

# Geometric series expansion of the Neumann–Poincaré operator: Application to composite materials

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The Neumann–Poincaré (NP) operator, a singular integral operator on the boundary of a domain, naturally appears when one solves a conductivity transmission problem via the boundary integral formulation. Recently, a series expression of the NP operator was developed in two dimensions based on geometric function theory [34]. In this paper, we investigate geometric properties of composite materials using this series expansion. In particular, we obtain explicit formulas for the polarisation tensor and the effective conductivity for an inclusion or a periodic array of inclusions of arbitrary shape with extremal conductivity, in terms of the associated exterior conformal mapping. Also, we observe by numerical computations that the spectrum of the NP operator has a monotonic behaviour with respect to the shape deformation of the inclusion. Additionally, we derive inequality relations of the coefficients of the Riemann mapping of an arbitrary Lipschitz domain using the properties of the polarisation tensor corresponding to the domain.

**Key words:** Homogenisation, integral equations methods in two dimensions, eigenvalues of operators, conformal mappings

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## 1 Introduction

Various interesting wave phenomena of composite materials can be modelled as transmission problems in partial differential equations. The Neumann–Poincaré (NP) operator naturally appears when one solves a conductivity transmission problem via the boundary integral formulation. One can then obtain essential properties of composites by analysing the NP operator. For a simply connected bounded Lipschitz domain  $\Omega \subset \mathbb{R}^2$  and a density function  $\varphi \in L^2(\partial\Omega)$ , the NP operator is defined by:

$$\mathcal{K}_{\partial\Omega}^*[\varphi](x) = p.v. \frac{1}{2\pi} \int_{\partial\Omega} \frac{\langle x-y, \nu_x \rangle}{|x-y|^2} \varphi(y) d\sigma(y). \quad (1.1)$$

Here,  $p.v.$  denotes the Cauchy principal value, and  $\nu_x$  is the unit outward normal vector at  $x \in \partial\Omega$ . In higher dimensions, the NP operator is defined as a boundary integral operator in a similar manner (see, e.g., [57]).

Recently, a series expression of the NP operator was developed in two dimensions based on geometric function theory [34]; see Subsections 2.2 and 2.3 for details. This was successfully applied to the study of inclusion problems [21, 22], and the decay properties of eigenvalues of the NP operator [33]. In the present paper, based on the geometric series expansion of the NP operator, we investigate geometric properties of composite materials by analysing the interrelation among the interior or exterior conformal mappings, the NP operator, the polarisation tensor (PT) and the effective conductivity, where each term is associated with the geometry of an inclusion.

For the conductivity transmission problem with an inclusion, one can find the solution using the ansatz  $u(x) = H(x) + \mathcal{S}_{\partial\Omega}[\varphi](x)$ , where  $\mathcal{S}_{\partial\Omega}$  indicates the single-layer potential associated with the fundamental solution to the Laplacian, and  $\varphi$  involves the inversion of  $\lambda I - \mathcal{K}_{\partial\Omega}^*$  for some constant  $\lambda$ . This boundary integral formulation provides us with the multipole expansion of the scattered field, whose coefficient of the leading term is expressed in terms of the so-called Pólya–Szegő PT, which may be defined in terms of the NP operator; we refer the reader to books [4, 5] and the references therein for detailed results on the PT based on the layer potential technique.

A similar integral formula holds for a periodic interface problem, where the layer potential is now defined in terms of the periodic Green's function. The determination of the effective property of composite materials is one of the classical topics in physics [26, 30, 31, 54]; we refer the reader to [15] and to more references therein. The literature most closely related to our result is [10] (see also [9]), in which the effective electrical conductivity  $\sigma^*$  of a two-phase medium was derived using the integral formulation as follows:

$$\sigma^* = I + fM \left( I - \frac{f}{2}M \right)^{-1} + O(f^3), \quad (1.2)$$

where  $f$  is the volume fraction of the inclusion, and  $M$  is the PT corresponding to the inclusion and the unit conductivity of the matrix.

As one of the main results, we obtain an explicit expression of the trace of the PT in terms of the exterior conformal mapping for a simply connected, bounded Lipschitz domain with extreme (zero or infinity) or near-extreme conductivity. As an application, we find upper and lower bounds on the trace of the PT as expressed by the diameter of the domain. Then, using (1.2), we derive an explicit asymptotic formula for the effective conductivity of cylindrical periodic composites with inclusions with extreme conductivity, in terms only of the logarithmic capacity and the coefficient of order  $-1$  of the associated exterior conformal mapping. We show that the asymptotic formula of the effective conductivity of a regular  $n$ -gon converges monotonically to that of a disc with unit area as  $n$  goes to infinity.

Additionally, we derive inequality relations between coefficients of the Riemann mapping. We remark that such inequalities are of interest in univalent function theory [23, 42, 44, 52, 56], for example, the Bieberbach conjecture, for which we refer the reader to [15] and to the comprehensive references therein. In the present paper, we derive relations between the coefficients of the Riemann mapping corresponding to arbitrary Jordan domains by employing the properties of the PTs corresponding to domains with extreme conductivities.

Spectral analysis of the NP operator has drawn significant attention in relation to plasmon resonances [3, 12, 48, 58]; the plasmon resonance happens at the spectrum of the NP operator [27, 37]. The NP operator  $\mathcal{K}_{\partial\Omega}^*$ , which is symmetric on  $L^2(\partial\Omega)$  only for a disc or a ball [46], can be symmetrised by Plemelj's symmetrisation principle [41]. As a result, the spectrum of the NP operator on  $H_0^{-1/2}(\partial\Omega)$  or  $L_0^2(\partial\Omega)$  lies on the real axis in the complex plane and, more specifically, is contained in  $(-1/2, 1/2)$  [40, 57] (see also [39, 43]), where  $H_0^{-1/2}(\partial\Omega)$  is the Sobolev space  $H^{-1/2}(\partial\Omega)$  with the mean-zero condition. The spectral structure of the NP operator is different depending on the regularity of the domain. If  $\Omega$  has a  $C^{1,\alpha}$  boundary, then  $\mathcal{K}_{\partial\Omega}^*$  is compact so that it admits the spectral decomposition:

$$\mathcal{K}_{\partial\Omega}^* = \sum_{j=1}^{\infty} \lambda_j \psi_j \otimes \psi_j,$$

where  $(\lambda_j, \psi_j)$  are pairs of eigenvalues and orthonormalised eigenfunctions of  $\mathcal{K}_{\partial\Omega}^*$ .

If  $\Omega$  is merely a Lipschitz domain, then  $\mathcal{K}_{\partial\Omega}^*$  also admits a continuous spectrum [28]; see also [37, 45, 49, 50]. The spectrum of the NP operator for contacting discs was analysed [32]. We refer the reader to a review article [14] and the references therein for more results on the spectral analysis of the NP operator.

In this paper, we numerically compute the spectrum of the NP operator for various smooth domains using the finite section method based on the geometric series expansion of the NP operator, including the algebraic domains given by  $\Psi(z) = z + \frac{a}{z^m}$  with  $a \in (0, \frac{1}{m})$ ,  $m \in \mathbb{N}$ ; see Subsection 6.2. It turns out that the eigenvalues are clustered in groups whose size is  $m + 1$ . This generalises the results in [11], where the eigenvalues for the algebraic domain with small  $a$  are obtained asymptotically. Furthermore, we observe a monotonic behaviour of eigenvalues of the NP operator depending on the shape deformation of the inclusion from a disc.

The remainder of this paper is organised as follows. Section 2 is devoted to reviewing the geometric series expansions of the layer potential operators. We then estimate the trace of the PT in Section 3 and obtain the asymptotic formula for the effective conductivity of dilute composites in Section 4. Inequality relations between the coefficients of the Riemann mapping is then derived in Section 5. By a numerical study, we investigate the shape dependence of the NP operator in Section 6. The paper ends with a conclusion in Section 7.

## 2 Preliminary

### 2.1 Layer potential formulation

Let  $\Omega$  be a simply connected bounded Lipschitz domain in two dimensions. Assume that  $\Omega$  is occupied by a homogeneous isotropic material with constant conductivity  $k$  satisfying  $0 \leq k \neq 1 \leq \infty$ . Consider the conductivity transmission problem:

$$\begin{cases} \nabla \cdot (k\chi(\Omega) + \chi(\mathbb{R}^2 \setminus \overline{\Omega})) \nabla u = 0 & \text{in } \mathbb{R}^2, \\ u(x) - H(x) = O(|x|^{-1}) & \text{as } |x| \rightarrow \infty, \end{cases} \quad (2.1)$$

where  $H$  is an arbitrary entire harmonic function and  $\chi(A)$  denotes the characteristic function of a set  $A$ . For the case  $k = 0$  or  $\infty$ , the zero Neumann boundary condition or the constant Dirichlet boundary condition is imposed on  $\partial\Omega$ .

One can find the solution to (2.1) from the layer potential formulation:

$$u = H + \mathcal{S}_{\partial\Omega}[\varphi], \tag{2.2}$$

where  $\varphi$  is the solution to

$$(\lambda I - \mathcal{K}_{\partial\Omega}^*)[\varphi] = \frac{\partial H}{\partial \nu} \quad \text{on } \partial\Omega$$

with  $\lambda = \frac{k+1}{2(k-1)}$ . The NP operator,  $\mathcal{K}_{\partial\Omega}^*$ , is given by (1.1), and  $\mathcal{S}_{\partial\Omega}[\varphi]$  denotes the single-layer potential:

$$\mathcal{S}_{\partial\Omega}[\varphi](x) := \frac{1}{2\pi} \int_{\partial\Omega} \ln|x-y|\varphi(y) d\sigma(y), \quad x \in \mathbb{R}^2.$$

We note that  $\lambda$  is in  $(-\infty, -\frac{1}{2}] \cup [\frac{1}{2}, \infty)$ , for which  $\lambda I - \mathcal{K}_{\partial\Omega}^*$  is invertible on  $L_0^2(\partial\Omega)$  [24, 57].

We define the Pólya–Szegő PT in  $\mathbb{R}^d$  with  $d = 2$  as the  $2 \times 2$  symmetric matrix  $M = (M_{ij})_{i,j=1}^2$  given by:

$$M_{i,j} := \int_{\partial\Omega} y_j(\lambda I - \mathcal{K}_{\partial\Omega}^*)^{-1} \left[ \frac{\partial x_i}{\partial \nu} \right] (y) d\sigma(y) \tag{2.3}$$

with  $\lambda = \frac{k+1}{2(k-1)}$ . Note that the PT depends on the domain  $\Omega$  as well as the conductivity  $k$ . We write the PT as  $M(\Omega, \lambda)$  to highlight its dependence on the domain and the material parameter. The PT of  $\Omega$  for  $d = 3$ , a  $3 \times 3$  symmetric matrix again denoted by  $M$ , is defined in the same way as (2.3).

The solution  $u$  to (2.1) admits a far-field expansion, which one easily finds by taking the Taylor series expansion for the integral formulation (2.2): for sufficiently large  $x = (x_1, x_2)$ ,

$$u(x) - H(x) = -\frac{1}{2\pi} \nabla H(0)M(\Omega, \lambda) \frac{x^T}{|x|^2} + \text{higher order terms.} \tag{2.4}$$

We refer the reader to [4, 5] and references therein for further details on the layer potential formulation for the conductivity transmission problem.

### 2.2 Exterior conformal mapping, Faber polynomials and Grunsky coefficients

We continue to assume that  $\Omega$  is a simply connected bounded Lipschitz domain in  $\mathbb{R}^2$ . We identify  $x = (x_1, x_2)$  in  $\mathbb{R}^2$  with  $z = x_1 + ix_2$  in  $\mathbb{C}$ . The symbols  $\text{Re}$  and  $\text{Im}$  indicate the real and imaginary parts of complex numbers, respectively. By the Riemann mapping theorem, there exists a unique pair  $(\gamma, \Psi)$  where  $\gamma > 0$  and  $\Psi : \{w \in \mathbb{C} : |w| > \gamma\} \rightarrow \mathbb{C} \setminus \bar{\Omega}$  is a conformal map satisfying  $\Psi(\infty) = \infty$  and  $\Psi'(\infty) = 1$ . The mapping then admits the Laurent series expansion:

$$\Psi(w) = w + a_0 + \sum_{n=1}^{\infty} \frac{a_n}{w^n} \tag{2.5}$$

for some complex coefficients  $a_n$ , which can be easily derived by reflecting the function with respect to a circle and applying the Taylor series expansion in a disc. The quantity  $\gamma$  is called the conformal radius of  $\Omega$  and it coincides with the logarithmic capacity of  $\bar{\Omega}$  (see (2.15)). From the well-known Bieberbach conjecture [15], it holds that

$$|a_1| \leq \gamma^2. \tag{2.6}$$

Note that simply connected bounded Lipschitz domains are Jordan domains. Indeed, for such a domain, there exists a constant  $C > 0$  such that for all  $x, y \in \Omega$ , there exists a smooth curve in  $\Omega$  from  $x$  to  $y$  whose length satisfies  $\leq C|x - y|$ . This fact, combined with Moore’s characterisation of Jordan domains, proves the assertion. As a result,  $\Psi$  extends to the boundaries as a homeomorphism by the Caratheodory extension theorem [19]. In particular, this gives a natural parametrisation of  $\partial\Omega$ :  $\theta \mapsto \Psi(\gamma e^{i\theta})$  for  $\theta \in [0, 2\pi)$ .

For a complex function,

$$g(w) = w + b_0 + \sum_{k=1}^{\infty} \frac{b_k}{w^k}$$

which is univalent on the exterior of a disc centred at the origin, there exists a unique degree  $m$  polynomial  $F_m$  such that  $F_m(g(w))$  has only one non-negative order term  $w^m$  for each  $m \in \mathbb{N} \cup \{0\}$ . In other words,

$$F_m(g(w)) = w^m + \text{negative order terms.} \tag{2.7}$$

Indeed,  $F(g(w))$  admits a Laurent series expansion with a finite number of non-negative order terms for any complex polynomial  $F$ . By appropriately choosing the coefficients for  $F$ , we have the property (2.7). Such  $F_m$  is called the  $m$ -th *Faber polynomial* of  $g$ . The concept of Faber polynomials, first introduced by G. Faber in [25], has been one of the essential elements in geometric function theory (see, e.g., [23]). The Faber polynomials can be also defined via the following generating function relation: for  $z \in \Omega$  and sufficiently large  $w$ ,

$$\frac{wg'(w)}{g(w) - z} = \sum_{m=0}^{\infty} \frac{F_m(z)}{w^m}.$$

In this study, we use the Faber polynomials, denoted again by  $F_m$ , of the exterior conformal map  $\Psi$  associated with  $\Omega$ . We denote by  $c_{m,k}$  the *Grunsky coefficients* of  $\Psi$  so that for each  $m$ ,  $F_m(\Psi(w))$  satisfies

$$F_m(\Psi(w)) = w^m + \sum_{k=1}^{\infty} c_{m,k} w^{-k}. \tag{2.8}$$

Each  $F_m$  is monic and is uniquely determined by  $a_0, a_1, \dots, a_{m-1}$ . For example, the first three polynomials are  $F_0(z) = 1$ ,  $F_1(z) = z - a_0$ ,  $F_2(z) = z^2 - 2a_0z + (a_0^2 - 2a_1)$ . The Grunsky identity holds for all  $m, n \in \mathbb{N}$ :  $nc_{m,n} = mc_{n,m}$ . It then holds that

$$\mu_{m,k} = \mu_{k,m}, \text{ where } \mu_{m,k} := \sqrt{\frac{m}{k}} \frac{c_{k,m}}{\gamma^{m+k}}. \tag{2.9}$$

### 2.3 Geometric series expansion of the layer potential operators

In this subsection, we additionally assume that  $\Omega$  is enclosed by a  $C^{1,\alpha}$  Jordan curve for some  $0 < \alpha < 1$ . As we assume  $C^{1,\alpha}$  regularity for the boundary curve  $\partial\Omega$ , we have further regularity for  $\Psi$ . That is, by the Kellogg–Warschawski theorem (see, e.g., [53]),  $\Psi'$  can be continuously extended to the boundary. Hence, the map

$$(\rho, \theta) \mapsto z = \Psi(e^{\rho+i\theta}), \tag{2.10}$$

from  $[\ln \gamma, \infty) \times \mathbb{R}$  onto  $\mathbb{C} \setminus \Omega$  is  $C^1$ . As  $\Psi$  is a conformal (angle preserving) map,  $(\rho, \theta)$  provides curvilinear orthogonal coordinates in  $\mathbb{C} \setminus \Omega$  via (2.10). The scale factors with respect to  $\rho$  and  $\theta$  coincide with each other. We denote them by

$$h(\rho, \theta) := \left| \frac{\partial \Psi}{\partial \rho} \right| = \left| \frac{\partial \Psi}{\partial \theta} \right|. \tag{2.11}$$

Set  $\rho_0 = \ln \gamma$ . The length element on  $\partial\Omega$  is  $d\sigma(z) = h(\rho_0, \theta)d\theta$ . For a smooth function  $g(z) = (g \circ \Psi)(e^{\rho+i\theta})$ , it holds that

$$\frac{\partial g}{\partial \nu} \Big|_{\partial\Omega}^+(z) = \frac{1}{h(\rho_0, \theta)} \frac{\partial}{\partial \rho} g(\Psi(e^{\rho+i\theta})) \Big|_{\rho \rightarrow \rho_0^+}. \tag{2.12}$$

We define density basis functions on  $\partial\Omega$  in terms of the coordinate system (2.10):

$$\begin{aligned} \zeta_0(z) &= \frac{1}{h(\rho_0, \theta)}, \\ \zeta_{\pm m}(z) &= |m|^{\frac{1}{2}} \frac{e^{\pm im\theta}}{h(\rho_0, \theta)} \quad \text{for } m \in \mathbb{N}. \end{aligned}$$

**Lemma 2.1** ([34]) *Let  $\Omega$  be a simply connected bounded domain having  $C^{1,\alpha}$  boundary with some  $0 < \alpha < 1$ . For  $z \in \mathbb{C} \setminus \bar{\Omega}$ , we set  $z = \Psi(w) = \Psi(e^{\rho+i\theta})$  with  $\rho > \rho_0 = \ln \gamma$ . We have*

$$\mathcal{S}_{\partial\Omega}[\zeta_0](z) = \begin{cases} \ln \gamma & \text{if } z \in \bar{\Omega}, \\ \ln |w| & \text{if } z \in \mathbb{C} \setminus \bar{\Omega}. \end{cases} \tag{2.13}$$

For each  $m \in \mathbb{N}$ , the single-layer potential associated with  $\Omega$  satisfies

$$\mathcal{S}_{\partial\Omega}[\zeta_m](z) = \begin{cases} -\frac{1}{2\sqrt{m}\gamma^m} F_m(z) & \text{if } z \in \bar{\Omega}, \\ -\frac{1}{2\sqrt{m}\gamma^m} \left( \sum_{k=1}^{\infty} c_{m,k} e^{-k(\rho+i\theta)} + \gamma^{2m} e^{m(-\rho+i\theta)} \right) & \text{if } z \in \mathbb{C} \setminus \bar{\Omega}. \end{cases}$$

The series converges uniformly for all  $(\rho, \theta)$  such that  $\rho \geq \rho_1$  for any fixed  $\rho_1 > \rho_0$ . Since the kernel function of  $\mathcal{S}_{\partial\Omega}$  is real-valued, we have  $\mathcal{S}_{\partial\Omega}[\zeta_{-m}](z) = \overline{\mathcal{S}_{\partial\Omega}[\zeta_m](z)}$ .

The NP operator  $\mathcal{K}_{\partial\Omega}^*$  admits the series expansion on  $\partial\Omega$ :

$$\begin{aligned} \mathcal{K}_{\partial\Omega}^*[\zeta_0] &= \frac{1}{2} \zeta_0, \\ \mathcal{K}_{\partial\Omega}^*[\zeta_m] &= \frac{1}{2} \sum_{k=1}^{\infty} \frac{\sqrt{m}}{\sqrt{k}} \frac{c_{k,m}}{\gamma^{m+k}} \zeta_{-k}, \quad \mathcal{K}_{\partial\Omega}^*[\zeta_{-m}] = \frac{1}{2} \sum_{k=1}^{\infty} \frac{\sqrt{m}}{\sqrt{k}} \frac{\overline{c_{k,m}}}{\gamma^{m+k}} \zeta_k. \end{aligned} \tag{2.14}$$

The series converges in the Sobolev space  $H^{-1/2}(\partial\Omega)$  sense.

In fact, Lemma 2.1 holds assuming that  $\Omega$  is enclosed by a piecewise  $C^{1,\alpha}$  Jordan curve possibly with a finite number of corner points without inward or outward cusps. Under this boundary regularity assumption, the series in (2.14) converges in the sense of  $l^2(\mathbb{C})$  space spanned by  $\zeta_{\pm m}$  and  $\zeta_0$ .



for  $d = 2$  [2]. The lower Hashin–Shtrikman bound is attained (i.e., the equality holds in (3.2)) if and only if  $\Omega$  is an ellipse, asserted as the Polya–Szegő conjecture [51]; this conjecture was proved by Kang and Milton [38].

As one of our main objectives of this paper, we investigate the property of  $\text{tr}(M)$  and  $\text{tr}(M^{-1})$  for a planar inclusion with extreme conductivity. We note that for  $k \rightarrow 0$  or  $\infty$ , (3.2) becomes

$$|\text{tr}(M^{-1})| \leq \frac{1}{|\Omega|}. \tag{3.3}$$

However, the upper Hashin–Shtrikman bound, (3.1), becomes vacuous as  $k \rightarrow 0$  or  $\infty$ . In the present paper, instead of using (3.1), we characterise the trace of the PT for an inclusion with extreme or near-extreme conductivity using the following explicit expression of the PT:

**Theorem 3.1** *Let  $\Omega$  be a simply connected, planar bounded Lipschitz domain with  $k = 0$  or  $\infty$  (i.e.,  $\lambda = -\frac{1}{2}$  or  $\frac{1}{2}$ ). We denote the exterior conformal mapping associated with  $\Omega$  as in (2.5). Then, the PT of  $\Omega$  is*

$$M(\Omega, \pm \frac{1}{2}) = 2\pi \begin{bmatrix} \pm\gamma^2 + \text{Re}(a_1) & \text{Im}(a_1) \\ \text{Im}(a_1) & \pm\gamma^2 - \text{Re}(a_1) \end{bmatrix}. \tag{3.4}$$

**Proof** For domains with  $C^{1,\alpha}$  boundary curves, (3.4) was derived in [22] using the series expansions of the layer potential operators described in Subsection 2.3 (see also [21] for the relations between higher-order terms). For general Lipschitz domains, this is a direct consequence of the exact relations between the coefficients of the conformal mapping and the generalised PTs obtained in [20, Proposition 2.3] or [36]. For the paper to be self-contained, we provide a derivation of the exact relation for the PT in [20, 36], as follows.

Let  $V_1(z)$  (or  $V_2(z)$ ) be an analytic function in  $\mathbb{C} \setminus \bar{\Omega}$  such that  $\text{Re}(V_1)$  (or  $\text{Re}(iV_2)$ ) is the solution to the transmission problem (2.1) corresponding to  $H(x) = \text{Re}(z)$  (or  $H(x) = \text{Re}(iz)$ ). It then follows from (2.4) that for sufficiently large  $|z|$ ,

$$V_1(z) = \text{const.} + z - \frac{1}{2\pi}(M_{1,1} + iM_{1,2})\frac{1}{z} + \sum_{n=2}^{\infty} \frac{b_{1n}}{z^n},$$

$$V_2(z) = \text{const.} + z - \frac{1}{2\pi}(-M_{2,2} + iM_{1,2})\frac{1}{z} + \sum_{n=2}^{\infty} \frac{b_{2n}}{z^n}$$

with some constant coefficients  $b_{1n}$  and  $b_{2n}$ . From (2.5), we have for sufficiently large  $|w|$  that

$$(V_1 + V_2) \circ \Psi(w) = \text{const.} + 2w + 2a_0 + \left(-\frac{1}{2\pi}(M_{1,1} - M_{2,2} + 2iM_{1,2}) + 2a_1\right) \frac{1}{w} + \text{h.o.t.}, \tag{3.5}$$

$$(V_1 - V_2) \circ \Psi(w) = \text{const.} - \frac{1}{2\pi}(M_{1,1} + M_{2,2})\frac{1}{w} + \text{h.o.t.}, \tag{3.6}$$

where the higher-order terms consist of  $w^{-n}$  with  $n \geq 2$ .



We now assume that  $k = 0$ . The zero Neumann condition for  $\text{Re}(V_1)$  and  $\text{Re}(iV_2)$  implies that  $\text{Im}(V_1 \circ \Psi)$  and  $\text{Im}(iV_2 \circ \Psi)$  are constant on  $|w| = \gamma$ . Hence, from the relation:

$$i \text{Im}(V_1 \circ \Psi) + \text{Im}(iV_2 \circ \Psi) = \frac{1}{2}(V_1 \circ \Psi + V_2 \circ \Psi) - \frac{1}{2}\overline{(V_1 \circ \Psi - V_2 \circ \Psi)},$$

it follows that

$$\begin{aligned} \frac{1}{2}(V_1 \circ \Psi + V_2 \circ \Psi)(w) &= \text{const.} + \frac{1}{2}\overline{(V_1 \circ \Psi - V_2 \circ \Psi)(w)} \\ &= \text{const.} + \frac{1}{2}\overline{(V_1 \circ \Psi - V_2 \circ \Psi)\left(\frac{\gamma^2}{\bar{w}}\right)} \quad \text{on } |w| = \gamma. \end{aligned} \tag{3.7}$$

Since the last term is analytic for  $|w| < \gamma$  (see (3.6)),  $(V_1 \circ \Psi + V_2 \circ \Psi)(w)$  has an entire extension. It then holds from (3.5) that

$$M_{1,1} - M_{2,2} + 2iM_{1,2} = 4\pi a_1. \tag{3.8}$$

Also, from (3.5), (3.6) and (3.7), one can derive that

$$M_{1,1} + M_{2,2} = -4\pi\gamma^2.$$

Hence, we obtain (3.4) for  $k = 0$ , that is,  $\lambda = -\frac{1}{2}$ .

One can extend the result for the case  $k = 0$  to the case  $k = \infty$  with minor changes in the proof. Specifically, assuming  $k = \infty$ , we have that  $\text{Re}(V_1 \circ \Psi)$  and  $\text{Re}(iV_2 \circ \Psi)$  are constant on  $|w| = \gamma$ . Hence, from the relation:

$$\text{Re}(V_1 \circ \Psi) - i \text{Re}(iV_2 \circ \Psi) = \frac{1}{2}(V_1 \circ \Psi + V_2 \circ \Psi) + \frac{1}{2}\overline{(V_1 \circ \Psi - V_2 \circ \Psi)},$$

it follows that

$$\begin{aligned} \frac{1}{2}(V_1 \circ \Psi + V_2 \circ \Psi)(w) &= \text{const.} - \frac{1}{2}\overline{(V_1 \circ \Psi - V_2 \circ \Psi)(w)} \\ &= \text{const.} - \frac{1}{2}\overline{(V_1 \circ \Psi - V_2 \circ \Psi)\left(\frac{\gamma^2}{\bar{w}}\right)} \quad \text{on } |w| = \gamma. \end{aligned} \tag{3.9}$$

One can derive that  $(V_1 \circ \Psi + V_2 \circ \Psi)(w)$  has an entire extension, which is the same as the case  $k = 0$ . Hence, (3.8) holds. From (3.5), (3.6) and (3.9), we have for  $k = \infty$  that

$$M_{1,1} + M_{2,2} = 4\pi\gamma^2.$$

Hence, we obtain (3.4) for  $k = \infty$ , that is,  $\lambda = \frac{1}{2}$ . This completes the proof. □

**Remark 3.1** *One can find more relations by considering the higher-order terms in (3.5). In [20, 36], relations between the conformal mapping coefficients and the generalised PTs were obtained for a simply connected Lipschitz domain  $\Omega$ , where the generalised PTs are defined as, for  $m, n \in \mathbb{N}$ ,*

$$\mathbb{N}_{mn}^{(1)}(\Omega, \lambda) = \int_{\partial\Omega} P_n(z)(\lambda I - \mathcal{K}_{\partial\Omega}^*)^{-1} \left[ \frac{\partial P_m}{\partial v} \right] (z) d\sigma(z), \tag{3.10}$$

$$\mathbb{N}_{mn}^{(2)}(\Omega, \lambda) = \int_{\partial\Omega} P_n(z)(\lambda I - \mathcal{K}_{\partial\Omega}^*)^{-1} \left[ \frac{\partial \overline{P_m}}{\partial v} \right] (z) d\sigma(z) \tag{3.11}$$

with  $P_k(z) = z^k$  for each  $k \in \mathbb{N}$ . We refer the reader to [20, 36] for the explicit formulas. Also, it was shown in [21], assuming  $\Omega$  has a  $C^2$  boundary, that

$$\gamma^2 = \frac{\mathbb{N}_{11}^{(2)}(\Omega, \lambda)}{8\pi\lambda}, \quad a_0 = \frac{\mathbb{N}_{12}^{(2)}(\Omega, \lambda)}{2\mathbb{N}_{11}^{(2)}(\Omega, \lambda)},$$

$$a_m = \frac{1}{4\pi m} \sum_{n=1}^m p_{mn} \mathbb{N}_{n1}^{(1)}(\Omega, \lambda) \quad \text{for } m \geq 1,$$

where  $p_{mn}$  are the expansion coefficients of the Faber polynomials into  $z^n$ , that is,  $F_m(z) = \sum_{n=0}^m p_{mn} z^n$ . The coefficients  $p_{mn}$  depend only on  $\{a_j\}_{0 \leq j \leq m-1}$ . As a result, one can recover the shape of the inclusion by knowing the values of the generalised PTs; see [21] for further details.

As a corollary of Theorem 3.1, we find that

$$\text{tr}(M^{-1}(\Omega, \pm 1/2)) = \pm \frac{1}{\pi} \frac{\gamma^2}{\gamma^4 - |a_1|^2}. \tag{3.12}$$

As the volume of  $\Omega$  satisfies

$$|\Omega| = \pi \left( \gamma^2 - \sum_{n \geq 1} n \frac{|a_n|^2}{\gamma^{2n}} \right), \tag{3.13}$$

equality holds in (3.3) if and only if  $a_n = 0$  for all  $n \geq 2$  (see [22]). This is an alternative proof of the Pólya–Szegő conjecture for a planar inclusion with extreme conductivity. Also note that the eigenvalues of  $M(\Omega, \pm \frac{1}{2})$ , namely  $\tau_1$  and  $\tau_2$ , are

$$\tau_1 = \pm 2\pi (\gamma^2 + |a_1|), \quad \tau_2 = \pm 2\pi (\gamma^2 - |a_1|). \tag{3.14}$$

Let us now analyse the trace of the PT for  $k \rightarrow 0$  or  $\infty$ . As a direct consequence of Theorem 3.1, it also holds for  $k = 0, \infty$  (i.e.,  $\lambda = \pm \frac{1}{2}$ ) that  $\text{tr}(M) = \pm 4\pi\gamma^2$ . Furthermore, we validate that this relation approximately holds also for the near-extreme conductivity case:

**Theorem 3.2** *Let  $\Omega$  be a simply connected bounded Lipschitz domain with the constant conductivity  $k$ . Let  $\gamma$  be the conformal radius of  $\Omega$ . For  $\lambda \approx \pm \frac{1}{2}$ , it holds that*

$$\text{tr}(M(\Omega, \lambda)) = \pm 4\pi\gamma^2 + O(|\lambda| - \frac{1}{2}). \tag{3.15}$$

**Proof** Let  $|\lambda_k| \geq \frac{1}{2}$  ( $k = 1, 2$ ) and define

$$\varphi_{\lambda_k} := (\lambda_k I - \mathcal{K}_{\partial\Omega}^*)^{-1} \left[ \frac{\partial x_i}{\partial \nu} \right].$$

Then, it holds from the spectral resolution of the NP operator (see the proof of Theorem 3.3 in [35]) that

$$\|\varphi_{\lambda_1} - \varphi_{\lambda_2}\|_{H^{-1/2}(\partial\Omega)} \leq C |\lambda_1 - \lambda_2|$$

for some constant independent of  $\lambda_1, \lambda_2$ . Since  $x_i \in H^{1/2}(\partial\Omega)$ , from the definition (2.3), this proves the theorem. □

### 3.2 Properties of the trace of the PT

One can estimate  $\gamma$  in terms of  $\text{diam}(\Omega)$  thanks to previous literature (e.g., [55]). We give the proof at the end of this subsection for the reader’s convenience.

**Lemma 3.3** *The conformal radius  $\gamma$  satisfies that  $\frac{1}{4} \text{diam}(\Omega) \leq \gamma \leq \text{diam}(\Omega)$ .*

From Theorem 3.2 and Lemma 3.3, we conclude that a large value of  $\text{diam}(\Omega)$  corresponds to a large value of  $|\text{tr}(M)|$ , as stated in the theorem below.

**Theorem 3.4** *For  $\lambda = \pm\frac{1}{2}$ , it holds for  $M = M(\Omega, \pm\frac{1}{2})$  that*

$$\frac{\pi}{4}(\text{diam}(\Omega))^2 \leq |\text{tr}(M)| \leq 4\pi(\text{diam}(\Omega))^2.$$

Theorem 3.4 encapsulates the intuition that domains which are close to the upper Hashin–Shtrikman bound are ‘thin’ and *vice versa*. The following proposition reinforces this idea by illustrating that ‘fat’ domains are close to the lower Hashin–Shtrikman bound for a planar domain with extreme conductivity, (3.3).

**Proposition 3.5** (Shape change towards a disc) *Let  $\Omega$  be a simply connected bounded Lipschitz domain with  $k = 0, \infty$ . We denote by  $\Omega_r$  the family of domains enclosed by the curve  $\{z = \Psi(w) : |w| = r\}$ ,  $r \geq \gamma$ .*

*Then, as the shape of the domain changes towards a disc, the trace of the PT corresponding to  $\Omega_r$  converges to the lower Hashin–Shtrikman bound:*

$$|\Omega_r| \left| \text{tr} \left( M^{-1}(\Omega_r, \pm 1/2) \right) \right| \rightarrow 1 \quad \text{as } r \rightarrow \infty. \tag{3.16}$$

*On the other hand, the difference between the two eigenvalues of  $M(\Omega_r, \pm 1/2)$  is constant independent of  $r$ :*

$$\left| \tau_1(\Omega_r) - \tau_2(\Omega_r) \right| = 4\pi |a_1| \quad \text{for all } r \geq \gamma. \tag{3.17}$$

**Proof** From (3.12), (3.13) and (3.14), one can easily find (3.16) and (3.17). □

While the family of domains  $\Omega_r$  has the PT gradually changes as  $r \rightarrow \infty$ , one can modify the shape of any domain to have a large diameter by stretching only a small portion of the domain; see, for example, Figure 1. Hence, we observe the following as a direct consequence of Theorems 3.2 and 3.4:

**Corollary 3.6** (Instability with respect to the shape perturbation) *Let  $\lambda \approx \pm\frac{1}{2}$ . For any simply connected, bounded Lipschitz domain  $\Omega$  and  $N \in \mathbb{N}$ , there exists a shape perturbation of  $\Omega$ , namely  $\tilde{\Omega}$ , such that*

$$\left| (\Omega \setminus \tilde{\Omega}) \cup (\tilde{\Omega} \setminus \Omega) \right| < \frac{1}{N} \quad \text{and} \quad \left| \text{tr}(M(\Omega, \lambda)) - \text{tr}(M(\tilde{\Omega}, \lambda)) \right| > N.$$

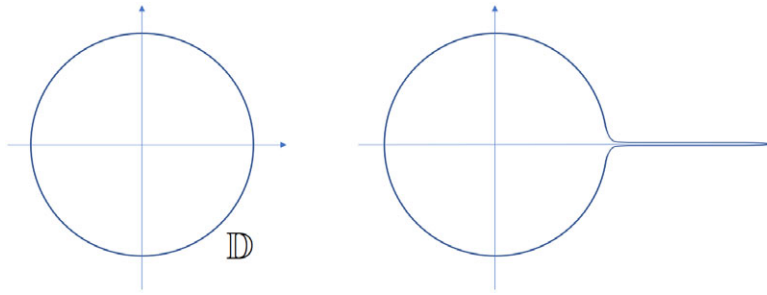


FIGURE 1. A disc (left figure) and its shape perturbation (right figure). While the perturbation is small in area sense, there is a big difference in the diameters of the two domains.

**Proof of Lemma 3.3** Set  $R := \text{diam}(\Omega) = \sup_{x,y \in \Omega} |x - y|$ . Fix a point  $z_0 \in \Omega$ . The map  $f : \mathbb{D} \rightarrow \mathbb{C}$  given by

$$f(z) := \frac{1}{\Psi\left(\frac{z}{R}\right) - z_0} = \frac{z}{\gamma} \left( 1 - \frac{a_0 - z_0}{\gamma} z + \text{higher-order terms} \right)$$

with  $f(0) = 0$  is a conformal map from the unit disc onto the range of  $f$  satisfying  $f'(0) = \frac{1}{\gamma}$ . From the Bieberbach’s conjecture for the second coefficient, we have  $|a_0 - z_0| \leq 2\gamma$ . Since  $z_0 \in \Omega$  is arbitrary, we conclude that  $R \leq 4\gamma$ .

Note that  $\left\{ \Psi\left(\frac{z}{R}\right) - z_0 : |z| < 1 \right\} = \mathbb{C} \setminus (\overline{\Omega} - z_0) \supset \{z : |z| > R\}$ . We then observe that the map  $z \mapsto f^{-1}\left(\frac{z}{R}\right)$  is a complex analytic function from the unit disc to the unit disc and sends 0 to 0. From the Schwarz lemma, we have  $|f'(0)R|^{-1} \leq 1$ , that is,  $\gamma \leq R$ .  $\square$

### 4 Effective conductivity of dilute composites

In this section, we determine the effective or macroscopic property of a two-phase medium using Theorem 3.1.

#### 4.1 Asymptotic formula for the effective conductivity

Let  $Y = ]-\frac{1}{2}, \frac{1}{2}[^2$  denote the unit cell in  $\mathbb{R}^2$  and  $D = \rho B$ ,  $0 < \rho < 1$ , satisfying  $\overline{D} \subset \subset Y$ , where  $B$  is a reference bounded Lipschitz domain containing the origin with  $|B| = 1$ . Let the conductivity  $\sigma$  be periodic with the periodic cell  $Y$  and, on  $Y$ ,

$$\sigma = 1 + (k - 1)\chi(D),$$

that is,  $Y \setminus \overline{D}$  and  $D$  have the constant conductivities 1 and  $k$  ( $0 < k \neq 1 < +\infty$ ), respectively. The effective conductivity, namely  $\sigma^* = (\sigma_{ij}^*)_{i,j=1,2}$ , of the periodic conductivity  $\sigma\left(\frac{x}{\varepsilon}\right)$  as  $\varepsilon \rightarrow 0$  is then defined to be (see, e.g., [31])

$$\sigma_{ij}^* = \int_Y \sigma \nabla u_i \cdot \nabla u_j \, dx, \tag{4.1}$$

where  $u_i$  ( $i = 1, 2$ ) is the unique solution to

$$\begin{cases} \nabla \cdot \sigma \nabla u_i = 0 & \text{in } Y, \\ u_i - x_i \text{ is periodic,} \\ \int_Y u_i \, dx = 0. \end{cases} \tag{4.2}$$

Using the divergence theorem, one can rewrite  $\sigma_{ij}^*$  as:

$$\sigma_{ij}^* = \int_{\partial Y} u_j \frac{\partial u_i}{\partial \nu} \, dS(x). \tag{4.3}$$

For the case  $k = 0$  or  $\infty$ , we impose the zero Neumann boundary condition or the constant Dirichlet boundary condition on  $\partial D$  and define the effective conductivity as (4.3).

When  $B$  is a ball, the corresponding effective conductivity is given by the Maxwell–Garnett formula [54]:

$$\sigma^* = 1 + \rho^2 \frac{2(k-1)}{k+1} + 2\rho^4 \frac{(k-1)^2}{(k+1)^2} + o(\rho^4).$$

This formula has been extended to inclusions of general shape with Lipschitz boundaries [10, Theorem 5.1]:

$$\sigma^* = \left( I + \rho^2 M + \frac{\rho^4}{2} M^2 \right) + O(\rho^6), \tag{4.4}$$

where  $M$  is the Pólya–Szegő PT corresponding to  $B$  and conductivity  $k$  with the background conductivity 1. The conductivity is assumed to be  $0 < k \neq 1 < \infty$  in [10], and the derivation in the paper can be applied to the extreme conductivity case as well.

By applying Theorem 3.1 to (4.4), we have the following.

**Theorem 4.1** *Let  $D = \rho\Omega$ ,  $0 \in \Omega$ , with the conductivity  $k = 0, \infty$  (i.e.,  $\lambda = -\frac{1}{2}, \frac{1}{2}$ ). Then,  $\sigma^* = \sigma^*(D, \pm 1/2)$  admits the asymptotic expansion for  $\rho \ll 1$ :*

$$\begin{aligned} \sigma^*(D, \pm 1/2) = & I + 2\pi\rho^2 \begin{bmatrix} \pm\gamma^2 + \operatorname{Re}(a_1) & \operatorname{Im}(a_1) \\ \operatorname{Im}(a_1) & \pm\gamma^2 - \operatorname{Re}(a_1) \end{bmatrix} \\ & + 2\pi^2\rho^4 \begin{bmatrix} \gamma^4 \pm 2\gamma^2 \operatorname{Re}(a_1) + |a_1|^2 & \pm 2\gamma^2 \operatorname{Im}(a_1) \\ \pm 2\gamma^2 \operatorname{Im}(a_1) & \gamma^4 \mp 2\gamma^2 \operatorname{Re}(a_1) + |a_1|^2 \end{bmatrix} + O(\rho^6 |\Omega|^3). \end{aligned} \tag{4.5}$$

**Proof** The scaled domain  $\tilde{\Omega} := \frac{\Omega}{|\Omega|^{\frac{1}{2}}}$  (hence,  $|\tilde{\Omega}| = 1$ ) has the exterior conformal mapping:

$$\tilde{\Psi}(w) = w + \frac{a_0}{|\Omega|^{\frac{1}{2}}} + \sum_{n=1}^{\infty} \frac{a_n}{|\Omega|^{\frac{n+1}{2}}} \frac{1}{w^n} \quad \text{for } |w| > \tilde{\gamma} := \frac{\gamma}{|\Omega|^{\frac{1}{2}}}.$$

The PT of  $\tilde{\Omega}$  is then given by:

$$\tilde{M} := M \left( \tilde{\Omega}, \pm \frac{1}{2} \right) = \frac{2\pi}{|\Omega|} \begin{bmatrix} \pm\gamma^2 + \operatorname{Re}(a_1) & \operatorname{Im}(a_1) \\ \operatorname{Im}(a_1) & \pm\gamma^2 - \operatorname{Re}(a_1) \end{bmatrix}. \tag{4.6}$$

From (4.4) with  $\rho$  replaced by  $|\Omega|^{\frac{1}{2}} \rho$ , this completes the proof. □

Let us denote the lower-order terms of  $\sigma^* - I$  as  $A = A(\rho, \gamma, a_1, \lambda)$  so that

$$\sigma^* = I + A + O(\rho^6 |\Omega|^3). \tag{4.7}$$

The matrix  $A$  has the trace and the determinant as follows:

$$\begin{aligned} \operatorname{tr}(A) &= \pm 4\pi \gamma^2 \rho^2 + 4\pi^2 (\gamma^4 + |a_1|^2) \rho^4, \\ \det(A) &= 4\pi^2 \rho^4 (\gamma^4 - |a_1|^2) \pm 8\pi^3 \gamma^2 \rho^6 (\gamma^4 - |a_1|^2) + 4\pi^4 \rho^8 (\gamma^4 - |a_1|^2)^2. \end{aligned}$$

In view of (2.6), in which the strict inequality holds for  $|\Omega| > 0$ ,  $A$  is invertible when  $\rho$  is sufficiently small. Recall that  $|\Omega|$  is given by  $|\Omega| = \pi \left( \gamma^2 - \sum_{n \geq 1} n \frac{|a_n|^2}{\gamma^{2n}} \right)$ . Assuming further that  $k = \infty$ , we have

$$\begin{aligned} \operatorname{tr}(A^{-1}) &= \frac{1 + 2\pi \gamma^2 \rho^2 - \rho^2 X}{\rho^2 X + 2\pi \gamma^2 \rho^4 X + \pi \rho^6 \gamma^2 X^2}, \quad X = \pi \left( \gamma^2 - \frac{|a_1|^2}{\gamma^2} \right) \\ &\leq \frac{1 + 2\pi \gamma^2 \rho^2 - \rho^2 |\Omega|}{\rho^2 |\Omega| + 2\pi \gamma^2 \rho^4 |\Omega| + \pi \rho^6 \gamma^2 |\Omega|^2}, \end{aligned}$$

where the equality holds if and only if  $a_n = 0$  for all  $n \geq 2$ . In other words, assuming that  $\gamma, \rho$  and  $|\Omega|$  are fixed,  $\operatorname{tr}(A^{-1})$  attains the maximum when  $\Omega$  is an ellipse. This can be considered an extension of the Polya–Szegő conjecture to the higher-order expansion of the effective conductivity.

### 4.2 Examples

We provide the asymptotic expansions of the effective conductivity for some simple polygons whose exterior conformal mapping can be easily expressed as the Schwarz–Christoffel integral.

#### 4.2.1 A rectangle

Consider a rectangle  $\Omega$  with width  $\sqrt{3}$  and height  $\frac{1}{\sqrt{3}}$  centred at the origin, where the vertices are given by  $\frac{\sqrt{3}}{2} \pm \frac{1}{2\sqrt{3}}i$  and  $-\frac{\sqrt{3}}{2} \pm \frac{1}{2\sqrt{3}}i$ . By numerically approximating the corresponding Schwarz–Christoffel mapping using the MATLAB SC Toolbox, we have  $a_1 \simeq 0.20439$  and  $\gamma \simeq 0.66273$ . This leads to the asymptotic expansion of the effective conductivity for  $D = \rho\Omega$ :

$$\begin{aligned} \sigma^*(D, 1/2) &= I + \begin{bmatrix} 4.0438 & 0 \\ 0 & 1.4754 \end{bmatrix} \rho^2 + \begin{bmatrix} 8.1763 & 0 \\ 0 & 1.0885 \end{bmatrix} \rho^4 + O(\rho^6), \\ \sigma^*(D, -1/2) &= I + \begin{bmatrix} -1.4754 & 0 \\ 0 & -4.0438 \end{bmatrix} \rho^2 + \begin{bmatrix} 1.0885 & 0 \\ 0 & 8.1763 \end{bmatrix} \rho^4 + O(\rho^6). \end{aligned}$$

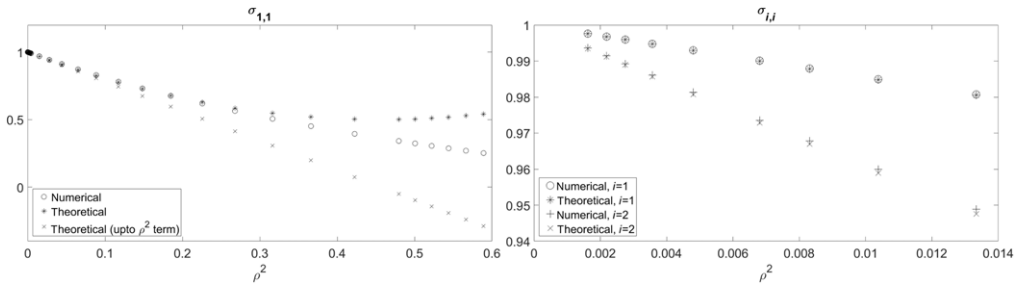


FIGURE 2. Numerical approximations of  $\sigma_{i,i}$ ,  $i = 1, 2$ , for a rectangle width  $\sqrt{3}$  and height  $\frac{1}{\sqrt{3}}$  (right figure) and a regular 4-gon (left figure). The edges are set parallel to the axes. In the left figure,  $\sigma_{1,1} = \sigma_{2,2}$  by symmetry.

In Figure 2, we illustrate the results on a rectangle and a regular 4-gon. We compare the leading term in the asymptotic expansion in (4.5) with the effective conductivity obtained by evaluating (4.3), for which we numerically solve the PDE problem (4.2) by a finite difference method. The conductivity is set to be  $k = 0$ . The left figure illustrates that the  $\rho^4$  term, although negligible for  $\rho \simeq 0$ , is indeed significant for mid-range  $\rho$  values.

4.2.2 Regular  $n$ -gons

**Lemma 4.2** For  $\Omega$  with rotation symmetry of angle  $\theta$  satisfying  $e^{2i\theta} \neq 1$ , we have  $a_1 = 0$  so that  $I + A$  is diagonal (see (4.7)).

**Proof** We may assume the rotational centre of the domain is at the origin. For such a domain  $\Omega$  and its associated exterior conformal mapping  $\Psi[\Omega]$ ,  $e^{i\theta} \Psi[\Omega](e^{-i\theta} z)$  is also an exterior conformal map. In particular, this map equals  $\Psi[\Omega](z)$  and it holds that  $a_1 e^{2i\theta} = a_1$ . This implies  $a_1 = 0$ . □

Lemma 4.2 holds in particular for domains with rotational symmetry of order greater than 2. Let  $\Omega$  be a regular  $n$ -gon domain with side length  $L$ , then  $a_1 = 0$  from Lemma 4.2. From [51] and [52, Corollary 11.1], we have

$$\gamma = \frac{1}{4\pi} \frac{\Gamma^2\left(\frac{1}{n}\right)}{\Gamma\left(\frac{2}{n}\right)} L, \quad \text{where } L = \sqrt{\frac{4 \tan\left(\frac{\pi}{n}\right)}{n}} \text{ assuming } |\Omega| = 1. \tag{4.8}$$

**Lemma 4.3** The conformal radius  $\gamma(n)$  of a regular  $n$ -gon with unit area, given by (4.8), strictly decreases to  $\frac{1}{\sqrt{\pi}}$  as  $n$  increases to  $\infty$ . Note that  $\frac{1}{\sqrt{\pi}}$  is the conformal radius of a disc with unit area.

**Theorem 4.4** Let  $\Omega$  be a regular  $n$ -gon with unit area centred at the origin with the conductivity  $k = 0, \infty$  ( $n \geq 3$ ). Then, the corresponding effective conductivity for  $D = \rho\Omega$  is

$$\sigma^* = \left( 1 \pm \frac{1}{2\pi} \frac{\Gamma^4\left(\frac{1}{n}\right) \tan\left(\frac{\pi}{n}\right)}{\Gamma^2\left(\frac{2}{n}\right) n} \rho^2 + \frac{1}{8\pi^2} \frac{\Gamma^8\left(\frac{1}{n}\right) \tan^2\left(\frac{\pi}{n}\right)}{\Gamma^4\left(\frac{2}{n}\right) n^2} \rho^4 \right) I + O(\rho^6), \tag{4.9}$$

where the symbols + and – correspond to  $k = \infty$  and  $k = 0$ , respectively. The asymptotic of  $\sigma^*$  up to  $\rho^4$ -term converges to that corresponding to the unit disc as  $n \rightarrow \infty$  in a strictly decreasing manner for  $k = \infty$ , and a strictly increasing manner for  $k = 0$  with sufficiently small  $\rho$ .

**Proof** From Theorem 4.1 and (4.8), we observe (4.9). Monotonicity properties follow from Lemma 4.3. □

For example, for the cases  $n = 3$  and  $n = 4$  we, respectively, have

$$\sigma^* = \left( 1 \pm \frac{3\sqrt{3} \Gamma^6(1/3)}{8\pi^3} \rho^2 + \frac{27 \Gamma^{12}(1/3)}{128\pi^6} \rho^4 \right) I + O(\rho^6),$$

$$\sigma^* = \left( 1 \pm \frac{\Gamma^4(1/4)}{8\pi^2} \rho^2 + \frac{\Gamma^8(1/4)}{128\pi^4} \rho^4 \right) I + O(\rho^6).$$

**Proof of Lemma 4.3** The digamma function admits the series expansion  $\frac{\Gamma'(z)}{\Gamma(z)} = -C - \frac{1}{z} + \sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+z} \right)$ ,  $z \neq 0, -1, -2, \dots$ , with the Euler–Mascheroni constant  $C$  (see [1, formulas 6.3.1, 5 and 16]). In particular, it holds for  $x > 2$  that

$$\gamma'(x) = \frac{1}{2\pi} \frac{\Gamma^2(\frac{1}{x})}{\Gamma(\frac{2}{x})} \frac{\sqrt{\tan(\frac{\pi}{x})}}{x^{\frac{3}{2}}} \left( \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{(nx+1)(nx+2)} - \frac{\pi}{x \sin(\frac{2\pi}{x})} \right),$$

where  $\gamma(x)$  is the function given by (4.8) with  $n$  replaced by a real variable  $x$ . We claim

$$\gamma'(x) < 0 \quad \text{for } x > 2. \tag{4.10}$$

Because of  $\sum_{n=1}^{\infty} \frac{2}{(nx+1)(nx+2)} \leq \sum_{n=1}^{\infty} \frac{2}{n^2x^2} = \frac{\pi^2}{3} \frac{1}{x^2}$ , we only need to show  $\frac{1}{2} + \frac{\pi^2}{3x^2} - \frac{\pi}{x \sin(\frac{2\pi}{x})} < 0$ . Note that

$$f(s) := (\pi^2 s^2 + 1) \sin(2\pi s) - 2\pi s \cos(2\pi s) > 0,$$

$$\frac{d}{ds} \left[ 6s^2 \frac{d}{ds} \left( \left( \frac{1}{2s} + \frac{\pi^2 s}{3} \right) \sin(2\pi s) \right) \right] = -8\pi^2 s f(s) < 0 \quad \text{for } s \in \left( 0, \frac{1}{2} \right).$$

Using this, one can easily find that  $\left( \frac{1}{2s} + \frac{\pi^2 s}{3} \right) \sin(2\pi s) < \pi$  for  $s \in (0, \frac{1}{2})$ , which is equivalent to  $\frac{1}{2} + \frac{\pi^2}{3x^2} - \frac{\pi}{x \sin(\frac{2\pi}{x})} < 0$  for  $x > 2$ . This proves (4.10). Hence, we conclude that  $\gamma(n)$  is strictly decreasing w.r.t.  $n \geq 3$ . Furthermore, as  $\Gamma(s) = \frac{1}{s} + \text{h.o.t.}$  near  $s = 0$ ,  $\gamma(n)$  converges to  $\frac{1}{\sqrt{\pi}}$  as  $n \rightarrow \infty$ . □

### 5 Inequality relations between the coefficients of the Riemann mapping

The following relations on the PT are well known:

**Lemma 5.1** ([6]) *Suppose that  $B$  is a bounded Lipschitz domain and  $0 \leq k \neq 1 \leq \infty$ . Let  $\lambda = \frac{k+1}{2(k-1)}$ . Assume that  $I$  is a finite set of multi-indices and  $\{a_\alpha\}_{\alpha \in I}$  are constants such that the function  $H(x) = \sum_{\alpha \in I} a_\alpha x^\alpha$  is harmonic. Then, the following inequality holds*



$$\left(1 - \frac{1}{k}\right) \int_B |\nabla H|^2 \leq \sum_{i,j \in I} a_i a_j M_{ij}(B, \lambda) \leq (k - 1) \int_B |\nabla H|^2.$$

**Lemma 5.2** ([8]) *Assume the same conditions for  $k, \lambda, I,$  and  $\{a_\alpha\}_{\alpha \in I}$  as in Lemma 5.1. Suppose that  $B$  and  $B'$  are two bounded Lipschitz domains satisfying  $B' \subseteq B$ . Then, the following inequalities hold*

$$\begin{aligned} \sum_{i,j \in I} a_i a_j M_{ij}(B, \lambda) &\geq \sum_{i,j \in I} a_i a_j M_{ij}(B', \lambda) \text{ if } k > 1, \\ \sum_{i,j \in I} a_i a_j M_{ij}(B, \lambda) &\leq \sum_{i,j \in I} a_i a_j M_{ij}(B', \lambda) \text{ if } k < 1. \end{aligned} \tag{5.1}$$

As Theorem 3.1 relates the PT for a domain with its exterior conformal mapping, an estimate of the PT  $M$  can be translated into an estimate of the coefficients of the Riemann mappings via this relation. In particular, we can obtain inequality relations of the coefficients of the interior conformal mapping using Lemmas 5.1 and 5.2.

In view of the fact that Theorem 3.1 relates the PT to the exterior map coefficients, it may be more natural to state the inequality relations directly in terms of exterior conformal maps. However, in univalent function theory, it is more common to consider functions defined on the unit disc  $\mathbb{D}$  so that, in the rest of this section, we derive inequality relations of the coefficients of the (interior) Riemann mappings.

Consider a Jordan domain (domain enclosed by a Jordan curve), namely  $\tilde{\Omega}$ , containing the origin. By the Riemann mapping theorem, there exists a unique (interior) conformal map  $\Phi[\tilde{\Omega}]$  from the unit disc onto  $\tilde{\Omega}$  with  $\Phi(0) = 0$  and  $\Phi'(0) > 0$ , which admits the Taylor series expansion:

$$\Phi[\tilde{\Omega}](z) = b_1 z + b_2 z^2 + \dots \tag{5.2}$$

for some complex coefficients  $b_n, n \geq 2$  and  $b_1 > 0$ . The reflection of the exterior of  $\tilde{\Omega}$  with respect to the unit circle centred at the origin defines a Jordan domain:

$$\Omega = \left\{ \frac{1}{z} : z \in \mathbb{C} \setminus \tilde{\Omega} \right\} \cup \{0\}.$$

The exterior conformal mapping  $\Psi$  associated with  $\Omega$  satisfies  $\gamma = \frac{1}{b_1}$  and is expanded as the following Laurent series:

$$\Psi[\Omega](z) = \frac{1}{\Phi[\tilde{\Omega}]\left(\frac{1}{b_1 z}\right)} = z - \frac{b_2}{b_1^2} + \frac{1}{b_1^3} \left( \frac{b_2^2}{b_1} - b_3 \right) \frac{1}{z} + \sum_{n \geq 2} \frac{a_n}{z^n} \tag{5.3}$$

for some complex coefficients  $a_n, n \geq 2$ . Assuming that  $\Omega$  has a Lipschitz boundary (see [29, Theorem 4.1]), we have from Theorem 3.1 that

$$M\left(\Omega, \pm \frac{1}{2}\right) = 2\pi \begin{bmatrix} \pm \frac{1}{b_1^2} + \frac{1}{b_1^3} \operatorname{Re}\left(\frac{b_2^2}{b_1} - b_3\right) & \frac{1}{b_1^3} \operatorname{Im}\left(\frac{b_2^2}{b_1} - b_3\right) \\ \frac{1}{b_1^3} \operatorname{Im}\left(\frac{b_2^2}{b_1} - b_3\right) & \pm \frac{1}{b_1^2} - \frac{1}{b_1^3} \operatorname{Re}\left(\frac{b_2^2}{b_1} - b_3\right) \end{bmatrix}. \tag{5.4}$$

Now, we derive the following theorem by translating Lemmas 5.1 and 5.2 into the coefficients of the interior Riemann mapping.

**Theorem 5.3** *Suppose that  $\tilde{\Omega}$  is a simply connected, bounded Lipschitz domain, containing 0. We denote its interior conformal mapping by  $\Phi[\tilde{\Omega}](z) = b_1z + b_2z^2 + \dots$ . Then, it holds that*

$$\frac{1}{b_1^2} \pm \frac{1}{b_1^3} \operatorname{Re} \left( \frac{b_2^2}{b_1} - b_3 \right) \geq \frac{|\Omega|}{2\pi}.$$

In particular, if  $|\Omega| = 1$ , then we have  $\frac{1}{b_1^2} \pm \frac{1}{b_1^3} \operatorname{Re} \left( \frac{b_2^2}{b_1} - b_3 \right) \geq \frac{1}{2\pi}$ .

Further, suppose that  $\tilde{V}$  is another domain satisfying  $\tilde{\Omega} \subseteq \tilde{V}$  (so that  $V \subseteq \Omega$ ). We denote its interior conformal mapping by  $\Phi[\tilde{V}](z) = c_1z + c_2z^2 + \dots$ . Then, the following monotonicity inequality holds

$$\frac{1}{b_1^2} \pm \frac{1}{b_1^3} \operatorname{Re} \left( \frac{b_2^2}{b_1} - b_3 \right) \geq \frac{1}{c_1^2} \pm \frac{1}{c_1^3} \operatorname{Re} \left( \frac{c_2^2}{c_1} - c_3 \right).$$

**Proof** From Lemmas 5.1 and 5.2 with  $k = 0, \infty$  and  $H(x) = c_1x_1 + c_2x_2$  for arbitrary constants  $c_1, c_2$ , we have

$$\begin{aligned} |\Omega|I_{2 \times 2} &\leq M \left( \Omega, +\frac{1}{2} \right), \\ -|\Omega|I_{2 \times 2} &\geq M \left( \Omega, -\frac{1}{2} \right), \\ M(V, +\frac{1}{2}) &\leq M \left( \Omega, +\frac{1}{2} \right), \\ M(V, -\frac{1}{2}) &\geq M \left( \Omega, -\frac{1}{2} \right), \end{aligned} \tag{5.5}$$

where we write  $A \leq B$  for real symmetric matrices  $A$  and  $B$  such that  $B - A$  is positive semi-definite. Combined with (5.4), the first or the second inequality in (5.5) proves the first assertion, and the third or the fourth inequality in (5.5) proves the next assertion. □

Recall that  $\tilde{\Omega} = \Phi[\tilde{\Omega}](\mathbb{D})$  and  $\Phi[\tilde{\Omega}](z) = b_1z + \text{higher-order terms}$  ( $b_1 > 0$ ), where  $\mathbb{D}$  is the unit disc centred at the origin. Therefore, it is quite reasonable to expect that large  $b_1$  implies large  $|\tilde{\Omega}|$  (so that small  $|\Omega|$ ). We may further establish more properties on the conformal mapping coefficients by exploiting our vast knowledge of the generalized polarization tensors (GPTs), for example, we may apply Lemma 5.2 with higher order  $H(x)$ .

It is worth remarking that inequalities between coefficients of the Riemann mappings are of interest in univalent function theory [23, 42, 44, 52, 56].

**Remark 5.1** *Theorem 5.3 can be extended to arbitrary Jordan domains (no smoothness assumptions on the boundary curves)  $\tilde{\Omega}$  and  $\tilde{V}$  with  $0 \in \tilde{\Omega} \subset \subset \tilde{V}$ . This is because given a Jordan domain  $0 \in \tilde{U}$ , we can approximate  $\tilde{U}$  by domains enclosed by  $\Phi[\tilde{U}](\{z : |z| = r\})$  ( $0 < r < 1$ ).*

### 6 Spectral monotonicity of the NP operator with respect to the shape deformation

In this section, we numerically compute eigenvalues of the NP operator for various Fourier modes in the shape of the inclusion. We find that the eigenvalues have a monotonic behaviour as the shape of the inclusion linearly deviates from a disc.

#### 6.1 Finite section method

For a  $C^{1,\alpha}$  domain  $\Omega$ , the NP operator on  $H_0^{-1/2}(\partial\Omega)$  is compact so that it admits only a sequence of real eigenvalues as its spectrum, and zero is the only possible accumulation point of the eigenvalues; see [39, 43] for the permanence of the spectrum for the NP operator with different norms. In two dimensions, the spectrum is symmetric with respect to zero [16]. The positive eigenvalues are enumerated in the following way, counting algebraic multiplicities:

$$0.5 > \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 \geq \dots \tag{6.1}$$

The  $l^2(\mathbb{C})$  space generated by the basis  $\{\zeta_{\pm m}\}_{m \in \mathbb{N}}$  corresponds to the Sobolev space  $H_0^{-1/2}(\partial\Omega)$  [34]. For each  $n \in \mathbb{N}$ , we set  $H_n = \text{span}\{\zeta_{-n}, \zeta_{-n+1}, \dots, \zeta_{-1}, \zeta_1, \dots, \zeta_{n-1}, \zeta_n\}$  and define  $P_n$  as the orthogonal projection to  $H_n$ . We then denote by  $[\mathcal{K}_{\partial\Omega}^*]_n$  the  $n$ -th section of  $[\mathcal{K}_{\partial\Omega}^*]$ , that is the  $2n \times 2n$  matrix corresponding to  $P_n \mathcal{K}_{\partial\Omega}^* P_n$ . For example, we have

$$[\mathcal{K}_{\partial\Omega}^*]_1 = \begin{bmatrix} 0 & \mu_{1,1} \\ \overline{\mu_{1,1}} & 0 \end{bmatrix}.$$

For a bounded self-adjoint operator on a separable complex (infinite-dimensional) Hilbert space, the spectrum outside the convex hull of the essential spectrum is a set of eigenvalues with finite algebraic multiplicity and can be approximated by eigenvalues of the finite section matrices [17, Theorem 3.1]. Combining this result with the fact that  $\mathcal{K}_{\partial\Omega}^*$  is self-adjoint on  $l^2(\mathbb{C})$  generated by  $\{\zeta_{\pm m}\}$ , we conclude that eigenvalues of the  $n$ -th section  $[\mathcal{K}_{\partial\Omega}^*]_n$  converge to the spectrum of  $\mathcal{K}_{\partial\Omega}^*$ ; see also [34, section 6.2]. Since  $[\mathcal{K}_{\partial\Omega}^*]_n$  is a self-adjoint finite-dimensional matrix, one can easily compute the eigenvalues.

In the following examples, we show positive eigenvalues  $\lambda_k$ ,  $1 \leq k \leq 30$ , of various smooth domains prescribed by their exterior conformal mapping, obtained by evaluating eigenvalues of  $[\mathcal{K}_{\partial\Omega}^*]_n$  with large  $n$ . The eigenvalue calculations were performed by MATLAB R2018a. More precisely, we gradually increase  $n$  from the initial value 100 with increment step size 100, namely  $n = 100 * \text{step}$ , until the following is satisfied:

**Stopping condition.** For all  $1 \leq k \leq 30$  with five consecutive values of  $\text{step}$  parameter, it holds that

$$r_k^{(\text{step})} = \frac{|\lambda_k^{(\text{step})} - \lambda_k^{(\text{step}-1)}|}{|\lambda_k^{(\text{step})}|} < 10^{-5}, \tag{6.2}$$

where  $\lambda_k^{(\text{step})}$  denotes the  $k$ -th eigenvalue of  $[\mathcal{K}_{\partial\Omega}^*]_n$  with  $n = 100 * \text{step}$ . Even if the above-mentioned condition does not hold, we terminate the calculation at  $\text{step} = 16$  to avoid excessive computing time and accept the computed value  $\lambda_k^{(16)}$  for  $k$  satisfying the relative error condition (6.2) for all  $\text{step} = 12, \dots, 16$ .

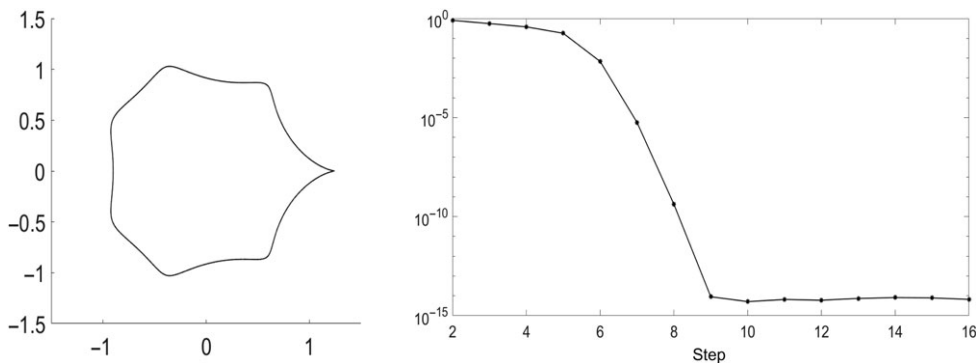


FIGURE 3. The maximum relative error,  $\max_{1 \leq k \leq 30} r_k^{(step)}$ , against  $step = 2, \dots, 16$  in log scale (right figure) for  $\Omega$  given by (6.3) (left figure).

Figure 3 shows the maximum relative error,  $\max_{1 \leq k \leq 30} r_k^{(step)}$ , against  $step = 2, \dots, 16$  for the smooth domain given by its exterior conformal mapping:

$$\Psi(z) = z + 0.01z^{-1} + 0.07z^{-2} + 0.01z^{-3} + 0.03z^{-4} + 0.05z^{-5} + 0.07z^{-6} \tag{6.3}$$

with  $\gamma = 1$ . The graph demonstrates that the relative error,  $r_k^{(step)}$ , decays at least exponentially, and the stopping condition is met when  $step = 11$  (i.e.,  $n = 1100$ ). Numerical experiments show that such ‘spiky’ domains tend to exhibit slower convergence for  $\lambda_k^{(step)}$ . Other domains in the examples below show faster convergence compared to that in Figure 3.

Note that for a smooth domain, the positive eigenvalue  $\lambda_k$  exponentially decays as  $k$  increases ([13, 33]), for example,  $\lambda_k = \pm \frac{1}{2} \left(\frac{a-b}{a+b}\right)^k$  for an ellipse with major axis  $a$  and minor axis  $b$  [7]. Some eigenvalues with large  $k$  tend to be small enough that they do not meet the stopping condition due to round-off errors. It turns out that for such cases in the following examples, the last approximated value  $\lambda_k^{(16)}$  satisfies  $0 < \lambda_k^{(16)} < 2.6161 \times 10^{-13}$ .

### 6.2 Examples

As the following examples will illustrate, a spectral monotonicity makes an appearance whenever the coefficients of  $\Psi$  vary linearly in a parameter. However, the spectral monotonicity does not hold in the general case where the coefficients vary non-linearly (see Remark 6.1).

**Example 6.1** We take  $\Omega = \Omega_{m,s}$  to be the domain given by its exterior conformal mapping:

$$\Psi(z) = z + \frac{s}{m} z^{-m}, \quad m = 1, \dots, 6, \quad s = 0.01, 0.02, \dots, 0.99, \tag{6.4}$$

with  $\gamma = 1$ ;  $s$  is set to be smaller than 1 for  $\Psi$  with  $|z|=1$  to define a Jordan curve. The eigenvalues  $\lambda_k$  of the NP operator are computed by the algorithm described above.

In Figure 4, in each sub-figure corresponding to one of the values of  $m$ , we plot the nine graphs of  $\lambda_k$  associated with the domain  $\Omega_{m,s}$  with  $s = 0.1, 0.2, \dots, 0.9$ . Note that, without any exception, the graphs corresponding to larger  $s$  are strictly above those corresponding to

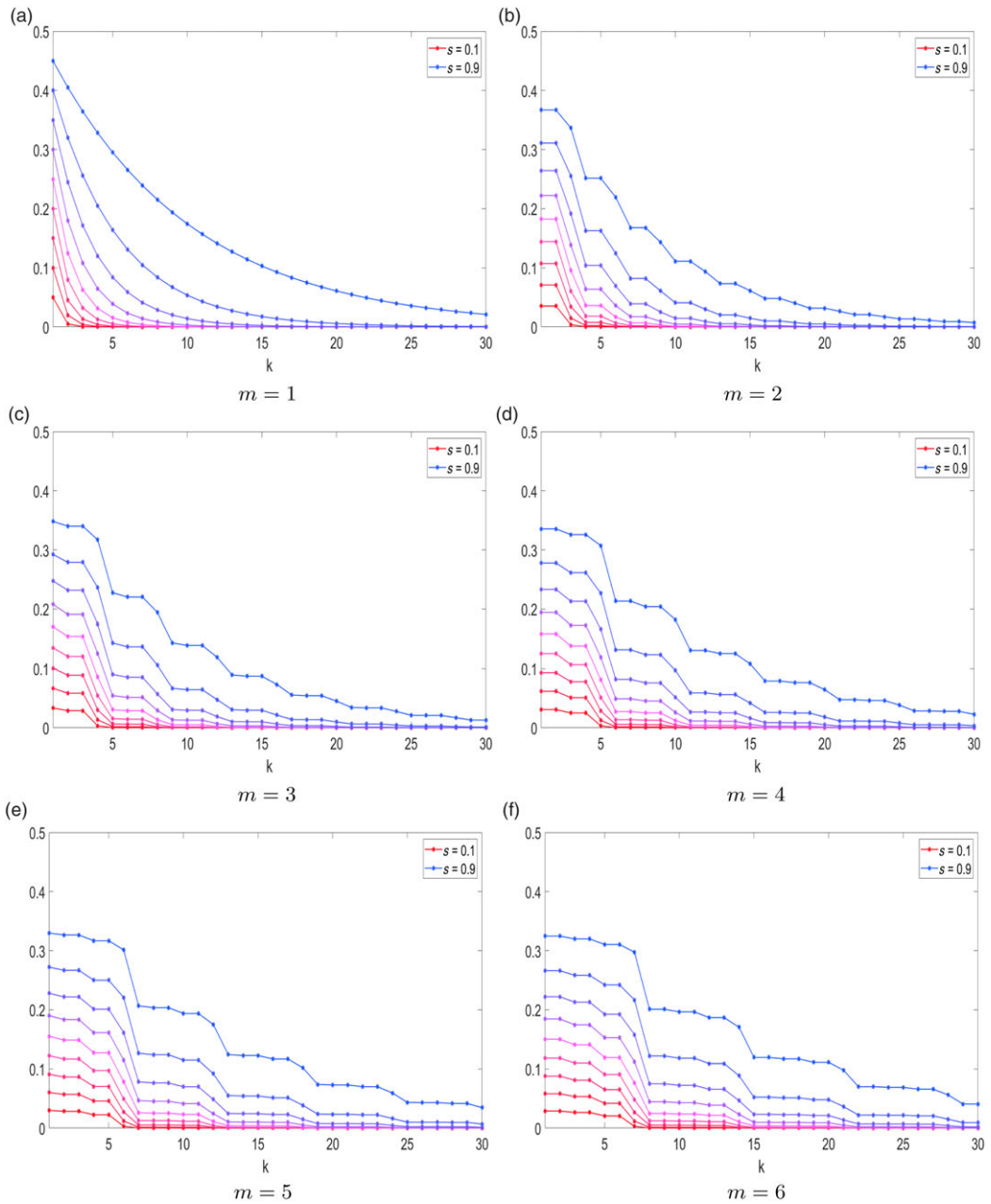


FIGURE 4. In each sub-figure corresponding to one of values of  $m$ , we plot the nine graphs of  $\lambda_k$  associated with the domain  $\Omega_{m,s}$  given by (6.4) with  $s = 0.1, 0.2, \dots, 0.9$ . For each  $k$ , the eigenvalues have a monotonic behaviour as the shape of the inclusion deviates from a disc (i.e., as  $s$  increases).

smaller  $s$ . That is, for each fixed  $m$  and  $k$ , the  $k$ -th eigenvalue corresponding to  $\Omega_{m,s}$  is strictly increasing in  $s$ .

Additionally, eigenvalues of the perturbation of a disc with a higher Fourier mode,  $m$ , tend to cluster; if  $m$  is larger, then the ‘clusters’ get bigger (their size being  $m + 1$ ). This behaviour

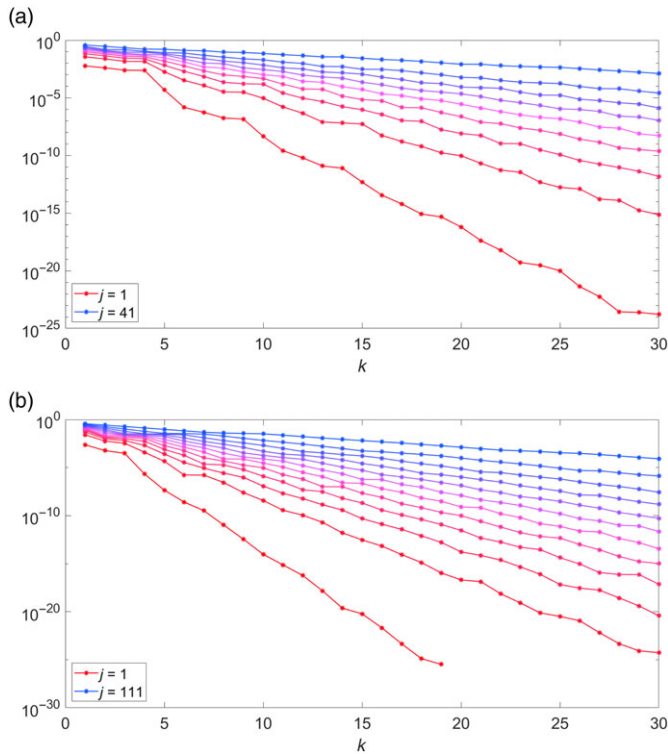


FIGURE 5. For each  $k$ , the eigenvalues have a monotonic behaviour as the shape of the inclusion deviates from a disc (i.e., as  $j$  increases). (a) Eigenvalues corresponding to the domain given by (6.6) in log scale. (b) Eigenvalues corresponding to the domain given by (6.7) in log scale.

accords with the analytic asymptotic results obtained in [11], where the significant eigenvalues of the NP operator of the algebraic domain with  $\Psi(z) = z + \frac{\delta}{z^m}$  ( $\delta > 0$  small) and  $\gamma = 1$  are shown to satisfy

$$\begin{cases} \frac{\delta}{2} \times \left\{ \pm\sqrt{1 \cdot m}, \pm\sqrt{2 \cdot (m-1)}, \dots, \pm\sqrt{k \cdot k} \right\} + O(\delta^2) & \text{if } m = 2k - 1, \\ \frac{\delta}{2} \times \left\{ \pm\sqrt{1 \cdot m}, \pm\sqrt{2 \cdot (m-1)}, \dots, \pm\sqrt{k \cdot (k+1)} \right\} + O(\delta^2) & \text{if } m = 2k. \end{cases} \quad (6.5)$$

**Example 6.2** In this example, we take  $\Omega = \Omega_j$  to be the domain given by its exterior conformal map:

$$\Psi(z) = z + j \left( \frac{1}{600}z^{-1} + \frac{1}{300}z^{-2} + \frac{1}{1200}z^{-3} + \frac{1}{320}z^{-4} \right), \quad 1 \leq j \leq 42. \quad (6.6)$$

For all  $1 \leq j \leq 42$  and  $1 \leq k \leq 30$ , the eigenvalues  $\lambda_k$  were successfully computed by following the stopping criterion (6.2). The obtained value of the  $k$ -th eigenvalue is strictly increasing in  $j$  for each  $k$  without any exception. Figure 5(a) shows the nine graphs of  $\lambda_k$  associated with the domain  $\Omega_j$  with  $j = 1, 6, \dots, 41$ .

**Example 6.3** We now take  $\Omega = \Omega_j$  to be the domain given by its exterior conformal map:

$$\Psi_j(z) = z + j \left( \frac{1}{400}z^{-1} + \frac{1}{600}z^{-2} + \frac{1}{1200}z^{-3} \right), \quad 1 \leq j \leq 119. \quad (6.7)$$

For  $1 \leq j \leq 119$  and  $1 \leq k \leq 30$ , the eigenvalues  $\lambda_k$  were successfully computed by following the stopping criterion (6.2) except some cases, in which  $\lambda_k^{(16)}$  satisfies  $0 < \lambda_k^{(16)} < 2.6161 \times 10^{-13}$ . The obtained value of  $\lambda_k$  is strictly increasing in  $j$ , for all  $k$  without any exception. Figure 5 (b) shows the graphs of  $\lambda_k$  against  $k$  for  $\Omega = \Omega_j$  with  $j = 1, 11, \dots, 111$ .

**Remark 6.1** Consider  $\Omega_j$ , the domain given by its exterior conformal map  $\Psi_j(z) = z + \frac{j^2}{600}z^{-1} + \frac{j}{300}z^{-2}$ . It turns out by the numerical computation that  $\lambda_2 = 4.7155 \cdot 10^{-3}$  for  $j = 5$  and  $\lambda_2 = 4.4402 \cdot 10^{-3}$  for  $j = 6$ . In other words,  $\lambda_2(\Omega_5) > \lambda_2(\Omega_6)$ , which demonstrates that the spectral monotonicity does not hold in the general case where the coefficients non-linearly vary.

## 7 Conclusion

We investigated the geometric dependence of properties of composite materials based on the series expansions of the layer potential operators obtained in [34]. We obtained an explicit relation between the trace of the PT and the exterior conformal mapping corresponding to a planar inclusion with extreme or near-extreme conductivity. We then derived an explicit asymptotic for the effective conductivity of periodic composites. Also, we observed a monotonic behaviour of the spectrum of the NP operator when the shape of the inclusion linearly varies from a disc, by numerically computing eigenvalues of the NP operator for various example shapes of inclusions. Additionally, we derived inequality relations between the coefficients of the Riemann mapping associated with a general simply connected bounded Lipschitz domain by employing the properties of the PT corresponding to the domain with extreme conductivity. It will be an interesting challenge to extend these results to inclusions with arbitrary conductivities.

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## Conflict of interest

None.

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