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MONOTONICITY PROPERTIES OF RESIDUAL LIFETIMES OF PARALLEL SYSTEMS AND INACTIVITY TIMES OF SERIES SYSTEMS WITH HETEROGENEOUS COMPONENTS

WEIYONG DING
School of Mathematics and Statistics
Lanzhou University
Lanzhou 730000, China

XIAOHU LI
School of Mathematical Science
Xiamen University
Xiamen 361005, China
E-mail: mathxhli@hotmail.com

NARAYANASWAMY BALAKRISHNAN
Department of Mathematics and Statistics
McMaster University
Hamilton, Ontario, Canada L8S 4K1

Here, we discuss the stochastic comparison of residual lifetimes of parallel systems and inactivity times of series systems by means of the reversed hazard rate order when the components of the systems are independent but not necessarily identically distributed. We also establish some monotonicity properties of such residual lifetimes of parallel systems and inactivity times of series systems. These results extend some of the recent results in this direction due to Zhao, Li, and Balakrishnan [21], Kochar and Xu [12], and Saledi and Asadi [16].

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1. INTRODUCTION

Order statistics have received considerable attention in theoretical and applied literature since they play an important role in reliability, data analysis, statistical inference, quality control, and applied probability. Let $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ be the order statistics from random variables $X_1, X_2, \ldots, X_n$. For example in the context of reliability, the lifetime of the k-out-of-n system corresponds to the $(n-k+1)$ th-order statistic $X_{n-k+1:n}$. In particular, the lifetimes of series and parallel systems are just the smallest order statistic $X_{1:n}$ and the largest order statistic $X_{n:n}$, respectively.

The residual life of a unit with lifetime X at time t,

$$X_t = X - t \mid (X > t),$$

describes the lifetime of the used unit with age time $t > 0$. In some situations, the times of failures of components of a system might not be available, but only the information on the number of failed components might be available. For this reason, many authors have studied the residual life of a k-out-of-n system under the condition that the number of failures by time t is greater than some fixed positive integer $l < k$; for example, one may refer to Bairamov, Ahsanullah, and Akhundov [3], Asadi and Bairamov [2], Li and Zhao [14], and Goliforushani, Asadi, and Balakrishnan [9]. It is of interest to mention here that Hu, Jin, and Khaledi [10], Zhao and Balakrishnan [20], and Balakrishnan, Belzunce, Hami, and Khaledi [5] have all discussed some stochastic ordering properties of the conditional distributions of generalized order statistics, which include the residual lifetimes of k-out-of-n systems as a special case, whereas Khaledi and Shaked [11] have studied the residual life of a coherent system given that at least $(n-k+1)$ components of the system are working. All of the above developments, however, are based on the critical assumption that the components in the system are all independent and identically distributed (i.i.d.). However, it might be more realistic to consider a reliability system consisting of independent and heterogeneous components. Due to the complicated nature of distributions of order statistics arising from independent and nonidentical components (see Balakrishnan [4]), not much work has been done on stochastic properties of reliability systems with heterogeneous components. In this regard, Sadegh [15] was the first to obtain some properties of the mean residual life function of a parallel system with independent but not necessarily identical (i.ni.d.) components. Zhao and Balakrishnan [21] subsequently considered the residual lifetime of an $(n-k+1)$-out-of-n system given that there are exactly l failures by time t (i.e., the lth failure occurred by time t and the $(l+1)$th failure has not occurred by time $t > 0$), given by

$$X_{k:n} - t \mid (X_{l:n} \leq t < X_{l+1:n}) \quad \text{for } 1 \leq l < k \leq n. \tag{1}$$

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Recently, Kochar and Xu [12] further investigated the residual lifetime of the $(n-k+1)$-out-of-n system with heterogeneous components under the condition that at least $(n-l+1)$ components of the system are working at time t; that is,

$$X_{k:n} - t \mid (X_{l:n} > t) \quad \text{for } 1 \leq l < k \leq n. \tag{2}$$

They also considered another interesting situation of the residual lifetime of the $(n-k+1)$-out-of-n system that is working at time t, but with at least l failed components, given by

$$X_{k:n} - t \mid (X_{l:n} \leq t, X_{k:n} > t) \quad \text{for } 1 \leq l < k \leq n. \tag{3}$$

They then showed that all these three types of residual lifetimes of an $(n-k+1)$-out-of-n system stochastically decreases in l in the sense of the usual stochastic order (which will be formally defined at the end of this section).

Another reliability characteristic that is of great interest in reliability and life-testing studies is the inactivity time of a unit with lifetime X at time t, given by

$$X_{(t)} = t - X \mid (X \leq t)\,,$$

which is simply the time that has elapsed since the failure of the unit. For a system that can be regarded as a black box in the sense that the exact failure times of its components cannot be observed, it is often of great importance for engineers and reliability analysts to make inference on the inactivity times of failed components in the system. For a system with i.i.d. components, Khaledi and Shaked [11] investigated the stochastic properties of the inactivity time of a coherent system, including

$$t - X_{k:n} \mid (X_{l:n} \leq t) \quad \text{for } 1 \leq k \leq l \leq n \tag{4}$$

as a special case. Recently, Saledi and Asadi [16] generalized the results of Khaledi and Shaked [11] under the assumption that the components of system are non-i.i.d.. In addition, for systems with heterogeneous components, Zhao et al. [21] considered the inactivity time of an $(n-k+1)$-out-of-n system given by

$$t - X_{k:n} \mid (X_{l:n} \leq t < X_{l+1:n}) \quad \text{for } 1 \leq k < l < n \tag{5}$$

and established that the above two types of inactivity times stochastically increase in l in the sense of the usual stochastic order. For more details, one may also refer to Asadi [1] and Tavangar and Asadi [18].

In the present work, we consider the three types of residual lifetimes of parallel systems defined in (1)–(3) and the two types of inactivity times of series systems

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defined in (4) and (5), and then prove some new stochastic comparison results. Moreover, analogous to the concept of the residual lifetime in (3), we can also consider

$$t - X_{k:n} \mid (X_{k:n} \le t, X_{l:n} > t) \quad \text{for } 1 \le k < l \le n \tag{6}$$

which corresponds to the inactivity time of an $(n-k+1)$-out-of-n system given that the system had failed by time t but with at least $(n-l+1)$ components still working.

The rest of this article is organized as follows. In Section 2, we present some results on the stochastic comparison of the three types of residual lifetimes of parallel systems and the three types of inactivity times of series systems in the sense of reversed hazard rate order. Then in Section 3, we establish some monotonicity results with respect to the time elapsed.

Note that throughout this article, "increasing" stands for "nondecreasing" and "decreasing" stands for "nonincreasing." Before proceeding to the main results, we now recall some stochastic orders that are most pertinent to subsequent developments. For more details, one can refer to Shaked and Shanthikumar [17].

Let X and Y be two random variables with absolutely continuous cumulative distribution functions F and G, probability density functions f and g, and survival functions $\bar{F} = 1 - F$ and $\bar{G} = 1 - G$, respectively.

Definition 1: *X is said to be smaller than Y in the*

(*i*) *usual stochastic order, denoted by $X \le_{\text{st}} Y$, if $\bar{F}(x) \le \bar{G}(x)$ for all x;*

(*ii*) *hazard rate order, denoted by $X \le_{\text{hr}} Y$, if $\bar{G}(x)/\bar{F}(x)$ is increasing in x;*

(*iii*) *reversed hazard rate order, denoted by $X \le_{\text{rh}} Y$, if $G(x)/F(x)$ is increasing in x.*

2. MONOTONICITY WITH RESPECT TO NUMBER OF FAILURES

Let $\boldsymbol{x} = (x_1, \ldots, x_n)$, with $n \ge 2$, be a real vector of positive components. For $j \in \{1, 2, \ldots, n\}$, let

$$S_j(\boldsymbol{x}) = \sum_{1 \le i_1 \le \cdots \le i_j \le n} x_{i_1} x_{i_2} \cdots x_{i_j}$$

be the jth elementary symmetrical function of $x_1, x_2, \ldots, x_n$. Now, for comparing elementary symmetrical functions of two different vectors, we introduce the following lemma, which will play a key role in proving our main results.

Lemma 2 (Bon and Păltănea **[7]**): *Let $\boldsymbol{p} = (p_1, \ldots, p_n)$ and $\boldsymbol{q} = (q_1, \ldots, q_n)$ be two vectors such that $0 < p_i \le q_i$ for all $i = 1, \ldots, n$. Then*

$$\frac{S_{r+1}(\boldsymbol{p})}{S_r(\boldsymbol{p})} \le \frac{S_{r+1}(\boldsymbol{q})}{S_r(\boldsymbol{q})} \quad \text{for any } r \in \{0, 1, \ldots, n-1\}.$$

Moreover, the inequality is strict whenever $p_i < q_i$ for some $i \in \{1, \ldots, n\}$.

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As a consequence of Lemma 2, we obtain the following result concerning the lifetime of a parallel system conditioned on the number of working components at time $t \geq 0$; that is, $N_t(\boldsymbol{X}) \equiv \sum_{i=1}^{n} I(X_i > t)$ with $I(A)$ taking 1 and 0 according to the occurrence of A and $\bar{A}$, respectively.

THEOREM 3: *Suppose nonnegative random variables $X_1, \ldots, X_n$ are i.ni.d. Then for any $t > 0$ and $r = 0, 1, \cdots, n-1$, we have*

$$X_{n:n} - t \mid (N_t(\boldsymbol{X}) = r+1) \geq_{\rm rh} X_{n:n} - t \mid (N_t(\boldsymbol{X}) = r)\,;$$

that is,

$$\frac{P\big(X_{n:n} \leq t+x, N_t(\boldsymbol{X}) = r+1\big)}{P\big(X_{n:n} \leq t+x, N_t(\boldsymbol{X}) = r\big)}$$

is increasing in $x \geq 0$.

PROOF: Note that for any $t > 0$ and $x \geq 0$, we can express

$$\begin{aligned} &\frac{P\big(X_{n:n} \leq t+x, N_t(\boldsymbol{X}) = r+1\big)}{P\big(X_{n:n} \leq t+x, N_t(\boldsymbol{X}) = r\big)} \\ &\quad = \frac{\sum_{C_{r+1}} \prod_{i \in C_{r+1}} \big(F_i(t+x) - F_i(t)\big) \prod_{j \notin C_{r+1}} F_j(t)}{\sum_{C_r} \prod_{i \in C_r} \big(F_i(t+x) - F_i(t)\big) \prod_{j \notin C_r} F_j(t)} \\ &\quad = \frac{\sum_{C_{r+1}} \prod_{i \in C_{r+1}} \Big(F_i(t+x)/F_i(t) - 1\Big)}{\sum_{C_r} \prod_{i \in C_r} \Big(F_i(t+x)/F_i(t) - 1\Big)}, \end{aligned} \qquad \textbf{(7)}$$

where the summation C_r extends over all subsets of $\{1, 2, \ldots, n\}$ with cardinality r and $C_0 \equiv \emptyset$.

Now, let

$$p_i(x) = \frac{F_i(t+x)}{F_i(t)} - 1 \quad \text{for } i = 1, 2, \ldots, n.$$

Then whenever $y \geq x \geq 0$, we have

$$p_i(x) \leq p_i(y) \quad \text{for } i = 1, 2, \ldots, n,$$

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and, consequently, it follows from Lemma 2 that for $y \geq x \geq 0$ and $r = 0, 1, \ldots, n-1$,

$$\frac{P\big(X_{n:n} \leq t+x, N_t(\boldsymbol{X}) = r+1\big)}{P\big(X_{n:n} \leq t+x, N_t(\boldsymbol{X}) = r\big)} \leq \frac{P\big(X_{n:n} \leq t+y, N_t(\boldsymbol{X}) = r+1\big)}{P\big(X_{n:n} \leq t+y, N_t(\boldsymbol{X}) = r\big)}.$$

This completes the proof of the theorem. ■

The next main result presents the reversed hazard rate ordering of the lifetime of a parallel system conditioned on the partial information on the number of observed working components.

THEOREM 4: *Suppose nonnegative random variables* $X_1, \cdots, X_n$ *are i.ni.d. Then for* $n > k \geq 1$ *and* $t \geq 0$*, we have*

$$X_{n:n} - t \mid (X_{k:n} > t) \geq_{\mathrm{rh}} X_{n:n} - t \mid (X_{k+1:n} > t)\,.$$

PROOF: Given that at least $n-k+1$ components are surviving at time $t \geq 0$, the residual lifetime of the parallel system with components $X_1, \cdots, X_n$ has its distribution function as

$$\begin{aligned} P(X_{n:n} - t \leq x \mid X_{k:n} > t) &\\ = \frac{P(X_{n:n} \leq t+x, X_{k:n} > t)}{P(X_{k:n} > t)} &\\ = \frac{P\big(X_{n:n} \leq t+x, N_t(\boldsymbol{X}) \geq n-k+1\big)}{P\big(N_t(\boldsymbol{X}) \geq n-k+1\big)}. & \qquad \textbf{(8)} \end{aligned}$$

According to Theorem 3, for any $t \geq 0$ and $k = 1, 2, \ldots, n$,

$$\frac{P\big(X_{n:n} \leq t+x, N_t(\boldsymbol{X}) = n-k+1\big)}{P\big(X_{n:n} \leq t+x, N_t(\boldsymbol{X}) = n-k\big)}$$

is increasing in $x \geq 0$. Thus, for any $t \geq 0$ and $1 \leq n-k < j \leq n$, we have

$$\begin{aligned} &\frac{P\big(X_{n:n} \leq t+x, N_t(\boldsymbol{X}) = j\big)}{P\big(X_{n:n} \leq t+x, N_t(\boldsymbol{X}) = n-k\big)} \\ &\quad = \frac{P\big(X_{n:n} \leq t+x, N_t(\boldsymbol{X}) = n-k+1\big)}{P\big(X_{n:n} \leq t+x, N_t(\boldsymbol{X}) = n-k\big)} \end{aligned}$$

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$$\times \frac{P\big(X_{n:n} \leq t+x, N_t(\boldsymbol{X}) = n-k+2\big)}{P\big(X_{n:n} \leq t+x, N_t(\boldsymbol{X}) = n-k+1\big)}$$

$$\vdots$$

$$\times \frac{P\big(X_{n:n} \leq t+x, N_t(\boldsymbol{X}) = j\big)}{P\big(X_{n:n} \leq t+x, N_t(\boldsymbol{X}) = j-1\big)}$$

is increasing in $x \geq 0$. As a result, for any $t \geq 0$,

$$\frac{\sum_{j=n-k+1}^{n} P\big(X_{n:n} \leq t+x, N_t(\boldsymbol{X}) = j\big)}{P\big(X_{n:n} \leq t+x, N_t(\boldsymbol{X}) = n-k\big)}$$

is increasing in $x \geq 0$.

In view of (8), we have for any $t \geq 0$,

$$\begin{aligned}\frac{P(X_{n:n} - t \leq x \mid X_{k+1:n} > t)}{P(X_{n:n} - t \leq x \mid X_{k:n} > t)} &\propto \frac{P\big(X_{n:n} \leq t+x, N_t(\boldsymbol{X}) \geq n-k\big)}{P\big(X_{n:n} \leq t+x, N_t(\boldsymbol{X}) \geq n-k+1\big)} \\ &= 1 + \frac{P\big(X_{n:n} \leq t+x, N_t(\boldsymbol{X}) = n-k\big)}{P\big(X_{n:n} \leq t+x, N_t(\boldsymbol{X}) \geq n-k+1\big)} \\ &= 1 + \frac{P\big(X_{n:n} \leq t+x, N_t(\boldsymbol{X}) = n-k\big)}{\sum_{j=n-k+1}^{n} P\big(X_{n:n} \leq t+x, N_t(\boldsymbol{X}) = j\big)}\end{aligned}$$

is decreasing in $x \geq 0$. This completes the proof of the theorem. ■

Theorem 3 of Li and Lu [13] claimed the hazard rate ordering between the parallel system of used components and a used parallel system, and this seems to be incorrect, as shown later in Example 6. The following corollary serves as a correction for this result.

COROLLARY 5: *Suppose nonnegative random variables $X_1, \ldots, X_n$ are i.ni.d. Then,*

$$(X_{(t)})_{n:n} \geq_{rh} \big(X_{n:n}\big)_{(t)} \quad \textit{for any } t \geq 0.$$

PROOF: From Theorem 4, we have for any $t \geq 0$,

$$\begin{aligned} X_{n:n} - t \mid (X_{1:n} > t) &\geq_{\mathrm{rh}} X_{n:n} - t \mid (X_{2:n} > t) \\ &\ \ \vdots \\ &\geq_{\mathrm{rh}} X_{n:n} - t \mid (X_{n:n} > t)\,. \end{aligned}$$

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Due to the independence among $X_1, \ldots, X_n$, we have

$$(X_{(t)})_{n:n} = \max_{1 \le i \le n} \{X_i - t \mid (X_i > t)\} = X_{n:n} - t \mid (X_{1:n} > t)\,,$$

from which we readily obtain $(X_{(t)})_{n:n} \ge_{\rm rh} \left(X_{n:n}\right)_{(t)}$ for any $t \ge 0$, as required. ■

Theorem 4 shows that the residual lifetime of a parallel system with independent but heterogenous components stochastically increases with respect to number of components surviving at some time point in terms of the reversed hazard rate order. This incidentally extends Theorem 4.2 of Kochar and Xu [12] from the usual stochastic order to the reversed hazard rate order for parallel systems. Naturally, one may wonder whether the hazard rate order actually holds, and, unfortunately, Example 6 below reveals that the answer is, in general, negative.

Example 6: Consider a parallel system with two independent components having their survival functions as

$$\bar{F}_1(x) = e^{-x} \quad \text{and} \quad \bar{F}_2(x) = \frac{1}{1+x} \quad \text{for } x \ge 0,$$

respectively. Then it is easy to verify that the ratio of the survival function of $X_{2:2} - t \mid (X_{1:2} > t)$ to that of $X_{2:2} - t \mid (X_{2:2} > t)$ is given by

$$\begin{aligned} g_t(x) &= \frac{P(X_{2:2} - t > x \mid X_{1:2} > t)}{P(X_{2:2} - t > x \mid X_{2:2} > t)} \\ &\propto \frac{1 + (t+x)e^{-(t+x)}}{1 + t + xe^{-x}}. \end{aligned}$$

As depicted in Figure 1, when $t = 10$, for example, $g_t(x)$ is not monotone. This means that there exists no hazard rate order and consequently no likelihood ratio order between $X_{2:2} - t \mid (X_{1:2} > t)$ and $X_{2:2} - t \mid (X_{2:2} > t)$.

In the context of non-i.i.d. components, Zhao et al. [21] showed that for $2 \le k < r \le n$ and $t \ge 0$,

$$X_{r:n} - t \mid (X_{k-1:n} < t \le X_{k:n}) \ge_{\rm st} X_{r:n} - t \mid (X_{k:n} < t \le X_{k+1:n})\,.$$

Subsequently, Kochar and Xu [12] showed that for $2 \le k < r \le n$ and $t \ge 0$,

$$X_{r:n} - t \mid (X_{k-1:n} \le t, X_{r:n} > t) \ge_{\rm st} X_{r:n} - t \mid (X_{k:n} \le t < X_{r:n})\,.$$

In Theorems 7 and 8, we improve these two results by strengthening the usual stochastic order to the reversed hazard rate order for parallel systems, respectively.

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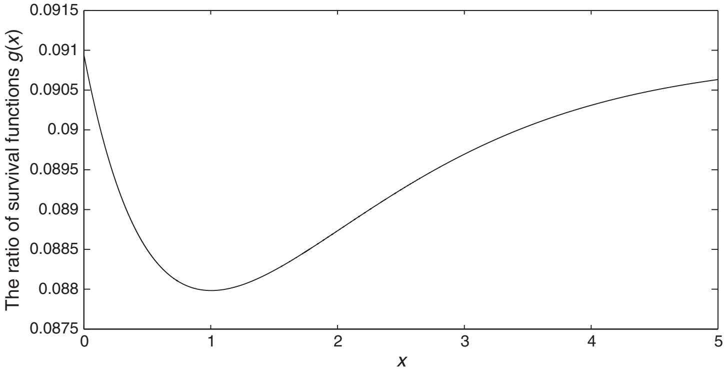

FIGURE 1. The ratio $g_t(x)$ with $t = 10$.

THEOREM 7: *Suppose nonnegative random variables $X_1, \ldots, X_n$ are i.ni.d. Then for $1 \le k < n-1$ and $t \ge 0$, we have*

$$X_{n:n} - t \mid (X_{k:n} \le t, X_{n:n} > t) \ge_{\mathrm{rh}} X_{n:n} - t \mid (X_{k+1:n} \le t, X_{n:n} > t)\,.$$

PROOF: We may present a proof along lines similar to those for Theorem 4. To begin, we find for any $t, x \ge 0$, and $1 \le k < n$,

$$\begin{aligned} &\frac{P(X_{n:n} - t \le x \mid X_{k:n} \le t, X_{n:n} > t)}{P(X_{n:n} - t \le x \mid X_{k+1:n} \le t, X_{n:n} > t)} \\ &\qquad \propto \frac{P\big(X_{n:n} \le t + x, 1 \le N_t(\boldsymbol{X}) \le n-k+1\big)}{P\big(X_{n:n} \le t+x, 1 \le N_t(\boldsymbol{X}) \le n-k\big)} \\ &\qquad = 1 + \frac{P\big(X_{n:n} \le t+x, N_t(\boldsymbol{X}) = n-k+1\big)}{\sum_{j=1}^{n-k} P\big(X_{n:n} \le t+x, N_t(\boldsymbol{X}) = j\big)}. \end{aligned} \tag{9}$$

It then suffices to prove that for $1 \le s \le n-k$ and $t \ge 0$,

$$\frac{P\big(X_{n:n} \le t+x, N_t(\boldsymbol{X}) = s+1\big)}{P\big(X_{n:n} \le t+x, N_t(\boldsymbol{X}) = s\big)}$$

is increasing in $x \ge 0$. This is actually guaranteed by Lemma 3, which completes the proof. ■

Theorem 8 below can be established in an analogous manner and therefore is presented here without a proof for the sake of brevity.

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THEOREM 8: *Suppose nonnegative random variables* $X_1, \ldots, X_n$ *are i.ni.d. Then for* $1 < k < n$ *and* $t \geq 0$*, we have*

$$X_{n:n} - t \mid (X_{k-1:n} < t \leq X_{k:n}) \geq_{\text{rh}} X_{n:n} - t \mid (X_{k:n} < t \leq X_{k+1:n}) .$$

In concluding this section, we present the dual versions for the inactivity times of series systems, and the basic idea of the proof comes from Zhao et al. [21].

COROLLARY 9: *Suppose nonnegative random variables* $X_1, \ldots, X_n$ *are i.ni.d. Then we have the following:*

(i) for $1 \leq k < n$ *and* $t \geq 0$,

$$t - X_{1:n} \mid (X_{k:n} \leq t) \leq_{\text{rh}} t - X_{1:n} \mid (X_{k+1:n} \leq t) ;$$

(ii) for $2 \leq k < n$ *and* $t \geq 0$,

$$t - X_{1:n} \mid (X_{1:n} \leq t, X_{k:n} > t) \leq_{\text{rh}} t - X_{1:n} \mid (X_{1:n} \leq t, X_{k+1:n} > t) ;$$

(iii) for $1 < k < n$ *and* $t \geq 0$,

$$t - X_{1:n} \mid (X_{k-1:n} \leq t < X_{k:n}) \leq_{\text{rh}} t - X_{1:n} \mid (X_{k:n} \leq t < X_{k+1:n}) .$$

PROOF: We only prove part (i); the other two parts can be proved in a similar manner. Although the random variables discussed in this article are all taken to be nonnegative, all results also hold for any random variables on the real line. Denote by $(-X)_{k:n}$ the k-th order statistic among $-X_1, -X_2, \ldots, -X_n$. Then it is clear that $(-X)_{k:n} = -X_{n-k+1:n}$. So, by Theorem 4, we have for $1 \leq k < n$ and $x \leq 0$,

$$(-X)_{n:n} - x \mid ((-X)_{k:n} > x) \geq_{\text{rh}} (-X)_{n:n} - x \mid ((-X)_{k+1:n} > x) ,$$

which is equivalent to

$$-X_{1:n} - x \mid (-X_{n-k+1:n} > x) \geq_{\text{rh}} -X_{1:n} - x \mid (-X_{n-k:n} > x) .$$

Now, setting $t = -x$, we obtain

$$t - X_{1:n} \mid (X_{n-k+1:n} \leq t) \geq_{\text{rh}} t - X_{1:n} \mid (X_{n-k:n} \leq t)),$$

which implies the required result. ■

Clearly, parts (i) and (iii) of Corollary 9 strengthen Theorem 2.9 of Saledi and Asadi [16] and Corollary 2.2(i) of Zhao et al. [21], respectively, by extending the usual stochastic order to the reversed hazard rate order.

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3. MONOTONICITY WITH RESPECT TO TIME ELAPSED

The reversed hazard rate as well as the reversed hazard rate order are rather useful in autopsy data analysis wherein, following the failure of a reliability system, one will be interested in inferring about failures of the components. Recall that X with distribution function F is said to have a *decreasing reversed hazard rate* (DRHR) if $F(x)$ is log-concave. As a dual notion, *increasing reversed hazard rate* (IRHR) is defined as the log-convexity of $F(x)$. One may refer to Block, Savits and Singh, [6] and Chandra and Roy [8] for more on this aging property.

Now, we establish some monotonicity results for the conditional inactivity times.

THEOREM 10: *Suppose $X_1, \ldots, X_n$ are i.ni.d. random variables. If X_i is DRHR for all $i = 1, \ldots, n$, then*

$$\frac{P\big(t - X_{k:n} < x \mid X_{n:n} \le t\big)}{P\big(t - X_{k+1:n} < x \mid X_{n:n} \le t\big)}$$

is decreasing in t, for any $x > 0$.

PROOF: In (7), let us set

$$q_i(t) = \frac{F_i(t+x)}{F_i(t)} - 1, \qquad i = 1, 2, \ldots, n.$$

Since X_i is DRHR, we have $q_i(t)$ to be decreasing with respect to $t \ge 0$ for any $x \ge 0$ and $i = 1, 2, \ldots, n$. Consequently, it follows from Lemma 2 once again that for any $x \ge 0$ and $r = 0, 1, \ldots, n-1$,

$$\begin{aligned}
\frac{P\big(X_{n:n} \le t + x, N_t(\boldsymbol{X}) = r+1\big)}{P\big(X_{n:n} \le t + x, N_t(\boldsymbol{X}) = r\big)}
&= \frac{\sum_{C_{r+1}} \prod_{i \in C_{r+1}} q_i(t)}{\sum_{C_r} \prod_{i \in C_r} q_i(t)} \\
&\le \frac{\sum_{C_{r+1}} \prod_{i \in C_{r+1}} q_i(s)}{\sum_{C_r} \prod_{i \in C_r} q_i(s)} \\
&= \frac{P\big(X_{n:n} \le s + x, N_s(\boldsymbol{X}) = r+1\big)}{P\big(X_{n:n} \le s + x, N_s(\boldsymbol{X}) = r\big)},
\end{aligned}$$

whenever $t \ge s \ge 0$.

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As a result, for any $x \geq 0$, we find

$$\begin{aligned}
\frac{P\big(X_{k+1:n} > t \mid X_{n:n} \leq t + x\big)}{P\big(X_{k:n} > t \mid X_{n:n} \leq t + x\big)}
&= \frac{P\big(X_{n:n} \leq t + x, X_{k+1:n} > t\big)}{P\big(X_{n:n} \leq t + x, X_{k:n} > t\big)} \\
&= \frac{P\big(X_{n:n} \leq t + x, N_t(\boldsymbol{X}) \geq n - k\big)}{P\big(X_{n:n} \leq t + x, N_t(\boldsymbol{X}) \geq n - k + 1\big)} \\
&= 1 + \frac{P\big(X_{n:n} \leq t + x, N_t(\boldsymbol{X}) = n - k\big)}{P\big(X_{n:n} \leq t + x, N_t(\boldsymbol{X}) \geq n - k + 1\big)} \\
&= 1 + \frac{P\big(X_{n:n} \leq t + x, N_t(\boldsymbol{X}) = n - k\big)}{\sum_{j=n-k+1}^{n} P\big(X_{n:n} \leq t + x, N_t(\boldsymbol{X}) = j\big)}
\end{aligned}$$

to be increasing in $t \geq 0$, from which the required result follows readily. ■

Remark 11: Quite interestingly, in the special case when X_i's are i.i.d. DRHR random variables, the ratio of the conditional probabilities considered in Theorem 10 can be shown to be equivalent to $1/(1 + h_{n-k}(t))$, where $h_{n-k}(t)$ is the hazard rate function at $n - k$ of Binomial $(n, 1 - F(t - x)/F(t))$, with fixed x. Here, the hazard rate function at $n - k$ of Binomial(n, p) distribution is defined as

$$\frac{\binom{n}{n-k} p^{n-k} (1 - p)^k}{\sum_{i=n-k+1}^{n} \binom{n}{i} p^i (1 - p)^{n-i}}.$$

It can be readily verified that $h_{n-k}(t)$ is increasing in t, for $t \geq x$, when the common distribution F is DRHR. This fact, together with its connection to the quantity considered in Theorem 10, provides an additional insight into the result.

THEOREM 12: *Suppose $X_1, \ldots, X_n$ are independent but not necessarily identically distributed variables. If X_i is IRHR for all $i = 1, \ldots, n$, then for $n > k \geq 1$ and $x > 0$,*

$$\frac{P(X_{n:n} - t \leq x \mid X_{k+1:n} > t)}{P(X_{n:n} - t \leq x \mid X_{k:n} > t)}$$

is decreasing in t.

PROOF: In a manner analogous to the proof of Theorem 10, the IRHR property of all X_i's can be shown to imply that for any $x > 0$ and $k = 1, 2, \ldots, n$,

$$\frac{P\big(X_{n:n} \leq t + x, N_t(\boldsymbol{X}) = n - k + 1\big)}{P\big(X_{n:n} \leq t + x, N_t(\boldsymbol{X}) = n - k\big)}$$

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is increasing in t. So, for any $x > 0$ and $1 \le n-k < j \le n$,

$$\frac{P\big(X_{n:n} \le t+x, N_t(\boldsymbol{X}) = j\big)}{P\big(X_{n:n} \le t+x, N_t(\boldsymbol{X}) = n-k\big)}$$

is also increasing in t. Accordingly, for any $x > 0$,

$$\frac{\sum_{j=n-k+1}^{n} P\big(X_{n:n} \le t+x, N_t(\boldsymbol{X}) = j\big)}{P\big(X_{n:n} \le t+x, N_t(\boldsymbol{X}) = n-k\big)}$$

is increasing in t. As a result, for any $x > 0$, we find

$$\begin{aligned}
&\frac{P\big(X_{n:n} \le t+x, X_{k+1:n} > t\big)}{P\big(X_{n:n} \le t+x, X_{k:n} > t\big)} \\
&\quad = \frac{P\big(X_{n:n} \le t+x, N_t(\boldsymbol{X}) \ge n-k\big)}{P\big(X_{n:n} \le t+x, N_t(\boldsymbol{X}) \ge n-k+1\big)} \\
&\quad = 1 + \frac{P\big(X_{n:n} \le t+x, N_t(\boldsymbol{X}) = n-k\big)}{P\big(X_{n:n} \le t+x, N_t(\boldsymbol{X}) \ge n-k+1\big)} \\
&\quad = 1 + \frac{P\big(X_{n:n} \le t+x, N_t(\boldsymbol{X}) = n-k\big)}{\sum_{j=n-k+1}^{n} P\big(X_{n:n} \le t+x, N_t(\boldsymbol{X}) = j\big)} \qquad \textbf{(10)}
\end{aligned}$$

to be decreasing in t.

On the other hand, according to Theorem 1.B.26 of Shaked and Shanthikumar [17], $X_{k+1:n} \ge_{hr} X_{k:n}$ for any $k = 0, 1, \ldots, n-1$; that is,

$$\frac{P(X_{k:n} > t)}{P(X_{k+1:n} > t)} \qquad \textbf{(11)}$$

is decreasing in t.

Now, by taking (8), (10), and (11) into account, we can conclude that for any $k = 0, 1, \ldots, n-1$,

$$\frac{P(X_{n:n} - t \le x \mid X_{k+1:n} > t)}{P(X_{n:n} - t \le x \mid X_{k:n} > t)} = \frac{P\big(X_{n:n} \le t+x, X_{k+1:n} > t\big)}{P\big(X_{n:n} \le t+x, X_{k:n} > t\big)} \cdot \frac{P(X_{k:n} > t)}{P(X_{k+1:n} > t)}$$

is decreasing in t, as required. ■

Remark 13: In the special case when X_i's are i.i.d. IRHR random variables, the ratio of the conditional probabilities considered in Theorem 12 can be shown to be equivalent

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to $(1 + r_{1,k}(t))/(1 + r_{2,k}(t))$, where $r_{1,k}(t)$ and $r_{2,k}(t)$ are the reversed hazard rate functions at k of Binomial $(n, F(t)/F(x+t))$ and Binomial $(n, F(t))$ distributions, respectively, with fixed x. Here, the reversed hazard rate function at k of Binomial(n, p) distribution is defined as

$$\frac{\binom{n}{k}p^k(1-p)^{n-k}}{\sum_{i=0}^{k-1}\binom{n}{i}p^i(1-p)^{n-i}}.$$

It is evident that $r_{2,k}(t)$ is increasing in t for any distribution F, whereas $r_{1,k}(t)$ is decreasing in t when the parent distribution F is IRHR. These interesting facts, together with their connection to the quantity considered in Theorem 12, gives an additional insight into the result.

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