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TRIGONOMETRIC INTERPOLATION*

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We consider interpolation at $2n$ equidistant nodes in $[0,\pi)$ from the space $\mathcal{T}_N$ spanned by sines and cosines of odd multiples of x. This interpolation problem is shown to be correct for an arbitrary sequence of derivatives specified at all the nodes. Explicit expressions for the fundamental polynomials are obtained and it is shown that under mild smoothness assumptions on the function f, interpolant from $\mathcal{T}_N$ converges uniformly to f as the node spacing goes to zero.

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1. Introduction

Generally in all problem of trigonometric interpolation, the underlying space of functions considered is spanned by

$$\{1, \cos x, \sin x, \ldots, \cos Nx, \sin Nx\}, \tag{1.1}$$

or by the functions

$$\left\{1, \cos x, \sin x, \ldots, \cos Nx, \sin Nx, \cos\left((N+1)x + \frac{\varepsilon\pi}{2}\right)\right\} \tag{1.2}$$

where $\varepsilon = 0$ or 1. The problem of $(0, m_1, \ldots, m_q)$ interpolation on equidistant nodes refers to the above spaces of trigonometric polynomials (**[3, 4, 6, 7, 8, 9]**).

However there seems to be no *a priori* reason to choose the above underlying spaces. Recently Goodman and Lee **[2]** have used a different system of trigonometric functions in their work related to trigonometric splines. They consider the space spanned by $\{\sin x, \cos x, \sin 3x, \cos 3x, \ldots, \sin(2N-1)x, \cos(2N-1)x\}$. These functions satisfy the condition $f(x+\pi) = -f(x)$ and so in this case it is natural to limit ourselves to the interval $[0,\pi)$ instead of $[0,2\pi)$ as we do in the earlier literature.

Let us denote the space spanned by $\{\sin(2j-1)x, \cos(2j-1)x\}_{j=1}^{N}$ by $\mathcal{T}_N$ and let

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$$x_k = \frac{k\pi}{2n}\ (k=0,1,\dots,2n-1)$$

be $2n$ equidistant nodes in $[0,\pi)$. We propose the following problems:

P_1. Given integers $0 \leqq m_1 < m_2 < \cdots < m_p$, find the conditions necessary and sufficient for the regularity of $(m_1,\dots,m_p)$ interpolation on the nodes $\{x_k\}_0^{2n-1}$ by trigonometric polynomials from the space $\mathcal{T}_N$.

It may be observed that since $1 \notin \mathcal{T}_N$, the smallest integer m_1 need not be zero, as happens to be the case in the classical situations.

P_2. Find the fundamental polynomials of interpolation.

P_3. If $f(x) \in C[0,\pi]$, find the convergence behaviour of the trigonometric interpolant to the function on $\{x_k\}_0^{2n-1}$ from $\mathcal{T}_N$ as $n \to \infty$.

In Section 2, we state some lemmas on determinants which will be required later. In Section 3, we state and prove the main result. It turns out that regularity (or unique solvability) of this interpolation from $\mathcal{T}_N$ is always possible for all choices of distinct integers $\{m_v\}_1^p$. This is in contrast with the case when the underlying space is (1.1) or (1.2), for then regularity depends upon the difference between the number of even and odd integers in the set $\{m_v\}_1^p$. In Section 4, we find explicit expressions for the fundamental polynomials. Section 5 is devoted to the properties of two determinants which occur in the expressions for the fundamental polynomials given in Section 4. Finally we state and prove the convergence theorem in Section 6.

2. Some lemmas

We shall need the following three lemmas:

Lemma 1. (cf. [1]). *If $0 < t_1 < t_2 < \cdots < t_s$ are given real numbers and if $0 \leqq m_1 < m_2 < \cdots < m_s$ are positive integers, then the generalized Vandermondian Δ is positive, where*

$$\Delta := K\begin{pmatrix} m_1,\dots,m_s \\ t_1,\dots,t_s \end{pmatrix} := \begin{vmatrix} t_1^{m_1} & t_2^{m_1} & \cdots & t_s^{m_1} \\ t_1^{m_2} & t_2^{m_2} & \cdots & t_s^{m_2} \\ \cdots & \cdots & \cdots & \cdots \\ t_1^{m_s} & t_2^{m_s} & \cdots & t_s^{m_s} \end{vmatrix}$$

If in the above, $t_i = t_{i+1}$, we shall be interested in the sign of the determinant Δ^* where

$$\Delta^* := K\begin{pmatrix} m_1,\dots,m_s \\ t_1,\dots,t_i,t_{i+2},\dots,t_s \end{pmatrix} = \begin{vmatrix} t_1^{m_1} & \cdots & t_i^{m_1} & m_1 t_i^{m_1-1} & t_{i+2}^{m_1} & \cdots & t_s^{m_1} \\ t_1^{m_2} & \cdots & t_i^{m_2} & m_2 t_i^{m_2-1} & t_{i+2}^{m_2} & \cdots & t_s^{m_2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ t_1^{m_s} & \cdots & t_i^{m_s} & m_s t_i^{m_s-1} & t_{i+2}^{m_s} & \cdots & t_s^{m_s} \end{vmatrix}. \quad (2.2)$$

Lemma 2. *If* $0<t_1<\cdots<t_i<t_{i+2}<\cdots<t_s$ *and* $0\leqq m_1<\cdots<m_s$, *then* $\Delta^*>0$.

Since we could not find this stated explicitly in the literature we sketch the proof.

Proof. Suppose $\Delta^*=0$. Then there exist $a_1,\ldots,a_s$ not all zero such that

$$\sum_{j=1}^{s} a_j t_i^{m_j}=0,\ j=1,\ldots,i,i+2,\ldots,p,$$

$$\sum_{j=1}^{s} m_j a_j t_i^{m_j-1}=0.$$

Thus the polynomial $Q(t)=\sum_{j=1}^{s} a_j t^{m_j}$ vanishes for $t=t_1,\ldots,t_i,t_{i+2},\ldots,t_s$ and $Q'(t_i)=0$. Thus Q has s zeros counting multiplicity in $(0,\infty)$. By Descartes' rule, $Q(t)$ cannot have more than $s-1$ zeros in $(0,\infty)$ and so $\Delta^*\neq 0$.

If in (2.1) we set $t_{i+1}=t_i+h$, $h>0$, then the corresponding $\Delta(h)>0$. Subtracting the ith column from the $(i+1)$th and dividing by h, the resulting determinant is still >0. Letting $h\to 0$, we see that $\Delta^*\geqq 0$. Thus $\Delta^*>0$, which completes the proof. □

Lemma 3. (cf. [9]). *Let* $0<t_1<t_2<\cdots<t_s$ *be given real numbers where* $s=p+q$. *Let* $0\leqq m_1<m_2<\cdots<m_p$ *and* $0\leqq m'_1<m'_2<\cdots<m'_q$ *be two sets of integers (not necessarily different from each other). If* D *denotes the determinant given below, then*

$$\operatorname{sgn} D=(-1)^{q(s+p-1)/2},$$

where

$$D:=\begin{vmatrix} t_1^{m_1} & t_2^{m_1} & \cdots & t_s^{m_1} \\ \vdots & \vdots & & \vdots \\ t_1^{m_p} & t_2^{m_p} & \cdots & t_s^{m_p} \\ t_1^{m'_1} & -t_2^{m'_1} & \cdots & (-1)^{s-1}t_s^{m'_1} \\ \vdots & \vdots & & \vdots \\ t_1^{m'_q} & -t_2^{m'_q} & \cdots & (-1)^{s-1}t_s^{m'_q} \end{vmatrix}.$$

3. Regularity of Problem P_1

The number of data in this case is $2N=2np$ and so we shall consider trigonometric polynomials from the class $\mathcal{T}_N$. More precisely, set

$$T_{np}(x):=\sum_{j=0}^{np-1}\{c_j e^{-(2j+1)ix}+d_j e^{(2j+1)ix}\}.$$

Equivalently, we may write

$$T_{np}(x)=\sum_{\lambda=0}^{p-1}\sum_{j=0}^{n-1}\{c_{j,\lambda}e^{-i(2j+2\lambda n+1)x}+d_{j,\lambda}e^{i(2j+2\lambda n+1)x}\}. \tag{3.1}$$

The homogeneous interpolation problem is given by

$$T_{np}^{(m_\nu)}(x_k)=0,\ \nu=1,\dots,p;\ k=0,1,\dots,2n-1, \tag{3.2}$$

where

$$x_k=\frac{k\pi}{2n},\ k=0,1,\dots,2n-1.$$

Theorem 1. *The problem of* $(m_1,m_2,\dots,m_p)$ *interpolation by trigonometric polynomials from the class* $\mathscr{T}_{np}$ *on* $2n$ *equidistant nodes* x_k $(k=0,1,\dots,2n-1)$ *is regular for all distinct positive integers* $m_1,m_2,\dots,m_p$.

Proof. Clearly it is enough to show that if $T_{np}(x)$ is given by (3.1) and satisfies (3.2), then it is identically zero. From (3.1) and (3.2), we see on putting $z_k=e^{ix_k}$, that

$$\sum_{\lambda=0}^{p-1}\sum_{j=0}^{n-1}c_{j,\lambda}(2j+2\lambda n+1)m_\nu(-1)^{(m_\nu)}z_k^{-(2j+1)}(-1)^{\lambda k}$$
$$+\sum_{\lambda=0}^{p-1}\sum_{j=0}^{n-1}d_{j,\lambda}(2j+2\lambda n+1)^{m_\nu}(-1)^{\lambda k}z_k^{2j+1}=0.$$

Multiplying both sides by z_k^{2n-1} and changing j into $n-j-1$ in the first summation, we obtain

$$\sum_{j=0}^{n-1}z_k^{2j}\left[\sum_{\lambda=0}^{p-1}(2\lambda n+2n-2j-1)^{m_\nu}(-1)^{m_\nu+\lambda k}c_{n-j-1,\lambda}\right.$$
$$\left.+z_k^{2n}\sum_{\lambda=0}^{p-1}(2\lambda n+2j+1)^{m_\nu}d_{j,\lambda}\right]=0,\quad (k=0,1,\dots,2n-1). \tag{3.3}$$

If k is even, $z_k^{2n}=1$, and when k is odd, $z_k^{2n}=-1$. The above conditions (3.3) show that when k is even (and similarly when k is odd), we have an equation of degree $n-1$ having n zeros. This observation leads to the following system of equations:

$$\sum_{\lambda=0}^{p-1}(2\lambda n+2n-2j-1)^{m_\nu}(-1)^{m_\nu}c_{n-j-1,\lambda}+\sum_{\lambda=0}^{p-1}(2\lambda n+2j+1)^{m_\nu}d_{j,\lambda}=0,$$

$$(\nu=1,\dots,p,\ j=0,1,\dots,n-1,\ \text{when } k \text{ is even}), \tag{3.4}$$

$$\sum_{\lambda=0}^{p-1}(2\lambda n+2n-2j-1)^{m_\nu}(-1)^{m_\nu+\lambda}c_{n-j-1,\lambda}-\sum_{\lambda=0}^{p-1}(2\lambda n+2j+1)^{m_\nu}(-1)^{\lambda}d_{j,\lambda}=0,$$

$$(\nu=1,\ldots,p,\ j=0,1,\ldots,n-1,\ \text{when } k \text{ is odd}). \tag{3.5}$$

Suppose there are s even and l odd integers in the set $0,1,\ldots,p-1$. Then $s+l=p$ and $s=l$ if p is even, and $s=l+1$ if p is odd. We now rewrite the equations (3.4) and (3.5) after replacing λ by 2λ or $2\lambda+1$ as needed. Then adding and subtracting, we get the following two systems of p equations for every j ($j=0,\ldots,n-1$):

$$\sum_{\lambda=0}^{s-1}c_{n-j-1,2\lambda}(-1)^{m_\nu}(4\lambda n+2n-2j-1)^{m_\nu}+\sum_{\lambda=0}^{l-1}d_{j,2\lambda+1}(4\lambda n+2n+2j+1)^{m_\nu}=0,\quad \nu=1,\ldots,p \tag{3.6}$$

and

$$\sum_{\lambda=0}^{l-1}c_{n-j-1,2\lambda+1}(-1)^{m_\nu}(4\lambda n+4n-2j-1)^{m_\nu}+\sum_{\lambda=0}^{s-1}d_{j,2\lambda}(4\lambda n+2j+1)^{m_\nu}=0,\quad \nu=1,\ldots,p. \tag{3.7}$$

The system of equations (3.6) (and similarly (3.7)) has p unknowns. Let us denote the determinants of systems (3.6) and (3.7) by $\Delta_{1,j}$ and $\Delta_{2,j}$ respectively.

We now examine $\Delta_{1,j}$ more closely and show that it cannot vanish. Any row of the determinant $\Delta_{1,j}$ is given by

$$\begin{aligned}&(2j+1-2n)^{m_\nu}\quad (2j+1-6n)^{m_\nu}\ldots(2j+1-4sn+2n)^{m_\nu}\\ &(2j+1+2n)^{m_\nu}\quad (2j+1+6n)^{m_\nu}\ldots(2j+1+4ln-2n)^{m_\nu}.\end{aligned}$$

So the first s terms of the row are decreasing successively and the next l terms are increasing by $4n$. The absolute values of these terms can be arranged in increasing order as follows:

$$(2n-2j-1)^{m_\nu}<(2n+2j+1)^{m_\nu}<(6n-2j-1)^{m_\nu}<(6n+2j+1)^{m_\nu}<\cdots$$

If all the m_ν's have the same parity, then $\Delta_{1,j}\neq 0$ by Lemma 1. If some of the m_ν's are even and some are odd, then if the columns are arranged so that the terms in each row are increasing in absolute value, then the sign of all terms in a row are positive if the terms have power m_ν (m_ν even) and are alternately positive and negative if m_ν is odd. It then follows from Lemma 3 that $\Delta_{1,j}\neq 0$. Similar reasoning shows that $\Delta_{2,j}\neq 0$. Indeed a close scrutiny of the determinants shows that

$$\Delta_{1,j}=(-1)^{\Sigma_1^p m_\nu}\Delta_{2,n-1-j}.$$

Since the determinants of (3.6) and (3.7) are not zero, $T_{np}(x)$ is identically zero which completes the proof. □

4. Fundamental polynomials

Let us denote the fundamental polynomials of this problem by $\rho_{k,m_\nu}(x)$, $(k=0,1,\ldots,2n-1)$. They will satisfy the following conditions:

$$\rho_{k,m_\nu}(x)\in\mathscr{T}_N,\ N=np, \tag{4.1}$$

$$\rho_{k,m_\nu}^{(m_\mu)}(x_l)=0,\ \mu\neq\nu,\ l=0,1,\ldots,2n-1, \tag{4.2}$$

$$\rho_{k,m_\nu}^{(m_\nu)}(x_l)=\delta_{kl}. \tag{4.3}$$

It is clear that $\rho_{k,m_\nu}(x)=\rho_{0,m_\nu}(x-x_k)$. In view of the equations (3.6) and (3.7) which are used in the proof of regularity, we set

$$\rho_{0,m_\nu}(x)=\sum_{\lambda=0}^{p-1}\sum_{j=0}^{n-1}\{c_{j,\lambda}z^{-(2j+2\lambda n+1)}+d_{j,\lambda}z^{2j+2\lambda n+1}\},$$

where $z=e^{ix}$. From (4.2) and (4.3), it follows that

$$\rho_{0,m_\nu}^{(m_\nu)}(x_k)=\begin{cases}0, & \mu\neq\nu,\ k=0,1,\ldots,2n-1\\ \delta_{0k}, & \mu=\nu.\end{cases} \tag{4.4}$$

The conditions (4.4) yield the conditions

$$\sum_{j=0}^{n-1}z_k^{2j}\sum_{\lambda=0}^{p-1}\{c_{n-j-1,\lambda}(2n+2\lambda n-2j-1)^{m_\nu}(-1)^{m_\nu}(-1)^{(\lambda+1)k}$$
$$+d_{j,\lambda}(2j+2\lambda n+1)^{m_\nu}(-1)^{m_\nu}\}$$
$$=(-1)^k e^{-i(k\pi/2n)}i^{-m_\nu}\delta_{0k},\quad z_k=e^{ik\pi/2n}(k=0,1,\ldots,2n-1).$$

Thus if k is even, we have from the above

$$\sum_{j=0}^{n-1}z^j\sum_{\lambda=0}^{p-1}\{c_{n-j-1,\lambda}(2n+2\lambda n-2j-1)^{m_\nu}(-1)^{m_\nu}+d_{j,\lambda}(2j+2\lambda n+1)^{m_\nu}\}=\frac{1}{n}\frac{z^n-1}{z-1}i^{-m_\nu}, \tag{4.5}$$

and if k is odd

$$\sum_{j=0}^{n-1}z^j\sum_{\lambda=0}^{p-1}\{c_{n-j-1,\lambda}(2n+2\lambda n-2j-1)^{m_\nu}(-1)^{m_\nu}(-1)^{\lambda+1}+d_{j,\lambda}(2j+2\lambda n+1)^{m_\nu}(-1)^{\lambda}\}=0. \tag{4.6}$$

Hence for every j, $(j=0,1,\ldots,n-1)$, we have from (4.6)

$$\left\{\sum_{\lambda=0}^{p-1} c_{n-j-1,\lambda}(2n+2\lambda n-2j-1)^{m_\nu}(-1)^{m_\nu}(-1)^{\lambda}-\sum_{\lambda=0}^{p-1} d_{j,\lambda}(2j+2\lambda n+1)^{m_\nu}(-1)^{\lambda}\right\}=0. \tag{4.7}$$

From (4.5), we have

$$\sum_{\lambda=0}^{p-1} c_{n-j-1,\lambda}(2n+2\lambda n-2j-1)^{m_\nu}(-1)^{m_\nu}+\sum_{\lambda=0}^{p-1} d_{j,\lambda}(2j+2\lambda n+1)^{m_\nu}=\frac{i^{-m_\nu}}{n}. \tag{4.8}$$

Adding and subtracting (4.7) and (4.8) yields

$$\sum_{\substack{\lambda=0\\ \lambda\,\mathrm{even}}}^{p-1} c_{n-j-1,\lambda}(2n+2\lambda n-2j-1)^{m_\nu}(-1)^{m_\nu}+\sum_{\substack{\lambda=0\\ \lambda\,\mathrm{odd}}}^{p-1} d_{j,\lambda}(2j+2\lambda n+1)^{m_\nu}=\frac{i^{-m_\nu}}{2n}. \tag{4.9}$$

and

$$\sum_{\substack{\lambda=0\\ \lambda\,\mathrm{odd}}}^{p-1} c_{n-j-1,\lambda}(2n+2\lambda n-2j-1)^{m_\nu}(-1)^{m_\nu}+\sum_{\substack{\lambda=0\\ \lambda\,\mathrm{even}}}^{p-1} d_{j,\lambda}(2j+2\lambda n+1)^{m_\nu}=\frac{i^{-m_\nu}}{2n}. \tag{4.10}$$

Similarly, if we use the property (4.2), we obtain

$$\sum_{\substack{\lambda=0\\ \lambda\,\mathrm{even}}}^{p-1} c_{n-j-1,\lambda}(2n+2\lambda n-2j-1)^{m_\mu}(-1)^{m_\mu}+\sum_{\substack{\lambda=0\\ \lambda\,\mathrm{odd}}}^{p-1} d_{j,\lambda}(2j+2\lambda n+1)^{m_\mu}=0, \tag{4.11}$$

$$\sum_{\substack{\lambda=0\\ \lambda\,\mathrm{odd}}}^{p-1} c_{n-j-1,\lambda}(2n+2\lambda n-2j-1)^{m_\mu}(-1)^{m_\mu}+\sum_{\substack{\lambda=0\\ \lambda\,\mathrm{even}}}^{p-1} d_{j,\lambda}(2j+2\lambda n+1)^{m_\mu}=0, \tag{4.12}$$

for $\mu=1,2,\dots,p,\mu\neq\nu$. From (4.9) and (4.11), we obtain the values of $c_{n-j-1,\lambda}$ (λ even) and of $d_{j,\lambda}$ (λ odd), $(\lambda=0,1,\dots,p-1)$. The values of $c_{n-j-1,\lambda}$ (λ odd) and $d_{j,\lambda}$ (λ even) are given by (4.10) and (4.12). The determinants of the system of equations (4.9) and (4.11) and of (4.10) and (4.12) agree with $\Delta_{1,j}$ and $\Delta_{2,j}$ respectively as defined in Section 3.

Let $\Delta_{1,j,\nu}(z)$ denote the determinant obtained from $\Delta_{1,j}$ by replacing the νth row by the following

$$(z^{2j+1-2n}\quad z^{2j+1-6n}\quad \dots\quad z^{2j+1-2n(2s-1)}\quad z^{2n+2j+1}\quad z^{6n+2j+1}\quad \dots\quad z^{2n(2l-1)+2j+1}).$$

Similarly, $\Delta_{2,j,\nu}(z)$ will denote the determinant obtain by $\Delta_{2,j}$ after replacing the νth row by the row

$$(z^{2j+1-4n}\quad z^{2j+1-8n}\quad \dots\quad z^{2j+1-4ln}\quad z^{2j+1}\quad z^{2j+1+4n}\quad \dots\quad z^{2j+1+4n(s-1)}).$$

If we set

$$\rho_{0,m_v}(x)=\sum_{j=0}^{n-1}(P_{j,m_v}(z)+Q_{j,m_v}(z)),$$

where

$$P_{j,m_v}(z)=\sum_{\substack{\lambda=0\\ \lambda\,\text{even}}}^{p-1} c_{n-j-1,\lambda}z^{2j+1-2\lambda n-2n}+\sum_{\substack{\lambda=0\\ \lambda\,\text{odd}}}^{p-1} d_{j,\lambda}z^{2j+1+2\lambda n},$$

$$Q_{j,m_v}(z)=\sum_{\substack{\lambda=0\\ \lambda\,\text{odd}}}^{p-1} c_{n-j-1,\lambda}z^{2j+1-2\lambda n-2n}+\sum_{\substack{\lambda=0\\ \lambda\,\text{even}}}^{p-1} d_{j,\lambda}z^{2j+1+2\lambda n},$$

then it follows from (4.9), (4.11) and (4.10), (4.12) that

$$P_{j,m_v}(z)=\frac{i^{-m_v}}{2n}\frac{\Delta_{1,j,v}(z)}{\Delta_{1,j}},$$

$$Q_{j,m_v}(z)=\frac{i^{-m_v}}{2n}\frac{\Delta_{2,j,v}(z)}{\Delta_{2,j}}.$$

It may be observed that for $1\leqq\alpha\leqq s$, the (i,α)th term of $\Delta_{1,n-1-j}$ equals the $(i,\alpha+l)$th term of $\Delta_{2,j}$ multiplied by $(-1)^{m_i}$, while the (v,α)th term of $\Delta_{1,n-1-j,v}(z)$ equals the $(v,\alpha+l)$th term of $\Delta_{2,j,v}(z)$ with z replaced by z^{-1}. Similarly for $1\leqq\alpha\leqq l$, the $(i,s+\alpha)$th term of $\Delta_{1,n-1-j}$ equals the (i,α)th term of $\Delta_{2,j}$ multiplied by $(-1)^{m_i}$, while the $(v,s+\alpha)$th term of $\Delta_{i,n-j-1,v}(z)$ equals the (v,α) term of $\Delta_{2,j,v}(z)$ with z replaced by z^{-1}. Thus

$$\begin{cases}\Delta_{1,n-1-j}=(-1)^{\Sigma_1^\rho m_k}\Delta_{2,j},\\[2mm] \dfrac{\Delta_{1,n-1-j,v}(z)}{\Delta_{1,j}}=(-1)^{m_v}\dfrac{\Delta_{2,j,v}(z^{-1})}{\Delta_{2,j}}.\end{cases}\tag{4.13}$$

Thus we have

$$\rho_{0,m_v}(z)=\begin{cases}\dfrac{(-1)^{m_v/2}}{n}\displaystyle\sum_{j=0}^{n-1}\frac{Re\,\Delta_{1,j,v}(z)}{\Delta_{1,j}}, & m_v\,\text{even},\\[4mm] \dfrac{(-1)^{[m_v/2]}}{n}\displaystyle\sum_{j=0}^{n-1}\frac{Im\,\Delta_{1,j,v}(z)}{\Delta_{1,j}}, & m_v\,\text{odd}.\end{cases}\tag{4.14}$$

5. Properties of $\Delta_{i,j}$ and $\Delta_{1,j,v}(z)$

In the following two sections we shall take $m_1=0$.

Determining the convergence behaviour of this type of interpolation depends upon examining the nature of the determinant $\Delta_{1,j}$ and the polynomial $\Delta_{1,j,v}(z)$ more carefully. In order to do so, we rearrange the columns of $\Delta_{1,j}$ and set $((2j+1)/2n)=t$. Then if $p=2r-1$ or $2r$, we set

$$\phi(t):=2^{-\Sigma_1^p m_j}\Delta_{1,j}=\begin{vmatrix} 1 & 1 & \cdots & 1 \\ (t-2r+1)^{m_2} & (t-2r+3)^{m_2} & \cdots & (t-2r+2p-1)^{m_2} \\ \cdots & \cdots & \cdots & \cdots \\ (t-2r+1)^{m_p} & (t-2r+3)^{m_p} & \cdots & (t-2r+2p-1)^{m_p} \end{vmatrix}. \tag{5.1}$$

If we denote by $\phi_{kl}(t)$ the co-factor of the (k,l) term in $\phi(t)$, we can show that

$$\rho_{0,m_v}(z)=\begin{cases} \dfrac{(-1)^{m_v/2}}{n} Re\displaystyle\sum_{j=0}^{n-1}\frac{1}{(2n)^{m_v}}\sum_{\lambda=0}^{p-1}\frac{\phi_{v,\lambda}((2j+1)/2n)}{\phi((2j+1)/2n)}z^{2j+1-2(2r-1)n+4\lambda n}, & m_v \text{ even,} \\[2ex] \dfrac{(-1)^{[m_v/2]}}{n} Im\displaystyle\sum_{j=0}^{n-1}\frac{1}{(2n)^{m_v}}\sum_{\lambda=0}^{p-1}\frac{\phi_{v,\lambda}((2j+1)/2n)}{\phi((2j+1)/2n)}z^{2j+1-2(2r-1)n+4\lambda n}, & m_v \text{ odd.} \end{cases} \tag{5.2}$$

It follows from Lemma 3, that $\phi(t)\neq 0$ for $0<t<1$. To see this one must rearrange the columns of (5.1) as is done in the proof of Theorem 1. The following Lemma describes the multiplicities of its zeros at 0 and 1.

Lemma 4. *If the polynomial $\phi(t)$ in (5.1) has zeros of multiplicity α and β at 0 and 1 respectively, then the polynomial $\phi_{kl}(t)$ has zeros of multiplicity at least $\alpha-1$ and $\beta-1$ at 0 and 1 respectively.*

Proof. Without loss of generality we may suppose that $m_1,m_2,\ldots,m_s$ are even and $m_{s+1},\ldots,m_p$ are odd. We shall prove the lemma for the case when p is even $(=2r)$ and $s\geqq r$. We then show that $\phi(t)$ has a zero of exact multiplicity $s-r$ at 0. To see this, we observe that the ith row of $\phi(t)$ is

$$(t-2r+1)^{m_i}\quad (t-2r+3)^{m_i}\quad \ldots\quad (t-1)^{m_i}\quad (t+1)^{m_i}\quad \ldots\quad (t+2r-1)^{m_i}.$$

Subtracting the $(p+1-j)$th column from the jth column for $j=1,2,\ldots,r$, then dividing the first r columns by t and multiplying the last $p-r$ rows by t, we see that $\phi(t)$ has a factor $t^{r-(p-s)}=t^{s-r}$. Thus

$$\phi(t)=t^{s-r}\psi(t)$$

where $\psi(t)$ is a determinant of order p (which is a polynomial in t). We claim that $\psi(0)\neq 0$.

Indeed we have

$$\psi(0)=(-2)^r\begin{vmatrix} A & B \\ C & 0 \end{vmatrix}$$

where A, B matrices of order $s\times r$, C is a matrix of order $(p-s)\times r$ and 0 is the zero matrix of order $(p-s)\times r$. More precisely if $A=(a_{ij})$, $B=(b_{ij})$ and $C=(c_{ij})$ then

$$a_{ij}=m_i(2r+1-2j)^{m_i-1},\quad i=1,\dots,s;\ j=1,\dots,r,$$

$$b_{ij}=(2j-1)^{m_i},\quad i=1,\dots,s;\ j=1,\dots,r,$$

$$c_{ij}=(2r+1-2j)^{m_i},\quad i=s+1,\dots,p;\ j=1,\dots,r.$$

We may arrange the first r column of $\psi(0)$ in increasing order and then use Laplace expansion in terms of the last $p-s$ rows. If we denote by

$$D_\psi\begin{pmatrix} i_1,\dots,i_k \\ j_1,\dots,j_k \end{pmatrix}$$

the determinant of order k obtained from the rearranged $\psi(0)$ with rows $i_1,\dots,i_k$ and columns $j_1,\dots,j_k$, then

$$\psi(0)=2^rC\sum(-1)^{j_1+\cdots+j_{p-s}}D_\psi\begin{pmatrix} s+1,\dots,p \\ j_1,\dots,j_{p-s} \end{pmatrix}D_\psi\begin{pmatrix} 1,\dots,s \\ j'_1,\dots,j'_{s-r},r+1,\dots,p \end{pmatrix}$$

where the summation runs over $1\leqq j_1<j_2<\cdots<j_{p-s}\leqq r$ and $\{j',\dots,j'_{s-r}\}=\{1,\dots,r\}\backslash\{j_1,\dots,j_{p-s}\}$ and C is a constant with absolute value 1. The determinant

$$D_\psi\begin{pmatrix} 1,\dots,s \\ j'_1,\dots,j'_{p-r},r+1,\dots,p \end{pmatrix}$$

is formed from the rows and columns of the matrix $[A \quad B]$ and its r columns from the $(r+1)$th to the last are identical with those of B. The columns $j'_1,\dots,j'_{s-r}$ are from A. The elements of these columns are the derivatives of corresponding elements in certain columns from $r+1$ to p. More precisely, column j'_ν is the derivative of the column $r+j'_\nu$ $(\nu=1,\dots,s-r)$. If therefore we juxtapose these columns into new positions so that a column is followed by its derivative column (if any), then by Lemma 2, the rearranged determinant will be positive. Thus

$$D_\psi\begin{pmatrix} 1,\dots,s \\ j'_1,\dots,j'_{s-r},r+1,\dots,p \end{pmatrix}=(-1)^{\Sigma_1^{s-r}j'_\nu}D^*_\psi,$$

where $D_\psi^* > 0$ is the juxtaposed matrix. Thus we see that

$$\psi(0) = C2^r \sum (-1)^{\Sigma_1^{p-s} j_\nu + \Sigma_1^{p-s} j'_\nu} D_\psi \begin{pmatrix} s+1, \dots, p \\ j_1, \dots, j_{p-s} \end{pmatrix} D_\psi^* \neq 0.$$

Consider now the polynomial $\phi_{kl}(t)$. For $1 \leqq j \leqq r$, if $1 \leqq l \leqq r$ and $j \neq l$ or $r+1 \leqq l \leqq 2r$ and $j \neq p+1-l$, then subtract the column with $(t+2r+1-2j)^{m_i}$ from the column with $(t-2r-1+2j)^{m_i}$, $(i=1, \dots, p, i \neq k)$ and divide each of the resulting columns by t. Then multiplying the last $p-1$ rows by t gives

$$\phi_{kl}(t) = t^{s-r-1} \psi_{kl}(t)$$

where $\psi_{kl}(t)$ is a determinant of order p which is a polynomial in t.

We have thus shown that at 0, $\phi(t)$ has a zero of exact multiplicity $s-r$ and $\phi_{kl}(t)$ has a zero of multiplicity at least $s-r-1$. Similarly, we can show that at 1, $\phi(t)$ has a zero of exact multiplicity $s-r-1$ and that $\phi_{kl}(t)$ has a zero of multiplicity at least $s-r-2$.

The same reasoning as above can be easily used *mutatis mutandis* to prove the lemma in the remaining cases.

As an example, consider $p=4$, $m_1=0$, $m_2=2$, $m_3=6$, $m_4=3$. Then $r=2$, $s=3$ and

$$\phi(t) = \begin{vmatrix} 1 & 1 & 1 & 1 \\ (r-3)^2 & (t-1)^2 & (t+1)^2 & (t+3)^2 \\ (t-3)^6 & (t-1)^6 & (t+1)^6 & (t+3)^6 \\ (t-3)^3 & (t-1)^3 & (t+1)^3 & (t+3)^3 \end{vmatrix}$$

$$= t \begin{vmatrix} 0 & 0 & 1 & 1 \\ -12 & -4 & (t+1)^2 & (t+3)^2 \\ -12.3^5 + O(t) & -12 + O(t) & (t+1)^6 & (t+3)^6 \\ -54 + O(t^2) & -2 + O(t^2) & t(t+1)^3 & t(t+3)^3 \end{vmatrix}$$

So

$$\psi(0) = 4 \begin{vmatrix} 0 & 0 & 1 & 1 \\ 6 & 2 & 1^2 & 3^2 \\ 6.3^5 & 6 & 1^6 & 3^6 \\ 27 & 1 & 0 & 0 \end{vmatrix} = 4 \left[27 \begin{vmatrix} 1 & 0 & 1 \\ 1^2 & 2.1 & 3^2 \\ 1^6 & 6.5 & 3^6 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 0 \\ 1^2 & 3^2 & 2.3 \\ 1^6 & 3^6 & 6.3^5 \end{vmatrix} \right] > 0,$$

by Lemma 2.

6. Convergence

The convergence problem requires estimates on the sum

$$\sum_{k=0}^{2n-1} |\rho_{0,m_\nu}(x-x_k)|$$

and to this effect we prove

Lemma 5. *If* $0=m_1, m_2, \ldots, m_p$ *are distinct integers then*

$$\sum_{k=0}^{2n-1} |\rho_{0,m_\nu}(x-x_k)| = O(n^{1-m_\nu}),\ \nu=1,\ldots,p. \tag{6.1}$$

Proof. From (4.14) and (5.2), we see that

$$|\rho_{0,m_\nu}(x)| \leqq \frac{1}{(2n)^{m_\nu}} \sum_{\lambda=0}^{p-1} \left| \frac{1}{n} \sum_{j=0}^{n-1} \frac{\phi_{\nu,\lambda}((2j+1)/2n)}{\phi((2j+1)/2n)} z^{2j+1} \right|,\ z=e^{ix}.$$

Since $\phi_{\nu,\lambda}(t) = t\phi_{\nu,\lambda}(t) + (1-t)\phi_{\nu,\lambda}(t)$, we have

$$|\rho_{0,m_\nu}(x)| \leqq \sum_{\lambda=0}^{p-1} \frac{1}{(2n)^{m_\nu}} (|S_{1,\lambda}| + |S_{2,\lambda}|), \tag{6.2}$$

where

$$S_{1,\lambda} = \sum_{j=0}^{n-1} r\left(\frac{2j+1}{2n}\right) \frac{z^{2j+1}}{2j+1},\quad S_{2,\lambda} = \sum_{j=0}^{n-1} r\left(\frac{2j+1}{2n}\right) \frac{z^{2j+1}}{2n-2j-1}$$

and

$$r(t) = \frac{2t(1-t)\phi_{\nu,\lambda}(t)}{\phi(t)}.$$

By Lemma 4, $r(t)$ is continuous on $[0,1]$ and hence of bounded variation. By summation by parts, we have

$$S_{1,\lambda} = \sum_{j=0}^{n-1} \left\{ r\left(\frac{2j+1}{2n}\right) - r\left(\frac{2j+3}{2n}\right) \right\} \sum_{\nu=0}^{j} \frac{z^{2\nu+1}}{2\nu+1} + r\left(\frac{2n+1}{2n}\right) \sum_{\nu=0}^{n-1} \frac{z^{2\nu+1}}{2\nu+1},$$

so that

$$|S_{1,\lambda}| \leqq C \max_{0 \leqq j \leqq n-1} \left| \sum_{\nu=0}^{j} \frac{z^{2\nu+1}}{2\nu+1} \right|. \tag{6.3}$$

Similarly, $|S_{2,\lambda}|$ is also bounded by the same expression with z replaced by $\bar{z}$.

In order to estimate $|S_{1,\lambda}|$ we see that for $0<x<\pi$, we have

$$\left|\sum_{v=0}^{j}\frac{\cos(2v+1)x}{2v+1}\right|=\left|\sum_{v=0}^{j}\int_{x}^{\pi/2}\sin(2v+1)t\,dt\right|$$

$$=\left|\int_{x}^{\pi/2}\sum_{v=0}^{j}\sin(2v+1)t\,dt\right|$$

$$\leqq\left|\int_{x}^{\pi/2}\frac{dt}{\sin t}\right|=\frac{1}{2}\left|\log\cot^2\frac{x}{2}\right|=:\frac{1}{2}g(x).$$

Since it is well known that

$$\left|\sum_{v=0}^{j}\frac{\sin(2v+1)x}{2v+1}\right|$$

is uniformly bounded on $[0,\pi)$, we see that

$$|S_{i,\lambda}|\leqq C+\min(\log n,\tfrac{1}{2}g(x)),\quad i=1,2.$$

Hence from (6.2) we obtain

$$\sum_{k=0}^{2n-1}|\rho_{0,m_v}(x-x_k)|\leqq\frac{p}{(2n)^{m_v}}\sum_{k=0}^{2n-1}[C+\min(2\log n,g(x-x_k))]$$

$$\leqq cn^{1-m_v}\left[2C+\sum_{k=0}^{2n-1}\min\left(\frac{2\log n}{n},\frac{1}{n}g(x-x_k)\right)\right].$$

Since it can be seen as in [3] that

$$\sum_{k=0}^{2n-1}\min\left(\frac{2\log n}{n},\frac{1}{n}g(x-x_k)\right)\leqq\frac{4\log n}{n}+\frac{2}{\pi}\int_0^{\pi}g(t)\,dt,$$

the proof of (6.1) is complete.

Now let $J_n(x;f)$ denote the Jackson polynomial of f of degree n. Note that if $f(x+\pi)=-f(x)$, then $J_n(x+\pi;f)=-J_n(x;f)$ and so $J_n(x;f)\in\mathcal{T}_n$.

Lemma 6. *If $f(x)$ in $C(\mathbb{R})$ satisfies the Zygmund condition*

$$f(x+h)-2f(x)+f(x-h)=o(h) \tag{6.4}$$

then

$$\sup_x |f(x)-J_n(x;f)|=o\left(\frac{1}{n}\right), \tag{6.5}$$

$$|J_n^{(j)}(x;f)|=o(n^{j-1}) \quad \textit{for} \quad j \geqq 2. \tag{6.6}$$

Moreover if f is in $C^1(\mathbb{R})$, then

$$|f'(x)-J_n'(x;f)|=o(1). \tag{6.7}$$

Proof. It is known ([**5**, p. 56]) that

$$\sup_x |f(x)-J_n(x;f)| \leqq A\omega_2\left(f, \frac{\pi}{n}\right). \tag{6.8}$$

So if f satisfies (6.4), then (6.8) gives (6.5). It is also known ([**10**, see 4.8.6 formula (18)]) that for any trigonometric polynomial T_n of degree n,

$$\sup_x |T_n''(x)| \leqq Bn^2 \sup_x |\Delta^2_{\pi/n} T_n(x)|. \tag{6.9}$$

From the definition of the Jackson polynomial, we have

$$\sup_x |\Delta^2_{\pi/n} J_n(x;t)| \leqq \omega_2\left(f, \frac{\pi}{n}\right)=o\left(\frac{1}{n}\right) \tag{6.10}$$

so that from (6.9) and (6.10), we have

$$J_n''(x;f)=o(n)$$

which yields (6.6).

If f is continuously differentiable, then

$$|f'(x)-J_n'(x;f)|=|f'(x)-J_n(x;f')|=o(1). \tag{6.11}$$

Theorem 3. *If $0=m_1<m_2<\cdots<m_p$ be integers, suppose that $f \in C(\mathbb{R})$ satisfies $f(x+\pi)=-f(x)$ and the Zygmund condition (6.4). Moreover if $m_2=1$, then suppose $f \in C^1(\mathbb{R})$.*

Let $T_{np} \in \mathscr{T}_N$ satisfy the interpolary conditions

$$T_{np}(x_k)=F(x_k), \quad k=0,1,\ldots,2n-1,$$

and for $k=0,1,\ldots,2n-1$, $j=2,\ldots,p$,

$$T_{np}^{(m_j)}(x_k)=\begin{cases} f'(x_k), & \text{if } m_j=1 \\ o(n^{m_j-1}) & \text{otherwise.}\end{cases}$$

Then T_{np} *converges uniformly to* f *on* $\mathbb{R}$.

Proof. From Theorem 2, we have

$$T_{np}(x)=\sum_{k=0}^{2n-1} f(x_k)\rho_{0,0}(x-x_k)$$

$$+\sum_{j=2}^{p}\sum_{k=0}^{2n-1} T_{np}^{(m_j)}(x_k)\rho_{0,m_j}(x-x_k).$$

Now

$$f(x)-T_{np}(x)=f(x)-J_n(x;f)+J_n(x;f)-T_{np}(x)$$

and

$$J_n(x;f)-T_{np}(x)=\sum_{k=0}^{n}(J_n(x_k;f)-f(x_k))\rho_{0,0}(x-x_k)$$

$$+\sum_{j=2}^{p}\sum_{k=0}^{2n-1} J_n^{(m_j)}(x_k;f)\rho_{0,m_j}(x-x_k)$$

$$-\sum_{j=2}^{p}\sum_{k=0}^{2n-1} T_{np}^{(m_j)}(x_k)\rho_{0,m_j}(x-x_k).$$

The result now follows from Lemmas 5 and 6.

7. Remarks

It seems natural to ask why we consider only the case when the number of nodes is $2n$. When the number of nodes is $2n+1$, we have to consider again two cases: (i) when p is even and (ii) when p is odd.

The case (i) when number of nodes is $2n+1$ and p is even can be treated the same way as above. The case (ii) when p is odd will be discussed elsewhere.

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REFERENCES

1. R. Gantmacher, *Matrix Theory II* (Chelsea Publications, N.Y., 1964).

2. T. N. T. Goodman and S. L. Lee, *B*-splines on the circle and trigonometric *B*-splines, in *Approximation Theory and Spline Functions* (Eds. S. P. Singh, J. W. H. Burry and B. Watson), D. Riedel Pub. Co., Holland 1983, 297–325.

3. O. Kis, Remarks on interpolation (Russian), *Acta Math. Acad. Sci. Hungar.* **11** (1960), 49–64.

4. G. G. Lorentz, K. Jetter and S. Riemenschneider, Birkhoff interpolation, in *Enclycopedia of Math. 19* (Addison-Wesley, Reading, MA 1983).

5. G. G. Lorentz, *Approximation of functions* (Chelsea Publications, N.Y. 1986).

6. A. Sharma and R. S. Varga, On a particular 2-periodic lacunary trigonometric interpolation problem on equidistant nodes, *Results in Mathematics* **16** (1989), 333–404.

7. A. Sharma, J. Szabados and R. S. Varga, 2-periodic lacunary trigonometric interpolation: the $(0, M)$ cases, *Constructive Theory of Functions '87* (House of Bulgar, Acad. Sciences Sofia 1988), 420–427.

8. A. Sharma and R. B. Saxena, Almost Hermitian trigonometric interpolation in three equidistant nodes, *Aequationes Math.* **41** (1991), 55–69.

9. A. Sharma, J. Szabados and R. S. Varga, Some two periodic trigonometric interpolation problems on equidistant nodes, *Analysis*, to appear.

10. A. F. Timan, *Theory of Approximation of Functions of a Real Variable* (English translation), (Hindustan Pub. Co., Delhi-7, 1986).

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