this foundation, we prove our Weil II analogue by combining ideas from several sources for the étale case. The first of these is of course Deligne's original argument, whose basic framework (the theory of global monodromy and determinantal weights) was imported into rigid cohomology by Crew [Cre92, Cre98]. To this we add the 'principle of stationary phase' introduced by Laumon [Lau87] (and redescribed in [KW01]) and incarnated in a geometric Fourier transform, which in the rigid cohomology setting is quite natural from the point of view of $\mathcal{D}$-modules (as was first noted by Mebkhout [Meb97], who indeed suggested using it to prove a p-adic Weil II theorem) and has been described in detail by Noot-Huyghe [Huy04]. We also mix in some variations described by Katz in his lectures at the 2000 Arizona Winter School [Kat01] (although not Katz's general approach of short-cutting the study of global monodromy using explicit families with big monodromy).

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1.2 Beyond this paper

Note that our analogue of Weil II is 'pointwise' in that it only applies to morphisms to a point. That is partly because the 'lisse' coefficients in rigid cohomology, namely the overconvergent F-isocrystals, are well understood, whereas the theory of the corresponding 'constructible' objects, which would be certain arithmetic $\mathcal{D}$-modules, is somewhat less complete (see Berthelot's survey [Ber02] for an overview). This is partly circumvented by Caro [Car05], who obtains an extension of our main theorem with more general coefficient objects on the source, but still a point as the base. Even with a fully functorial set of coefficient objects, though, some technical issues remain: see Remark 5.3.3 for more on these.

However, already pointwise Weil II in rigid cohomology is of some significance. It demonstrates that one can recover a proof of the Weil conjectures purely within the framework of a Weil cohomology theory which is 'explicitly constructible'. (This is not demonstrated by two earlier forms of p-adic Weil II: the sketch given by Faltings [Fal90], which relies on unverified properties of relative crystalline cohomology, or the theorem of Chiarellotto [Chi98] using Katz and Messing's crystalline version of the Weil conjectures [KM74], which ultimately relies on Deligne's original theorem.) It should also suggest techniques for further extending the analogy between rigid cohomology and étale cohomology, which may ultimately lead to finding some provable statements in rigid cohomology where only conjectural analogues exist in étale cohomology (one promising candidate seems to be the weight-monodromy conjecture). We leave additional thoughts on this to the imagination of the interested reader.

We also note that Olsson [Ols05a, Ols05b] has incorporated the purity theorem of this paper into a program for building a p-adic version of nonabelian Hodge theory.

1.3 Structure of the paper

We conclude this introduction by describing the contents of the various sections of the paper.

In § 2, we set notation concerning rigid cohomology and overconvergent F-isocrystals.

In § 3, we recall the results of [Ked05b] on higher direct images in rigid cohomology, along some simple morphisms of relative dimension 1. We then work out some more precise results along these lines, particularly concerning degeneration in families. That is, we must understand how the cohomology of a single member of a family is controlled by the cohomology of the other members of the family.

In § 4, we introduce, in a limited context, the geometric Fourier transform in the p-adic setting and its $\mathcal{D}$-module interpretation. We also formulate an analogue of the Grothendieck–Ogg–Shafarevich formula, which constrains the Euler–Poincaré characteristic of (the cohomology of) an overconvergent F-isocrystal in terms of local monodromy. This formula is needed to show that the Fourier transform of certain overconvergent F-isocrystals are again isocrystals.

In § 5, we assemble the proof of p-adic Weil II. We also give an estimate in the same spirit for the p-adic valuations of eigenvalues in cohomology.

2. Rigid cohomology

We start by setting notation regarding rigid cohomology. Our notation follows [Ked05b], to which we defer for additional details and further references.

Let q be a fixed power of the prime p. Let k be a perfect field of characteristic p containing $\mathbb{F}_q$, let $\mathfrak{o}$ be a finite totally ramified extension of the ring of Witt vectors $W(k)$, let $\mathfrak{m}$ be the maximal ideal of $\mathfrak{o}$, and let K be the fraction field of $\mathfrak{o}$. We will assume throughout that $\mathfrak{o}$ is equipped with an automorphism σ_K lifting the qth power map. For instance, if $\mathfrak{o} = W(k)$, then there is a unique choice of σ_K; it coincides with a power of the Witt vector Frobenius. Also, if $k = \mathbb{F}_q$, we may of course take σ_K to be the identity map whatever $\mathfrak{o}$ happens to be.

We will frequently consider modules over various rings equipped with a linear or semilinear endomorphism. If M is such a module equipped with F, we write $M(-i)$ to denote M equipped with $q^i F$, and call this the *$(-i)$th Tate twist* of M.

2.1 The formalism of rigid cohomology

We first recall some of the formalism of rigid cohomology, following [Ber86] and [Ked05b, § 4] (which serve as blanket references except when other references are specified); we postpone defining anything until the next section. For shorthand, we abbreviate 'reduced separated scheme of finite type over (the field) k' to 'variety over k'.

The coefficient objects in rigid cohomology are called *overconvergent F-isocrystals* (with respect to K); they form a category fibred in symmetric tensor categories over the category of k-varieties. In other words:

- the fibre over each variety X admits direct sums, tensor products (which commute), duals, internal Homs, and an identity object $\mathcal{O}_X$ for tensoring (the 'constant sheaf');
- to each morphism $f : X \to Y$ of k-varieties is associated a pullback functor f^* that commutes with the aforementioned operations; these pullback functors compose up to natural isomorphism.

This particular category has the following additional properties.

- One also has pullback functors associated to automorphisms of K.
- The category carries a natural isomorphism F ('Frobenius') between the identity functor and the composition of the pullback functor associated to σ_K with the relative Frobenius. (By definition, the action of F on the dual of an overconvergent F-isocrystal is the *inverse* transpose of its action on the original.)
- The fibre over $\operatorname{Spec} k'$, for k' a finite extension of k, is equivalent to the category of finite-dimensional K'-vector spaces, for K' the unramified extension of K with residue field k', equipped with a bijective $\sigma_{K'}$-linear transformation F. (Here $\sigma_{K'}$ is the unique extension of σ_K to an automorphism of K' lifting the q-power Frobenius.)
- There are Tate twist functors which pointwise multiply F by the appropriate power of q.

Associated to a k-variety X and an overconvergent F-isocrystal $\mathcal{E}$ over X are its rigid cohomology spaces $H^i_{\mathrm{rig}}(X/K, \mathcal{E})$ and its rigid cohomology spaces with compact supports $H^i_{c,\mathrm{rig}}(X/K, \mathcal{E})$. These are vector spaces over K equipped with Frobenius actions, which coincide if X is proper; they vanish for $i < 0$ and for $i > 2 \dim X$ and are finite dimensional in general by [Ked05b, Theorems 1.1 and 1.2]. If X is smooth of pure dimension n, by [Ked05b, Theorem 1.3] there is a canonical perfect pairing (Poincaré duality)

$$H^i_{\mathrm{rig}}(X/K, \mathcal{E}) \times H^{2n-i}_{c,\mathrm{rig}}(X/K, \mathcal{E}^\vee) \to \mathcal{O}_X(-n).$$

The cohomology spaces are functorial in the following senses. Given overconvergent F-isocrystals $\mathcal{E}_1, \mathcal{E}_2$ on X and a morphism $h : \mathcal{E}_1 \to \mathcal{E}_2$, we obtain morphisms

$$H^i_{\mathrm{rig}}(X/K, \mathcal{E}_1) \to H^i_{\mathrm{rig}}(X/K, \mathcal{E}_2), \quad H^i_{c,\mathrm{rig}}(X/K, \mathcal{E}_1) \to H^i_{c,\mathrm{rig}}(X/K, \mathcal{E}_2)$$

which compose as expected. Given a morphism $f : X \to Y$ of varieties and an overconvergent F-isocrystal $\mathcal{E}$ on Y, we obtain morphisms

$$H^i_{\mathrm{rig}}(Y/K, \mathcal{E}) \to H^i_{\mathrm{rig}}(X/K, f^*\mathcal{E})$$

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which again compose as expected; the same is true if we allow f to lie over a nontrivial power of Frobenius, using the lift σ_K of Frobenius to K. If $f : X \to Y$ is finite étale, there is a push-forward functor f_* from overconvergent F-isocrystals on X to those on Y, and we have canonical isomorphisms

$$H^i_{\mathrm{rig}}(X/K,\mathcal{E}) \cong H^i_{\mathrm{rig}}(Y/K, f_*\mathcal{E}), \quad H^i_{c,\mathrm{rig}}(X/K,\mathcal{E}) \cong H^i_{c,\mathrm{rig}}(Y/K, f_*\mathcal{E}).$$

In cohomology with compact supports, for $Z \hookrightarrow X$ a closed immersion, we have an excision exact sequence

$$\cdots \to H^i_{c,\mathrm{rig}}(X \setminus Z/K,\mathcal{E}) \to H^i_{c,\mathrm{rig}}(X/K,\mathcal{E}) \to H^i_{c,\mathrm{rig}}(Z/K,\mathcal{E}) \to H^{i+1}_{c,\mathrm{rig}}(X \setminus Z/K,\mathcal{E}) \to \cdots, \tag{2.1.1}$$

where the maps 'at one level' (i.e. from one H^i to another) are Frobenius-equivariant. There is also an excision sequence in ordinary cohomology, but it involves relative cohomology which we will not discuss here.

For any closed point x of a variety X, we can pull back an overconvergent F-isocrystal $\mathcal{E}$ along the embedding $x \hookrightarrow X$ to obtain an object we denote as $\mathcal{E}_x$. As noted above, the data of $\mathcal{E}_x$ amounts to a vector space over the unramified extension K' of K with residue field $\kappa(x)$, equipped with a $\sigma_{K'}$-linear bijection induced by F. We call either object the *fibre* of $\mathcal{E}$ at x.

Now suppose $k = \mathbb{F}_q$ and that σ_K is the identity morphism. Then $F^{\deg(x)}$ induces a *linear* transformation F_x on $\mathcal{E}_x$. (However, the natural action of $F^{\deg(x)}$ on $\mathcal{E}_x \otimes_{K'} L$, for L a finite extension of K', is typically *not* linear.) By a theorem of Étesse and le Stum [EL93, Théorème 6.3], we have a Lefschetz trace formula for Frobenius, given by the following equality of formal power series:

$$\prod_{x \in X} \det(1 - F_x t^{\deg(x)}, \mathcal{E}_x)^{-1} = \prod_i \det(1 - Ft, H^i_{c,\mathrm{rig}}(X/K,\mathcal{E}))^{(-1)^{i+1}}. \tag{2.1.2}$$

Note that in (2.1.2), the determinant of $1 - F_x t^{\deg(x)}$ is being taken over K', but actually has coefficients in K. If one prefers to work exclusively over K, one may write (2.1.2) in the form given in [EL93]:

$$\prod_{x \in X} \det_K(1 - F_x t^{\deg(x)}, \mathcal{E}_x)^{-1/\deg(x)} = \prod_i \det(1 - Ft, H^i_{c,\mathrm{rig}}(X/K,\mathcal{E}))^{(-1)^{i+1}}.$$

2.2 Affinoid and dagger algebras

We compute in rigid cohomology not using its general definition, but using a construction special to the smooth affine case, due to Monsky and Washnitzer. This theory looks like algebraic de Rham cohomology except that the coordinate ring of the original affine scheme is replaced by a 'dagger algebra'. In this section, we recall the construction and properties of dagger algebras, following [Ked05b, § 2] (which again we treat as a blanket reference).

We first recall the notion of an affinoid algebra. Define the ring

$$T_n = K\langle x_1, \dots, x_n \rangle = \left\{ \sum_I a_I x^I : a_I \in K, \lim_{\sum I \to \infty} |a_I| = 0 \right\}.$$

Here $I = (i_1, \dots, i_n)$ denotes an n-tuple of nonnegative integers, $x^I = x_1^{i_1} \cdots x_n^{i_n}$, and $\sum I = i_1 + \cdots + i_n$. An *affinoid algebra* over K is any K-algebra isomorphic to a quotient of T_n for some n. If A is a reduced affinoid algebra, there is a canonical power-multiplicative norm $|\cdot|_{\sup,A}$ on A, called the *spectral norm*, with respect to which A is complete. We also define the *spectral valuation* v_A by

$$v_A(x) = -\log_p |x|_{\sup,A}.$$

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We now proceed to dagger algebras. Let Γ^* be the (multiplicative) value group of K^{alg}. For $\rho > 1$ in Γ^*, define the ring

$$T_{n,\rho} = \left\{ \sum_I a_I x^I : a_I \in K, \lim_{\sum I \to \infty} |a_I| \rho^{\sum I} = 0 \right\};$$

it is a reduced affinoid algebra with spectral norm given by $|\sum a_I x^I|_\rho = \max_I\{|a_I|\rho^{\sum I}\}$. Define the ring of *overconvergent power series* in $x_1, \dots, x_n$ by

$$W_n = K\langle x_1, \dots, x_n\rangle^\dagger = \bigcup_{\rho>1} T_{n,\rho}.$$

We note in passing that any finite projective module over T_n or W_n is free, by an analogue of the Quillen–Suslin theorem; see [Ked04b, Theorem 6.7]. A *dagger algebra* over K is any K-algebra isomorphic to a quotient of W_n for some n. Topologizing W_n as a subspace of T_n, we induce a topology on any dagger algebra, called the *affinoid topology*.

If A is a dagger algebra, we define a *fringe algebra* of A as a subalgebra of the form $f(T_{n,\rho})$ for some surjection $f : W_n \to A$ and some $\rho > 1$ in Γ^*; note that any fringe algebra is an affinoid algebra, and so has a natural topology under which it is complete. We can retopologize A as the direct limit of its fringe algebras (i.e. a sequence converges to a limit if and only if it does so in some fringe algebra); we call this topology the *fringe topology*. The fringe topology is crucial for constructing Robba rings over dagger algebras in § 3.1.

Let T_n^{int} or W_n^{int} be the subring of T_n or W_n, respectively, consisting of series with integral coefficients. Then the image of T_n^{int} or W_n^{int} under a surjection $f : T_n \to A$ or $f : W_n \to A$ is independent of f, because the elements of this image can be characterized topologically (as those $x \in A$ such that cx^d is topologically nilpotent for any positive integer d and any $c \in \mathfrak{m}$). We call this image the *integral subring* of A, denoted A^{int}. More generally, any homomorphism $g : A \to B$ of affinoid or dagger algebras carries A^{int} into B^{int}.

In the same vein, it turns out that the image under a surjection $f : W_n \to A$ of the ideal of W_n^{int} consisting of series whose coefficients all lie in $\mathfrak{m}$ is independent of f. The elements of this ideal are the topologically nilpotent elements of A^{int}; the quotient of A^{int} by this ideal, which is finitely generated as a k-algebra, is called the *reduction* of A. If R is the reduction of A, we call $\operatorname{Spec} R$ the *special fibre* of A.

Given a dagger algebra $A = W_n/\mathfrak{a}$, write

$$A\langle t\rangle^\dagger = W_{n+1}/\mathfrak{a}W_{n+1},$$

identifying t with x_{n+1}. This construction does not depend on the presentation of A. For $f \in A$ with $|f|_{\sup,A} = 1$, write

$$A\langle f^{-1}\rangle^\dagger = A\langle t\rangle^\dagger/(tf-1);$$

this is called the *localization* of A at f.

2.3 Cohomology of affine schemes

We now construct Monsky–Washnitzer cohomology, our main computational tool in studying rigid cohomology on smooth affine varieties. Our blanket reference now is [Ked05b, § 3].

The *module of continuous differentials* $\Omega^1_{A/K}$ of a dagger algebra can be constructed as follows. For $A = W_n$, take it to be the free module generated by $dx_1, \dots, dx_n$ equipped with the K-linear derivation $d : W_n \to \Omega^1_{W_n/K}$ given by

$$\sum_I c_I x^I \mapsto \sum_I \sum_{j=1}^n i_j c_I (x^I/x_j)\, dx_j.$$

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For $A \cong W_n/\mathfrak{a}$, let $\Omega^1_{A/K}$ be the quotient of $\Omega^1_{W_n/K} \otimes_{W_n} A$ by the submodule generated by dr for $r \in \mathfrak{a}$. This construction ends up being universal for A-linear derivations into finitely generated A-modules; in particular, it yields a well-defined A-module $\Omega^1_{A/K}$ and K-linear derivation $d : A \to \Omega^1_{A/K}$. If A is a subring of the dagger algebra B, we define the relative module of differentials $\Omega^1_{B/A}$ as the quotient of $\Omega^1_{B/K}$ by the images of da for $a \in A$. We also put $\Omega^i_{B/A} = \wedge^i_A \Omega^1_{B/A}$.

At this point, we restrict to a special class of dagger algebras. We say that a dagger algebra A is *of MW type* if the ideal of topologically nilpotent elements of A^{int} is generated by a uniformizer of $\mathfrak{o}$ and the special fibre of A is smooth. In the terminology of [MW68], B is a formally smooth, weakly complete, weakly finitely generated algebra over $(\mathfrak{o}, \mathfrak{m})$.

A *Frobenius lift* on a dagger algebra A of MW type is a ring endomorphism $\sigma : A \to A$ acting on K via σ_K and acting on $A^{\mathrm{int}} \otimes_{\mathfrak{o}} k$ as the qth power map $x \mapsto x^q$. Such a map exists for any A; for example, if $A = W_n$, we can define a *standard Frobenius* σ by the formula

$$\left(\sum_I c_I t^I\right)^{\sigma} = \sum_I c_I^{\sigma_K} t^{qI}.$$

Given a dagger algebra A equipped with a Frobenius lift σ, we define a *σ-module* over A as a finite locally free A-module equipped with:

(a) *a Frobenius structure*, an additive, σ-linear map $F : M \to M$ (that is, $F(a\mathbf{v}) = a^{\sigma} F(\mathbf{v})$ for $a \in A$ and $\mathbf{v} \in M$) which induces an isomorphism $\sigma^* M \to M$.

We define a *(σ, ∇)-module* over A as a σ-module additionally equipped with:

(b) *an integrable connection*, an additive, K-linear map $\nabla : M \to M \otimes_A \Omega^1_{A/K}$ satisfying the Leibniz rule $\nabla(a\mathbf{v}) = a\nabla(\mathbf{v}) + \mathbf{v} \otimes da$ for $a \in A$ and $\mathbf{v} \in M$, and such that, if we write ∇_n for the induced map $M \otimes_A \Omega^n_{A/K} \to M \otimes_A \Omega^{n+1}_{A/K}$, we have $\nabla_{n+1} \circ \nabla_n = 0$ for all $n \geqslant 0$;

subject to the compatibility condition:

(c) the isomorphism $\sigma^* M \to M$ induced by F is horizontal for the corresponding connections, in other words, the following diagram commutes.

$$\begin{tikzcd} M \arrow[r, "\nabla"] \arrow[d, "F"] & M \otimes_A \Omega^1_{A/K} \arrow[d, "F \otimes d\sigma"] \\ M \arrow[r, "\nabla"] & M \otimes_A \Omega^1_{A/K} \end{tikzcd}$$

For example, the module $M = A$, with F acting by σ and ∇ acting by d, is a (σ, ∇)-module, called the *trivial (σ, ∇)-module*. More generally, if M is spanned over A by the kernel of ∇, we say that M is *constant*.

Given a (σ, ∇)-module M over A, we define the cohomology spaces as the cohomology of the de Rham complex tensored with M. That is,

$$H^i(M) = \frac{\ker(\nabla_i : M \otimes_A \Omega^i_{A/K} \to M \otimes_A \Omega^{i+1}_{A/K})}{\mathrm{im}(\nabla_{i-1} : M \otimes_A \Omega^{i-1}_{A/K} \to M \otimes_A \Omega^i_{A/K})}.$$

If M is a (σ, ∇)-module over A, we call an A-submodule N of M a *(σ, ∇)-submodule* if it is closed under F and ∇ (the latter meaning that $\nabla(N) \subseteq N \otimes \Omega^1_{A/K}$); it turns out [Ked05b, Lemma 3.3.4] that this forces N to be a direct summand of M as an A-module, so the quotient M/N is also a (σ, ∇)-module. (Note that N need not be a direct summand of M in the category of (σ, ∇)-modules over A, because the exact sequence $0 \to N \to M \to M/N \to 0$ may not have a horizontal splitting.) This gives us a notion of irreducibility of a (σ, ∇)-module.

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We now summarize the relationship of this construction to rigid cohomology; for more details, see [Ber97, Proposition 1.10] in the constant coefficient case, or [EL93, § 3]. If X is a smooth affine k-variety, there exists a dagger algebra A of MW type with special fibre X (unique up to noncanonical isomorphism). Given the choice of A and a Frobenius lift σ on A, the category of overconvergent F-isocrystals on X is canonically equivalent to the category of (σ, ∇)-modules over A (in particular, the latter category is independent of the choice of σ). If $\mathcal{E}$ is an overconvergent F-isocrystal corresponding to a (σ, ∇)-module M, there is a canonical isomorphism

$$H^i_{\mathrm{rig}}(X/K, \mathcal{E}) \cong H^i(M),$$

which matches up the Frobenius actions. Moreover, this isomorphism is compatible with maps of k-varieties on the left and corresponding lifts (not necessarily Frobenius-equivariant) to maps of dagger algebras on the right.

2.4 The Robba ring and p-adic local monodromy

We next want to make the cohomology of curves more explicit, but first we need to introduce an auxiliary ring from the theory of p-adic differential equations.

The *Robba ring* $\mathcal{R}_K = \mathcal{R}_K^t$ (the latter notation being used when we need to name the series parameter) is defined as the ring of formal Laurent series $\sum_{n=-\infty}^{\infty} c_n t^n$, with $c_n \in K$, such that for all sufficiently small $r > 0$ (where the meaning of 'sufficiently small' depends on the series),

$$\lim_{n \to \pm\infty} (v_p(c_n) + rn) = \infty.$$

That is, such a series converges for $t \in K^{\mathrm{alg}}$ satisfying $\eta < |t| < 1$, for some η depending on the series.

We denote by $\mathcal{R}_K^{\mathrm{int}}$ the subring of $\mathcal{R}_K$ of series with $v_p(c_n) \geqslant 0$ for all n, and by $\mathcal{R}_K^+$ the subring of series with $c_n = 0$ for $n < 0$. We denote by $\mathcal{R}_K^{+,\mathrm{int}}$ the intersection of these two subrings; it coincides with $\mathfrak{o}[\![t]\!]$.

Given $r > 0$ rational, for those elements $x = \sum c_n t^n \in \mathcal{R}_K$ for which $v_p(c_n) + rn \to \infty$ as $n \to \pm\infty$, we put

$$w_r(x) = \inf_n \{v_p(c_n) + rn\};$$

this function is a discrete valuation on the subring where it is defined, and in fact it is the valuation corresponding to the supremum norm on the circle $v_p(t) = r$. Note that for any fixed $x \in \mathcal{R}_K$, $w_r(x)$ is defined for all sufficiently small $r > 0$.

We define (σ, ∇)-modules over $\mathcal{R}_K$ or $\mathcal{R}_K^+$ as in the dagger algebra setting, taking Ω^1 to be the free module generated by dt. Note that finite locally free modules over $\mathcal{R}_K$ or $\mathcal{R}_K^+$ are automatically free, because a theorem of Lazard [Laz62] implies that $\mathcal{R}_K$ and $\mathcal{R}_K^+$ are Bézout rings (rings in which every finitely generated ideal is principal).

A technique due to Dwork (analytic continuation via Frobenius) leads to the following result; see [Dej98, Lemma 6.3] for its proof, or see Proposition 3.1.1 below for a generalization.

Lemma 2.4.1. *Let M be a (σ, ∇)-module over $\mathcal{R}_K^+$. Then there exists a basis $\mathbf{w}_1, \dots, \mathbf{w}_n$ of M such that $\nabla \mathbf{w}_i = 0$ for each i. (Note that on any such basis, F acts via a matrix over K.)*

A weaker form of Lemma 2.4.1 holds for M over $\mathcal{R}_K$, but is much deeper. It is the so-called 'p-adic local monodromy theorem', and underpins this entire article as well as [Ked05b]. Proofs have been given by André [And02], Mebkhout [Meb02], and the present author [Ked04a].

Proposition 2.4.2. *Let M be a (σ, ∇)-module over $\mathcal{R}_K$. Then there exist a finite étale extension $\mathcal{R}'$ of $\mathcal{R}_K^{\mathrm{int}}$ and a basis $\mathbf{w}_1, \dots, \mathbf{w}_n$ of $M \otimes_{\mathcal{R}_K^{\mathrm{int}}} \mathcal{R}'$ such that for $i = 1, \dots, n$, the span M_i of $\mathbf{w}_1, \dots, \mathbf{w}_i$ is carried into $M_i \otimes dt$ by ∇ and the image of $\mathbf{w}_i$ in M_i/M_{i-1} is killed by ∇.*

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We say that M is *unipotent* if it satisfies the conclusion of the p-adic local monodromy theorem with $\mathcal{R}' = \mathcal{R}_K^{\mathrm{int}}$. In that case, one can find a basis $\mathbf{v}_1, \dots, \mathbf{v}_n$ of M whose K-span is preserved by the operator $E : M \to M$ defined by $\nabla(\mathbf{v}) = E(\mathbf{v}) \otimes dt/t$ (see [Ked05b, Proposition 5.2.6]).

2.5 Cohomology of curves

A detailed study of the cohomology of overconvergent F-isocrystals on curves has been made by Crew [Cre98]; we summarize his results in this section. Note that Crew's hypothesis that the connection is strict is superfluous in this setting: it follows from the p-adic local monodromy theorem thanks to [Cre98, Proposition 10.2].

Let X be a smooth irreducible affine curve, let $\overline{X}$ be its smooth compactification, and let A be a dagger algebra of MW type with special fibre X. Then for each closed point $x \in \overline{X}$, one gets a (noncanonical) embedding $A \hookrightarrow \mathcal{R}_x$, where $\mathcal{R}_x$ is a copy of the Robba ring over the unramified extension K' of K with residue field $\kappa(x)$; we can and will take this embedding to map into $\mathcal{R}_x^+$ if $x \in X$. Observe that given such an embedding, any Frobenius lift on A can be extended compatibly to $\mathcal{R}_x$.

Define

$$A_{\mathrm{loc}} = \bigoplus_{x \in \overline{X} \setminus X} \mathcal{R}_x, \quad \Omega^1_{\mathrm{loc}} = \Omega^1_{A/K} \otimes_A A_{\mathrm{loc}}$$
$$A_{\mathrm{qu}} = A_{\mathrm{loc}}/A, \quad \Omega^1_{\mathrm{qu}} = \Omega^1_{A/K} \otimes_A A_{\mathrm{qu}} = \Omega^1_{\mathrm{loc}}/\Omega^1_{A/K},$$

where the last equality holds because $\Omega^1_{A/K}$ is a flat A-module. (Note that A_{loc} is a ring but A_{qu} is only an A-module.) For M a (σ, ∇)-module over A corresponding to an overconvergent F-isocrystal $\mathcal{E}$ on X, we have already defined

$$H^0(M) = \ker(\nabla : M \to M \otimes_A \Omega^1_{A/K})$$
$$H^1(M) = \operatorname{coker}(\nabla : M \to M \otimes_A \Omega^1_{A/K}),$$

and observed that $H^i(M) \cong H^i_{\mathrm{rig}}(X/K, \mathcal{E})$. We now define

$$H^0_{\mathrm{loc}}(M) = \ker(\nabla : M \otimes_A A_{\mathrm{loc}} \to M \otimes_A \Omega^1_{\mathrm{loc}})$$
$$H^1_{\mathrm{loc}}(M) = \operatorname{coker}(\nabla : M \otimes_A A_{\mathrm{loc}} \to M \otimes_A \Omega^1_{\mathrm{loc}})$$
$$H^1_c(M) = \ker(\nabla : M \otimes_A A_{\mathrm{qu}} \to M \otimes_A \Omega^1_{\mathrm{qu}})$$
$$H^2_c(M) = \operatorname{coker}(\nabla : M \otimes_A A_{\mathrm{qu}} \to M \otimes_A \Omega^1_{\mathrm{qu}});$$

Crew [Cre98] has shown that $H^i_c(M) \cong H^i_{c,\mathrm{rig}}(X/K, \mathcal{E})$. (This identification and the previous one become F-equivariant once we specify that F acts on Ω^1 via the linearization $d\sigma$ of the Frobenius lift.) For $x \in \overline{X} \setminus X$, we write $H^0_{\mathrm{loc},x}(M)$ for the kernel of $\nabla : M \otimes_A \mathcal{R}_x \to M \otimes_A \Omega^1_{\mathcal{R}_x/K}$, so that $H^0_{\mathrm{loc}}(M) = \bigoplus_x H^0_{\mathrm{loc},x}(M)$; we also write $H^i_{\mathrm{loc}}(X/K, \mathcal{E})$ for $H^i_{\mathrm{loc}}(M)$.

All of the $H^i(M)$, $H^i_{\mathrm{loc}}(M)$, and $H^i_c(M)$ are finite-dimensional vector spaces over K, by [Cre98, Theorem 9.5 and Proposition 10.2] and the p-adic local monodromy theorem. Because the rows of the diagram

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d] & M \otimes_A A_{\mathrm{loc}} \arrow[r] \arrow[d] & M \otimes_A A_{\mathrm{qu}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \otimes_A \Omega^1_{A/K} \arrow[r] & M \otimes_A \Omega^1_{\mathrm{loc}} \arrow[r] & M \otimes_A \Omega^1_{\mathrm{qu}} \arrow[r] & 0
\end{tikzcd}$$

are exact, the snake lemma produces the canonical exact sequence

$$0 \to H^0(M) \to H^0_{\mathrm{loc}}(M) \to H^1_c(M) \to H^1(M) \to H^1_{\mathrm{loc}}(M) \to H^2_c(M) \to 0. \tag{2.5.1}$$

Moreover, there are F-equivariant perfect pairings

$$H^i(M) \times H_c^{2-i}(M^\vee) \to H_c^2(K) = K(-1)$$

which correspond to Poincaré duality of overconvergent F-isocrystals; there is also an F-equivariant perfect pairing

$$H^0_{\mathrm{loc}}(M) \times H^1_{\mathrm{loc}}(M^\vee) \to K(-1).$$

We will consider these further in § 3.2.

3. Higher direct images in rigid cohomology

The notion of a higher direct image is the relative version of the notion of the cohomology of a single space. Picking up a thread from [Ked05b, § 7], we consider some simple higher direct images in relative dimension 1.

3.1 Robba rings over dagger algebras

For the calculations in this chapter, we need to extend the definition of the Robba ring by allowing coefficients not just in K, but in a more general dagger algebra. The correct procedure for doing this is given in [Ked05b, § 2.5]; we quickly review it here.

For A a reduced dagger algebra, the *Robba ring* $\mathcal{R}_A = \mathcal{R}_A^t$ is defined as the ring of formal Laurent series $\sum_{n=-\infty}^{\infty} c_n t^n$, with $c_n \in A$, such that for all sufficiently small $r > 0$ (depending on the series), $c_n p^{\lfloor rn \rfloor} \to 0$ as $n \to \pm\infty$ in the fringe topology of A (that is, within some fringe algebra depending on r). By [Ked05b, Corollary 2.5.5], it is equivalent to require that

$$\lim_{n \to \pm\infty} (v_A(c_n) + rn) = \infty$$

for all sufficiently small $r > 0$ and that $c_n p^{\lfloor rn \rfloor} \to 0$ in the fringe topology of A for *one* value of r.

We define $\Omega^1_{\mathcal{R}_A/A}$ as the free module over $\mathcal{R}_A$ generated by dt, equipped with the derivation

$$d : \mathcal{R}_A \to \Omega^1_{\mathcal{R}_A/A}, \quad \sum_i c_i t^i \mapsto \sum_i i c_i t^{i-1}\, dt.$$

A quick calculation [Ked05b, Proposition 3.1.4] shows that the kernel and cokernel of this derivation are isomorphic to A in the expected manner. In particular, we define the residue map $\mathrm{Res} : \Omega^1_{\mathcal{R}_A/A} \to A$ by sending $\sum_i c_i t^i\, dt$ to c_{-1}; then $\omega \in \Omega^1_{\mathcal{R}_A/A}$ is in the image of d if and only if $\mathrm{Res}(\omega) = 0$.

We define (σ, ∇)-modules over $\mathcal{R}_A$ (or $\mathcal{R}_A^+$) relative to A as expected, using the relative module of differentials $\Omega^1_{\mathcal{R}_A/A}$ and requiring that the connection ∇ be A-linear. Then the Dwork trick admits the following relative version.

Proposition 3.1.1. *Let M be a free (σ, ∇)-module over $\mathcal{R}_A^+$ relative to A. Then there exists a basis $\mathbf{v}_1, \dots, \mathbf{v}_n$ of M such that $\nabla \mathbf{v}_i = 0$ for $i = 1, \dots, n$. (Note that on any such basis, F acts via a matrix over A.)*

Proof. Choose any basis $\mathbf{e}_1, \dots, \mathbf{e}_n$ of M, and define the $n \times n$ matrix N over $\mathcal{R}_A^+$ by

$$\nabla \mathbf{e}_j = \sum_i N_{ij} \mathbf{e}_i \otimes dt.$$

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Write $N = \sum_{l=0}^{\infty} N_l t^l$; then a straightforward induction shows that there is a unique invertible $n \times n$ matrix $U = I + \sum_{l=1}^{\infty} U_l t^l$ over $A[\![t]\!]$ such that $NU + dU/dt = 0$. Namely, for each $l > 0$, we have

$$lU_l + \sum_{i=0}^{l-1} N_i U_{l-1-i} = 0 \tag{3.1.2}$$

and this lets us solve for U_l in terms of the U_i for $i < l$.

Since N has entries in $\mathcal{R}_A^+$, for some $s > 0$ we can choose a fringe algebra B such that $v_B(N_l) + sl \to \infty$ as $l \to \infty$. By taking s large enough, we can ensure that in fact $v_B(N_l) + s(l+1) > 0$ for all l. Then (3.1.2) implies easily that $v_B(l!U_l) + sl > 0$ for all l, and so $v_B(U_l) + rl \to \infty$ for $r > s + 1/(p-1)$.

Define the invertible $n \times n$ matrix C over $\mathcal{R}_A^+$ by

$$F\mathbf{e}_j = \sum_i C_{ij}\mathbf{e}_i.$$

Then the condition that $NU + dU/dt = 0$ forces the matrix $D = U^{-1}CU^{\sigma}$ to have entries in A. Writing $U = CU^{\sigma}D^{-1}$, we deduce by induction on h that for each nonnegative integer h, there is a fringe algebra B_h such that $v_{B_h}(U_l) + rl \to \infty$ for $r > q^{-h}(s + 1/(p-1))$. Therefore, U indeed has entries in $\mathcal{R}_A^+$.

The same argument applied to the basis of $M^{\vee}$ dual to $\mathbf{e}_1, \dots, \mathbf{e}_n$ shows that the inverse transpose of U has entries in $\mathcal{R}_A^+$. Consequently the elements $\mathbf{v}_1, \dots, \mathbf{v}_n$ of M defined by

$$\mathbf{v}_j = \sum_i U_{ij}\mathbf{e}_i$$

form a basis with the desired property. □

There is also a relative version of the p-adic local monodromy theorem [Ked05b, Theorem 5.1.3], which underlies the pushforward construction of the next section; however, we will not use it explicitly.

3.2 Direct images with and without supports

Let X be a smooth irreducible k-variety, let $\mathcal{E}$ be an overconvergent F-isocrystal on $\mathbb{A}^1 \times X$, and let $f : \mathbb{A}^1 \times X \to X$ denote the implicit projection. In [Ked05b, § 7], 'generic' higher direct images $R^i f_* \mathcal{E}$ and $R^i f_! \mathcal{E}$ of f are constructed over an open dense subscheme of X; this is the best one can do within a category of locally free modules, since the rank of the corresponding cohomology space may jump at particular fibres. We now review this construction, which follows the setup of [Cre98] as presented earlier in § 2.5.

Let A be a dagger algebra of MW type with special fibre X, and let M be a (σ, ∇)-module over $A\langle x\rangle^{\dagger}$ corresponding to $\mathcal{E}$. Then we get a map

$$\nabla_v : M \to M \otimes \Omega^1_{A\langle x\rangle^{\dagger}/K} \to M \otimes \Omega^1_{A\langle x\rangle^{\dagger}/A}$$

from the projection $\Omega^1_{A\langle x\rangle^{\dagger}/K} \to \Omega^1_{A\langle x\rangle^{\dagger}/A}$.

Embed $A\langle x\rangle^{\dagger}$ into $\mathcal{R}_A = \mathcal{R}_A^t$ by mapping $\sum c_i x^i$ to $\sum c_i t^{-i}$. By analogy with the notations of § 2.5, we put

$$M^{\mathrm{loc}} = M \otimes_{A\langle x\rangle^{\dagger}} \mathcal{R}_A, \quad M^{\mathrm{qu}} = M^{\mathrm{loc}}/M$$

and let

$$\nabla_v^{\mathrm{loc}} : M^{\mathrm{loc}} \to M^{\mathrm{loc}} \otimes_{\mathcal{R}_A} \Omega^1_{\mathcal{R}_A/A}$$
$$\nabla_v^{\mathrm{qu}} : M^{\mathrm{qu}} \to (M^{\mathrm{loc}} \otimes_{\mathcal{R}_A} \Omega^1_{\mathcal{R}_A/A})/(M \otimes_{A\langle x\rangle^{\dagger}} \Omega^1_{A\langle x\rangle^{\dagger}/A})$$

be the maps induced by ∇_v. We then define

$$\begin{aligned} R^0 f_* \mathcal{E} &= \ker(\nabla_v), \quad & R^1 f_* \mathcal{E} &= \operatorname{coker}(\nabla_v) \\ R^0_{\mathrm{loc}} f_* \mathcal{E} &= \ker(\nabla_v^{\mathrm{loc}}), \quad & R^1_{\mathrm{loc}} f_* \mathcal{E} &= \operatorname{coker}(\nabla_v^{\mathrm{loc}}) \\ R^1 f_! \mathcal{E} &= \ker(\nabla_v^{\mathrm{qu}}), \quad & R^2 f_! \mathcal{E} &= \operatorname{coker}(\nabla_v^{\mathrm{qu}}); \end{aligned}$$

we take $R^i f_* \mathcal{E}$, $R^i_{\mathrm{loc}} f_* \mathcal{E}$, $R^i f_! \mathcal{E}$ to be zero for values of i not covered by the above list. We also sometimes write M in place of $\mathcal{E}$ in this notation.

By the snake lemma, we have an F-equivariant exact sequence of A-modules

$$0 \to R^0 f_* \mathcal{E} \to R^0_{\mathrm{loc}} f_* \mathcal{E} \to R^1 f_! \mathcal{E} \to R^1 f_* \mathcal{E} \to R^1_{\mathrm{loc}} f_* \mathcal{E} \to R^2 f_! \mathcal{E} \to 0. \tag{3.2.1}$$

Moreover, there are canonical A-linear, F-equivariant Poincaré duality pairings

$$R^i f_* \mathcal{E} \times R^{2-i} f_! \mathcal{E}^\vee \to A(-1) \tag{3.2.2}$$

$$R^i_{\mathrm{loc}} f_* \mathcal{E} \times R^{1-i}_{\mathrm{loc}} f_* \mathcal{E}^\vee \to A(-1) \tag{3.2.3}$$

obtained from the canonical pairing $[\cdot,\cdot] : \mathcal{E} \times \mathcal{E}^\vee \to A\langle x \rangle^\dagger$ and the residue map $\operatorname{Res} : \Omega^1_{\mathcal{R}_A/A} \to A$.

By [Ked05b, Theorem 7.3.2, Proposition 7.5.2, and Proposition 8.6.1], we have the following result.

Theorem 3.2.4. *There exists a localization B of A in the category of dagger algebras of MW type, such that $R^i f_* M_B$, $R^i_{\mathrm{loc}} f_* M_B$, $R^i f_! M_B$ are overconvergent F-isocrystals for all i (where $M_B = M \otimes B\langle x \rangle^\dagger$), and the Poincaré duality pairings are perfect. Moreover, the formation of these objects commutes with a subsequent flat base change (e.g., further localization).*

One can relate the cohomology of an overconvergent F-isocrystal to that of its direct images (by a Leray spectral sequence); the particular instance of this relationship that we need is precisely [Ked05b, Proposition 7.4.1].

Proposition 3.2.5. *Let X be a smooth irreducible affine k-variety, let $f : \mathbb{A}^1 \times X \to X$ be the canonical projection and let $\mathcal{E}$ be an overconvergent F-isocrystal on $\mathbb{A}^1 \times X$ for which $R^0 f_* \mathcal{E}$, $R^1 f_* \mathcal{E}$ are overconvergent F-isocrystals. Then there are canonical, F-equivariant exact sequences*

$$H^i_{\mathrm{rig}}(X/K, R^0 f_* \mathcal{E}) \to H^i_{\mathrm{rig}}(\mathbb{A}^1 \times X/K, \mathcal{E}) \to H^{i-1}_{\mathrm{rig}}(X/K, R^1 f_* \mathcal{E})$$

for each i.

These short exact sequences actually come from a long exact sequence, but the connecting maps are not F-equivariant (they are off by a Tate twist).

3.3 Degeneration in families

Our strategy for studying the cohomology of an isocrystal on a curve is to embed that isocrystal into a family, most of whose fibres are easy to control. For this to return a result on the original isocrystal, we need a theorem that specifies how the cohomology of an isocrystal behaves under specialization. A corresponding statement in [Kat01] is the 'degeneration lemma'.

We will need to work over a certain auxiliary ring. Let $\mathcal{S}$ denote the ring of formal double Laurent series $\sum_{i,j \in \mathbb{Z}} c_{ij} s^i t^j$ over K in two variables s and t with the following property: for each $\delta \in (0,1)$ sufficiently close to 1, there exists $\epsilon \in (0,1)$ such that the series converges for $s, t \in K^{\mathrm{alg}}$ with $|s| = \delta$ and $|t| \in (\epsilon, 1)$. We use the superscripts $s+$ and $t-$ to denote the subrings of $\mathcal{S}$ where s occurs only with positive powers and where t occurs only with negative powers, respectively.

The value of the ring $\mathcal{S}$ is that it is defined using a very mild convergence restriction on series, so many other rings naturally embed into it. Specifically, we can and will identify $\mathcal{R}^{s,+}_{K\langle x \rangle^\dagger}$ and $\mathcal{R}^{s}_{K\langle x \rangle^\dagger}$

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with subrings of $\mathcal{S}^{s+,t-}$ and $\mathcal{S}^{t-}$, respectively, by identifying x with t^{-1}. In particular, this gives an embedding of $A\langle x\rangle^\dagger$ into $\mathcal{S}$ for any localization A of $K\langle s\rangle^\dagger$, since $A\langle x\rangle^\dagger \subset \mathcal{R}^s_{K\langle x\rangle^\dagger}$.

Given a (σ,∇)-module M (which by definition is finite, hence free) over $K\langle s,x\rangle^\dagger$, for any localization A of $K\langle s\rangle^\dagger$, we write M_A for $M\otimes A\langle x\rangle^\dagger$. We also write ∇_s and $\nabla_x = \nabla_t$ for the components of ∇ mapping to $M\otimes ds$ and $M\otimes dx$, respectively.

THEOREM 3.3.1. *Let M be a (σ,∇)-module over $K\langle s,x\rangle^\dagger$, and let $f : K\langle s\rangle^\dagger \to K\langle s,x\rangle^\dagger$ denote the canonical inclusion. Let A be a localization of $K\langle s\rangle^\dagger$ such that the conclusion of Theorem 3.2.4 holds for M_A and $M_A^\vee$. Then there is a canonical F-equivariant injection*

$$H^1_c(M/sM) \hookrightarrow H^0_{\mathrm{loc},s=0}(R^1 f_! M_A).$$

Proof. By Poincaré duality, it is equivalent to exhibiting a canonical F-equivariant pairing

$$H^1_c(M/sM) \times H^1_{\mathrm{loc},s=0}(R^1 f_* M_A^\vee) \to K(-2) \tag{3.3.2}$$

which is nondegenerate on the left. Using the relative Dwork's trick (Proposition 3.1.1), we get a K-linear map $g : M/sM \to M\otimes \mathcal{R}^{s,+}_{K\langle x\rangle^\dagger}$ such that for all $\mathbf{v}\in M/sM$, $g(\mathbf{v})$ reduces to $\mathbf{v}$ modulo s, $\nabla_s g(\mathbf{v}) = 0$ and $\nabla_t g(\mathbf{v}) = g(\nabla_t \mathbf{v})$. In particular, g induces an F-equivariant inclusion

$$H^1_c(M/sM) \hookrightarrow \frac{\{\mathbf{v}\in M\otimes\mathcal{S} : \nabla_s\mathbf{v} = 0, \nabla_t\mathbf{v}\in M\otimes\mathcal{S}^{t-}\otimes dx\}}{\{\mathbf{v}\in M\otimes\mathcal{S}^{t-} : \nabla_s\mathbf{v}=0\}}. \tag{3.3.3}$$

Note that the initial hypothesis on A (that the conclusion of Theorem 3.2.4 holds for M_A and $M_A^\vee$) is preserved by further localization of A, whereas the conclusion is insensitive to a localization on A since it only concerns $H^0_{\mathrm{loc},s=0}(R^1 f_! M_A)$. Hence, we may as well assume that $s^{-1}\in A$. Then we have a natural F-equivariant nondegenerate pairing

$$\frac{M\otimes\mathcal{S}}{M\otimes\mathcal{S}^{t-}} \times (M_A^\vee \otimes (ds\wedge dx)) \to K(-2)$$

given by the residue map on $\mathcal{S}\otimes(ds\wedge dt)$ (i.e. extracting the coefficient of $(ds/s)\wedge(dt/t)$). If we restrict on the left to the classes of those $\mathbf{v}$ with $\nabla_t\mathbf{v}\in M\otimes\mathcal{S}^{t-}\otimes dx$, then the pairing vanishes when the right member is in $(\nabla_t M_A^\vee)\otimes ds$. We thus obtain a second pairing

$$\frac{\{\mathbf{v}\in M\otimes\mathcal{S} : \nabla_t\mathbf{v}\in M\otimes\mathcal{S}^{t-}\otimes dx\}}{M\otimes\mathcal{S}^{t-}} \times ((R^1 f_* M_A^\vee)\otimes ds) \to K(-2)$$

which is again nondegenerate on the left, and remains so after tensoring the right member over A with $\mathcal{R}^s_K$. If we restrict further on the left to the classes of those $\mathbf{v}\in M\otimes\mathcal{S}$ with $\nabla_s\mathbf{v}=0$, then the pairing vanishes when the right member is in $\nabla_s(\mathcal{R}^s_K\otimes_A R^1 f_* M_A^\vee)$. We thus obtain a third pairing

$$\frac{\{\mathbf{v}\in M\otimes\mathcal{S} : \nabla_s\mathbf{v} = 0, \nabla_t\mathbf{v}\in M\otimes\mathcal{S}^{t-}\otimes dx\}}{\{\mathbf{v}\in M\otimes\mathcal{S}^{t-} : \nabla_s\mathbf{v}=0\}} \times H^1_{\mathrm{loc},s=0}(R^1 f_* M_A^\vee) \to K(-2) \tag{3.3.4}$$

which is again nondegenerate on the left. Combining this pairing with the inclusion (3.3.3) yields the desired result. □

Remark 3.3.5. In an earlier version of this paper, we attempted to construct the embedding of Theorem 3.3.1 more directly, rather than deduce it from Poincaré duality. This ran into trouble because the convergence regions defining the rings $\mathcal{R}^x_A$, for A a localization of $K\langle s\rangle^\dagger$, and $\mathcal{R}^s_{K\langle x\rangle^\dagger}$ do not share any common territory. Thus, we must avoid $R^1 f_! M_A$ and work with its dual instead, which can be computed in the context of dagger algebras.

3.4 More degeneration in families

We continue to consider the situation of the previous section, particularly in the case when the injection of Theorem 3.3.1 is actually a bijection. We retain all notation from the previous section.

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In addition, for K' a finite extension of K, we write $M' = M \otimes_K K'$, and for μ in the ring of integers $\mathfrak{o}'$ of K' fixed by a power of σ_K, we write $M_\mu = M'/(s-\mu)M'$.

Lemma 3.4.1. *Let W be an $n \times n$ invertible matrix over $\mathcal{R}^s_K$. Then there exist $n \times n$ matrices U and V such that U is invertible over $\mathcal{R}^{s,+}_K$, V is invertible over a localization of $K\langle s^{-1}\rangle^\dagger$ and $W = UV$.*

Proof. Choose $r > 0$ such that $w_r(W)$ is defined. (Note that, in this argument, applying w_r to a matrix means taking its minimum over entries, i.e. the L^∞ operator norm.) Choose a matrix X over $K[s, s^{-1}]$ with nonzero determinant such that $w_r(X - W^{-1}) > -w_r(W)$; then $w_r(WX - I) > 0$. By [Ked04a, Proposition 6.5], we can factor WX as YZ, with Y invertible over $\mathcal{R}^{s,+}_K$ and Z invertible over $K\langle s^{-1}\rangle^\dagger$. Since $\det(X) \in K[s, s^{-1}]$, $\det(X)$ is a unit in some localization A of $K\langle s^{-1}\rangle^\dagger$. Thus, we may put $U = Y$ and $V = ZX^{-1}$. □

Lemma 3.4.2. *Let M be a (σ, ∇)-module over $K\langle s, x\rangle^\dagger$ such that $H^0(M'_\mu) = 0$ for all K' and μ. Let $A \subseteq B$ be localizations of $K\langle s\rangle^\dagger$, and suppose that $\mathbf{v} \in M \otimes B\langle x\rangle^\dagger$ satisfies $\nabla_t \mathbf{v} \in M \otimes A\langle x\rangle^\dagger \otimes dx$. Then $\mathbf{v} \in M \otimes A\langle x\rangle^\dagger$.*

Proof. Put $C = B \cap \mathcal{R}^{s,+}_K$; it suffices to check that if s is not invertible in A, then $\mathbf{v} \in M \otimes C\langle x\rangle^\dagger$. Namely, once this is done, we can repeat the argument after translating (and enlarging K as needed) to deduce that $\mathbf{v} \in M \otimes A\langle x\rangle^\dagger$.

By Proposition 3.1.1, we can find a basis $\mathbf{v}_1, \ldots, \mathbf{v}_n$ of $M \otimes \mathcal{R}^{s,+}_{K\langle x\rangle^\dagger}$ such that $\nabla_s \mathbf{v}_i = 0$ for $i = 1, \ldots, n$. From the integrability of ∇, if we write $\nabla_t \mathbf{v}_j = \sum_i D_{ij} \mathbf{v}_i \otimes dx$, then $D_{ij} \in K\langle x\rangle^\dagger$ for all i, j.

Write $\mathbf{v} = \sum_i a_i \mathbf{v}_i$ with $a_i \in \mathcal{R}^s_{K\langle x\rangle^\dagger}$ (that is possible because $\mathbf{v} \in M \otimes B\langle x\rangle^\dagger$ and $B\langle x\rangle^\dagger \subset \mathcal{R}^s_{K\langle x\rangle^\dagger}$), write formally $a_i = \sum_l b_{il} s^l$ with each $b_{il} \in K\langle x\rangle^\dagger$, and put $\mathbf{w}_l = \sum_i b_{il} \mathbf{v}_i$. Then the series $\sum_l s^l \nabla_t \mathbf{w}_l$ converges (in the fringe topology of $M \otimes A\langle x\rangle^\dagger$) to $\nabla_t \mathbf{v}$, and the fact that $\nabla_t \mathbf{v} \in M \otimes \mathcal{R}^{s,+}_{K\langle x\rangle^\dagger} \otimes dx$ implies that $\nabla_t \mathbf{w}_l = 0$ for $l < 0$.

However, the $K\langle x\rangle^\dagger$-span of the $\mathbf{v}_i$ is a (σ, ∇)-module isomorphic to M/sM, which by assumption has no horizontal sections. Hence, $\nabla_t \mathbf{w}_l = 0$ if and only if $\mathbf{w}_l = 0$. We conclude that $\mathbf{w}_l = 0$ for $l < 0$, and so

$$\mathbf{v} \in M \otimes (B\langle x\rangle^\dagger \cap \mathcal{R}^{s,+}_{K\langle x\rangle^\dagger}) = M \otimes C\langle x\rangle^\dagger,$$

which, as noted above, suffices to yield the desired result. □

Proposition 3.4.3. *Let M be a free (σ, ∇)-module over $K\langle s, x\rangle^\dagger$, and let $f : K\langle s\rangle^\dagger \to K\langle s, x\rangle^\dagger$ denote the canonical inclusion. Suppose that for some nonnegative integer m,*

$$\dim_{K'} H^0(M_\mu) = \dim_{K'} H^0(M_\mu^\vee) = 0, \quad \dim_{K'} H^1(M_\mu) = \dim_{K'} H^1(M_\mu^\vee) = m$$

for all K' and μ. Then $R^1 f_* M$, $R^1 f_* M^\vee$, $R^1 f_! M$, $R^1 f_! M^\vee$ are free of rank m over $K\langle s\rangle^\dagger$.

Proof. It is enough to show that $R^1 f_* M$ and $R^1 f_! M$ are locally free of rank m over $K\langle s\rangle^\dagger$, or likewise after replacing K by a finite extension; by translation, it suffices to check in a neighborhood of $s = 0$. Let A be a localization of $K\langle s\rangle^\dagger$ such that the conclusion of Theorem 3.2.4 holds for M_A and $M_A^\vee$; we may as well assume that s is invertible in A, else we are already done.

We first treat $R^1 f_* M$ and $R^1 f_* M^\vee$. Under our hypothesis, the pairing (3.3.2) must be perfect, as must be (3.3.4). In fact, the pairing

$$[\cdot,\cdot] : \frac{\{\mathbf{v} \in M \otimes \mathcal{S}^{s+} : \nabla_s \mathbf{v} = 0, \nabla_t \mathbf{v} \in M \otimes \mathcal{S}^{s+,t-} \otimes dx\}}{\{\mathbf{v} \in M \otimes \mathcal{S}^{s+,t-} : \nabla_s \mathbf{v} = 0\}} \times H^1_{\mathrm{loc},s=0}(R^1 f_* M_A^\vee) \to K(-2)$$

must then also be perfect: it is nondegenerate on the left, and the left factor has K-dimension at least as large as does the second factor.

Choose a basis $\mathbf{v}_1, \ldots, \mathbf{v}_m$ of $H^1_c(M/sM)$. Then the map $h : (R^1 f_* M_A^\vee) \otimes_A \mathcal{R}^s_K \to (\mathcal{R}^s_K)^m$ defined by

$$\mathbf{w} \mapsto \left(\sum_{l \in \mathbb{Z}} s^l [g(\mathbf{v}_i), s^{-l}\mathbf{w} \otimes ds] \right)_{1 \leqslant i \leqslant m}$$

with g and $[\cdot, \cdot]$ as in Theorem 3.3.1, is an isomorphism of $\mathcal{R}^s_K$-modules. By Lemma 3.4.1, for some localization B of A, we can find elements $\mathbf{w}_1, \ldots, \mathbf{w}_m$ of $R^1 f_* M_B^\vee$ such that the images $h(\mathbf{w}_1), \ldots, h(\mathbf{w}_m)$ lie in $(\mathcal{R}^{s,+}_K)^m$ and freely generate $(\mathcal{R}^{s,+}_K)^m$ over $\mathcal{R}^{s,+}_K$. Choose $\mathbf{x}_j \in M_B^\vee \otimes dx$ whose image in $R^1 f_* M_B^\vee$ is equal to $\mathbf{w}_j$. Then $[g(\mathbf{v}_i), s^{-l}\mathbf{x}_j \otimes ds] = 0$ for $l < 0$, so each $\mathbf{x}_j$ lies in

$$M^\vee \otimes (B\langle x \rangle^\dagger \cap \mathcal{S}^{s+}) \otimes dx = M^\vee \otimes C\langle x \rangle^\dagger \otimes dx,$$

where we write $C = B \cap \mathcal{R}^{s,+}_K$; this is a localization of $K\langle s \rangle^\dagger$ in which s is not invertible.

Now given any $\mathbf{w} \in R^1 f_* M_C^\vee$ which is the image of some $\mathbf{x} \in M_C^\vee \otimes dx$, we can uniquely write $\mathbf{w} = \sum_j b_j \mathbf{w}_j$ with $b_j \in B$. On the other hand, in the equality $h(\mathbf{w}) = \sum_j b_j h(\mathbf{w}_j)$, $h(\mathbf{w})$ belongs to $(\mathcal{R}^{s,+}_K)^m$, and the $h(\mathbf{w}_j)$ generate $(\mathcal{R}^{s,+}_K)^m$ freely over $\mathcal{R}^{s,+}_K$. Therefore, each b_j in fact belongs to $B \cap \mathcal{R}^{s,+}_K = C$.

That is, given $\mathbf{x} \in M_C^\vee \otimes dx$, $\mathbf{x} - \sum_j b_j \mathbf{x}_j$ is an element of $M_C^\vee \otimes dx$ which vanishes in $R^1 f_* M_B^\vee$. By Lemma 3.4.2 (and using the hypothesis on the vanishing of H^0), $\mathbf{x} - \sum_j b_j \mathbf{x}_j$ already vanishes in $R^1 f_* M_C^\vee$. Hence, $R^1 f_* M_C^\vee$ is freely generated by the $\mathbf{w}_j$. As noted above, this suffices to imply that $R^1 f_* M^\vee$ is free of rank m; by the same argument, $R^1 f_* M$ is free of rank m.

We next consider $R^1 f_! M$. Choose a basis of $R^1 f_* M^\vee$, and let $\mathbf{v}_1, \ldots, \mathbf{v}_m$ be the dual basis of $R^1 f_! M_A$. Then under the residue pairing

$$\frac{M \otimes \mathcal{R}^t_A}{M \otimes A\langle x \rangle^\dagger} \times (M^\vee \otimes dx) \to A,$$

each $\mathbf{v}_i$ always pairs into $K\langle s \rangle^\dagger$. Hence, $\mathbf{v}_i$ is represented by an element of $M \otimes \mathcal{R}^t_{K\langle s \rangle^\dagger}$, and so belongs to $R^1 f_! M$. Similarly, given any element of $R^1 f_! M$, we can write it uniquely as an A-linear combination of the $\mathbf{v}_i$ and then observe that the coefficients actually lie in $K\langle s \rangle^\dagger$. Thus, $R^1 f_! M$ (and likewise $R^1 f_! M^\vee$) is also free of rank m, as desired. $\square$

4. A p-adic Fourier transform

Laumon's 'principle of stationary phase' asserts that one can gain information about a single coefficient object by putting it into a family with twists by certain characters, i.e. by applying a Fourier transform. The p-adic Fourier transform was first constructed algebraically by Mebkhout [Meb97], who originated the idea of using it to imitate Laumon's proof of Weil II in p-adic cohomology; much of its subsequent study is due to Huyghe [Huy95a, Huy95b, Huy95c, Huy04]. In particular, Huyghe gave a geometric interpretation of this construction using a notion of a sheaf of overconvergent differential operators, as proposed by Berthelot.

As one might expect from the ℓ-adic situation, the Fourier transform acts not on the class of overconvergent F-isocrystals, but on a broader class of $\mathcal{D}^\dagger$-modules. For our purposes, we will need to know some conditions under which the Fourier transform of an overconvergent F-isocrystal is again an overconvergent F-isocrystal; this is provided by an analogue of the Grothendieck–Ogg–Shafarevich formula.

Throughout this section, we assume that K contains a primitive pth root of unity. It is equivalent to assume that K contains a $(p-1)$th root of $-p$, which we will call π.

4.1 The algebraic Fourier transform

Let $A(K)$ be the Weyl algebra in the variable x, whose elements we write as finite sums $\sum_{i,j=0}^{\infty} a_{ij} x^i \partial^{[j]}$, with $\partial^{[j]} = \frac{1}{j!}\frac{\partial^j}{\partial x^j}$. By virtue of the commutation relation $\partial^{[j]} x = x\partial^{[j]} + \partial^{[j-1]}$, we may form the weak completion $A(K)^\dagger$, consisting of those sums $\sum_{i,j=0}^{\infty} a_{ij} x^i \partial^{[j]}$ for which $\sum a_{ij} x^i y^j \in K\langle x, y\rangle^\dagger$, and this set will inherit a ring structure by continuity from $A(K)$ (this is worked out in detail in both [Meb97, § 2.2] and [Huy95a, Chapitre 1]). Similarly, $A(K)^\dagger$ inherits from $A(K)$ an automorphism ρ sending x and ∂ to $-\partial/\pi$ and $x\pi$, respectively; define the *algebraic Fourier transform* of a left $A(K)^\dagger$-module M to be its pullback $\widehat{M}$ by this automorphism. We think of $\widehat{M}$ as having the same underlying set as M but a different module structure.

Remark 4.1.1. Huyghe also defines an n-dimensional Weyl algebra $A_n(K)^\dagger$; our $A(K)^\dagger$ is her $A_1(K)^\dagger$.

Let M be a (σ, ∇)-module over $K\langle x\rangle^\dagger$. Then M carries an action of $A(K)$ in which $\nabla \mathbf{v} = (\partial \mathbf{v}) \otimes dx$. It is not immediately obvious that this extends to a left action of $A(K)^\dagger$ by continuity, but this was shown to be the case by Berthelot [Ber96, Théorème 4.4.5]. Thus, we may define the Fourier transform of a (σ, ∇)-module over $K\langle x\rangle^\dagger$, but the result is only *a priori* a left $A(K)^\dagger$-module. In order to again make it a (σ, ∇)-module (i.e. to get it to be finitely generated over $K\langle x\rangle^\dagger$) we must impose further conditions, and use the geometric interpretation of the next section.

4.2 The geometric Fourier transform

The *Dwork isocrystal* on the x-line $\mathbb{A}^1$ is the overconvergent F-isocrystal of rank one corresponding to the (σ, ∇)-module $\mathcal{L}$ over $K\langle x\rangle^\dagger$ with a generator $\mathbf{e}$ satisfying

$$F\mathbf{e} = \exp(\pi x - \pi x^\sigma)\mathbf{e}, \quad \nabla \mathbf{e} = \pi \mathbf{e} \otimes dx.$$

This isocrystal becomes trivial after adjoining u such that $u^p - u = x$; this implies that $\mathcal{L}^{\otimes p}$ is trivial.

For any dagger algebra A and any $f \in A^{\mathrm{int}}$, we identify f with the map $K\langle x\rangle^\dagger \to A$ mapping x to f, and write $\mathcal{L}_f$ for $f^*\mathcal{L}$. Note that $\mathcal{L}_{f+g} = \mathcal{L}_f \otimes \mathcal{L}_g$ and that the isomorphism class of $\mathcal{L}_f$ depends only on f modulo π.

Let M be a (σ, ∇)-module over $K\langle x\rangle^\dagger$, and let $f : K\langle s\rangle^\dagger \to K\langle s, x\rangle^\dagger$ and $g : K\langle x\rangle^\dagger \to K\langle s, x\rangle^\dagger$ be the canonical embeddings. Then g^*M and $\mathcal{L}_{sx}$ are (σ, ∇)-modules over $K\langle s, x\rangle^\dagger$, as is

$$N = g^*M \otimes_{K\langle s,x\rangle^\dagger} \mathcal{L}_{sx}.$$

We can decompose $\Omega^1_{K\langle s,x\rangle^\dagger}$ into two rank-one submodules, generated by ds and dx. Let ∇_s and ∇_x be the components of the connection on N mapping to these two submodules.

We define the *geometric Fourier transform* of M as $\widehat{M}_{\mathrm{geom}} = \operatorname{coker} \nabla_x$; it carries a natural left $A(K)^\dagger$-module structure (inherited from the $A_2(K)^\dagger$-module structure of N, where $A_2(K)^\dagger$ is as in Remark 4.1.1).

Proposition 4.2.1. *Let M be a (σ, ∇)-module over $K\langle x\rangle^\dagger$. Then there is a canonical isomorphism $\widehat{M} \to \widehat{M}_{\mathrm{geom}}$ of left $A(K)^\dagger$-modules.*

Proof. The map in question is defined as follows. Remember that we are thinking of M and $\widehat{M}$ as having the same underlying set. We then identify M with a subset of g^*M via g, and in turn identify g^*M with N by identifying $\mathbf{w} \in g^*M$ with $\mathbf{w} \otimes \mathbf{e}$ (where $\mathbf{e}$ is the distinguished generator used in the definition of $\mathcal{L}$ or, more precisely, its image in $\mathcal{L}_{sx}$). The desired map is now constructed by tracing through these identifications, then composing with the map $N \to \operatorname{coker} \nabla_x$ induced by $\mathbf{v} \mapsto \mathbf{v} \otimes dx$. The fact that it is bijective follows from [Huy95c, Théorème 4]. ∎

For our purposes, the main significance of this result is the following, which we state in the notation of § 3.4.

PROPOSITION 4.2.2. *Let M be a (σ, ∇)-module over $K\langle x\rangle^\dagger$. Suppose that there exists a nonnegative integer d such that for each K' and μ,*

$$\dim_{K'} H^0(M \otimes \mathcal{L}_{\mu x}) = \dim_{K'} H^0(M^\vee \otimes \mathcal{L}_{\mu x}) = 0,$$
$$\dim_{K'} H^1(M \otimes \mathcal{L}_{\mu x}) = \dim_{K'} H^1(M^\vee \otimes \mathcal{L}_{\mu x}) = d.$$

Then $\widehat{M}$ is a free (σ, ∇)-module of rank d over $K\langle s\rangle^\dagger$. If, in addition, M is irreducible as a (σ, ∇)-module, then $\widehat{M}$ is irreducible as a (σ, ∇)-module.

Proof. The first assertion follows immediately from Proposition 3.4.3 (which shows that $\widehat{M}_{\text{geom}}$ is free of rank d) and Proposition 4.2.1 (which shows that $\widehat{M} \cong \widehat{M}_{\text{geom}}$), so we focus on the second. If $\widehat{M}$ is reducible as a (σ, ∇)-module, it has a Frobenius-stable $A(K)^\dagger$-submodule $\widehat{N}$ such that $\widehat{N}$ and $\widehat{M}/\widehat{N}$ are infinite-dimensional K-vector spaces. Undoing the Fourier transform gives a Frobenius-stable $A(K)^\dagger$-submodule N of M such that N and M/N are infinite-dimensional K-vector spaces (note that we are using the geometric interpretation to get the Frobenius stability). However, then N is a (σ, ∇)-submodule of M such that N and M/N are nontrivial, so M is reducible. Hence, if M is irreducible, then so is $\widehat{M}$. □

4.3 An Euler characteristic formula

In order to apply the results of the previous section to a (σ, ∇)-module M over $K\langle x\rangle^\dagger$, we need to establish conditions under which the dimension of $H^1(M \otimes \mathcal{L}_{\mu x})$ does not depend on μ. This requires a formula for this dimension; in this section, we establish such a formula using some recent results in p-adic cohomology.

The ℓ-adic analogue of the formula we seek is the Grothendieck–Ogg–Shafarevich formula [Gro77] (see also [Ray66]), which relates the Euler–Poincaré characteristic of a lisse sheaf on a curve to the local monodromy at the missing points. Naturally, its p-adic analogue will also be given in terms of local monodromy.

Let C be a smooth irreducible affine curve over k, let $\overline{C}$ be the smooth compactification of k, and let $\mathcal{E}$ be an overconvergent F-isocrystal on C. Let x be a closed point of $\overline{C} \setminus C$, let E be the fraction field of the completed local ring of C at x, and let F be a Galois extension of E over which the local monodromy of $\mathcal{E}$ at x becomes unipotent (or, more precisely, over which the module obtained from a (σ, ∇)-module corresponding to $\mathcal{E}$ by tensoring up to a Robba ring $\mathcal{R}_x$ corresponding to x becomes unipotent). Let f be the residue field degree of F/E, and let G be the Galois group $\mathrm{Gal}(F/E)$.

Let $\mathrm{Swan}_x(\mathcal{E})$ denote the Swan conductor (see [Ser79] and/or [Kat88, ch. 1] for definitions and properties) of the representation of G on the local monodromy of $\mathcal{E}$ at x, i.e. on the horizontal sections of $\mathcal{E}$ over the extension of $\mathcal{R}_x$ corresponding to F (and the horizontal sections of the quotient of $\mathcal{E}$ by the span of all horizontal sections, and so on).

In terms of the Swan conductor, the desired analogue of the Grothendieck–Ogg–Shafarevich formula is as follows.

THEOREM 4.3.1. *Let $\mathcal{E}$ be an overconvergent F-isocrystal on a smooth irreducible affine curve C over k, and write*

$$\chi(C/K, \mathcal{E}) = \dim_K H^0_{\text{rig}}(C/K, \mathcal{E}) - \dim_K H^1_{\text{rig}}(C/K, \mathcal{E}).$$

Then

$$\chi(C/K, \mathcal{E}) = \chi(C/K, \mathcal{O}_C)\,\mathrm{rank}(\mathcal{E}) - \sum_{x \in \overline{C} \setminus C} [\kappa(x) : k]\, \mathrm{Swan}_x(\mathcal{E}). \tag{4.3.2}$$

Proof. Recall the following two results.

(a) A theorem of Christol and Mebkhout [CM01, Corollaire 5.0–12] states that

$$\chi(C/K,\mathcal{E}) = \chi(C/K,\mathcal{O}_C)\operatorname{rank}(\mathcal{E}) - \sum_{x\in\overline{C}\setminus C}[\kappa(x):k]\operatorname{Irr}_x(\mathcal{E}),$$

where Irr is the 'irregularity' of $\mathcal{E}$ at x (a generalized form of a definition of Robba).

(b) A theorem of Crew [Cre00, Theorem 5.4], Matsuda [Mat02, Theorem 8.6], and Tsuzuki [Tsu98, Theorem 7.2.2] states that

$$\operatorname{Irr}_x(\mathcal{E}) = \operatorname{Swan}_x(\mathcal{E}).$$

Combining these results yields the theorem. □

Using Theorem 4.3.1, we obtain the following result.

PROPOSITION 4.3.3. *Let M be a (σ,∇)-module over $K\langle x\rangle^\dagger$. Then there exists an integer N with the following property: for any integer $d>N$ not divisible by p, and for any monic polynomial $P\in\mathfrak{o}[x]$ of degree d, we have*

$$\dim_K H^0_{\mathrm{loc}}(M\otimes\mathcal{L}_P) = \dim_K H^1_{\mathrm{loc}}(M\otimes\mathcal{L}_P) = 0,$$
$$\dim_K H^0(M\otimes\mathcal{L}_P) = 0, \quad \dim_K H^1(M\otimes\mathcal{L}_P) = (d-1)\operatorname{rank}(M).$$

Proof. Let $\rho: G\to \mathrm{GL}(V)$ be the local monodromy representation of M at infinity; then the local monodromy representation of $M\otimes\mathcal{L}_P$ at infinity is equal to $\rho\otimes\psi_P$, for ψ_P a suitable nontrivial character of the Galois group $\mathrm{Gal}(L/k((t^{-1})))$, with $L = k((t^{-1}))[u]/(u^p-u-P(t))$. Let N be the largest break of ρ; this will turn out to be a good choice.

Suppose $d>N$; then $\rho\otimes\psi_P$ has all ramification breaks equal to d. That first implies that $\rho\otimes\psi_P$ has no trivial subrepresentations, so $\dim H^0_{\mathrm{loc}}(M\otimes\mathcal{L}_P) = \dim H^0_{\mathrm{loc}}(M^\vee\otimes\mathcal{L}_{-P}) = 0$; by Poincaré duality, we also have $\dim H^1_{\mathrm{loc}}(M\otimes\mathcal{L}_P)=0$. It next implies that $\operatorname{Swan}_\infty(\rho\otimes\psi_P) = d\operatorname{rank}(M)$, so by Theorem 4.3.1 we compute

$$\begin{aligned}\chi(M\otimes\mathcal{L}_P) &= \chi(\mathbb{A}^1)\operatorname{rank}(M) - \operatorname{Swan}_\infty(M)\\ &= \operatorname{rank}(M) - d\operatorname{rank}(M) = (1-d)\operatorname{rank}(M).\end{aligned}$$

Since $H^0(M\otimes\mathcal{L}_P)$ injects into $H^0_{\mathrm{loc}}(M\otimes\mathcal{L}_P)$ by the exactness of (2.5.1), it also vanishes, and we conclude that $\dim_K H^1(M\otimes\mathcal{L}_P) = (d-1)\operatorname{rank}(M)$, as desired. □

In particular, when n is sufficiently large and not divisible by p, for any r,s in the ring of integers $\mathfrak{o}'$ of a finite extension K' of K, with r not in the maximal ideal of $\mathfrak{o}'$, we have that $\dim_{K'} H^0(M\otimes\mathcal{L}_{rx^n+s}) = \dim_{K'} H^0(M^\vee\otimes\mathcal{R}_{-rx^n-s}) = 0$, while $\dim_{K'} H^1(M\otimes\mathcal{L}_{rx^n+s})$ and $\dim_{K'} H^1(M^\vee\otimes\mathcal{L}_{-rx^n-s})$ are equal to each other and to a common value not depending on r or s. Hence, we may apply Proposition 4.2.2 to deduce that the Fourier transform of $M\otimes\mathcal{L}_{rx^n}$ is also a (σ,∇)-module over $K\langle s\rangle^\dagger$, which is irreducible if and only if $M\otimes\mathcal{L}_{rx^n}$ is irreducible.

5. Cohomology over finite fields

With the geometric setup in place, we now introduce the archimedean considerations that will yield our analogue of Weil II, following the trail blazed by Crew [Cre92, Cre98].

In this section, we again take $k=\mathbb{F}_q$ and σ_K to be the identity map. Also, unless otherwise specified, all curves will be smooth, geometrically irreducible, affine, and defined over $\mathbb{F}_q$.

We will always let ι denote an embedding $K^{\mathrm{alg}} \hookrightarrow \mathbb{C}$. As in [Del80], this contrivance is really just a technical convenience, but one whose removal would make the exposition substantially more awkward.

5.1 Weights

In this section, we introduce the notions of weights in rigid cohomology, following Crew [Cre98].

Suppose $q' = q^a$, and K' is the smallest unramified extension of K whose residue field contains $\mathbb{F}_{q'}$. For $T : V \to V$ an endomorphism of a finite-dimensional K'-vector space, we say that:

- T is *ι-pure of weight w* if for each eigenvalue α of T, we have $|\iota(\alpha)| = q^{(w/2)a}$;
- T is *ι-mixed of weight at least w* (respectively at most w) if for each eigenvalue α of T, we have $|\iota(\alpha)| = q^{((w+i)/2)a}$ for some integer $i = i(\alpha) \geqslant 0$ (respectively $i \leqslant 0$);
- T is *ι-real* if the characteristic polynomial of T has coefficients which map under ι into $\mathbb{R}$; in other words, the eigenvalues of $T : V \otimes_\iota \mathbb{C} \to V \otimes_\iota \mathbb{C}$ occur in complex conjugate pairs.

If $\mathcal{E}$ is an overconvergent F-isocrystal on a smooth $\mathbb{F}_q$-variety X, then we say that $\mathcal{E}$ has one of the above properties *pointwise* (or *punctually*) if the linear transformation F_x on $\mathcal{E}_x$ has that property for each closed point x of X, when we take $\mathbb{F}_{q'} = \kappa(x)$. This immediately implies that $H^0_{\mathrm{rig}}(X/K, \mathcal{E})$ has the same property, since the action of Frobenius on its elements can be read off by restricting them to any fibre of $\mathcal{E}$.

We say that $\mathcal{E}$ is *ι-realizable* if $\mathcal{E}$ is a direct summand of a pointwise ι-real overconvergent F-isocrystal. Note that if $\mathcal{E}$ is pointwise ι-pure of some weight w, then $\mathcal{E}$ is ι-realizable, since $\mathcal{E} \oplus \mathcal{E}^\vee(-w)$ is pointwise ι-real.

Remark 5.1.1. It is conventional to omit the word 'pointwise' when referring to ι-purity or ι-reality. However, one cannot do this with mixedness, as Deligne's notion of ι-mixedness is genuinely global: it requires that $\mathcal{E}$ have a filtration whose successive quotients are each ι-pure. (Over a point, this coincides with our pointwise ι-mixedness.)

THEOREM 5.1.2. *The Jordan–Hölder constituents of an ι-real overconvergent F-isocrystal on a curve X are all ι-pure. In particular, any irreducible ι-realizable overconvergent F-isocrystal X is ι-pure of some weight.*

Proof. The assertion, which parallels [Del80, Théorème 1.5.1], is [Cre98, Theorem 10.5] once one verifies the hypothesis of quasi-unipotence. However, this follows from the p-adic local monodromy theorem, as shown in [MT04, Corollaire 8]. □

Remark 5.1.3. In the ℓ-adic context, one can deduce the same result for more general X by restricting to a suitable curve; this amounts to an application of Bertini's theorem. (See, for instance, [KW01, Theorem I.4.3].) We expect a similar result to hold in the p-adic setting, but its proof will be a bit more technical; in its absence, we will have to be a bit careful in order to work around it.

PROPOSITION 5.1.4. *Let $\mathcal{E}$ be an overconvergent F-isocrystal on a curve X which is ι-pure of weight w. Then $H^0_{\mathrm{loc}}(X/K, \mathcal{E})$ is ι-mixed of weight at most w.*

Proof. We may replace X by a finite cover without loss of generality; in particular, by the p-adic local monodromy theorem (Proposition 2.4.2), we may reduce to the case where $\mathcal{E}$ is unipotent at each point x of $\overline{X} \setminus X$, for $\overline{X}$ a smooth compactification of X. The result, which parallels [Del80, Théorème 1.8.4], now follows from [Cre98, Theorem 10.8]. □

As in the ℓ-adic situation, Proposition 5.1.4 can be viewed as affirming a form of the weight-monodromy conjecture in equal characteristics; we omit further details. One may also deduce an equidistribution theorem for Frobenius eigenvalues; see [Cre98, Theorem 10.11].

5.2 Weil II on the affine line

We can now deduce our main result on the affine line. We first use the Fourier transform to derive a key special case, in which one gets a stronger conclusion.

Theorem 5.2.1. *Let M be an irreducible (σ,∇)-module over $K\langle x\rangle^\dagger$ which is ι-pure of weight w. Suppose there is a nonnegative integer d with the following property: for any finite extension K' of K and any a in the ring of integers of K', if we put $M' = M\otimes_K K'$, we then have $\dim_{K'} H^0_{\mathrm{loc},\infty}(M'\otimes\mathcal{L}_{ax}) = \dim_{K'} H^1_{\mathrm{loc},\infty}(M'\otimes\mathcal{L}_{ax}) = 0$ and $\dim_{K'} H^1(M'\otimes\mathcal{L}_{ax}) = d$. Then $H^1(M)$ and $H^1_c(M)$ are ι-pure of weight $w+1$.*

Proof. Without loss of generality, assume that K contains π such that $\pi^{p-1}=-p$, and that $w=0$, so that M and $M^\vee$ have complex conjugate trace functions; then the same is true of $M'\otimes\mathcal{L}_{ax}$ and $(M')^\vee\otimes\mathcal{L}_{-ax}$. By Poincaré duality, we have a perfect pairing for each a:

$$H^1_c(M'\otimes\mathcal{L}_{ax})\times H^1((M')^\vee\otimes\mathcal{L}_{-ax})\to H^2_c(K'\langle x\rangle^\dagger)\cong K'(-1).$$

Given the assumption $\dim_{K'} H^0_{\mathrm{loc},\infty}(M'\otimes\mathcal{L}_{ax}) = \dim_{K'} H^1_{\mathrm{loc},\infty}(M'\otimes\mathcal{L}_{ax}) = 0$, the 'forget supports' map $H^1_c(M'\otimes\mathcal{L}_{ax})\to H^1(M'\otimes\mathcal{L}_{ax})$ in (2.5.1) must be an isomorphism. We thus have an F-equivariant perfect pairing

$$H^1(M'\otimes\mathcal{L}_{ax})\times H^1((M')^\vee\otimes\mathcal{L}_{-ax})\to K'(-1). \tag{5.2.2}$$

In particular, $\dim_{K'} H^1((M')^\vee\otimes\mathcal{L}_{-ax}) = d$ by duality.

By Proposition 3.4.3, $\widehat{M}$ and $\widehat{M^\vee}$ are (σ,∇)-modules over $K\langle x\rangle^\dagger$. Moreover, $\widehat{M}$ and the pullback of $\widehat{M^\vee}$ by the map $x\to -x$ have pointwise complex conjugate trace functions, so their direct sum is ι-real. By Proposition 4.2.2, each of them is irreducible, and hence by Theorem 5.1.2 is ι-pure of some weight j, necessarily the same for both (again since they have pointwise complex conjugate traces).

In the pairing (5.2.2), each factor on the left is ι-pure of weight j, and the object on the right is ι-pure of weight 2. This is only possible if $j+j=2$, so $j=1$. Thus, $\widehat{M}$ is ι-pure of weight 1, as then is any fibre, including $H^1(M)\cong H^1_c(M)$. $\square$

By degenerating this purity result, we get the desired statement on $\mathbb{A}^1$.

Theorem 5.2.3. *Let $\mathcal{E}$ be an overconvergent F-isocrystal on $\mathbb{A}^1$ which is ι-realizable and ι-mixed of weight $\geqslant w$. Then $H^1_{\mathrm{rig}}(\mathbb{A}^1/K,\mathcal{E})$ is ι-mixed of weight at least $w+1$.*

Proof. Let M be a (σ,∇)-module over $K\langle x\rangle^\dagger$ corresponding to $\mathcal{E}$. By a snake lemma argument, there is no loss of generality in assuming that M is irreducible; in particular, by Theorem 5.1.2, M is ι-pure of some weight, which we take to be w.

Choose an integer N satisfying the conclusion of Proposition 4.3.3 for M and for $M^\vee$. Choose $n>N$ not divisible by p, let $f: K\langle x\rangle^\dagger\to K\langle s,x\rangle^\dagger$ and $g: K\langle s\rangle^\dagger\to K\langle s,x\rangle^\dagger$ be the canonical embeddings, and define a (σ,∇)-module on $K\langle s,x\rangle^\dagger$ by

$$Q = f^*M^\vee\otimes_{K\langle s,x\rangle^\dagger}\mathcal{L}_{sx^n}.$$

(Geometrically, this corresponds to pulling back $M^\vee$ from $\mathbb{A}^1$ to $\mathbb{A}^2$ and twisting by a certain line bundle, as in the Fourier transform.) By Theorem 3.2.4, there exists a localization A of $K\langle s,s^{-1}\rangle^\dagger$ over which $R^1g_!Q_A$ and $R^1g_*Q_A$ are (σ,∇)-modules, where $Q_A = Q\otimes A\langle x\rangle^\dagger$. By Theorem 3.3.1, we have an F-equivariant injection

$$H^1_{c,\mathrm{rig}}(M^\vee)\hookrightarrow H^0_{\mathrm{loc}}(R^1g_!Q_A).$$

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By Proposition 4.3.3 and the choice of N, for K' a finite extension of K and a, c integers in K' with a not reducing to zero in the residue field, $\dim_{K'} H^0_{\mathrm{loc}}((M')^\vee \otimes \mathcal{L}_{ax^n+cx}) = \dim_{K'} H^1_{\mathrm{loc}}((M')^\vee \otimes \mathcal{L}_{ax^n+cx}) = 0$ and the K'-dimension of $H^1((M')^\vee \otimes \mathcal{L}_{ax^n+cx})$ does not depend on c. By Theorem 5.2.1, $H^1_c(M^\vee \otimes \mathcal{L}_{ax^n})$ is ι-pure of weight $-w+1$; in other words, $R^1 g_! Q_A$ is ι-pure of weight $-w+1$. By Proposition 5.1.4, $H^0_{\mathrm{loc}}(R^1 g_! Q_A)$ is ι-mixed of weight at most $-w+1$, so $H^1_{c,\mathrm{rig}}(M^\vee)$ is as well. By Poincaré duality, $H^1_{\mathrm{rig}}(M)$ is ι-mixed of weight at least $w+1$, as desired. □

5.3 Rigid Weil II and the Weil conjectures

To apply our results to arbitrary smooth varieties, we employ the formalism of rigid cohomology. In so doing, we recover the Riemann hypothesis component of the Weil conjectures.

As in [Ked05b], we use the following geometric lemma proved in [Ked05a]. This allows us to reduce consideration of a complicated isocrystal on a complicated variety to a more complicated isocrystal on a less complicated variety, namely affine space.

Proposition 5.3.1. *Let X be a smooth k-variety of dimension n, for k a field of characteristic $p > 0$, and let S be a zero-dimensional closed subscheme of X. Then X contains an open dense affine subvariety containing S and admitting a finite étale morphism to affine n-space.*

Theorem 5.3.2 (Rigid Weil II over a point). *Let X be a variety (reduced separated scheme of finite type) over $\mathbb{F}_q$, and let $\mathcal{E}$ be an ι-realizable overconvergent F-isocrystal on X.*

(a) *If $\mathcal{E}$ is ι-mixed of weight at most w, then for each i, $H^i_{c,\mathrm{rig}}(X/K, \mathcal{E})$ is ι-mixed of weight at most $w+i$.*

(b) *If X is smooth and $\mathcal{E}$ is ι-mixed of weight at least w, then for each i, $H^i_{\mathrm{rig}}(X/K, \mathcal{E})$ is ι-mixed of weight at least $w+i$.*

Proof. We prove the result (for all q) by induction primarily on $n = \dim X$ and secondarily on $\operatorname{rank} \mathcal{E}$. Before proceeding to the main argument, we give a number of preliminary reductions.

Note that part (b) follows from part (a) by Poincaré duality. On the other hand, using the excision exact sequence (2.1.1) and the induction hypothesis, we may assume in part (a) that X is affine and smooth of pure dimension n. By Poincaré duality again, we may now reduce to proving just part (b) for X, or for any one open dense subset of X.

There is no loss of generality in enlarging K, so we assume $w = 0$ by twisting as necessary. By Proposition 5.3.1, X admits an open dense affine subscheme U which in turn admits a finite étale morphism $f : U \to \mathbb{A}^n$. As noted earlier, we may replace X by U by excision; since $H^i_{\mathrm{rig}}(U/K, \mathcal{E}) \cong H^i_{\mathrm{rig}}(\mathbb{A}^n/K, f_* \mathcal{E})$, we may in turn reduce to the case $X = \mathbb{A}^n$.

Finally, note that we may assume that $\mathcal{E}$ is irreducible: if $0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2 \to 0$ is a short exact sequence of overconvergent F-isocrystals on X, then proving part (b) for $\mathcal{E}_1$ and $\mathcal{E}_2$ implies part (b) for $\mathcal{E}$ by the evident long exact sequence in homology.

With these reductions in hand, we proceed to the main argument. Choose a decomposition $\mathbb{A}^n \cong \mathbb{A}^1 \times \mathbb{A}^{n-1}$ and let $f : \mathbb{A}^n \to \mathbb{A}^{n-1}$ be the associated projection. By Theorem 3.2.4, there is an open dense subset W of $\mathbb{A}^{n-1}$ on which the kernel $\mathcal{F}_0$ and cokernel $\mathcal{F}_1$ of the vertical connection ∇_v on $\mathcal{E}$ are overconvergent F-isocrystals (and similarly for $\mathcal{E}^\vee$). By applying excision, we may reduce to the case $X = \mathbb{A}^1 \times W$.

Note that $f^* \mathcal{F}_0$ is canonically isomorphic to a sub-F-isocrystal of $\mathcal{E}$. By the irreducibility hypothesis on $\mathcal{E}$, if $\mathcal{F}_0$ is nonzero, then $\mathcal{E} = f^* \mathcal{F}_0$. In this case, $H^i_{\mathrm{rig}}(X/K, \mathcal{E}) \cong H^i_{\mathrm{rig}}(W/K, \mathcal{F}_0)$ by the Künneth decomposition [Ked05b, Theorem 1.2.4] and so the desired result follows by the induction hypothesis. We may thus assume that $\mathcal{F}_0 = 0$.

Since $\mathcal{E}$ is ι-realizable, we may choose an overconvergent F-isocrystal $\mathcal{E}'$ on X such that $\mathcal{E} \oplus \mathcal{E}'$ is ι-real; we may assume that $\mathcal{E}'$ is semisimple, since passing from $\mathcal{E}'$ to its semisimplification does not change traces. By shrinking W if needed, we may ensure by Theorem 3.2.4 again that the kernel $\mathcal{F}'_0$ and cokernel $\mathcal{F}'_1$ of the vertical connection ∇_v on $\mathcal{E}'$ are overconvergent F-isocrystals (and similarly for $(\mathcal{E}')^\vee$). Again, $f^*\mathcal{F}'_0$ is canonically isomorphic to a sub-F-isocrystal of $\mathcal{E}'$; in particular, it is ι-realizable, as then is its restriction to $\{0\} \times W$. In other words, $\mathcal{F}'_0$ itself is ι-realizable.

The trace formula (2.1.2) shows that the trace of Frobenius on a fibre of $\mathcal{F}_1$, plus on that fibre of $\mathcal{F}'_1$, minus on that fibre of $\mathcal{F}'_0$, is ι-real. Since $\mathcal{F}'_0$ is ι-realizable, we deduce that $\mathcal{F}_1 \oplus \mathcal{F}'_1$ is ι-realizable.

In particular, $\mathcal{F}_1$ is ι-realizable. By Theorem 5.2.3 applied to each fibre, $\mathcal{F}_1$ is ι-mixed of weight at least 1. Thus by the induction hypothesis, $H^i_{\mathrm{rig}}(W/K, \mathcal{F}_1)$ is ι-mixed of weight at least $i+1$ for each i. By Proposition 3.2.5, we have an F-equivariant exact sequence

$$0 \to H^i_{\mathrm{rig}}(X/K, \mathcal{E}) \to H^{i-1}_{\mathrm{rig}}(W/K, \mathcal{F}_1).$$

(In the application of Proposition 3.2.5, the space X there is our W, and the $R^i f_* \mathcal{E}$ there are our $\mathcal{F}_i$; remember that $\mathcal{F}_0$ vanishes.) Hence, $H^i_{\mathrm{rig}}(X/K, \mathcal{E})$ is ι-mixed of weight at least i for all i. That is, part (b) holds on X, which, thanks to the reductions, completes the induction. □

Remark 5.3.3. We describe Theorem 5.3.2 as rigid Weil II 'over a point' because Deligne's theorem also treats the relative case. We do not attempt to give a relative theorem here for two reasons. The more serious reason is that we do not have a category containing the overconvergent F-isocrystals admitting Grothendieck's six operations, so we are unable to even formulate a proper analogue. (As noted earlier, the correct setting is Berthelot's theory of arithmetic $\mathcal{D}$-modules; Caro's work [Car04] provides a potential shortcut around some technical difficulties in Berthelot's theory.) The other reason is that we are using a pointwise definition of ι-mixedness, whereas Deligne's theorem uses a global definition; see Remark 5.1.1. Remedying this discrepancy will require extending Theorem 5.1.2 to general varieties; see Remark 5.1.3.

For completeness, we point out how Theorem 5.3.2 plus the formalism of rigid cohomology imply the Weil conjectures in the following form.

(a) *Analytic continuation.* For X a variety over $\mathbb{F}_q$, the generating function

$$\zeta_X(t) = \exp\biggl(\sum_{n=1}^{\infty} \#X(\mathbb{F}_{q^n}) \frac{t^n}{n}\biggr)$$

can be written as a product $\prod_{i=0}^{2\dim X} P_i(t)^{(-1)^{i+1}}$, where each $P_i(t)$ is a polynomial with integer coefficients and constant coefficient 1.

(b) *Functional equation.* If X is smooth, proper, and purely of dimension n, then the product representation can be chosen so that

$$P_{2n-i}(t) = ct^j P_i(t^{-1})$$

for some integer j and some nonzero rational number c.

(c) *Riemann hypothesis.* If X is smooth, proper, and purely of dimension n, then the product representation can be chosen so that each complex root of P_i has reciprocal absolute value $q^{i/2}$.

Part (a) follows from the Lefschetz trace formula (2.1.2) and the finite dimensionality of rigid cohomology (with constant coefficients), taking $P_i(t) = \det(1 - Ft, H^i_{c,\mathrm{rig}}(X/K))$. Part (b) follows from Poincaré duality and the fact that $H^i_{c,\mathrm{rig}}(X/K) \cong H^i_{\mathrm{rig}}(X/K)$ when X is proper. Part (c) follows from Theorem 5.3.2: it implies, on the one hand, that for any ι, $H^i_{c,\mathrm{rig}}(X/K)$ is ι-mixed of weight at most i and, on the other hand, that $H^i_{c,\mathrm{rig}}(X/K)^\vee \cong H^{2n-i}_{\mathrm{rig}}(X/K)(n) \cong H^{2n-i}_{c,\mathrm{rig}}(X/K)(n)$ is ι-mixed of weight at most $(2n-i) - 2n = -i$. Hence, $H^i_{c,\mathrm{rig}}(X/K)$ is ι-pure of weight i.

5.4 The p-adic situation

In closing, it is worth pointing out that one can set up the same sort of framework using the p-adic valuation on K^{alg} as the weight formalism using the archimedean valuation on $\mathbb{C}$. This points out one of the benefits of having p-adic Weil II at hand: one can treat both archimedean and p-adic valuations within the same formalism. (The fact that Dwork-style techniques lead to control on p-adic valuations of coefficients of zeta functions is well known: only the cohomological interpretation is novel.)

We say that an element $\alpha \in K^{\mathrm{alg}}$ has *slope* s if $|\alpha| = |q^s|$; note that this means s can be any rational number, not just an integer. (The term 'slope' derives from the fact that the valuations of roots of polynomials are typically computed as slopes of certain Newton polygons.) An overconvergent F-isocrystal $\mathcal{E}$ on an $\mathbb{F}_q$-scheme X is said to *have slopes in the interval* $[r,s]$ if for each closed point x of X of degree d over $\mathbb{F}_q$, the eigenvalues of F_x on $\mathcal{E}_x$ have slopes in the interval $[dr, ds]$.

One cannot expect to precisely determine the slopes of the cohomology of an overconvergent F-isocrystal, nor to limit them to integral values; for instance, the cohomology of an elliptic curve can have slopes 0 and 1 (in the ordinary case) or $\frac{1}{2}$ and $\frac{1}{2}$ (in the supersingular case). The best we can do is limit the range of the variation as follows.

THEOREM 5.4.1. *Let X be an $\mathbb{F}_q$-variety of dimension n, and let $\mathcal{E}$ be an overconvergent F-isocrystal on X which has slopes in the interval $[r,s]$. Then for $0 \leqslant i \leqslant 2n$, $H^i_{c,\mathrm{rig}}(X/K,\mathcal{E})$ has slopes in the interval $[r+\max\{0,i-n\}, s+\min\{i,n\}]$. If X is smooth, then $H^i_{\mathrm{rig}}(X/K,\mathcal{E})$ also has slopes in the interval $[r+\max\{0,i-n\}, s+\min\{i,n\}]$.*

The key step in the proof of this theorem is the following innocuous-looking lemma.

LEMMA 5.4.2. *Let $\mathcal{E}$ be an overconvergent F-isocrystal on an affine curve C which has slopes in the interval $[0,\infty)$. Then $H^1_{c,\mathrm{rig}}(C/K,\mathcal{E})$ also has slopes in the interval $[0,\infty)$.*

Proof. By excision, there is no harm in shrinking C; hence, by Proposition 5.3.1, we may assume that C admits a finite étale map to $\mathbb{A}^1$. By pushing forward, we may then assume that, in fact, $C = \mathbb{A}^1$. Then by the usual long exact sequence in homology, we may reduce to the case of $\mathcal{E}$ irreducible; in particular, we may assume that $H^0_{\mathrm{rig}}(\mathbb{A}^1/K,\mathcal{E}) = H^0_{\mathrm{rig}}(\mathbb{A}^1/K,\mathcal{E}^\vee) = 0$, since there is nothing to check whether $\mathcal{E}$ is spanned by horizontal sections (as $H^1_{c,\mathrm{rig}}(\mathbb{A}^1/K,\mathcal{E})$ vanishes in this case).

It suffices to show that $\mathrm{Trace}(F^i, H^1_{c,\mathrm{rig}}(\mathbb{A}^1/K,\mathcal{E}))$ has nonnegative p-adic valuation. By the Lefschetz trace formula (2.1.2), we have

$$\mathrm{Trace}(F, H^1_{c,\mathrm{rig}}(\mathbb{A}^1/K,\mathcal{E})) = \mathrm{Trace}(F, H^2_{c,\mathrm{rig}}(\mathbb{A}^1/K,\mathcal{E})) - \sum_{x\in\mathbb{A}^1(\mathbb{F}_q)} \mathrm{Trace}(F_x,\mathcal{E}_x).$$

By Poincaré duality, $H^2_{c,\mathrm{rig}}(\mathbb{A}^1/K,\mathcal{E})$ vanishes, while the trace of F_x on each $\mathcal{E}_x$ has nonnegative p-adic valuation. This yields the desired integrality for $i=1$; the general result follows by repeating the argument over $\mathbb{F}_{q^i}$. $\square$

Proof of Theorem 5.4.1. We first verify the desired result for $X = \mathbb{A}^1$; for brevity, let M be a (σ,∇)-module over $K\langle x\rangle^\dagger$ corresponding to $\mathcal{E}$. The case $i=0$ is straightforward, again since $H^0(M)$ embeds F-equivariantly into any fibre of M; similarly for $i=2$ via Poincaré duality. As for the case $i=1$, Lemma 5.4.2 (applied after a twist) implies that $H^1_c(M)$ has slopes in the interval $[r,\infty)$, and that $H^1_c(M^\vee)$ has slopes in the interval $[-s,\infty)$. By Grothendieck's specialization theorem (see, e.g., [Ked04a, Proposition 5.14]), $H^0_{\mathrm{loc}}(M)$ has slopes in the interval $[r,s]$; by Poincaré duality, $H^1_{\mathrm{loc}}(M^\vee)$ has slopes in the interval $[-s+1,-r+1]$. Since $H^1(M^\vee)$ sits between $H^1_c(M^\vee)$ and $H^1_{\mathrm{loc}}(M^\vee)$ in the exact sequence (2.5.1), it has slopes in the interval $[-s,\infty)$. By Poincaré duality, $H^1_c(M)$ has slopes in the interval $[r,\infty)\cap(-\infty,s+1]=[r,s+1]$. Similarly for $H^1(M)$ by the same arguments applied to $M^\vee$ plus Poincaré duality.

We now proceed to the general case, where we proceed by induction on $n=\dim X$; we may assume that X is irreducible. By Poincaré duality, it suffices to consider the case of cohomology with compact supports. If U is an open subset of X and $Z=X\setminus U$, the excision sequence (2.1.1) traps $H^i_{c,\mathrm{rig}}(X/K,\mathcal{E})$ between $H^i_{c,\mathrm{rig}}(U/K,\mathcal{E})$ and $H^i_{c,\mathrm{rig}}(Z/K,\mathcal{E})$. Assuming the induction hypothesis and the fact that the claim holds over U, the terms surrounding $H^i_{c,\mathrm{rig}}(X/K,\mathcal{E})$ have slopes in the intervals

$$[r+\max\{0,i-n\},s+\min\{i,n\}] \quad \text{and} \quad [r+\max\{0,i-\dim(Z)\},s+\min\{i,\dim(Z)\}].$$

Since $\dim(Z)\leqslant n$ and $i-\dim(Z)\geqslant i-n$, the union of these intervals is $[r+\max\{0,i-n\},s+\min\{i,n\}]$. In other words, to prove the desired result over X, it suffices to prove it over U.

Now apply Proposition 5.3.1 and excision again, as in Theorem 5.3.2, to reduce consideration to the case where $X=\mathbb{A}^1\times W$, $f:X\to W$ is the canonical projection, and $R^jf_*\mathcal{E}$ and $R^jf_*\mathcal{E}^\vee$ are overconvergent F-isocrystals on W for $j=0,1$. Now we switch to considering cohomology without supports (since we no longer need excision). Applying the affine line case fibrewise, we see that $R^jf_*\mathcal{E}$ has slopes in the interval $[r,s+j]$ for $j=0,1$. By the induction hypothesis, $H^i_{\mathrm{rig}}(W/K,R^0f_*\mathcal{E})$ has slopes in the interval $[r+\max\{0,i-n+1\},s+\min\{i,n-1\}]$ and $H^{i-1}_{\mathrm{rig}}(W/K,R^1f_*\mathcal{E})$ has slopes in the interval $[r+\max\{0,i-n\},s+\min\{i,n\}]$. By Proposition 3.2.5, $H^i_{\mathrm{rig}}(X/K,\mathcal{E})$ thus has slopes in the interval $[r+\max\{0,i-n\},s+\min\{i,n\}]$, as desired. $\square$

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