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SWAN CONDUCTORS FOR p-ADIC DIFFERENTIAL MODULES. II GLOBAL VARIATION

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Abstract Using a local construction from a previous paper, we exhibit a numerical invariant, the differential Swan conductor, for an isocrystal on a variety over a perfect field of positive characteristic overconvergent along a boundary divisor; this leads to an analogous construction for certain p-adic and ℓ-adic representations of the étale fundamental group of a variety. We then demonstrate some variational properties of this definition for overconvergent isocrystals, paying special attention to the case of surfaces.

Keywords: p-adic differential modules; Swan conductors; overconvergent isocrystals; wild ramification

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Introduction

This paper is a sequel to [**15**], which defines a numerical invariant, called the differential Swan conductor, for certain differential modules on a rigid analytic annulus over a p-adic field. In that paper, the key application of the construction is the definition of a sensible numerical invariant for Galois representations with finite local monodromy over a complete discretely valued field of equal characteristic, without any assumption of perfectness of the residue field.

In this paper, we adopt a more geometric viewpoint, taking the construction back to its roots in the theory of p-adic cohomology. We define differential Swan conductors for an overconvergent isocrystal on a variety over a perfect field of positive characteristic. The definition depends on the choice of a boundary divisor along which one measures the conductor; we are particularly interested in understanding how the conductor can vary as a function of this boundary divisor. We give special attention to the case of surfaces; one of the variational properties loosely resembles subharmonicity for functions on Berkovich analytic curves, in the sense of Thuillier [**29**]. Another resembles a semicontinuity property of étale cohomology [**23**].

The variational properties of differential Swan conductors seem analogous to properties of the irregularity of a holomorphic differential module on a complex surface along a boundary divisor; indeed, a complex analogue of the semicontinuity property mentioned above has recently been proved by André [**3**], extending an old result of Deligne (see [**25**]). Variation of the boundary divisor has been studied in that setting by Sabbah [**27**]; our study was motivated by questions in the p-adic realm analogous to Sabbah's questions about Stokes decompositions. These arise in the study of semistable reduction for overconvergent F-isocrystals, which is the subject of an ongoing series of papers by the author [**14**, **16**–**18**]; in fact, some of the constructions used in [**15**] and in this paper already appear in [**17**]. We have begun carrying these techniques over to Sabbah's setting [**19**].

As in [**15**], there is a mechanism for converting certain p-adic representations of the étale fundamental group of a smooth variety into F-isocrystals. This makes it possible to define differential Swan conductors, and (with some effort) to prove some of the corresponding properties, also for lisse ℓ-adic étale sheaves.

We end this introduction by cautioning that this paper is not intended to be read independently from [**15**]. In particular, we freely use notation and terminology introduced in [**15**], without explicit reintroduction except in a few places for emphasis.

1. Relative annuli

In this section, we gather some facts about the rigid geometry of relative annuli (products of annuli with other spaces), in the vein of [**14**, § 3].

Hypothesis 1.0.1. Throughout this paper, let K be a complete nonarchimedean field of characteristic 0 equipped with m commuting continuous derivations $\partial_1, \dots, \partial_m$, for some nonnegative integer m. Assume that K is of *rational type* in the sense of [**22**, Definition 1.5.3], i.e. there exist elements $u_1, \dots, u_m \in K$ such that

- for $i, j \in \{1, \dots, m\}$ with $i \neq j$, $\partial_i(u_i) = 1$ and $\partial_i(u_j) = 0$;
- for n a positive integer, $i \in \{1, \dots, m\}$, and $x \in K$, $|u_i^n \partial_i^n(x)/n!| \leqslant |x|$.

Let k be the residue field of K, and assume that k is of characteristic $p > 0$. Let $\mathfrak{o}_K$ denote the valuation subring of K, let $\mathfrak{m}_K$ denote the maximal ideal of $\mathfrak{o}_K$, and let Γ^* be the divisible closure of $|K^\times|$. Let K_0 be the joint kernel of $\partial_1, \dots, \partial_m$ on K.

Remark 1.0.2. Hypothesis 1.0.1 differs from the running hypothesis [**15**, Hypothesis 2.1.3] from the previous paper, which required one of the following. (Beware that as written, [**15**, Hypothesis 2.1.3(b)] is missing the hypothesis that k is separable over k_0.)

(a) K is a finite unramified extension of the completion of $K_0(u_1, \dots, u_n)$ for the $(1, \dots, 1)$-Gauss norm.

(b) K_0 and K are discretely valued with the same value group, k is separable over the residue field k_0 of K_0, and k admits a finite p-basis over k_0.

By [**22**, Remark 1.5.10], both of these are special cases of Hypothesis 1.0.1. On the other hand, we can bypass most of the results of [**15**, § 2] which depend on [**15**, Hypothesis 2.1.3] by citing results from [**22**] instead.

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Hypothesis 1.0.3. Throughout this section,

- let P denote a smooth affine irreducible formal scheme over $\operatorname{Spf} \mathfrak{o}_K$, with generic fibre P_K and special fibre $Z = P_k$;
- let L denote the completion of $\operatorname{Frac} \Gamma(P, \mathcal{O})$ for the topology induced by the supremum norm on P_K;
- let U denote an open dense subscheme of Z.

Notation 1.0.4. For $Z' \hookrightarrow Z$ an immersion, we denote by $]Z'[_P$ the inverse image of Z' under the specialization map $P_K \to Z$; we also refer to $]Z'[_P$ as the *tube* of Z' in P_K.

Definition 1.0.5. We say a subinterval of $[0, +\infty)$ is *aligned* if each endpoint at which it is closed belongs to $\Gamma^* \cup \{0\}$. This is consistent with [**15**, Notation 2.4.1], which only applied to intervals not containing 0, and with [**14**, Definition 3.1.1].

1.1. Relative annuli

Lemma 1.1.1. *Let Y be a rigid subspace of $P_K \times A_K[0,1)$. Then the following conditions are equivalent.*

(a) *There exists $\epsilon \in (0,1)$ such that $P_K \times A_K(\epsilon, 1) \subseteq Y$.*

(b) *There exists an affinoid subspace V of $P_K \times A_K[0,1)$ such that $\{Y, V\}$ forms an admissible covering of $P_K \times A_K[0,1)$.*

(c) *There exist $\rho \in (0,1) \cap \Gamma^*$ and an affinoid subspace V of $P_K \times A_K[\rho, 1)$ such that $\{Y \cap (P_K \times A_K[\rho, 1)), V\}$ forms an admissible covering of $P_K \times A_K[\rho, 1)$.*

Proof. The implication (a) $\Longrightarrow$ (b) is clear: take $V = P_K \times A_K[0, \rho]$ for any $\rho \in (\epsilon, 1) \cap \Gamma^*$. The implication (b) $\Longrightarrow$ (c) is trivial. For (c) $\Longrightarrow$ (a), note that the maximum modulus principle [**6**, Proposition 6.2.1/4] implies that t achieves its supremum η on V, so η must be less than 1; we can thus satisfy (a) by choosing any $\epsilon \in (\eta, 1)$. □

Definition 1.1.2. Define a *relative annulus* over P_K to be a subspace of $P_K \times A_K[0,1)$ satisfying one of the equivalent conditions of Lemma 1.1.1.

Definition 1.1.3. Given a coherent (locally free) sheaf $\mathcal{E}$ on a relative annulus X containing $P_K \times A_K(\epsilon, 1)$, there is a unique coherent (locally free) sheaf $\mathcal{F}$ on $A_L(\epsilon, 1)$ such that for each closed aligned subinterval I of $(\epsilon, 1)$, we have an identification

$$\Gamma(A_L(I), \mathcal{F}) \cong \Gamma(P_K \times A_K(I), \mathcal{E}) \otimes_{\Gamma(P_K \times A_K(I), \mathcal{O})} \Gamma(A_L(I), \mathcal{O}),$$

and these identifications commute with restriction maps. We call $\mathcal{F}$ the *generic fibre* of $\mathcal{E}$. (See [**17**, Definition 5.3.3] for more details.)

The following lemma will be useful in consideration of generic fibres.

Lemma 1.1.4. *For $\rho \in (0,1) \cap \Gamma^*$, suppose that $f \in \operatorname{Frac} \Gamma(P_K \times A_K[\rho,\rho], \mathcal{O})$ can be written as a ratio of two elements of $\Gamma(P_K \times A_K[\rho,\rho], \mathcal{O})$, neither of which has a zero in $A_L[\rho,\rho]$. Then the open dense subscheme U of Z can be chosen so that for any $x \in]U[_P \times A_K[\rho,\rho]$, $|f(x)| = |f|_\rho$.*

Proof. It suffices to consider $f \in \Gamma(P_K \times A_K[\rho,\rho], \mathcal{O})$ having no zero in $A_L[\rho,\rho]$, since by hypothesis we can write the original f as a quotient of two such functions. Since f has no zero in $A_L[\rho,\rho]$, its Newton polygon (in the sense of Lazard [**24**]) has no segment of the corresponding slope; that is, if we write $f = \sum_{i \in \mathbb{Z}} c_i t^i$ with $c_i \in \Gamma(P_K, \mathcal{O})$, then there is a unique index i with $|c_i|\rho^i = |f|_\rho$. It thus suffices to check the given assertion for $f = c_i t^i$, for which it is evident: choose a scalar $\lambda \in K^\times$ such that λc_i belongs to $\Gamma(P, \mathcal{O})$ and has nonzero image in $\Gamma(Z, \mathcal{O})$, then take U not meeting the zero locus of said image. □

Definition 1.1.5. Let X be a relative annulus over P_K containing $P_K \times A_K(\epsilon, 1)$, let $\mathcal{E}$ be a ∇-module on X relative to K_0, and let $\mathcal{F}$ be the generic fibre of $\mathcal{E}$. Then $\mathcal{F}$ naturally admits the structure of a ∇-module on $A_L(\epsilon, 1)$ relative to K_0, in the sense of [**15**, Definition 2.4.5]. We say that $\mathcal{E}$ is *solvable at* 1 if $\mathcal{F}$ is, in which case we define the *highest break*, *break multiset*, and *differential Swan conductor* of $\mathcal{E}$ as the corresponding items associated to $\mathcal{F}$.

Remark 1.1.6. By the results of [**15**, § 2.6], the constructions in Definition 1.1.5 are invariant under pullback along an automorphism of $P_K \times A_K[\epsilon, 1)$ for $\epsilon \in (0,1) \cap \Gamma^*$, even if the automorphism does not preserve P_K or the projection onto P_K.

1.2. Fringed relative annuli

We will have use for a variant of the concept of a relative annulus; the resulting objects are related to relative annuli in the same way that the weak formal schemes of Meredith [**26**] are related to ordinary formal schemes, or the dagger spaces of Grosse-Klönne [**9**] are related to ordinary rigid spaces.

Definition 1.2.1. A *strict neighborhood* of $]U[_P$ in P_K is a rigid subspace $W \subseteq P_K$ such that $\{W,]Z \setminus U[_P\}$ is an admissible covering of P_K.

Lemma 1.2.2. *Suppose that $Z \setminus U$ has pure codimension 1 in Z. Let W be a strict neighborhood of $]U[_P$ in P_K. Then for each $c \in (0,1)$, there exists a strict neighborhood $W' \subseteq W$ of $]U[_P$ in P_K such that for each $f \in \Gamma(W, \mathcal{O})$,*

$$|f|_{\sup,W'} \leqslant |f|_{\sup,W}^c |f|_L^{1-c}.$$

Proof. See [**14**, Proposition 3.5.2]. □

Definition 1.2.3. We say a subspace $Y \subseteq P_K \times A_K[0,1)$ is a *fringed relative annulus* over $]U[_P$ (or better, over the inclusion $]U[_P \hookrightarrow P_K$) if Y satisfies the following property for some $\epsilon \in (0,1)$: for every closed aligned subinterval I of $(\epsilon, 1)$, there is a strict neighborhood W of $]U[_P$ in P_K such that $W \times A_K(I) \subset Y$.

Remark 1.2.4. We can use the same definition to define a fringed relative annulus over $]U[_P$ for P smooth *proper* over Spf $\mathfrak{o}_K$. We will have occasion to do this in § 4.1.

Definition 1.2.5. Let Y be a fringed relative annulus over $]U[_P$ in P_K. Then the intersection $Y_0 = Y \cap (]U[_P \times A_K[0,1))$ is a relative annulus over $]U[_P$; we call Y_0 the *core* of Y. We will extend various properties of relative annuli, or sheaves on relative annuli, to fringed relative annuli by restriction to the core.

Lemma 1.2.6. *Suppose that $Z \setminus U$ has pure codimension 1 in Z. Let W be a strict neighborhood of $]U[_P$. Let I be a closed aligned interval, let I' be a closed aligned subinterval of the interior of I, and choose $\rho \in I' \cap \Gamma^*$. Then there exists a strict neighborhood W' of $]U[_P$ in P_K such that within $\Gamma(A_L[\rho,\rho], \mathcal{O})$,*

$$\Gamma(W \times A_K[\rho,\rho], \mathcal{O}) \cap \Gamma(A_L(I), \mathcal{O}) \subseteq \Gamma(W' \times A_K(I'), \mathcal{O}).$$

Proof. Write $I = [a,b]$ and $I' = [a',b']$, so that $a < a' \leqslant \rho \leqslant b' < b$. Note that for $c \in (0,1)$ sufficiently close to 0, we have

$$\rho^c a^{1-c} < a', \qquad b' < \rho^c b^{1-c}. \tag{1.1}$$

Fix one such c; by Lemma 1.2.2, we can choose the strict neighborhood W' so that for any $f \in \Gamma(W, \mathcal{O})$,

$$|f|_{\sup,W'} \leqslant |f|^c_{\sup,W} |f|^{1-c}_L. \tag{1.2}$$

For $f \in \Gamma(A_L[\rho,\rho], \mathcal{O})$, we can write $f = \sum_{i \in \mathbb{Z}} f_i t^i$ with $f_i \in L$. If $f \in \Gamma(W \times A_K[\rho,\rho], \mathcal{O})$, then $f_i \in \Gamma(W, \mathcal{O})$ for each i, and $|f_i|_{\sup,W} \rho^i \to 0$ as $i \to \pm\infty$; if $f \in \Gamma(A_L(I), \mathcal{O})$, then for each $\eta \in I$, $|f_i|_L \eta^i \to 0$ as $i \to \pm\infty$. If both containments hold, then by (1.2),

$$\lim_{i \to \pm\infty} |f_i|_{\sup,W'} (\rho^c \eta^{1-c})^i = 0 \quad (\eta \in I);$$

by (1.1), this implies that for any $\eta \in I'$, $|f_i|_{\sup,W'} \eta^i \to 0$ as $i \to \pm\infty$. This proves the claim. □

Lemma 1.2.7. *Let*

$$\begin{array}{ccc} R & \longrightarrow & S \\ \downarrow & & \downarrow \\ T & \longrightarrow & U \end{array}$$

be a commuting diagram of inclusions of integral domains, such that the intersection $S \cap T$ within U is equal to R. Let M be a finite locally free R-module. Then the intersection of $M \otimes_R S$ and $M \otimes_R T$ within $M \otimes_R U$ is equal to M.

Proof. See [**22**, Lemma 2.3.1]. □

Lemma 1.2.8. *Suppose that $Z \setminus U$ has pure codimension 1 in Z. Let Y be a fringed relative annulus over $]U[_K$, choose $\epsilon \in (0,1)$ as in Definition 1.2.3, choose $\rho \in (\epsilon, 1) \cap \Gamma^*$, and choose a strict neighborhood W of $]U[_K$ in P_K such that $W \times A_K[\rho,\rho] \subset Y$. Let*

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$\mathcal{E}$ be a coherent locally free sheaf on Y, and let $\mathcal{F}$ be the generic fibre of $\mathcal{E}$ on $A_L(\epsilon, 1)$. Suppose that $\boldsymbol{v} \in \Gamma(A_L[\rho, \rho], \mathcal{F})$ satisfies

$$\boldsymbol{v} \in \Gamma(W \times A_K[\rho, \rho], \mathcal{E}) \cap \Gamma(A_L(\epsilon, 1), \mathcal{F}).$$

Then there exists a fringed relative annulus Y' over $]U[_K$ such that $\boldsymbol{v} \in \Gamma(Y', \mathcal{E})$.

Proof. It suffices to show that for each closed aligned subinterval I' of $(\epsilon, 1)$ containing ρ, there exists a strict neighborhood W' of $]U[_K$ in P_K such that $\boldsymbol{v} \in \Gamma(W' \times A_K(I'), \mathcal{E})$. Choose a closed aligned interval I of $(\epsilon, 1)$ containing I' in its interior; by Lemma 1.2.6, we can choose W' so that within $\Gamma(A_L[\rho, \rho], \mathcal{O})$,

$$\Gamma(W \times A_K[\rho, \rho], \mathcal{O}) \cap \Gamma(A_L(I), \mathcal{O}) \subseteq \Gamma(W' \times A_K(I'), \mathcal{O}).$$

We may then apply Lemma 1.2.7 to deduce the claim. □

1.3. Globalizing the break decomposition

The main result of this subsection (Theorem 1.3.2) is a globalized version of [**15**, Theorem 2.7.2]. To prove it, we use the following relative version of [**15**, Lemma 2.7.10].

Proposition 1.3.1. *Suppose $u_{m+1}, \dots, u_n \in \Gamma(P, \mathcal{O})$ are such that $\mathrm{d}u_{m+1}, \dots, \mathrm{d}u_n$ freely generate $\Omega^1_{P/\mathfrak{o}_K}$ over $\mathfrak{o}_K$. Write $\partial_1, \dots, \partial_{n+1}$ for the basis of derivations on $P_K \times A_K[0, 1)$ over K_0 dual to $\mathrm{d}u_1, \dots, \mathrm{d}u_n, \mathrm{d}t$. Let Y be a fringed relative annulus over $]U[_P$. Let $\mathcal{E}$ be a ∇-module on Y which is solvable at 1. Let $\mathcal{F}$ denote the generic fibre of $\mathcal{E}$, viewed as a ∇-module relative to K_0, and choose $i \in \{1, \dots, n+1\}$ such that ∂_i is eventually dominant for $\mathcal{F}$. Suppose that there exist $\rho \in (0, 1)$ arbitrarily close to 1 such that the scale multiset for ∂_i on $\mathcal{F}_\rho$ contains more than one element. Then after shrinking U (to another open dense subscheme of Z) and Y (to a fringed relative annulus over $]U[_P$), $\mathcal{E}$ becomes decomposable.*

Proof. We first treat the case $i = n + 1$. Let b be the highest break of $\mathcal{F}$. By [**22**, Theorem 2.3.9, Theorem 2.6.1] (replacing [**15**, Theorem 2.7.2, Remark 2.7.7]), we may choose $\epsilon \in (0, 1)$ such that $\mathcal{F}$ admits a break decomposition over $A_L(\epsilon, 1)$, and for all $\rho \in (\epsilon, 1)$, ∂_{n+1} is dominant for $\mathcal{F}_\rho$ and $T(\mathcal{F}, \rho) = \rho^b$. Pick a closed aligned interval $I \subset (\epsilon, 1)$ of positive length for which there exists a nonnegative integer m such that

$$|p|^{p^{-m+1}/(p-1)} < T(\mathcal{F}, \rho) < |p|^{p^{-m}/(p-1)} \quad (\rho \in I).$$

Let $\mathcal{F}_m$ be the ∇-module on $A_L(I^{p^m})$ which is the m-fold Frobenius antecedent of $\mathcal{F}$ in the t-direction, as produced by [**12**, Theorem 6.15], so that

$$T(\mathcal{F}_m, \rho^{p^m}) = T(\mathcal{F}, \rho)^{p^m} < |p|^{1/(p-1)}.$$

Since the defining inequality for Frobenius antecedents is strict, Lemma 1.2.2 allows us to correspondingly construct an m-fold Frobenius antecedent $\mathcal{E}_m$ of $\mathcal{E}$ on $W \times A_K(I^{p^m})$ for some strict neighborhood W of $]U[_P$ in P_K.

Choose a cyclic vector for $\mathcal{E}_m$ with respect to ∂_{n+1} over $\operatorname{Frac} \Gamma(W \times A_K(I^{p^m}), \mathcal{O})$, and let $Q = T^d + \sum_{i=0}^{d-1} a_i T^i$ be the corresponding twisted polynomial. Pick $\rho \in I \cap \Gamma^*$ such that each a_i can be written as a ratio of two elements of $\Gamma(W \times A_K(I^{p^m}), \mathcal{O})$, neither having any zeroes in $A_L[\rho^{p^m}, \rho^{p^m}]$; the restriction excludes only finitely many ρ. By Lemma 1.1.4, after shrinking U, each of the a_i becomes invertible on $]U[_P \times A_K[\rho^{p^m}, \rho^{p^m}]$, and the norm of $a_i(x)$ for each $x \in]U[_P \times A_K[\rho^{p^m}, \rho^{p^m}]$ equals the supremum norm of a_i in $A_L[\rho^{p^m}, \rho^{p^m}]$.

Let a_j be the coefficient at which the Newton polygon of Q with respect to the supremum norm on $]U[_P \times A_K[\rho^{p^m}, \rho^{p^m}]$ has its first breakpoint (i.e. the one separating the segment of least slope). By a suitable application of Lemma 1.2.2, we see that after shrinking W, a_j is also a breakpoint (though maybe not the first) when computing slopes of Q using the supremum norm on $W \times A_K[\rho^{p^m}, \rho^{p^m}]$. Using Christol's factorization theorem [**20**, Theorem 2.2.2] for the supremum norm on $W \times A_K[\rho^{p^m}, \rho^{p^m}]$ (and otherwise arguing as in [**20**, Theorem 6.4.4]), we deduce that the factorization of Q provided by [**15**, Proposition 1.1.10] that splits off the least $d-j$ slopes (counting multiplicity) is defined over $W \times A_K[\rho^{p^m}, \rho^{p^m}]$. By performing the same argument again in the opposite twisted polynomial ring (as in the proof of [**17**, Proposition 3.3.10]), we obtain a projector in $\mathcal{E}_m^\vee \otimes \mathcal{E}_m$ on $W \times A_K[\rho^{p^m}, \rho^{p^m}]$. This pulls back to a projector in $\mathcal{E}^\vee \otimes \mathcal{E}$ on $W \times A_K[\rho, \rho]$; since the projector is already defined on $A_L(\epsilon, 1)$, by Lemma 1.2.8 it becomes defined on Y after shrinking Y. This proves the desired decomposability.

We now suppose $i \neq n+1$. By Lemma 1.2.7, we may check the claim after enlarging the constant subfield of K; by adjoining an element of the same norm as u_i and then rescaling, we may reduce to the case $|u_i| = 1$. In this case, we may perform rotation as in the proof of [**15**, Lemma 2.7.10]; that is, first pull back along a map effecting $t \mapsto t^{p^N}$ for N a suitably large integer, then along a map effecting $u_i \mapsto u_i + t$. (Note that both of these extend to maps between suitable fringed relative annuli: by Lemma 1.2.2, the series in [**15**, Definition 2.6.2] converges on some fringed relative annulus.) As in the proof of [**15**, Lemma 2.7.10], the decomposition obtained after rotation descends back to $\mathcal{E}$. □

Theorem 1.3.2. *Let Y be a fringed relative annulus over $]U[_P$. Let $\mathcal{E}$ be a ∇-module on Y which is solvable at 1. Then after shrinking U (to another open dense subscheme of Z) and Y (to a fringed relative annulus over $]U[_P$), there exists a unique decomposition*

$$\mathcal{E} = \bigoplus_{b \in \mathbb{Q}_{\geqslant 0}} \mathcal{E}_b$$

of ∇-modules on Y, such that $\mathcal{E}_b$ has uniform break b.

Proof. The claim is local on Z, so we may reduce to the case where there exist $u_{m+1}, \dots, u_n \in \Gamma(P, \mathcal{O})$ such that $\mathrm{d}u_{m+1}, \dots, \mathrm{d}u_n$ freely generate $\Omega^1_{P/\mathfrak{o}_K}$ over $\mathfrak{o}_K$. After shrinking U and Y, we may reduce to the case where $\mathcal{E}$ remains indecomposable after further shrinking of U and Y. In this case, $\mathcal{E}$ is forced to have a uniform break by Proposition 1.3.1. □

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Definition 1.3.3. For $z \in Z$, we say that *the break decomposition of $\mathcal{E}$ extends across z* if we can choose U in Theorem 1.3.2 to contain z.

We will need a criterion for detecting when the break decomposition extends across z.

Lemma 1.3.4. *With notation as in Theorem 1.3.2, let K_ρ, L_ρ be the completions of $K(t)$, $L(t)$ for the ρ-Gauss norm. Suppose that $z \in U$ and that for each $\rho \in (\epsilon, 1)$ sufficiently close to 1, the restriction of $\mathcal{E}$ to $]U[_K \times_K K_\rho$ admits a decomposition whose restriction to L_ρ coincides with the restriction of the break decomposition of the generic fibre of $\mathcal{E}$. Then the break decomposition of $\mathcal{E}$ extends across z.*

Proof. For each closed interval $I \subset (\epsilon, 1)$ containing ρ, inside L_ρ we have

$$\Gamma(]U[_K \times_K K_\rho, \mathcal{O}) \cap \Gamma(A_L(I), \mathcal{O}) = \Gamma(]U[_K \times A_K(I), \mathcal{O}).$$

We may thus deduce the claim from Lemma 1.2.7. □

2. Representations, isocrystals, and conductors

In this section, we define the differential highest break and Swan conductor associated to an isocrystal on a k-variety X and a boundary divisor in some compactification of X along which the isocrystal is overconvergent. We then show how a special class of overconvergent isocrystals, those admitting unit-root Frobenius actions, relate closely to representations of the étale fundamental group of X. This allows us to define differential ramification breaks and Swan conductors for an appropriate class of p-adic representations, including discrete representations (those with open kernel).

Convention 2.0.1. For the rest of this paper, a *variety over k* will be a reduced separated (but not necessarily irreducible) scheme of finite type over k, and *points* of a variety will always be closed points unless otherwise specified.

2.1. Convergent and overconvergent isocrystals

This is not the place to reintroduce the full theory of convergent and overconvergent isocrystals; we give here merely a quick summary. See [**14**] for a less hurried review, or [**5**] for a full development.

Definition 2.1.1. Let P be an affine formal scheme of finite type over $\operatorname{Spf} \mathfrak{o}_K$ with special fibre Y. Let X be an open dense subscheme of Y such that $Z = Y \setminus X$ is of pure codimension 1 in Y, and P is smooth over $\mathfrak{o}_K$ in a neighborhood of X; let Q be the open formal subscheme of P with special fibre X. An *isocrystal on X overconvergent along Z* is a ∇-module $\mathcal{E}$ relative to K_0 on a strict neighborhood of $]X[_P$ in P_K, whose formal Taylor isomorphism converges on a strict neighborhood of $]X[_{P \times P}$ in $P_K \times P_K$; morphisms between these should likewise be defined on some strict neighborhood. This definition turns out to be canonically independent of the choices of P, so extends to arbitrary pairs (X, Y) where X is an open dense subscheme of Y smooth over k, and $Y \setminus X$ is of pure codimension 1 in Y. (The codimension 1 condition can be eased with a bit more work.) If Y is proper, then the category of isocrystals on X overconvergent along $Y \setminus X$ is independent of the choice of Y; we call such objects *overconvergent isocrystals* on X. If on the other hand $Y = X$, we say $\mathcal{E}$ is a *convergent isocrystal* on X.

Remark 2.1.2. The usual definition of an isocrystal involves a ∇-module relative to K, not K_0. In fact, there is no harm in adding this extra data: the Taylor isomorphism is determined by the connection relative to K, so it is harmless to carry the extra components of the connection through the arguments in [**5**]. The construction relative to a subfield is useful for certain arguments where one wants to reduce the dimension of a variety without losing critical data about the connection. See Theorem 3.4.3 for an argument of this form.

Definition 2.1.3. Let ϕ_K be a q-power Frobenius lift on K acting on K_0; that is, ϕ_K is an isometric endomorphism of K acting on K_0, and its action on k is the q-power absolute Frobenius. With notation as in Definition 2.1.1, a *Frobenius structure* on an isocrystal $\mathcal{E}$ on X overconvergent along Z is an isomorphism $F : \phi^*\mathcal{E} \cong \mathcal{E}$, for ϕ a ϕ_K-semilinear q-power Frobenius lift on Q; note that ϕ extends to a strict neighborhood of Q_K in P_K, so that it makes sense to require F to be an isomorphism of overconvergent isocrystals. The word *F-isocrystal* is shorthand for *isocrystal with Frobenius structure*.

Proposition 2.1.4. *Assume that K is discretely valued. Let $X \hookrightarrow Y$ be an open immersion of k-varieties with dense image, with X smooth and $Y \setminus X$ of pure codimension* 1 *in Y. Then the restriction functor from the category of F-isocrystals on X overconvergent along $Y \setminus X$ to the category of convergent F-isocrystals on X is fully faithful.*

Proof. It suffices to check relative to K, in which case this assertion becomes [**16**, Theorem 4.2.1]. □

Definition 2.1.5. With notation as in Definition 2.1.3, we say that $\mathcal{E}$ is *unit-root* if for each closed point $x \in X$, the pullback of $\mathcal{E}$ to x, which we may view as a finite-dimensional K-vector space V_x equipped with a ϕ_K-semilinear endomorphism ϕ, admits a $\mathfrak{o}_K$-lattice T such that ϕ induces an isomorphism $\phi_K^*(T) \cong T$.

2.2. Globalizing the Swan conductor

Much as the calculations on relative annuli in [**14**, § 3] were used later therein to define notions of constant/unipotent local monodromy for overconvergent isocrystals, we can define differential Swan conductors for overconvergent isocrystals as follows. (See [**14**, § 4] for a similar construction.)

Definition 2.2.1. Let $\bar{X}$ be a smooth k-variety, let Z be a smooth irreducible divisor on $\bar{X}$, and let $\mathcal{E}$ be an isocrystal on $X = \bar{X} \setminus Z$ overconvergent along Z. Suppose for the moment that there exists a smooth irreducible affine formal scheme Q over $\operatorname{Spf} \mathfrak{o}_K$ with $Q_k \cong \bar{X}$; then $\mathcal{E}$ can be realized as a ∇-module on some strict neighborhood V of $]X[_Q$ in Q_K, as in Definition 2.1.1. Moreover, $]Z[_Q$ is a relative annulus by Berthelot's fibration theorem [**5**, Théorème 1.3.7], [**14**, Proposition 2.2.9], as then is $W = V \cap\,]Z[_Q$ after appropriately shrinking V. The overconvergence property forces the restriction of

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$\mathcal{E}$ to W to be solvable at 1, so $\mathcal{E}$ admits a break multiset and Swan conductor (relative to K_0) via Definition 1.1.5.

Now go back and note that the construction persists under restricting from X to an open neighborhood of any given point of Z. Moreover, by Remark 1.1.6, there is no dependence on how $]Z[_Q$ is viewed as a relative annulus. (This implies independence from the choice of Q itself, since Q is unique up to noncanonical isomorphism by [**4**, Proposition 1.4.3].) Consequently, the definitions extend unambiguously even if $\bar{X}$ is reducible or does not lift globally. We write $b_i(\mathcal{E}, Z)$ and $\mathrm{Swan}(\mathcal{E}, Z)$ for the differential ramification breaks (listed in decreasing order as i increases) and differential Swan conductor of $\mathcal{E}$ along Z.

Remark 2.2.2. If k is perfect, X is a smooth irreducible k-variety, $\mathcal{E}$ is an overconvergent isocrystal on X, and v is any divisorial valuation on the function field $k(X)$ over k, then we can also define the break multiset and Swan conductor of $\mathcal{E}$ along v, by blowing up into the case where v is centered on a generically smooth divisor, then applying Definition 2.2.1. If k is imperfect, then the previous discussion applies unless blowing up gives a divisor which is geometrically nonreduced. If $\mathcal{E}$ is only overconvergent along the boundary of some partial compactification $\bar{X}$ of X, then the previous discussion applies to divisorial valuations which are centered on $\bar{X}$. (That is, there must exist some blowup of $\bar{X}$ on which the valuation corresponds to the order of vanishing along an irreducible divisor.)

2.3. Étale fundamental groups and unit-root isocrystals

Hypothesis 2.3.1. Throughout this subsection, fix a power q of p, and assume that the field $k = k_0$ is perfect and contains $\mathbb{F}_q$. Assume also that $K = K_0$ is discretely valued, and comes equipped with a q-power Frobenius lift ϕ_K. Let K^ϕ denote the fixed field of K under ϕ; it is a complete discretely valued field with residue field $\mathbb{F}_q$.

Hypothesis 2.3.2. Throughout this subsection, let X be a smooth irreducible k-variety and let $\bar{x}$ be a geometric point of X. We write $\pi_1(X, \bar{x})$ for the étale fundamental group of X with basepoint $\bar{x}$.

Convention 2.3.3. By a *p-adic representation* of $\pi_1(X, \bar{x})$, we will mean a continuous homomorphism $\rho : \pi_1(X, \bar{x}) \to \mathrm{GL}(V)$ for $V = V(\rho)$ a finite-dimensional K^ϕ-vector space.

The following result is due to Crew [**7**, Theorem 2.1].

Theorem 2.3.4. *There is a natural equivalence of categories (functorial in X) between the category of p-adic representations of $\pi_1(X, \bar{x})$ and the category of convergent unit-root F-isocrystals on X.*

Crew also posed the question of identifying which p-adic representations correspond to overconvergent unit-root F-isocrystals on X. For X a curve, this was answered by Tsuzuki [**30**]; the hard work in the general case is already present in Tsuzuki's work. All we need to add is a bit of analysis of extendability for overconvergent isocrystals, from [**14**].

Definition 2.3.5. Let v be a divisorial valuation on the function field $k(X)$ over k, and let $k(X)_v$ be the completion of $k(X)$ under v. Fix a separable closure $k(X)_v^{\mathrm{sep}}$ of $k(X)_v$ and a perfect closure $k(X)_v^{\mathrm{alg}}$ of $k(X)_v^{\mathrm{sep}}$, and let $\bar{x}$ be the geometric point of X corresponding to the inclusion $k(X) \hookrightarrow k(X)_v^{\mathrm{alg}}$. Put $\eta = \operatorname{Spec} k(X)_v$; then the morphism $\eta \to X$ corresponding to the inclusion $k(X) \hookrightarrow k(X)_v$ induces a homomorphism $\iota : \pi_1(\eta, \bar{x}) \to \pi_1(X, \bar{x})$, and the former group may be canonically identified with $\mathrm{Gal}(k(X)_v^{\mathrm{sep}}/k(X)_v)$. Let I_v be the inertia subgroup of $\mathrm{Gal}(k(X)_v^{\mathrm{sep}}/k(X)_v)$, i.e. the subgroup acting trivially on the residue field of $k(X)_v^{\mathrm{sep}}$; we refer to any subgroup of $\pi_1(X, \bar{x})$ conjugate to $\iota(I_v)$ as an *inertia subgroup* corresponding to v.

Definition 2.3.6. We say a p-adic representation ρ of $\pi_1(X, \bar{x})$ is *unramified* if every inertia subgroup of $\pi_1(X, \bar{x})$ lies in the kernel of ρ. If X admits a dense open immersion into a smooth proper irreducible k-variety $\bar{X}$ (as would be ensured by a suitably strong form of resolution of singularities in positive characteristic), then by Zariski–Nagata purity [**10**, Exposé X, Théorème 3.1], ρ is unramified if and only if ρ factors through $\pi_1(\bar{X}, \bar{x})$. We say ρ is *potentially unramified* if there exists a finite étale cover Y of X such that for any geometric point $\bar{y}$ of Y over $\bar{x}$, the restriction of ρ to $\pi_1(Y, \bar{y})$ is unramified (it suffices to check for a single $\bar{y}$).

Theorem 2.3.7. *The functor of Theorem 2.3.4 induces an equivalence between the category of potentially unramified p-adic representations of $\pi_1(X, \bar{x})$, and the category of overconvergent unit-root F-isocrystals on X.*

Proof. We first show that every representation ρ corresponding to an overconvergent unit-root F-isocrystal is potentially unramified. Choose a ρ-stable $\mathfrak{o}_{K^\phi}$-lattice T in $V = V(\rho)$; then there is a unique finite étale Galois cover Y of X such that for any geometric point $\bar{y}$ of Y over $\bar{x}$, $\pi_1(Y, \bar{y})$ equals the kernel of the action of ρ on $T/2pT$. By [**31**, Proposition 7.2.1], the intersection of $\pi_1(Y, \bar{y})$ with any inertia subgroup of $\pi_1(X, \bar{x})$ belongs to the kernel of ρ; hence ρ is potentially unramified.

We next show that every potentially unramified ρ corresponds to an overconvergent unit-root F-isocrystal. Let $\mathcal{E}$ be the convergent unit-root F-isocrystal on X corresponding to ρ. Choose a finite étale Galois cover $f : Y \to X$ such that for any geometric point $\bar{y}$ of Y over $\bar{x}$, the restriction of ρ to $\pi_1(Y, \bar{y})$ is unramified. By de Jong's alterations theorem [**8**, Theorem 4.1], there exists an open dense subscheme U of X and a finite étale cover $g : Z \to f^{-1}(U)$ such that Z admits a dense open immersion into a smooth proper k-variety $\bar{Z}$. There is no harm in moving the basepoints $\bar{x}$ and $\bar{y}$ so that $\bar{x} \in U$; then for any geometric point $\bar{z}$ of Z over $\bar{x}$, the restriction of ρ to $\pi_1(Z, \bar{z})$ is again unramified, so factors through $\pi_1(\bar{Z}, \bar{z})$.

By Theorem 2.3.4, this restriction of ρ corresponds to a convergent unit-root F-isocrystal $\mathcal{F}$ on $\bar{Z}$. Since $\bar{Z}$ is proper, there is no distinction between convergent and overconvergent on $\bar{Z}$, so we may restrict $\mathcal{F}$ to an overconvergent F-isocrystal on Z. Now put $\mathcal{G} = f_* g_* \mathcal{F}$, which is an overconvergent unit-root F-isocrystal on U (see [**31**, § 5] for the pushforward construction). Let σ be the p-adic representation of $\pi_1(U, \bar{x})$ corresponding to $\mathcal{G}$; then adjunction and trace give π_1-equivariant maps $V(\rho) \to V(\sigma) \to V(\rho)$ whose

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composition is the identity. Composing the other way gives a projector on $V(\sigma)$, corresponding to a projector on $\mathcal{G}$ in the category of convergent unit-root F-isocrystals on U. By Proposition 2.1.4, this projector actually exists in the overconvergent category; its image is an overconvergent unit-root F-isocrystal on U which becomes isomorphic to $\mathcal{E}$ in the convergent category. By [**14**, Proposition 5.3.7], that isomorphism ensures that $\mathcal{E}$ is the restriction to U of an overconvergent unit-root F-isocrystal on X, as desired. □

Theorem 2.3.7 can also be stated for partially overconvergent isocrystals.

Definition 2.3.8. Let $X \hookrightarrow \bar{X}$ be an open immersion of k-varieties with dense image, with X smooth irreducible. We say a p-adic representation ρ of $\pi_1(X, \bar{x})$ is *unramified on* $\bar{X}$ if every inertia subgroup of $\pi_1(X, \bar{x})$ corresponding to a divisorial valuation centered on $\bar{X}$ lies in the kernel of ρ. We say ρ is *potentially unramified on* $\bar{X}$ if there exists a connected finite cover $f : \bar{Y} \to \bar{X}$ étale over X, such that for any geometric point $\bar{y}$ of $Y = f^{-1}(X)$, the restriction of ρ to $\pi_1(Y, \bar{y})$ is unramified on $\bar{Y}$.

Theorem 2.3.9. *The functor of Theorem 2.3.4 induces an equivalence between the category of p-adic representations of $\pi_1(X, \bar{x})$ potentially unramified on $\bar{X}$, and the category of unit-root F-isocrystals on X overconvergent along $\bar{X} \setminus X$.*

Proof. The proof is as in Theorem 2.3.7. Note that the case $X = \bar{X}$ is Theorem 2.3.4 itself, while the case where $\bar{X}$ is proper over k is Theorem 2.3.7. □

Remark 2.3.10. One can also use the construction of Abbes and Saito [**1**, **2**] to define Swan conductors for p-adic representations. It has been shown recently by Xiao [**32**] that this construction agrees with the differential Swan conductor. Consequently, the results we obtain about differential Swan conductors will apply also to Abbes–Saito conductors. This agreement also occurs in the ℓ-adic setting, as discussed in § 5. (In [**33**], Xiao gives an analogue of differential Swan conductors in mixed characteristic, and obtains an analogous comparison theorem with Abbes–Saito conductors.)

2.4. Normalization of conductors

When studying variation of differential Swan conductors, it will be useful to normalize as follows.

Definition 2.4.1. Let X be a smooth irreducible k-variety, let $X \hookrightarrow \bar{X}$ be an open immersion of k-varieties with dense image, and let $\mathcal{E}$ be an isocrystal on X overconvergent along $\bar{X} \setminus X$. As noted in Remark 2.2.2, we can define the differential ramification breaks and the differential Swan conductor of $\mathcal{E}$ with respect to a suitable divisorial valuation v on $k(X)$ over k centered on $\bar{X}$ (which may be arbitrary if k is perfect); we refer to these as being in their *natural normalization*. For $t \in k(X)^*$ with $v(t) \neq 0$, we define the *normalization with respect to* t of the differential ramification breaks, or the differential Swan conductor, with respect to v as the natural normalization divided by the index of $v(t)\mathbb{Z}$ in the value group of v.

For an easy example, we return to the Dwork isocrystals of [**15**, Example 3.5.10], but this time in a global setting.

Definition 2.4.2. Assume that K contains an element π with $\pi^{p-1} = -p$ (a *Dwork pi*). Let $\mathcal{L}$ be the ∇-module of rank 1 on $\mathbb{A}^1_K$ with ∇-action given on a generator $\boldsymbol{v}$ by

$$\nabla(\boldsymbol{v}) = \pi\boldsymbol{v} \otimes \mathrm{d}t.$$

One shows by a direct calculation that $\mathcal{L}$ gives an overconvergent F-isocrystal on $\mathbb{A}^1_k$, called the (*standard*) *Dwork isocrystal*; it is in fact the image under the functor of Theorem 2.3.7 of a nontrivial character of the Artin–Schreier cover $\operatorname{Spec} k[z,t]/(z^p - z - t) \to \operatorname{Spec} k[t] = \mathbb{A}^1_k$. For X any variety over k and $f \in \Gamma(X, \mathcal{O})$, we may identify f with a regular map $X \to \mathbb{A}^1_k$, and define $\mathcal{L}_f$ as the pullback $f^*\mathcal{L}$, as an overconvergent F-isocrystal on X.

Example 2.4.3. Assume $k = k_0$, let $\mathcal{E}$ be the Dwork isocrystal $\mathcal{L}_{xy}$ on $\mathbb{A}^2_k$, and compute conductors using [**15**, Example 3.5.10]. For positive integers a, b with $\gcd(a,b) = 1$, let $x^{-a} \sim y^{-b}$ denote the exceptional divisor of the blowup of the ideal sheaf on $\mathbb{P}^1_k \times \mathbb{P}^1_k$ concentrated at (∞,∞) generated by x^{-a}, y^{-b}. We extend this notation to the case $(a,b) = (1,0), (0,1)$, meaning the divisors on $\mathbb{P}^1_k \times \mathbb{P}^1_k$ cut out by y^{-1}, x^{-1}, respectively.

For $r \in \mathbb{Q}_{\geqslant 0}$, write $r = b/a$ in lowest terms and write $x^{-1} \sim y^{-r}$ for $x^{-a} \sim y^{-b}$. Along $x^{-1} \sim y^{-r}$, the Swan conductor in its natural normalization is $a + b$, which behaves erratically as r varies. However, the normalization with respect to y is $1 + r$, which is an affine function of r. This behavior will prove to be typical; see Theorem 4.2.7.

3. ∇-modules on polyannuli

The easiest setting in which to study the variation of differential highest breaks and Swan conductors is on polyannuli, or more conveniently on the generalized polyannuli of [**17**, § 4]. Using some analysis of differential modules on such spaces carried out in [**22**] (jointly with Liang Xiao), we obtain a strong result on the variation of differential Swan conductors (Theorem 3.4.6). In fact, all results in this section should be considered to be joint work with Xiao, as explained in Remark 3.3.4.

3.1. Convex functions

We need some basic definitions and theorems about convex functions from [**17**, § 2] and [**22**, § 3]. For stronger results along these lines, see [**21**].

Definition 3.1.1. Let C be a convex subset of $\mathbb{R}^n$. A function $f : C \to \mathbb{R}$ is *convex* if for all $x, y \in C$ and $t \in [0,1]$,

$$tf(x) + (1-t)f(y) \geqslant f(tx + (1-t)y);$$

such a function is continuous on the interior of C.

Definition 3.1.2. An *affine functional* on $\mathbb{R}^n$ is a function $\lambda : \mathbb{R}^n \to \mathbb{R}$ of the form $\lambda(x) = a_1x_1 + \cdots + a_nx_n + b$ for some $a_1, \ldots, a_n, b \in \mathbb{R}$. We say λ is *transintegral* if $a_1, \ldots, a_n \in \mathbb{Z}$ and *integral* if also $b \in \mathbb{Z}$.

Definition 3.1.3. A subset C of $\mathbb{R}^n$ is *(trans)rational polyhedral*, or *(T)RP*, if there exist (trans)integral affine functionals $\lambda_1, \dots, \lambda_m$ such that

$$C = \{x \in \mathbb{R}^n : \lambda_i(x) \geqslant 0 \ \ (i = 1, \dots, m)\}.$$

In particular, any TRP set is convex and closed (but not necessarily bounded).

Definition 3.1.4. Let C be a (T)RP subset of $\mathbb{R}^n$. A function $f : C \to \mathbb{R}$ is *polyhedral* if there exist affine functionals $\lambda_1, \dots, \lambda_m$ such that

$$f(x) = \sup_i \{\lambda_i(x)\} \quad (x \in C).$$

Such a function is continuous and convex. We say that f is *(trans)integral polyhedral* if the λ_i can be taken to be (trans)integral.

The following result is [**17**, Theorem 2.4.2].

Theorem 3.1.5. *Let C be a bounded RP subset of $\mathbb{R}^n$. Then a continuous convex function $f : C \to \mathbb{R}$ is integral polyhedral if and only if*

$$f(x) \in \mathbb{Z} + \mathbb{Z}x_1 + \cdots + \mathbb{Z}x_n \quad (x \in C \cap \mathbb{Q}^n). \tag{3.1}$$

The following result is [**22**, Theorem 3.2.4].

Theorem 3.1.6. *Let C be a TRP subset of $\mathbb{R}^n$. Then a function $f : C \to \mathbb{R}$ is transintegral polyhedral if and only if its restriction to the intersection of C with every one-dimensional TRP subset of $\mathbb{R}^n$ is transintegral polyhedral.*

3.2. Generalized polyannuli

We set notation as in [**17**, § 4].

Notation 3.2.1. For $* = (*_1, \dots, *_n)$ and $J = (J_1, \dots, J_n)$, we interpret $*^J$ to mean $*_1^{J_1} \cdots *_n^{J_n}$.

Definition 3.2.2. A subset S of $(0, +\infty)^n$ is *log-(T)RP* if $\log(S) \subseteq \mathbb{R}^n$ is a (trans)rational polyhedral set. We say S is *ind-log-(T)RP* if it is a union of an increasing sequence of log-(T)RP sets; for instance, any open subset of $(0, +\infty)^n$ is covered by ind-log-RP subsets.

Definition 3.2.3. For S an ind-log-TRP set, let $A_K(S)$ be the subspace of the rigid analytic n-space with coordinates $t_1, \dots, t_n$ defined by the condition

$$(|t_1|, \dots, |t_n|) \in S.$$

The elements of $\Gamma(A_K(S), \mathcal{O})$ can be represented by formal series $\sum_{J \in \mathbb{Z}^n} c_J t^J$; for $R = (R_1, \dots, R_n) \in S$, write $|\cdot|_R$ for the R-Gauss norm

$$\Big| \sum c_J t^J \Big|_R = \sup_J \{|c_J| R^J\}.$$

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Lemma 3.2.4. *Let $S \subseteq (0,+\infty)^n$ be an ind-log-TRP subset. For $A, B \in S$ and $c \in [0,1]$, put $R = A^c B^{1-c}$; that is, $r_i = a_i^c b_i^{1-c}$ for $i = 1, \dots, n$. Then for any $f \in \Gamma(A_K(S), \mathcal{O})$,*

$$|f|_R \leqslant |f|_A^c |f|_B^{1-c}.$$

Proof. See [**14**, Lemma 3.1.6(b)] or [**17**, Lemma 4.1.7]. □

The following corollary is loosely analogous to Lemma 1.2.6.

Corollary 3.2.5. *Let S_1, S_2 be log-TRP subsets of $(0,+\infty)^n$ with nonempty intersection, and let*

$$S = \{A^c B^{1-c} : A \in S_1,\ B \in S_2,\ c \in [0,1]\}$$

be the log-convex hull of S_1, S_2. Then inside $\Gamma(A_K(S_1 \cap S_2), \mathcal{O})$, we have

$$\Gamma(A_K(S_1), \mathcal{O}) \cap \Gamma(A_K(S_2), \mathcal{O}) = \Gamma(A_K(S), \mathcal{O}).$$

Definition 3.2.6. Let S be a log-TRP set, and let $\mathcal{E}$ be a ∇-module on $A_K(S)$ relative to K_0. For $R \in S$, let F_R be the completion of $\operatorname{Frac} \Gamma(A_K(S), \mathcal{O})$ under $|\cdot|_R$, viewed as a differential field of order $m + n$ with respect to

$$\partial_1, \dots, \partial_{m+n} = \frac{\partial}{\partial u_1}, \dots, \frac{\partial}{\partial u_m}, \frac{\partial}{\partial t_1}, \dots, \frac{\partial}{\partial t_n}.$$

Put

$$\mathcal{E}_R = \Gamma(A_K(S), \mathcal{E}) \otimes_{\Gamma(A_K(S), \mathcal{O})} F_R,$$

viewed as a differential module over F_R. Let $S(\mathcal{E}, R)$ be the multiset of reciprocals of the scale multiset of $\mathcal{E}_R$. Let $T(\mathcal{E}, R)$ be the least element of $S(\mathcal{E}, R)$, i.e. the reciprocal of the scale of $\mathcal{E}_R$. These constructions are stable under shrinking S, so they make sense even if S is only ind-log-TRP.

The main result we need about differential modules on generalized polyannuli is [**22**, Theorem 3.3.8].

Theorem 3.2.7. *Let S be an ind-log-TRP subset of $(0,+\infty)^n$, and let $\mathcal{E}$ be a ∇-module of rank d on $A_K(S)$ relative to K_0. For $r \in -\log S$, write*

$$S(\mathcal{E}, \mathrm{e}^{-r}) = \{\mathrm{e}^{-f_1(\mathcal{E}, r)}, \dots, \mathrm{e}^{-f_d(\mathcal{E}, r)}\}$$

with $f_1(\mathcal{E}, r) \geqslant \cdots \geqslant f_d(\mathcal{E}, r)$, and put $F_i(\mathcal{E}, r) = f_1(\mathcal{E}, r) + \cdots + f_i(\mathcal{E}, r)$. Then the following hold for $i = 1, \dots, d$.

(a) *(Continuity.) The functions $f_i(\mathcal{E}, r)$ and $F_i(\mathcal{E}, r)$ are continuous.*

(b) *(Convexity.) The function $F_i(\mathcal{E}, r)$ is convex.*

(c) *(Polyhedrality.) The functions $d! F_i(\mathcal{E}, r)$ and $F_d(\mathcal{E}, r)$ are transintegral polyhedral on any TRP subset of $-\log S$.*

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3.3. Solvable modules on polyannuli

Hypothesis 3.3.1. Throughout this subsection, let S be an ind-log-RP set of the form $\{R^c : R \in T,\ c \in (0,1]\}$ for T a log-RP set. Let $\mathcal{E}$ be a ∇-module of rank d on $A_K(S)$ relative to K_0.

Definition 3.3.2. We say that $\mathcal{E}$ is *solvable at* 1 if for each $R \in T$, we have $T(\mathcal{E}, R^c) \to 1$ as $c \to 0^+$. In case $-\log T$ is bounded, it is the log-convex hull of its vertices, which we write as $-\log R_1, \dots, -\log R_l$ for suitable $R_1, \dots, R_l \in T$. Then by the convexity in Theorem 3.2.7 (or an argument using Lemma 3.2.4, as in [**17**, Proposition 4.2.6]), to check solvability, it suffices to do so for $R = R_1, \dots, R_l$.

Theorem 3.3.3. *Suppose that $\mathcal{E}$ is solvable at 1. Then there exist a constant $\epsilon \in (0,1]$ and functions $b_1(\mathcal{E}, r) \geqslant \cdots \geqslant b_d(\mathcal{E}, r)$ on $-\log T$ such that*

$$S(\mathcal{E}, \mathrm{e}^{-cr}) = \{\mathrm{e}^{-cb_1(\mathcal{E},r)}, \dots, \mathrm{e}^{-cb_d(\mathcal{E},r)}\} \quad (c \in (0,\epsilon];\ r \in -\log T). \tag{3.2}$$

Moreover, the functions $d!(b_1(\mathcal{E}, r) + \cdots + b_i(\mathcal{E}, r))$ and $b_1(\mathcal{E}, r) + \cdots + b_d(\mathcal{E}, r)$ are convex and integral polyhedral.

Proof. Extend $d!F_i(\mathcal{E}, r)$ and $F_d(\mathcal{E}, r)$ to $U = \{cr : r \in -\log T,\ c \in [0,1]\}$ by forcing them to take the value 0 at $0 \in U$. By Theorem 3.2.7, the functions are convex and transintegral polyhedral on any one-dimensional TRP subset of U not containing 0. We claim that the same is true for a one-dimensional TRP subset of U passing through 0; the missing assertion is that the functions are affine in a neighborhood of 0 on any line with rational slopes. This holds by virtue of [**22**, Theorem 2.6.1] (replacing [**15**, Theorem 2.7.2]).

We may thus apply Theorem 3.1.6 to deduce that $d!F_i(\mathcal{E}, r)$ and $F_d(\mathcal{E}, r)$ are transintegral polyhedral on U. This gives the existence of ϵ and the b_i, as well as the convexity and polyhedrality of $d!(b_1(\mathcal{E}, r) + \cdots + b_i(\mathcal{E}, r))$ and $b_1(\mathcal{E}, r) + \cdots + b_d(\mathcal{E}, r)$. We may deduce the integral polyhedrality by then applying Theorem 3.1.5. □

Remark 3.3.4. In the original version of this paper, the results of this section were only proved assuming that $\mathcal{E}$ admits a Frobenius structure. This was needed to ensure the existence of ϵ such that (3.2) holds, as we were unable to prove this otherwise. It is the more careful analysis of differential modules on p-adic polyannuli in the joint paper [**22**] with Xiao that makes the stronger result possible; consequently, we consider all results in this section to be joint work with Xiao.

Remark 3.3.5. One can also obtain a decomposition theorem in case one of the functions $b_1(\mathcal{E}, r) + \cdots + b_i(\mathcal{E}, r)$ is affine, by using [**22**, Theorem 3.4.2]. However, the conclusion will only hold on the interior of S.

3.4. Geometric interpretation

We now interpret the previous calculation in terms of Swan conductors.

Hypothesis 3.4.1. Let $\bar{X}$ be a smooth irreducible k-variety. Let $D_1, \dots, D_n$ be smooth irreducible divisors on $\bar{X}$ meeting transversely at a closed point x. Choose local coordinates $t_1, \dots, t_n$ at x such that t_i vanishes along D_i. Put $D = D_1 \cup \dots \cup D_n$ and $X = \bar{X} \setminus D$. Let $\mathcal{E}$ be an isocrystal of rank d on X overconvergent along D.

We next state an analogue of Theorem 3.2.7, with a similar proof.

Hypothesis 3.4.2. Assume Hypothesis 3.4.1, but suppose further that $\bar{X}$ is affine and that the common zero locus of $t_1, \dots, t_n$ on $\bar{X}$ consists solely of x. Let P be a smooth affine irreducible formal scheme over $\mathrm{Spf}\, \mathfrak{o}_K$ with $P_k \cong \bar{X}$, and choose $\tilde{t}_1, \dots, \tilde{t}_n \in \Gamma(P, \mathcal{O})$ lifting $t_1, \dots, t_n$. Realize $\mathcal{E}$ as a ∇-module relative to K_0 on the space

$$\{y \in P_K : \epsilon \leqslant |\tilde{t}_i(y)| \leqslant 1 \ \ (i = 1, \dots, n)\}.$$

For $R \in [\epsilon, 1]^n$, let $|\cdot|_R$ be the supremum norm on the space

$$\{y \in P_K : |\tilde{t}_i(y)| = R_i \ \ (i = 1, \dots, n)\},$$

then define $S(\mathcal{E}, R)$ as in Definition 3.2.6.

Theorem 3.4.3. *Under Hypothesis 3.4.2, for $r \in [0, -\log \epsilon]^n$, write*

$$S(\mathcal{E}, \mathrm{e}^{-r}) = \{\mathrm{e}^{-f_1(\mathcal{E}, r)}, \dots, \mathrm{e}^{-f_d(\mathcal{E}, r)}\}$$

with $f_1(\mathcal{E}, r) \geqslant \dots \geqslant f_d(\mathcal{E}, r)$, and put $F_i(\mathcal{E}, r) = f_1(\mathcal{E}, r) + \dots + f_i(\mathcal{E}, r)$. Then the following hold for $i = 1, \dots, d$.

(a) *(Continuity.) The functions $f_i(\mathcal{E}, r)$ and $F_i(\mathcal{E}, r)$ are continuous.*

(b) *(Convexity.) The function $F_i(\mathcal{E}, r)$ is convex.*

(c) *(Polyhedrality.) The functions $d!F_i(\mathcal{E}, r)$ and $F_d(\mathcal{E}, r)$ are transintegral polyhedral on $[0, -\log \epsilon]^n$.*

Proof. By Theorem 3.1.6, it suffices to check that $d!F_i(\mathcal{E}, r)$ and $F_d(\mathcal{E}, r)$ are transintegral polyhedral on any transrational line segment L contained in $[0, -\log \epsilon]^n$. Let L be such a segment parallel to the vector $a = (a_1, \dots, a_n) \in \mathbb{Z}^n$ with $\gcd(a_1, \dots, a_n) = 1$. For any indices $i \neq j$, we may replace a_i by $a_i \pm a_j$ by blowing up or down on $\bar{X}$; we may thus reduce to the case where $a = (1, 0, \dots, 0)$.

We now reduce the problem to a corresponding problem in dimension 1, using an analogue of the generic fibre construction of Definition 1.1.3. (Here it is important that we are working relative to a subfield K_0 of K; see Remark 2.1.2.) Let R be the Fréchet completion of

$$\Gamma(P, \mathcal{O}) \otimes_{K[\tilde{t}_2, \dots, \tilde{t}_n]} K(\tilde{t}_2, \dots, \tilde{t}_n)$$

for the norms $|\cdot|_{\mathrm{e}^{-r}}$ for $r \in L$. Let K' be the integral closure in R of the completion of $K(\tilde{t}_2, \dots, \tilde{t}_n)$ for the $(\mathrm{e}^{-r_2}, \dots, \mathrm{e}^{-r_n})$-Gauss norm for some $r \in L$. (This does not depend on r because the elements of L only differ in their first components.) Then R is an affinoid algebra over K' in which $|\tilde{t}_1|_R \leqslant 1$. Moreover, if we put $Y = \mathrm{Maxspec}\, R$, then the subspace $\{y \in Y : |\tilde{t}_1(y)| < 1\}$ is isomorphic to the open unit disc over K' with coordinate $\tilde{t}_1$.

For some $\delta > 0$, $\mathcal{E}$ gives rise to a ∇-module $\mathcal{F}$ relative to K_0 on the space $\{y \in Y : \delta \leqslant |\tilde{t}_1(y)| \leqslant 1\}$. On this space, we may carry out a computation analogous to [**22**, Theorem 2.4.4] to deduce that $d!F_i(\mathcal{E}, r)$ and $F_d(\mathcal{E}, r)$ are transintegral polyhedral on L. □

This in turn leads to an analogue of Theorem 3.3.3.

Definition 3.4.4. Under Hypothesis 3.4.1, let T be the simplex $\{(r_1, \dots, r_n) \in [0,1]^n : r_1 + \cdots + r_n = 1\}$. For $r \in T$, define the valuation v_r on $k(\bar{X})$ to be the restriction from the $(r_1, \dots, r_n)$-Gauss valuation on $\operatorname{Frac} k[\![t_1, \dots, t_n]\!]$; this valuation is divisorial if and only if $r \in T \cap \mathbb{Q}^n$.

Theorem 3.4.5. *Under Hypothesis 3.4.1, there exist $\epsilon \in (0,1]$ and functions $b_1(\mathcal{E}, r) \geqslant \cdots \geqslant b_d(\mathcal{E}, r)$ on $-\log T$ such that*

$$S(\mathcal{E}, \mathrm{e}^{-cr}) = \{\mathrm{e}^{-cb_1(\mathcal{E},r)}, \dots, \mathrm{e}^{-cb_d(\mathcal{E},r)}\} \quad (c \in (0, \epsilon];\ r \in -\log T). \tag{3.3}$$

Moreover, for $i = 1, \dots, d$, the functions $d!(b_1(\mathcal{E}, r) + \cdots + b_i(\mathcal{E}, r))$ and $b_1(\mathcal{E}, r) + \cdots + b_d(\mathcal{E}, r)$ are convex and integral polyhedral.

Proof. Given Hypothesis 3.4.1, we can achieve Hypothesis 3.4.2 by shrinking $\bar{X}$ to a suitable open affine neighborhood of x. We then deduce the claim by replacing Theorem 3.2.7 with Theorem 3.4.3 in the proof of Theorem 3.3.3. (The analogue of the solvability hypothesis is the hypothesis that $\mathcal{E}$ arises from an isocrystal on X overconvergent along D.) □

Reinterpreting Theorem 3.4.5 in terms of Swan conductors gives the following.

Theorem 3.4.6. *Under Hypothesis 3.4.1, for $i = 1, \dots, d$ and $r \in T \cap \mathbb{Q}^n$, let $b_i(\mathcal{E}, r)$ denote the ith largest differential ramification break of $\mathcal{E}$ along v_r, normalized with respect to $t_1 \cdots t_n$. Put $B_i(\mathcal{E}, r) = b_1(\mathcal{E}, r) + \cdots + b_i(\mathcal{E}, r)$. Then the functions $d!B_i(\mathcal{E}, r)$ and $B_d(\mathcal{E}, r)$ are continuous, convex, and integral polyhedral on T.*

Proof. It suffices to check that the quantities $b_i(\mathcal{E}, r)$ as defined in the statement of the theorem coincide with those defined in Theorem 3.4.5, as then that theorem implies the claims. For this, impose Hypothesis 3.4.2 as in the proof of Theorem 3.4.5. We may blow up or down on $\bar{X}$ as needed to reduce the claim for general $r \in T \cap \mathbb{Q}^n$ to the claim for $r = (1, 0, \dots, 0)$, in which case it is evident from Definition 1.1.5. □

Remark 3.4.7. It may be possible to use Theorem 3.4.6 to give a new proof of local semistable reduction of overconvergent F-isocrystals at monomial valuations [**17**, Theorem 6.3.1]. Such an argument would likely give some results without having to assume that K is discretely valued, as is necessary in [**17**] due to the use of Frobenius slope filtrations.

4. Variation near a surface divisor

We now make a more careful study of the variation of differential Swan conductors on a surface, in the vicinity of a single irreducible divisor.

4.1. A raw calculation

Hypothesis 4.1.1. Throughout this subsection,

- assume that k is algebraically closed and $m = 0$ (so $K = K_0$);
- let P be a smooth irreducible formal scheme over $\operatorname{Spf} \mathfrak{o}_K$, such that $Z = P_k$ is an open dense subscheme of a curve of genus $g = g(Z)$;
- let U denote an open dense affine subscheme of Z;
- let L be the completion of $\operatorname{Frac} \Gamma(U, \mathcal{O})$ for the supremum norm on $]U[_P$ (this does not depend on U);
- let Y denote a fringed relative annulus over $]U[_P$ (as in Remark 1.2.4);
- let $\mathcal{E}$ be a ∇-module on Y of rank d, which is solvable at 1.

Definition 4.1.2. Choose $\epsilon \in (0, 1)$ as in Definition 1.2.3. For a closed point $z \in Z$, choose a local uniformizer $\bar{x} \in \mathcal{O}_{Z,z}$ of Z at z. Choose a lift x of $\bar{x}$ to $\Gamma(Q, \mathcal{O})$ for some open dense formal subscheme Q of P containing z. This choice gives an isomorphism $]z[_P \times A_K[0, 1) \cong A_K[0, 1)^2$; for each $\rho \in (\epsilon, 1)$, for $r \in (0, +\infty)$ in some neighborhood of 0 (depending on ρ), we may then compute $S(\mathcal{E}, (\rho^r, \rho))$ and $T(\mathcal{E}, (\rho^r, \rho))$ in the sense of Definition 3.2.6. To indicate the dependence on z, we write these as $S(\mathcal{E}, z, (\rho^r, \rho))$ and $T(\mathcal{E}, z, (\rho^r, \rho))$. We extend the definitions to $r = 0$ by putting $S(\mathcal{E}, (1, \rho)) = S(\mathcal{F}_\rho)$ and $T(\mathcal{E}, (1, \rho)) = T(\mathcal{F}_\rho)$, for $\mathcal{F}$ the generic fibre of $\mathcal{E}$.

Note that we have omitted the dependence on $\bar{x}$ and x from the notation. That is because we are only interested here in behavior as r approaches 0, in which limit the choice of x (or $\bar{x}$) does not matter. To see this, suppose $x' \in \Gamma(Q, \mathcal{O})$ also lifts a local uniformizer of Z at z. We can then write $x' = \sum_{i=0}^{\infty} c_i x^i$ with $|c_0| < 1$, $|c_1| = 1$, and $|c_i| \leqslant 1$ for $i > 1$. If $\rho^r \geqslant |c_0|$, then $|x|_{\rho^r} = |x'|_{\rho^r}$. Hence for each $\rho \in (0, 1)$, for $r \in (0, +\infty)$ sufficiently close to 0, the quantities $S(\mathcal{E}, z, (\rho^r, \rho))$ and $T(\mathcal{E}, z, (\rho^r, \rho))$ are the same regardless of whether we use x or x' to define the isomorphism $]z[_P \times A_K[0, 1) \cong A_K[0, 1)^2$. (The definitions for $r = 0$ visibly do not depend on this choice.)

Proposition 4.1.3. *We can choose a subset R of $(0, 1)$ of the form $(\epsilon, 1) \setminus R'$, where R' is a set with discrete limit points, such that the following statements hold.*

(a) *For each $z \in Z$ and $\rho \in R$, there exist affine functions $b_1(\rho, r), \dots, b_d(\rho, r)$ on $[0, a]$, for some $a > 0$, such that*

$$S(\mathcal{E}, z, (\rho^r, \rho)) = \{\rho^{b_1(\rho,r)}, \dots, \rho^{b_d(\rho,r)}\} \quad (r \in [0, a]).$$

(b) *For $z \in Z$ and $\rho \in R$, put*

$$f(\rho, z, r) = \sum_{\alpha \in S(\mathcal{E}, z, (\rho^r, \rho))} \log_\rho \alpha$$

and write $f'(\rho, z)$ for the right slope of $f(\rho, z, r)$ at $r = 0$. Then there exist $\ell \in \{0, 1, \dots, d\}$ (independent of ρ) and a choice of the open dense subscheme U of Z (dependent on ρ) such that $f'(\rho, z) = -\ell$ for all $z \in U$.

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(c) *Assume that Z is proper. With notation as in (b), we have*

$$\sum_{z\in Z}(f'(\rho,z)+\ell) \geqslant (2-2g(Z))\ell. \tag{4.1}$$

Proof. There is no harm in shrinking U or Y, so we may assume that $\mathcal{E}$ is indecomposable and remains so upon further shrinking of U or Y. We may also assume that we can choose $u \in (\operatorname{Frac}\Gamma(P,\mathcal{O}))\cap\Gamma(]U[,\mathcal{O})$ such that $\mathrm{d}u$ freely generates $\Omega^1_{P/\mathfrak{o}_K}$ over $]U[_P$; put

$$\partial_1,\partial_2 = \frac{\partial}{\partial u},\frac{\partial}{\partial t}.$$

Let $s_{i,1}(\rho,z,r) \leqslant \cdots \leqslant s_{i,d}(\rho,z,r)$ be the reciprocals of the elements (counted with multiplicity) of the scale multiset of ∂_i on $\mathcal{E}_{(\rho^r,\rho)}$ in the bidisc $]z[_P\times A_K[0,1)$. Choose $\epsilon\in(0,1)$ as in Definition 1.2.3, and also satisfying $T(\mathcal{E},\rho)=\rho^b$ for all $\rho\in(\epsilon,1)$, where b is the highest break of $\mathcal{E}$.

Set notation as in the proof of Proposition 1.3.1. Choose i such that ∂_i is eventually dominant for $\mathcal{E}$. Then for all $\rho\in I$ except for a discrete subset R_I', we can read off the $s_{i,j}(\rho,z,r)$ from the Newton polygon of the twisted polynomial Q: for $r=0$ they are all equal to $T(\mathcal{E},\rho)=\rho^b$ by the conclusion of Proposition 1.3.1, so for r close to zero, we do not cross the threshold set by [**15**, Proposition 1.1.9] for reading off scales from slopes of the Newton polygon. We deduce that for each $\rho\in I\setminus R_I'$, we can choose $a>0$ such that each function $r\mapsto\log s_{i,j}(\rho,z,r)$ is affine for $r\in[0,a]$. (That is because these functions measure the slopes of a Newton polygon whose vertices vary linearly in r when r is sufficiently close to 0.) In particular, we may apply [**15**, Proposition 1.1.9] or [**22**, Theorem 2.3.5] to perform a simultaneous scale decomposition of $\mathcal{E}$ for ∂_i over $A_K(S)$, for $S=\{(\rho^r,\rho): r\in(0,a)\}$. Let $m_{i,j}(\rho,z)$ be the right slope of $\log_\rho s_{i,j}(\rho,z,r)$ at $r=0$.

Consider the case $i=2$. Given $h\in\{0,\dots,d-1\}$, write $a_h=\sum_j f_jt^j$; by the choice of ρ, there is a unique $j=j(h)$ which minimizes $|f_j|_L\rho^j$. Choose $\lambda_j\in K^\times$ with $|\lambda_j|=|f_j|_L$; then if we shrink U so as not to meet the zero locus of the reduction of $\lambda_{j(h)}^{-1}f_{j(h)}$ for any h, then the $m_{2,j}(\rho,z)$ vanish for all $z\in U$ by Lemma 1.1.4. Also, $\sum_{j=1}^d m_{2,j}(\rho,z)$ equals the order of vanishing at z of the reduction of $f_{j(0)}\lambda_{j(0)}^{-1}$, so its sum over $z\in Z$ equals 0 if Z is proper.

Consider the case $i=1$. Rotate as in the proof of Proposition 1.3.1, i.e. first pull back along $t\mapsto t^{p^N}$ for a large integer N, then along $u\mapsto u+t$. The effect of the first step is to pull back the action of ∂_1 unchanged, while replacing the action of ∂_2 by the pullback action of ∂_2 times $p^Nt^{p^N-1}$. The effect of the second step is to pull back the action of ∂_1 unchanged, while replacing the action of ∂_2 by the pullback action of $\partial_2+\partial_1$. Consequently, after rotation with N sufficiently large, for r sufficiently small the reciprocals of the scale multiset of ∂_2 on $\mathcal{E}_{(\rho^r,\rho)}$ in the bidisc $]z[_P\times A_K[0,1)$ consist of

$$\rho^{rc+r-1}s_{1,1}(\rho,z,r),\dots,\rho^{rc+r-1}s_{1,d}(\rho,z,r),$$

where c equals the order of vanishing of the differential $\mathrm{d}u$ on Z at the point z. (The factor ρ^{r-1} comes from the change of normalization in measuring the scale of ∂_2 rather than ∂_1. The factor ρ^{rc} comes from the fact that for x a local parameter of Z at z, ∂_1 equals $x^{-c}(\partial/\partial x)$ times a unit in $\mathcal{O}_{Z,z}$.) In particular, each $m_{1,j}(\rho, z)$ equals -1 for all but finitely many $z \in Z$, and the sum of $d + \sum_{j=1}^{d} m_{1,j}(\rho, z)$ over all $z \in Z$ equals $(2 - 2g(Z))d$ if Z is proper.

If ∂_i is eventually dominant for only one i, then for each z, we have $S(\mathcal{E}, z, (\rho^r, \rho)) = \{s_{i,1}(\rho, z, r), \dots, s_{i,d}(\rho, z, r)\}$ for r close to zero, so all the desired results follow with

$$\ell = \begin{cases} d & (i = 1), \\ 0 & (i = 2). \end{cases}$$

(The excluded set R' consists of those ρ not appearing in $I \setminus R'_I$ for any I; the only limit points of this set are those ρ for which $T(\mathcal{E}, \rho) = |p|^{p^{-m}/(p-1)}$ for some $m \in \mathbb{Z}$.) Assume hereafter that both ∂_1, ∂_2 are eventually dominant; we will again prove the claims with $\ell = 0$.

To deduce (a), note that for each z, we can choose $a > 0$ such that for $S = \{(\rho^r, \rho) : \rho \in I,\ r \in (0, a)\}$, we obtain a simultaneous scale decomposition of $\mathcal{E}$ for both ∂_1 and ∂_2 over $A_K(S)$. (Compare this with [**22**, Theorem 3.4.2].)

To deduce (b), note that by shrinking U, we can ensure that for all $z \in U$, $s_{2,j}(\rho, z, r)$ is constant for small r, and $m_{2,j}(\rho, z) = 0$; by rotation, we can also ensure that for all $z \in U$, $m_{1,j}(\rho, z) = -1$. Consequently, for $z \in U$, for r close to 0, $S(\mathcal{E}, z, (\rho^r, \rho))$ consists of $T(\mathcal{E}, \rho)$ with multiplicity d.

To deduce (c) if Z is proper, note that $f'(\rho, z) \geqslant \sum_{j=1}^{d} m_{2,j}(\rho, z)$, and summing the right-hand side over z yields 0. □

Remark 4.1.4. One might like to prove Proposition 4.1.3 directly by reading off the Swan conductor from a twisted polynomial, without having to decompose into indecomposables. There are two reasons why this will not work. One is the fact that different derivations may be dominant on different components of the break decomposition. The other is the limitation on slopes in [**15**, Proposition 1.1.9]: the presence of some λ in a radius multiset masks the presence of any $\lambda' > \lambda^{1/p}$ when viewing Newton polygons. By working in the indecomposable case, we fail to encounter this masking for r sufficiently small because we have a uniform break at $r = 0$.

Remark 4.1.5. The arguments in [**22**, § 2.4] are in a similar spirit. Using ideas from there, it should be possible to remove the restriction to the set R in Proposition 4.1.3.

4.2. Subharmonicity

We now obtain a subharmonicity theorem for differential Swan conductors on a surface.

Hypothesis 4.2.1. Assume that $m = 0$, so $K = K_0$. Let $\bar{X}$ be a smooth irreducible projective surface over k, let Z be a smooth irreducible divisor on $\bar{X}$, and let v_0 be the divisorial valuation on $k(\bar{X})$ measuring order of vanishing along Z. Let W be a divisor not containing Z, and put $Y = \bar{X} \setminus W$; note that $Y \cap Z$ is open dense in Z. Let X be an open dense subscheme of Y, and let $\mathcal{E}$ be an isocrystal of rank d on X overconvergent along $Y \setminus X$.

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Definition 4.2.2. Let P be a smooth formal scheme over $\operatorname{Spf} \mathfrak{o}_K$ with special fibre $Z \cap Y$. As in Definition 2.2.1, for any open affine subscheme Z_0 of $Z \cap Y$, we obtain from $\mathcal{E}$ a ∇-module on a fringed relative annulus over $]U[_P$, for some open dense subscheme U of Z_0. Moreover, any two such ∇-modules so obtained become isomorphic on a suitably small fringed relative annulus, so the construction glues to give a ∇-module on a fringed relative annulus over $]U[_P$, for some open dense subscheme U of $Z \cap Y$; we will also use the symbol $\mathcal{E}$ to refer to this ∇-module.

Definition 4.2.3. Given $z \in Z \cap Y$, choose $x \in \mathcal{O}_{\bar{X},z}$ whose zero locus has a single component at z, which is smooth of multiplicity 1 and meets Z transversely. For $r \in \mathbb{Q} \cap [0,1]$, let $v_r(z;x)$ be the valuation on $k(\bar{X})$ corresponding to the divisor $x \sim t^r$ (in the sense of Example 2.4.3) on a suitable blowup of $\bar{X}$ at z, for t a local parameter of Z at z. If we identify the completion of the local ring $\mathcal{O}_{\bar{X},z}$ with $k[\![x,t]\!]$, then $v_r(z;x)$ is induced by the $(r,1)$-Gauss valuation on $k[\![x,t]\!]$. The latter valuation is invariant under any continuous automorphism of $k[\![x,t]\!]$ of the form $t \mapsto ut$, $x \mapsto \lambda x + w$ where u is a unit in $k[\![x,t]\!]$, $\lambda \in k^\times$, and w belongs to the ideal (t, x^2). This allows replacing x by any other $x' \in \mathcal{O}_{\bar{X},z}$ whose zero locus has a single component at z, which is smooth of multiplicity 1 and meets Z transversely. It also allows replacing t by another local parameter of Z at z. Consequently, those replacements do not affect the definition of $v_r(z;x)$.

Let $b_1(\mathcal{E},z,x,r) \geqslant \cdots \geqslant b_d(\mathcal{E},z,x,r)$ and $\operatorname{Swan}_Z(\mathcal{E},z,x,r)$ be the differential highest breaks and Swan conductor of $\mathcal{E}$ along $v_r(z;x)$, normalized with respect to t. By Theorem 3.4.6, the function $r \mapsto b_j(\mathcal{E},z,x,r)$ is affine in a neighborhood of 0. It thus extends continuously to all $r \in [0,a]$ for some $a > 0$.

Lemma 4.2.4. *With notation as in Definition 4.1.2, there exist $\epsilon \in (0,1)$ and $a > 0$ (depending on z) such that for $r \in [0,a]$ and $\rho \in (\epsilon,1)$, $S(\mathcal{E},z,(\rho^r,\rho))$ is defined and*

$$S(\mathcal{E},z,(\rho^r,\rho)) = \{\rho^{b_1(\mathcal{E},z,x,r)},\dots,\rho^{b_d(\mathcal{E},z,x,r)}\}.$$

Proof. Apply Theorem 3.4.5. □

The value of ϵ in Lemma 4.2.4 depends on the choice of z. However, we can use the following argument to make a uniform choice.

Lemma 4.2.5. *With notation as in Definition 4.1.2, suppose that for some $\rho_0 < \rho_1 \in (\epsilon,1)$ and some $c \in \mathbb{R}$,*

$$S(\mathcal{E},z,(\rho_j^r,\rho_j)) = \{\rho_j^{b_1(\mathcal{E},z,x,0)+cr},\dots,\rho_j^{b_d(\mathcal{E},z,x,0)+cr}\}$$

for $j = 0,1$ and $r \in [0,a]$. Then there exists $b > 0$ such that for all $\rho \in [\rho_0,1)$ and all $r \in [0,b]$, $S(\mathcal{E},z,(\rho^r,\rho))$ is defined and

$$S(\mathcal{E},z,(\rho^r,\rho)) = \{\rho^{b_1(\mathcal{E},z,x,0)+cr},\dots,\rho^{b_d(\mathcal{E},z,x,0)+cr}\}.$$

Proof. Choose $b \in [0,a]$ so that $S(\mathcal{E},z,(\rho^r,\rho))$ is defined for all $\rho \in [\rho_0,1)$ and all $r \in [0,b]$. For $i = 1,\dots,d$, and $r \in [0,b] \cap \mathbb{Q}$, the function $F_i(\mathcal{E},(rs,s))$ is convex for $s \in (0,-\log\rho_0]$ by Theorem 3.4.3, and extends continuously to $s = 0$ with the value 0

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because $\mathcal{E}$ is overconvergent. On the other hand, it agrees with a linear function at the three values $s = 0, -\log \rho_1, -\log \rho_0$, so it must be linear on all of $[0, -\log \rho_0]$. This proves the claim for $r \in [0, b] \cap \mathbb{Q}$; the full claim follows by continuity (Theorem 3.4.3). □

Corollary 4.2.6. *In Lemma 4.2.4, the value of ϵ can be chosen independent of $z \in Z \cap Y$. Moreover, for all but finitely many $z \in Z \cap Y$, either $b_i(\mathcal{E}, z, x, r)$ or $b_i(\mathcal{E}, z, x, r) + r$ (depending on whether ∂_2 is or is not eventually dominant on the corresponding component of $\mathcal{E}$) is constant for r in some neighborhood of 0 (depending on z).*

Proof. By the proof of Proposition 4.1.3, the hypothesis of Lemma 4.2.5 holds for all but finitely many $z \in Z \cap Y$. The assertion is then clear from the proof of Proposition 4.1.3. □

Theorem 4.2.7. *Under Hypothesis 4.2.1, we have the following.*

(a) *For each $z \in Z \cap Y$, the functions $b_j(\mathcal{E}, z, x, r)$ for $j = 1, \dots, d$ and $\mathrm{Swan}_Z(\mathcal{E}, z, x, r)$ are affine in a neighborhood of $r = 0$.*

(b) *Let $\mathrm{Swan}'_Z(\mathcal{E}, z)$ be the right slope of $\mathrm{Swan}_Z(\mathcal{E}, z, x, r)$ at $r = 0$. Then there exists $\ell = \ell(\mathcal{E}, Z) \in \{0, 1, \dots, d\}$ such that $\mathrm{Swan}'_Z(\mathcal{E}, z) = -\ell$ for all but finitely many $z \in Z \cap Y$.*

(c) *Assume that $Z \subset Y$. With notation as in (b), we have*

$$\sum_{z \in Z} (\mathrm{Swan}'_Z(\mathcal{E}, z) + \ell) \geqslant (2 - 2g(Z))\ell - Z^2 \,\mathrm{Swan}(\mathcal{E}, Z), \tag{4.2}$$

where $g(Z)$ denotes the genus of Z, and Z^2 denotes the self-intersection of Z on $\bar{X}$.

Proof. We deduce (a) from Lemma 4.2.4 and (b) from Corollary 4.2.6. For (c), we must account for the fact that we cannot necessarily choose the local parameter t uniformly for all $z \in Z$. Pick $t \in k(\bar{X})$ with $v_0(t) = 1$, and let D denote the principal divisor defined by t; then $D \cdot Z = 0$, so $(D - Z) \cdot Z = -Z^2$.

For $z \in Z$, let t_z be a local parameter for Z at z, and let c_z be the order of vanishing at z of the restriction of t/t_z to Z. Then c_z is equal to the local intersection multiplicity $((D - Z) \cdot Z)_z$, so $\sum_{z \in Z} c_z = -Z^2$. Let $x_z \in k(\bar{X})$ cut out a divisor with a single component at z, which is smooth of multiplicity 1 and meets Z transversely. For s close to 0, the valuation $v_s(z; x_z)$ corresponds to the divisor $x_z \sim t_z^s$, or $x_z \sim t^r$ with $r = s/(1 + sc_z)$. (Again, the notation $\sim$ is used as in Example 2.4.3.)

Define $f(\rho, z, r)$ as in Proposition 4.1.3; by Corollary 4.2.6, it is independent of ρ for r in some neighborhood of 0 and ρ in some neighborhood of 1, so we may call the resulting value $f(z, r)$. This quantity is the Swan conductor along $x_z \sim t^r$ normalized with respect to t; renormalizing with respect to t_z, we obtain

$$\mathrm{Swan}_Z(\mathcal{E}, z, x, s) = f(z, r) \frac{v_s(x; z)(t)}{v_s(x; z)(t_z)} = \frac{s}{r} f(z, r) = (1 + sc_z) f(z, r).$$

Differentiating with respect to s at $r = s = 0$ yields

$$\mathrm{Swan}'_Z(\mathcal{E}, z) = c_z f(z, 0) + f'(z).$$

We now deduce (c) by summing over $z \in Z$ and invoking Proposition 4.1.3 (c). □

Example 4.2.8. Here is a typical example where Theorem 4.2.7 holds with $\ell \neq 0$: take Z to be the x-axis in the x, t-plane $\mathbb{A}^2_k \subset \mathbb{P}^2_k$, take $X = \mathbb{A}^2_k \setminus Z$, and take $\mathcal{E}$ to be the Dwork isocrystal $\mathcal{L}_{xt^{-p}}$.

Remark 4.2.9. As is apparent in the proof of Theorem 4.2.7, the self-intersection number in (4.2) is a side effect of normalizing with respect to a different parameter at each point of Z; it drops out if one normalizes everything with respect to a single function.

Remark 4.2.10. It is reasonable to ask whether equality necessarily holds in (4.2) as long as the ramification breaks along Z are all nonzero. Unfortunately, the proof of Proposition 4.1.3 does not suffice to establish this; what is missing is a proof that if $\partial/\partial t$ and $\partial/\partial x$ are both dominant on $\mathcal{E}_\rho$, then $\partial/\partial t$ is dominant on $\mathcal{E}_{(\rho^r,\rho)}$ for $r > 0$ small.

4.3. Monotonicity

We now use some refined results on p-adic differential modules on discs, to gain some further control over differential Swan conductors. In the original version of this paper, this was done using results on rigid cohomology to imitate what one does in the ℓ-adic setting (compare Laumon's proof of the semicontinuity theorem [**23**]); that method was limited to fully overconvergent F-isocrystals, with K discrete.

Definition 4.3.1. Under Hypothesis 4.2.1, for $i \in \{1, \dots, d\}$ such that either $i = d$ or $b_i(\mathcal{E}, Z) > b_{i+1}(\mathcal{E}, Z)$, let $\ell_i(\mathcal{E}, Z)$ be the sum of the ranks of the components of the break decomposition of $\mathcal{E}$ contributing to $b_1(\mathcal{E}, Z) + \cdots + b_i(\mathcal{E}, Z)$ on which ∂_2 is not eventually dominant. In particular, $\ell_d(\mathcal{E}, Z) = \ell(\mathcal{E}, Z)$.

Theorem 4.3.2. *Assume Hypothesis 4.2.1. Suppose that $z \in Z \cap Y$ is a smooth point of $Z \cup (\bar{X} \setminus X)$. Let $b'_i(\mathcal{E}, z)$ be the right slope of $b_i(\mathcal{E}, z, x, r)$ at $r = 0$. Then for $i \in \{1, \dots, d\}$ such that either $i = d$ or $b_i(\mathcal{E}, Z) > b_{i+1}(\mathcal{E}, Z)$, we have $b'_1(\mathcal{E}, z) + \cdots + b'_i(\mathcal{E}, z) + \ell_i(\mathcal{E}, Z) \leqslant 0$, with equality for all but finitely many z.*

The proof is again by rotation, but this time in the opposite direction from the arguments of [**15**]: we use a result about ∂_1 to prove something about ∂_2.

Proof. The equality for all but finitely many z follows from Corollary 4.2.6, so it suffices to check the inequality. We first treat the case $i = d$.

Take x, t as in Definition 4.2.3. Because z is a smooth point of $\bar{X} \setminus X$, we may restrict $\mathcal{E}$ to a space of the form $A_{K,x}[0, 1) \times A_{K,t}(\epsilon, 1)$ for some $\epsilon \in (0, 1)$. By Lemma 4.2.4, we can choose $a > 0$ and $\epsilon \in (0, 1)$ so that for $r \in (0, a)$ and $\rho \in (\epsilon, 1)$,

$$S(\mathcal{E}, z, (\rho^r, \rho)) = \{\rho^{b_1(\mathcal{E},z,x,r)}, \dots, \rho^{b_d(\mathcal{E},z,x,r)}\}.$$

By Theorem 3.4.6, we can choose a so that each of $b_1(\mathcal{E}, z, x, r), \dots, b_d(\mathcal{E}, z, x, r)$ is affine in r for $r \in [0, a]$.

Pick any $\rho \in (\epsilon, 1)$, and let K_ρ be the completion of $K(t)$ for the ρ-Gauss norm. We may then restrict $\mathcal{E}$ to obtain a ∇-module $\mathcal{F}$ on $A_{K_\rho,x}[0,1)$. As in the proof of [**22**, Theorem 2.4.4], for a suitable choice of a, we may decompose $\mathcal{F} = \bigoplus_j \mathcal{F}_j$ over $A_{K_\rho,x}(T)$ for $T = \{\rho^r : r \in (0, a)\}$, so that for each $h \in \{1, 2\}$, either ∂_h is not dominant on $(\mathcal{F}_j)_{\rho^r}$ for each $r \in (0, a)$, or ∂_h is dominant on $(\mathcal{F}_j)_{\rho^r}$ for each $r \in (0, a)$ with scale multiset consisting of a single element. (We abbreviate this by saying that ∂_h is or is not dominant on $\mathcal{F}_j$.)

Write the scale of $(\mathcal{F}_j)_{\rho^r}$ as $\rho^{-\alpha r - \beta}$, where we write $\alpha = \alpha(\mathcal{F}_j)$ and $\beta = \beta(\mathcal{F}_j)$ if it is necessary to disambiguate. Then

$$\sum_j (\alpha(\mathcal{F}_j) r + \beta(\mathcal{F}_j)) \operatorname{rank}(\mathcal{F}_j) = \operatorname{Swan}_Z(\mathcal{E}, z, x, r)$$

and so

$$\sum_j \alpha(\mathcal{F}_j) \operatorname{rank}(\mathcal{F}_j) = \operatorname{Swan}'_Z(\mathcal{E}, z). \tag{4.3}$$

Put $\ell(\mathcal{F}_j) = 0$ if the limit of the scale of ∂_2 on $(\mathcal{F}_j)_{\rho^r}$ as $r \to 0^+$ equals $\rho^{-\beta}$, and $\ell(\mathcal{F}_j) = \operatorname{rank}(\mathcal{F}_j)$ otherwise. Then

$$\sum_j \ell(\mathcal{F}_j) = \ell(\mathcal{E}, Z). \tag{4.4}$$

Let K_1 be the completion of $K_\rho(u)$ for the 1-Gauss norm. Let $f : A_{K_1}[0,1) \to A_{K_\rho}[0,1)$ be the K_0-linear map of locally G-ringed spaces acting on global sections via $f^*(x) = x$, $f^*(t) = t(1 + ux)$. This has the effect

$$\mathrm{d}x \mapsto \mathrm{d}x, \qquad \mathrm{d}t \mapsto (1 + ux)\,\mathrm{d}t + ut\,\mathrm{d}x.$$

Writing ∂'_1, ∂'_2 for the actions of $\partial/\partial x$, $\partial/\partial t$ before pulling back, the actions of ∂_1, ∂_2 are given by

$$\partial'_1 + ut\partial'_2, \qquad (1 + ux)\partial'_2.$$

In particular, the scale of ∂_1 on $(f^*\mathcal{F}_j)_{\rho^r}$ is bounded below by the greater of the following quantities: the scale of ∂'_1 on $(\mathcal{F}_j)_{\rho^r}$, and ρ^r times the scale of ∂'_2 on $(\mathcal{F}_j)_{\rho^r}$. We obtain the reverse inequality from the interpretation of scales in terms of convergence of Taylor series [**15**, § 2.2]. (See the appendix for a similar argument.)

Write the scale of ∂_1 on $(f^*\mathcal{F}_j)_{\rho^r}$ as $\rho^{-g(r)}$. Since $\mathcal{F}$ extends to an affinoid space containing the annulus $A_{K_\rho,x}(T)$, the proof of [**20**, Theorem 11.3.2] shows that each $g(r)$ extends continuously to $[0, a)$, and is affine in a neighborhood of $r = 0$ (as in Theorem 3.4.3). Let $m = m(\mathcal{F}_j)$ be the right slope of g at $r = 0$.

From the calculation of the scale of ∂_1 on $(f^*\mathcal{F}_j)_{\rho^r}$ above, we have the following.

- If ∂_1 is dominant on $\mathcal{F}_j$, then $g(r) = \alpha r + \beta$, so $m = \alpha$.
- If ∂_1 is not dominant on $\mathcal{F}_j$, then $\alpha r + \beta > g(r) \geqslant (\alpha - 1) r + \beta$, so $\alpha > m \geqslant \alpha - 1$.

We say that $\mathcal{F}_j$ is *negligible* if $\alpha = \beta = 0$. By [**20**, Theorem 11.3.2(d)] applied on $A_{K_1}[0, 1-\epsilon]$ for some small $\epsilon > 0$, we have

$$\sum_j (m(\mathcal{F}_j) + 1)\operatorname{rank}(\mathcal{F}_j) \leqslant 0, \tag{4.5}$$

provided we take the sum over those j for which $\mathcal{F}_j$ is not negligible. For each such j, we have the following.

- If $\ell(\mathcal{F}_j) = 0$, then $m \geqslant \alpha - 1$ whether or not ∂_1 is dominant on $\mathcal{F}_j$, so $m + 1 \geqslant \alpha$.
- If $\ell(\mathcal{F}_j) = \operatorname{rank}(\mathcal{F}_j)$, then ∂_2 cannot be dominant on $\mathcal{F}_j$ for $r > 0$ small, so ∂_1 must be dominant on $\mathcal{F}_j$. We thus must have $m = \alpha$.

In both cases, we have

$$(m(\mathcal{F}_j) + 1)\operatorname{rank}(\mathcal{F}_j) \geqslant \alpha(\mathcal{F}_j)\operatorname{rank}(\mathcal{F}_j) + \ell(\mathcal{F}_j),$$

so by (4.5) we have

$$\sum_j (\alpha(\mathcal{F}_j)\operatorname{rank}(\mathcal{F}_j) + \ell(\mathcal{F}_j)) \leqslant 0 \tag{4.6}$$

provided that we only sum over j for which $\mathcal{F}_j$ is not negligible. However, the left-hand side of (4.6) does not change if we include summands for which $\mathcal{F}_j$ is negligible (as those have $\alpha(\mathcal{F}_j) = \ell(\mathcal{F}_j) = 0$), so (4.6) holds even if we sum over all j. By (4.3) and (4.4), this yields the desired inequality in the case $i = d$.

We now treat the case where $i < d$ but $b_i(\mathcal{E}, Z) > b_{i+1}(\mathcal{E}, Z)$. Pick a rational number $c/m \in (b_{i+1}(\mathcal{E}, Z), b_i(\mathcal{E}, Z))$ with denominator m coprime to p. Let $\mathcal{F}$ be the direct sum of the Dwork isocrystals $\mathcal{L}_{t^{c/m}}$ (in the sense of Definition 2.4.2) for $t^{c/m}$ running over all of the mth roots of t^c. This isocrystal is initially only defined on an m-fold cover of X, but it descends to an overconvergent isocrystal of rank m such that for r near 0,

$$b_1(\mathcal{F}, z, x, r) = \cdots = b_m(\mathcal{F}, z, x, r) = \frac{c}{m}$$

by [**15**, Example 3.5.10]. Consequently,

$$b_{(j-1)m+1}(\mathcal{E} \otimes \mathcal{F}, z, x, r) = \cdots = b_{jm}(\mathcal{E} \otimes \mathcal{F}, z, x, r) = \begin{cases} b_i(\mathcal{E}, z, x, r), & j \leqslant i, \\ \dfrac{c}{m}, & j > i. \end{cases}$$

Thus we may obtain the desired result for $\mathcal{E}$ by applying the previously shown case for $\mathcal{E} \otimes \mathcal{F}$. □

Equality in Theorem 4.3.2 has a special meaning.

Theorem 4.3.3. *With notation as in Theorem 4.3.2, suppose that for each* $i \in \{1, \dots, d\}$ *such that* $b_i(\mathcal{E}, Z) > b_{i+1}(\mathcal{E}, Z)$, *we have* $b'_1(\mathcal{E}, z) + \cdots + b'_i(\mathcal{E}, z) + \ell_i(\mathcal{E}, Z) = 0$. *Then the break decomposition of* $\mathcal{E}$ *along* Z *extends over* z.

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Proof. Set notation as in the proof of Theorem 4.3.2; then we must have equality in (4.5) for each i such that $b_i(\mathcal{E}, Z) > b_{i+1}(\mathcal{E}, Z)$. By [**20**, Theorem 12.2.2], $f^*\mathcal{F}$ admits a direct sum decomposition over all of $A_{K_\rho}[0,1)$ such that over $A_{K_\rho}(T)$, the $\mathcal{F}_j$ which are grouped into the same summand all have the same value of $\beta(\mathcal{F}_j)$. Over $A_{K_\rho}(T)$, this decomposition coincides with the decomposition obtained by pulling back the break decomposition of $\mathcal{E}$; in particular, it descends to a decomposition of $\mathcal{F}$ itself.

The projectors onto the summands in this decomposition of $\mathcal{F}$ are horizontal sections of $\mathcal{F}^\vee \otimes \mathcal{F}$. Since these match the projectors over $A_{K_\rho}(T)$ defined by the break decomposition, we may apply Lemma 1.3.4 to deduce that the break decomposition of $\mathcal{E}$ along Z extends over z. □

4.4. Turning points

We investigate some potential notions of *turning points*, analogous to the corresponding objects in the holomorphic setting. However, we stop short of giving a completely satisfactory definition.

Hypothesis 4.4.1. Let $\bar{X}$ be a smooth irreducible projective surface over k, and let $K_{\bar{X}}$ denote a canonical divisor on $\bar{X}$. Let D be a strict normal crossings divisor on $\bar{X}$, and put $X = \bar{X} \setminus D$. Let $\mathcal{E}$ be an overconvergent isocrystal of rank d on X.

Definition 4.4.2. Let z be a nonsmooth point of D, and let Z_1, Z_2 be the components of D containing z. Let t_1, t_2 be local parameters for Z_1, Z_2 at z. Define the functions $B_1(\mathcal{E}, r), \dots, B_d(\mathcal{E}, r)$ as in Theorem 3.4.6; for $s \in [0,1]$, put $f_i(s) = B_i(\mathcal{E}, (1-s, s))$. We say that z is a *hidden turning point* if $f_i(s)$ fails to be affine in s for some $i \in \{1, \dots, d\}$.

Proposition 4.4.3. *In Definition 4.4.2, let $f_i'(0)$ denote the right slope of f_i at $s = 0$. Then $f_i'(0) \leqslant f_i(1) - f_i(0)$, with equality if and only if $f_i(s)$ is affine in s.*

Proof. This is evident from the fact that f_i is convex (Theorem 3.4.6). □

Definition 4.4.4. Let z be a smooth point of D, and let Z be the component of D containing z. By Theorem 4.3.2, for each $i \in \{1, \dots, d\}$ such that either $i = d$ or $b_i(\mathcal{E}, Z) > b_{i+1}(\mathcal{E}, Z)$, we have $b_1'(\mathcal{E}, z) + \cdots + b_i'(\mathcal{E}, z) + \ell_i(\mathcal{E}, Z) \leqslant 0$. We say z is an *exposed turning point* if this inequality is strict for at least one i.

It is natural to mention a variant of Theorem 4.2.7 phrased in terms of intersection theory rather than valuations.

Definition 4.4.5. For each component Z of D, let $\mathrm{Swan}(\mathcal{E}, Z)$ denote the differential Swan conductor of $\mathcal{E}$ along Z, and define $\ell(\mathcal{E}, Z)$ as in Theorem 4.2.7. Define the *Swan divisor* of $\mathcal{E}$ on $\bar{X}$ as the divisor

$$\mathrm{Swan}(\mathcal{E}) = \sum_{Z \in D} \mathrm{Swan}(\mathcal{E}, Z) Z.$$

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Lemma 4.4.6. *Under Hypothesis 4.4.1, for each component Z of D,*

$$\begin{aligned} Z \cdot (\mathrm{Swan}(\mathcal{E}) + \ell(\mathcal{E}, Z)(K_{\bar{X}} + D)) \\ \geqslant (2g(Z) - 2)\ell(\mathcal{E}, Z) + Z^2 \,\mathrm{Swan}(\mathcal{E}, Z) + \sum_{z \in Z} (\mathrm{Swan}'_Z(\mathcal{E}, z) + \ell(\mathcal{E}, Z)). \end{aligned}$$

Moreover, equality holds if $\mathcal{E}$ has no turning points on Z.

Proof. Rewrite the left-hand side as

$$Z^2 \,\mathrm{Swan}(\mathcal{E}, Z) + \ell(\mathcal{E}, Z)(Z \cdot K_{\bar{X}} + Z^2) + \sum_{Z'} (\mathrm{Swan}(\mathcal{E}, Z') + \ell(\mathcal{E}, Z))(Z \cdot Z'),$$

where Z' runs over the components of D other than Z. By adjunction, $Z \cdot K_{\bar{X}} + Z^2 = 2g(Z) - 2$.

Since we assumed D is a strict normal crossings divisor, $Z \cap Z'$ never contains more than one point. For each $z \in Z$ occurring as $Z \cap Z'$ for some Z', by Proposition 4.4.3, we have $\mathrm{Swan}(\mathcal{E}, Z') \geqslant \mathrm{Swan}'_Z(\mathcal{E}, z)$ with equality if z fails to be a hidden turning point. More explicitly, if we identify Z, Z' with the divisors Z_1, Z_2 of Proposition 4.4.3, then $f_d(s) = (1 - s)\,\mathrm{Swan}(\mathcal{E}, x, z, s/(1 - s))$, so $\mathrm{Swan}'_Z(\mathcal{E}, z) = f'_d(0) + \mathrm{Swan}(\mathcal{E}, Z) \leqslant f_d(1) - f_d(0) + \mathrm{Swan}(\mathcal{E}, Z) = \mathrm{Swan}(\mathcal{E}, Z')$.

For each $z \in Z$ not occurring as $Z \cap Z'$ for any Z', we have by Theorem 4.3.2 that $\mathrm{Swan}'_Z(\mathcal{E}, z) + \ell(\mathcal{E}, Z) \leqslant 0$, with equality if z fails to be an exposed turning point. This yields the claimed results. □

Theorem 4.4.7. *Under Hypothesis 4.4.1, for each component Z of D,*

$$Z \cdot (\mathrm{Swan}(\mathcal{E}) + \ell(\mathcal{E}, Z)(K_{\bar{X}} + D)) \geqslant 0. \tag{4.7}$$

Proof. This holds by combining Lemma 4.4.6 with Theorem 4.2.7. □

Remark 4.4.8. Our present notion of turning points has two unfortunate defects, which suggest that a modified definition may be needed. One defect is that it does not quite match the definition in characteristic 0, which can be formulated by imposing an affinity condition on the Swan conductors of both $\mathcal{E}$ and $\mathcal{E}^\vee \otimes \mathcal{E}$ [**19**]. A more serious defect is that our definition does not line up with the notion of *cleanness* for abelian ℓ-adic characters introduced by Kato [**11**]. Namely, one expects $\mathcal{E}$ to be clean if and only if it has no turning points. However, Kato's notion of cleanness (defined using a refined Swan conductor) is stable under introducing additional components into D, whereas ours is not. We suspect that our definition includes too many points as turning points; fixing this would require an analogue of refined Swan conductors for differential modules. Such an analogue has been introduced very recently by Xiao [**34**].

Our next two questions refer to cleanness, even though we have not pinned this down exactly. For concrete questions that are still reasonable, one may read 'has no turning points' for 'is clean'.

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Question 4.4.9. If $\mathcal{E}$ is clean, can one assert an Euler characteristic formula for $\mathcal{E}$ analogous to the Grothendieck–Ogg–Shafarevich formula for curves? (For the p-adic version for curves, see for instance [**13**, Theorem 4.3.1].) Such a formula would involve not only contributions from the components of D, but also from the pairwise intersections of components.

Question 4.4.10. If $\mathcal{E} = f_* \mathcal{O}_Y$ for $f : Y \to X$ a finite étale morphism, and $\mathcal{E}$ is clean, can one form a finite cover $\bar{f} : \bar{Y} \to \bar{X}$ extending f such that $\bar{Y}$ has only mild singularities? For instance, if f is Galois and abelian, it should be possible to ensure that $\bar{Y}$ has only quotient singularities; something along these lines has been established by Kato [**11**], although some work may be needed to compare our construction with his.

5. Results for lisse ℓ-adic sheaves

In this section, we describe how to define differential ramification breaks and Swan conductors for lisse ℓ-adic étale sheaves, and how some of the variational results in the p-adic case may be carried over. Throughout this section, retain Hypotheses 2.3.1 and 2.3.2.

Hypothesis 5.0.1. Throughout this section, let ℓ be a prime different from p, and let E be a finite extension of $\mathbb{Q}_\ell$.

5.1. Defining the ramification breaks

Definition 5.1.1. Let v be a divisorial valuation on $k(X)$ over k, and let I_v be an inertia subgroup of the absolute Galois group of $k(X)$ corresponding to v. The *wild inertia subgroup* W_v of I_v is the absolute Galois group of the maximal tamely ramified extension of $k(X)_v$. The group W_v is a pro-p-group, whereas the quotient I_v/W_v is congruent to $\prod_{\ell \neq p} \mathbb{Z}_\ell$.

Definition 5.1.2. Let $\rho : \pi_1(X, \bar{x}) \to \mathrm{GL}(V)$ be a continuous homomorphism for $V = V(\rho)$ a finite-dimensional E-vector space, corresponding to a lisse E-sheaf $\mathcal{E}$ on X. Filter the inertia group I_v as in [**15**, Definition 3.5.12]. For ρ irreducible, define the *differential highest break* $b(\rho, v)$ of ρ along v to be the maximal r such that $I_v^r \not\subset \ker(\rho)$. For general ρ, let $\rho_1, \dots, \rho_n$ be the irreducible constituents of ρ, and define the *differential ramification breaks* $b_1(\rho, v) \geqslant \cdots \geqslant b_d(\rho, v)$ (or $b_1(\mathcal{E}, v) \geqslant \cdots \geqslant b_d(\mathcal{E}, v)$) of ρ to be the elements of the multiset consisting of $b(\rho_i, v)$ with multiplicity $\dim(\rho_i)$. Define the *differential Swan conductor* $\mathrm{Swan}(\rho, v)$ (or $\mathrm{Swan}(\mathcal{E}, v)$) of ρ along v to be the sum $\sum_{i=1}^{d} b_i(\rho, v)$.

As in Definition 2.4.1, the previous definition gives the differential ramification breaks and differential Swan conductor in their *natural normalization*. If desired, we may instead normalize with respect to any $t \in k(X)$ for which $v(t) \neq 0$.

Unlike in the p-adic case, the differential ramification breaks of an ℓ-adic representation of $\pi_1(X, \bar{x})$ are not obtained by first constructing a corresponding isocrystal. Consequently, it is not immediate that variational properties of differential ramification breaks of representations can be transferred to the ℓ-adic case. The remainder of this section is devoted to making such transfers; we start with a few useful remarks.

Remark 5.1.3. With notation as in Definition 5.1.2, choose a ρ-stable $\mathfrak{o}_E$-lattice T of V, and let $\bar{\rho} : \pi_1(X, \bar{x}) \to \mathrm{GL}(T/\mathfrak{m}_E T)$ be the resulting residual representation. Then the image in $\mathrm{GL}(T)$ of the pro-p-group W_v has trivial intersection with the pro-ℓ-group $\ker(\mathrm{GL}(T) \to \mathrm{GL}(T/\mathfrak{m}_E T))$, and so injects into $\mathrm{GL}(T/\mathfrak{m}_E T)$. Consequently, if we use the same procedure as in Definition 5.1.2 to define the differential ramification breaks and Swan conductor of a mod ℓ representation of $\pi_1(X, \bar{x})$, then these quantities are the same for a $\mathfrak{o}_E$-representation as for its mod ℓ reduction.

Remark 5.1.4. In Remark 5.1.3, if the representation $\bar{\rho}$ lifts to a *discrete* representation $\pi_1(X, \bar{x}) \to \mathrm{GL}(T)$ (i.e. a representation with open kernel), then we can generate an overconvergent F-isocrystal which computes the differential ramification breaks of $\bar{\rho}$, using Theorem 2.3.7.

5.2. Integral polyhedrality

In this section, we establish an analogue of Theorem 3.4.6 for ℓ-adic sheaves.

Theorem 5.2.1. *Under Hypothesis 3.4.1, let $\mathcal{E}$ be a lisse étale E-sheaf on X. For $i = 1, \dots, d$ and $r \in T \cap \mathbb{Q}^n$, let $b_i(\mathcal{E}, r)$ denote the ith largest differential ramification break of $\mathcal{E}$ along v_r, normalized with respect to $t_1 \cdots t_n$. Put $B_i(\mathcal{E}, r) = b_1(\mathcal{E}, r) + \cdots + b_i(\mathcal{E}, r)$. Then the functions $d!B_i(\mathcal{E}, r)$ and $B_d(\mathcal{E}, r)$ are continuous, convex, and integral polyhedral on T.*

Proof. By Remark 5.1.3, we may replace $\mathcal{E}$ by a locally constant étale $\mathbb{F}$-sheaf, where $\mathbb{F}$ is the residue field of E, and prove the same result. Let G be the image of $\pi_1(X, \bar{x})$ in $\mathrm{GL}_d(\mathbb{F})$, and let H be a p-Sylow subgroup of G. Let $f : Y \to X$ be a finite étale cover such that for some geometric point $\bar{y}$ of Y over $\bar{x}$, $\pi_1(Y, \bar{y}) = \bar{\rho}^{-1}(H)$. Put $\mathcal{F} = f_* f^* \mathcal{E}$, which corresponds to the representation $\tau = \mathrm{Ind}_H^G \mathrm{Res}_H^G \bar{\rho}$. Put $m = [G : H]$. Then for each divisor Z on X,

$$b_{(m-1)i+1}(\mathcal{F}, Z) = \cdots = b_{mi}(\mathcal{F}, Z) = b_i(\mathcal{E}, Z)$$

since the differential ramification breaks only depend on the action of H. On the other hand, $\mathrm{Res}_H^G \bar{\rho}$ is a mod ℓ representation of the group H whose order is prime to ℓ. It is thus liftable to $\mathfrak{o}_E$, as then is its induction τ. We may thus apply Remark 5.1.4 to deduce from Theorem 3.4.6 that $md!B_i(\mathcal{E}, r) = d!B_{mi}(\mathcal{F}, r)$ and $mB_d(\mathcal{E}, r) = B_{md}(\mathcal{F}, r)$ are continuous, convex, and integral polyhedral.

To conclude, note that on one hand, $d!B_i(\mathcal{E}, r)$ and $B_d(\mathcal{E}, r)$ are continuous, convex, and polyhedral by the previous paragraph. On the other hand, for each $r \in T \cap \mathbb{Q}^n$, $d!B_i(\mathcal{E}, r)$ and $B_d(\mathcal{E}, r)$ take values in $\mathbb{Z} + \mathbb{Z}r_1 + \cdots + \mathbb{Z}r_n$ by the Hasse–Arf property of differential Swan conductors [**15**, Theorem 2.8.2] (or [**22**, Theorem 2.6.1]). Hence by Theorem 3.1.5 (or an elementary argument), $d!B_i(\mathcal{E}, r)$ and $B_d(\mathcal{E}, r)$ are integral polyhedral. □

Remark 5.2.2. Although the above argument suffices for our purposes, it is worth mentioning another lifting construction that may occasionally be useful. Let $\mathcal{E}$ be a locally constant étale $\mathbb{F}$-sheaf on X, where $\mathbb{F}$ is the residue field of E. Let G be the image of

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$\pi_1(X, \bar{x})$ in $\mathrm{GL}_d(\mathbb{F})$. For S any ring, let $R_S(G)$ denote the Grothendieck ring of finite $S[G]$-modules. Then the canonical map $R_{\mathfrak{o}_E}(G) \to R_{\mathbb{F}}(G)$ is surjective by [**28**, Chapter 16, Theorem 33], so the given $\mathbb{F}$-representation of G lifts to a virtual $\mathfrak{o}_E$-representation of G. We may then convert each factor of the virtual representation into an overconvergent F-isocrystal as in Remark 5.1.4. Unfortunately, since this representation is only virtual, one cannot use this argument to deduce convexity or polyhedrality.

5.3. Subharmonicity and monotonicity

We may also obtain subharmonicity and monotonicity results for surfaces, by using the same technique as in Theorem 5.2.1 to reduce to Theorems 4.2.7 and 4.3.2, respectively. (Initially one only proves $\ell(\mathcal{E}, Z) \in \mathbb{Q} \cap [0, d]$ because of the division by m in the argument of Theorem 5.2.1, but the integral polyhedrality of Theorem 5.2.1 forces $\ell(\mathcal{E}, Z) \in \mathbb{Z}$, so there is no problem.)

Theorem 5.3.1. *Assume that k is algebraically closed. Let $\bar{X}$ be a smooth irreducible projective surface over k, let Z be a smooth divisor on $\bar{X}$, and let v_0 be the divisorial valuation on $k(\bar{X})$ measuring order of vanishing along Z. Let X be an open dense subscheme of $\bar{X}$, and let $\mathcal{E}$ be a lisse étale E-sheaf on X. Define $b_j(\mathcal{E}, z, x, r)$ for $j = 1, \ldots, d$ and $\mathrm{Swan}_Z(\mathcal{E}, z, x, r)$ as in Definition 4.2.3.*

(a) *For each $z \in Z$, the functions $b_j(\mathcal{E}, z, x, r)$ for $j = 1, \ldots, d$ and $\mathrm{Swan}_Z(\mathcal{E}, z, x, r)$ are affine in a neighborhood of $r = 0$.*

(b) *Let $\mathrm{Swan}'_Z(\mathcal{E}, z)$ be the right slope of $\mathrm{Swan}_Z(\mathcal{E}, z, x, r)$ at $r = 0$. Then there exists $\ell(\mathcal{E}, Z) \in \{0, 1, \ldots, d\}$ such that $\mathrm{Swan}'_Z(\mathcal{E}, z) = -\ell(\mathcal{E}, Z)$ for all but finitely many $z \in Z$.*

(c) *With notation as in (b), we have*

$$\sum_{z \in Z} (\mathrm{Swan}'_Z(\mathcal{E}, z) + \ell(\mathcal{E}, Z)) \geqslant (2 - 2g(Z))\ell(\mathcal{E}, Z) - Z^2 \,\mathrm{Swan}(\mathcal{E}, Z),$$

where $g(Z)$ denotes the genus of Z, and Z^2 denotes the self-intersection of Z on $\bar{X}$.

(d) *If z is a smooth point of $Z \cup (\bar{X} \setminus X)$, then $\mathrm{Swan}'_Z(\mathcal{E}, z) + \ell(\mathcal{E}, Z) \leqslant 0$.*

Theorem 5.3.2. *With hypotheses as in Theorem 5.3.1, for each component Z of D,*

$$Z \cdot (\mathrm{Swan}(\mathcal{E}) + \ell(\mathcal{E}, Z)(K_{\bar{X}} + D)) \geqslant 0.$$

Proof. This follows from Theorem 5.3.1 by the same argument as in Theorem 4.4.7. □

Remark 5.3.3. It should be possible to use Theorem 5.3.1 to give an independent derivation of the semicontinuity theorem in étale cohomology [**23**]. We leave this as an exercise for the interested reader.

Appendix A. Errata for [15]

We take the opportunity here to record a few errata from [**15**].

As noted earlier (see Remark 1.0.2), [**15**, Hypothesis 2.1.3(b)] is missing the hypothesis that k is separable over k_0.

More seriously, the proof of [**15**, Proposition 2.7.11] is not correct as written. There are two problems with the given proof. One is that the formulae given for the action of $\partial/\partial u_i$, $\partial/\partial v_i$, $\partial/\partial t$ are only valid on sections of $\mathcal{E}'$ which are pulled back from $\mathcal{E}$, since $\partial'_1, \dots, \partial'_{n+1}$ are only defined for such sections. The other is that the summands making up $\partial/\partial t$ do not commute, so one cannot argue that the scale of the sum is bounded above by the the maximum of the scales of the summands.

What is valid to deduce from the construction given is that the highest break of $\mathcal{E}'$ is at least $pb - b + 1$, and that if equality occurs, $\partial/\partial t$ is eventually dominant for $\mathcal{E}'$. To complete the proof, we must argue another way that the highest break of $\mathcal{E}'$ is at most $pb - p + 1$. Using the interpretation of scales in terms of convergence of Taylor isomorphisms [**15**, § 2.2], this follows from the following fact: for $\rho \in (0, 1)$ sufficiently close to 1, if $t, u_1, \dots, u_n, v_1, \dots, v_n$ and $t', u'_1, \dots, u'_n, v'_1, \dots, v'_n$ satisfy

$$|t| = \rho, \qquad |u_i| = 1, \qquad |v_i| = 1,$$
$$|t - t'| \leqslant \rho^{pb-p+2}, \qquad |u'_i - u_i| \leqslant \rho^{pb-p+1}, \qquad |v'_i - v_i| \leqslant \rho^{pb-p+1},$$

then

$$\left| \frac{t^p}{1 - t^{p-1}} - \frac{(t')^p}{1 - (t')^{p-1}} \right| \leqslant \rho^{p(b+1)}, \qquad |(u_i^p + v_i t^{p-1}) - ((u'_i)^p + v'_i (t')^{p-1})| \leqslant \rho^{pb}.$$

The proof of this is straightforward.

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