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A GALTON–WATSON PROCESS WITH A THRESHOLD

K. B. ATHREYA,* *Iowa State University*
H.-J. SCHUH,** *Johannes Gutenberg-Universität*

Abstract

In this paper we study a special class of size dependent branching processes. We assume that for some positive integer K as long as the population size does not exceed level K, the process evolves as a discrete-time supercritical branching process, and when the population size exceeds level K, it evolves as a subcritical or critical branching process. It is shown that this process does die out in finite time T. The question of when the mean value $\mathbb{E}(T)$ is finite or infinite is also addressed.

Keywords: Branching process; size dependence; threshold; extinction time

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1. Introduction

Let $\mathbb{N} \equiv \{1, 2, 3, \ldots\}$ and $\mathbb{N}_0 \equiv \{0, 1, 2, 3, \ldots\}$. Size dependent branching processes have been well studied in the literature; see, e.g. [3], [4], and [5]. In this paper we treat a special class of such processes. In particular, we focus on the behaviour of T and $\mathbb{E}(T)$, where T is the extinction time of a Galton–Watson branching process $(Z_n)_{n\in\mathbb{N}_0}$ with a threshold K, a positive integer, i.e. $(Z_n)_{n\in\mathbb{N}_0}$ behaves according to a supercritical process as long as $Z_n \leq K$ and when $Z_n \geq K+1$ as a subcritical process or as a critical process, for all n. Specifically:

(i) if $Z_{n-1} = i \leq K$ then each of these i individuals creates offspring independently of all the others according to a distribution $(\pi_j)_{j\in\mathbb{N}_0}$ and with mean $M \equiv \sum_{j=1}^{\infty} j\pi_j > 1$, which could be ∞;

(ii) if $Z_{n-1} = k \geq K+1$ then each of these k individuals creates offspring independently of all the others according to a distribution $(p_j)_{j\in\mathbb{N}_0}$ with mean $m \equiv \sum_{j=1}^{\infty} jp_j \leq 1$ (if $m = 1$, we further assume that $p_1 < 1$). (The distributions $(\pi_j)_{j\in\mathbb{N}_0}$ and $(p_j)_{j\in\mathbb{N}_0}$ do not change with n.)

2. Main results

Theorem 1. *Let* $T \equiv \min\{n \colon n \in \mathbb{N},\ Z_n = 0\}$ *be the extinction time of the process* $(Z_n)_{n\in\mathbb{N}_0}$. *Then for each* $j \in \mathbb{N}$,

$$Q_j \equiv \mathbb{P}(T < \infty \mid Z_0 = j) = \mathbb{P}(\textit{there exists } n, n < \infty, Z_n = 0 \mid Z_0 = j) = 1.$$

Theorem 2. *Let* $T \equiv \min\{n \colon n \in \mathbb{N},\ Z_n = 0\}$ *and* $f(s) \equiv \sum_{j=0}^{\infty} p_j s^j, 0 \leq s \leq 1$, *be the probability generating function (PGF) of the offspring distribution* $(p_j)_{j\in\mathbb{N}_0}$.

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* Postal address: Departments of Mathematics and Statistics, Iowa State University, Ames, IA 50010, USA. Email address: kba@iastate.edu
** Postal address: Fachbereich 8, Physik, Mathematik und Informatik, Johannes Gutenberg-Universität, D-55099 Mainz, Germany. Email address: schuh@mathematik.uni-mainz.de

Let $1 < M = \sum_{j=1}^{\infty} j\pi_j < \infty$ *and either* $m < 1$ *or* $m = 1$ *and*

$$I \equiv \int_0^1 \frac{1-s}{f(s)-s}\,\mathrm{d}s < \infty.$$

Then $\mathbb{E}(T) < \infty$.

Remark 1. Let $m = 1$ and the distribution $(p_j)_{j\in\mathbb{N}_0}$ be of the Slack type (see [7]); i.e. for some $0 < \alpha < 1$,

$$f(s) \equiv \sum_{j=0}^{\infty} p_j s^j = s + (1-s)^{1+\alpha} L(1-s), \qquad 0 \le s \le 1,$$

where $L(x)$ is slowly varying as $x \to 0$. Then, the integral I in Theorem 2 is finite. In particular, this includes the following case:

$$f(s) = s + \frac{a}{1+\alpha}(1-s)^{1+\alpha}, \qquad 0 < \alpha < 1,\ 0 < a \le 1,$$

i.e.

$$f(s) = \frac{a}{1+\alpha} + (1-a)s + \frac{\alpha(1+\alpha)}{2}s^2 + \sum_{j=3}^{\infty} \left| \binom{1+\alpha}{j} \right| s^j.$$

Here, $0 < p_0 = f(0) = a/(1+\alpha) < 1$, $m = 1$; see [6].

Remark 2. If $m = 1$ and $\sum_{j=0}^{\infty} p_j s^j = f(s) = s + (1-s)^2 L(1-s)$, $L(x)$ is slowly varying at $x = 0$, then $I = \infty$.

All critical processes with finite offspring variance σ^2 belong to this class. In Theorem 4 below we show that if $I = \infty$ then $\mathbb{E}(T) = \infty$.

Theorem 3. *Let* $p_0 + p_1 = 1$ *and* $p_1 < 1$.

(i) *If* $L_M \equiv \sum_{j=2}^{\infty} (\log j)\pi_j < \infty$ *then* $\mathbb{E}(T) < \infty$.

(ii) *If* $L_M = \infty$ *then* $\mathbb{E}(T) = \infty$.

Remark 3. Thus, for the special case (necessarily subcritical) when $p_0 + p_1 = 1$, $p_1 < 1$, a necessary and sufficient condition for $\mathbb{E}(T) < \infty$ is $L_M < \infty$, i.e. we need only a logarithmic moment condition on the supercritical offspring distribution. Thus, M could be ∞.

Theorem 4. *Let* $\pi_0 = 0$. *If* $(p_j)_{j\in\mathbb{N}_0}$ *is a critical offspring distribution and*

$$I = \int_0^1 \frac{1-s}{f(s)-s}\,\mathrm{d}s = \infty,$$

then $\mathbb{E}(T) = \infty$. *(This class includes all critical processes with offspring variance* $\sigma^2 < \infty$, *i.e. finite second moment.)*

Remark 4. From Theorems 2 and 4, when $\pi_0 = 0$ and $m = 1$, a necessary and sufficient condition for $\mathbb{E}(T) = \infty$ is $I = \infty$.

3. Proofs

The following result on the extinction probability for subcritical or critical Galton–Watson processes is well known.

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Lemma 1. *Let $(\zeta_n)_{n\in\mathbb{N}_0}$ be a subcritical or critical Galton–Watson process with offspring distribution $(p_j)_{j\in\mathbb{N}_0}$ and $p_1 < 1$. Then for any $j \in \mathbb{N}$, $q_j \equiv \mathbb{P}(\zeta_n = 0$ for some $1 \le n < \infty \mid \zeta_0 = j) = 1$.*

Proof. See [1, p. 7]. □

Lemma 2. *Let $(\zeta_n)_{n\in\mathbb{N}_0}$ be a subcritical process as in Lemma 1, i.e. $m = \sum_{j=1}^{\infty} jp_j < 1$. Let $\tau \equiv \min\{n : n \in \mathbb{N}$ and $\zeta_n = 0\}$ be the extinction time of the process. Then*

$$\mathbb{E}(\tau \mid \zeta_0 = 1) \le \frac{1}{1-m} < \infty.$$

Proof. The proof follows from

$$\begin{aligned}\mathbb{E}(\tau \mid \zeta_0 = 1) &= \sum_{j=0}^{\infty} \mathbb{P}(\tau > j \mid \zeta_0 = 1)\\ &= \sum_{j=0}^{\infty} \mathbb{P}(\zeta_j > 0 \mid \zeta_0 = 1)\\ &= \sum_{j=0}^{\infty}(1 - f_j(0))\\ &\le \sum_{j=0}^{\infty} m^j\\ &= \frac{1}{1-m},\end{aligned}$$

where $f_j(s)$ is the PGF of ζ_j (since we have $1 - f_j(0) \le f_j'(1-) = m^j$ by the mean value theorem). □

Lemma 3. *Let $(\zeta_n)_{n\in\mathbb{N}_0}$ be a critical process as in Lemma 1 and $\zeta_0 = 1$. Then*

$$\mathbb{E}(\tau) = \infty \quad \textit{when } I = \int_0^1 \frac{1-s}{f(s)-s}\,\mathrm{d}s = \infty,$$

or

$$\mathbb{E}(\tau) < \infty \quad \textit{when } I = \int_0^1 \frac{1-s}{f(s)-s}\,\mathrm{d}s < \infty.$$

Proof. See [6]. □

Lemma 4. *Let $(\zeta_n)_{n\in\mathbb{N}_0}$ be a supercritical process with $\zeta_0 \equiv 1$, $g(s) \equiv \sum_{i=0}^{\infty}\pi_j s^j$ its offspring PGF, and $q \equiv \mathbb{P}(\zeta_n = 0$ for some $1 < n < \infty)(< 1)$ its extinction probability.*

The underlying probability space Ω disintegrates almost surely into

$$Q \equiv \{\omega \mid (\zeta_n(\omega))_{n\in\mathbb{N}_0} \text{ dies out}\}$$

and $Q^c \equiv \{\omega \mid (\zeta_n(\omega))_{n\in\mathbb{N}_0}$ has an infinite line of descent$\}$ with $\mathbb{P}(Q) = q$ and $\mathbb{P}(Q^c) = 1-q$.

If $\pi_0 > 0$ (and, hence, $q > 0$) then on Q^c $(\zeta_n)_{n\in\mathbb{N}_0}$ can be decomposed into $\zeta_n(\omega) = \zeta_n^{(A)}(\omega) + \tilde{\zeta}(\omega)$, where $(\zeta_n^{(A)})_{n\in\mathbb{N}_0}$ is the process of all descendants with infinite lines of descent and $(\tilde{\zeta}_n)_{n\in\mathbb{N}_0}$ is a nonnegative process. Then $(\zeta_n^{(A)})_{n\in\mathbb{N}_0}$ is a supercritical process with PGF $\hat{g}(s) = (g((1-q)s+q)-q)/(1-q)$ and values in $\mathbb{N}$. Hence, $\hat{g}(s) = \sum_{j=1}^{\infty}\hat{\pi}_j s^j$, i.e. $\hat{\pi}_0 = 0$ and $\hat{M} \equiv \hat{g}'(1-m) = M(> 1)$.

On Q $\zeta_n = \zeta_n^{(B)}$, $n \in \mathbb{N}_0$, is a subcritical process with PGF $g^*(s) \equiv g(sq)/q$ and $m^* \equiv g'(q) < 1$. If $\pi_0 = 0$ then $q = 0$ and, hence, $Q^c = \Omega$ and $(\zeta_n)_{n\in\mathbb{N}_0} = (\zeta_n^{(A)})_{n\in\mathbb{N}_0}$ almost surely.

Proof. See [1, pp. 47–53]. □

Lemma 5. *Let $Z_0 \equiv j$, $j \in \{1, \dots, K\}$ and $T_1 := \min\{n \colon n \in \mathbb{N},\ Z_n \geq K+1$ or $Z_n = 0\}$. Then*

$$\mathbb{E}(T_1) \leq \frac{K}{1-m^*} + 2K + C(K)\frac{\sqrt{\hat{\pi}_1}}{1-\sqrt{\hat{\pi}_1}} =: C_1 < \infty,$$

where m^* and $\hat{\pi}_1$ are as in Lemma 4 and $C(K) := \max_{n\in\mathbb{N}}(n^K \hat{\pi}_1^{n/2}) < \infty$.

Proof. For each $1 \leq i \leq K$, let $(\zeta_{n,i}^{(A)})_{n\in\mathbb{N}_0}$, $(\zeta_{n,i}^{(B)})_{n\in\mathbb{N}_0}$, Q_i, and Q_i^c be as defined in Lemma 4.

Case 1. Let H be the event $Q_1 \cap \dots \cap Q_j$. Then $Z_n = \zeta_{n,1}^{(B)} + \cdots + \zeta_{n,j}^{(B)}$ for $0 \leq n \leq T_1$ almost surely on H, where the processes $(\zeta_{n,i}^{(B)})_{n\in\mathbb{N}_0}$ are independent and identically distributed (i.i.d.) subcritical with PGF $g^*(s)$ and $\zeta_{0,i}^{(B)} = 1$. Let $\tau_i^{(B)}$ be the extinction time of $(\zeta_{n,i}^{(B)})_{n\in\mathbb{N}_0}$. Then $\mathbb{E}(\tau_i^{(B)}) \leq 1/(1-m^*) < \infty$. Here, $Z_{T_1} = 0$ with probability less than or equal to q^j, namely, when $\max_{1\leq i\leq j} \tau_i^{(B)} = T_1$. Otherwise, $Z_{T_1} \geq K+1$. In either case $T_1 \leq \tau_1^{(B)} + \cdots + \tau_j^{(B)}$. Hence, $\mathbb{E}(T_1, H) \leq j\mathbb{E}(\tau_1^{(B)}) \leq K(\tau_1^{(B)}) \leq K/(1-m^*) < \infty$.

Case 2. Consider the event $H^c = \Omega \setminus H$. Almost surely on H^c there exists at least one $1 \leq i \leq j$ with $Z_n \geq \zeta_{n,i}^{(A)}$ for all $0 \leq n \leq T_1$, where the $(\zeta_{n,i}^{(A)})_{n\in\mathbb{N}_0}$ are i.i.d., supercritical processes with PGF $\hat{g}(s)$ and $\zeta_{0,i}^{(A)} = 1$.

Let $\tau_i^{(A)} := \min\{n \mid n \in \mathbb{N},\ \zeta_{n,i}^{(A)} \geq K+1\}$. The paths $(\zeta_{l,i}^{(A)})_{0\leq l\leq n}$ are increasing as $\hat{\pi}_0 = 0$. These paths can attain not more than K values before they reach the level $K+1$. As $\mathbb{P}(\zeta_{l+1,i}^{(A)} = k \mid \zeta_{l,i} = k) = \hat{\pi}_1^k \leq \hat{\pi}_1$ and $\mathbb{P}(\zeta_{l+1,i} \geq k+1 \mid \zeta_{l,i} = k) \leq 1$, the probability of every path with $\tau_i^{(A)} \geq n$ is not greater than $1^K \hat{\pi}_1^{n-K}$ and there are not more than n^K such paths. Hence, $\mathbb{P}(\tau_i^{(A)} \geq n \mid \zeta_{0,i}^{(A)} = 1) \leq n^K \hat{\pi}_1^{n-K} = (n^K \hat{\pi}_1^{n/2})\hat{\pi}_1^{(n-2K)/2}$ for $n \geq 2K$. Then $\lim_{n\to\infty} n^K \hat{\pi}_1^{n/2} = 0$ and, therefore, $C(K) := \max_{n\in\mathbb{N}}(n^K \hat{\pi}_1^{n/2}) < \infty$. Hence,

$$\begin{aligned}
\mathbb{E}(\tau_i^{(A)}) &= \sum_{n=1}^{\infty} \mathbb{P}(\tau_i^{(A)} \geq n) \\
&\leq \sum_{n=1}^{2K} 1 + \sum_{n=2K+1}^{\infty} \mathbb{P}(\tau_i^{(A)} \geq n) \\
&\leq 2K + C(K)\sum_{l=1}^{\infty} \sqrt{\hat{\pi}_1}^{\,l} \\
&= 2K + C(K)\frac{\sqrt{\hat{\pi}_1}}{1-\sqrt{\hat{\pi}_1}} \\
&< \infty.
\end{aligned}$$

We have $T_1 \leq \tau_i^{(A)}$ on H^c and, therefore,

$$\mathbb{E}(T_1) = \mathbb{E}(T_1, H) + \mathbb{E}(T_1, H^c) \leq \frac{K}{1-m^*} + 2K + C(K)\frac{\sqrt{\hat{\pi}_1}}{1-\sqrt{\hat{\pi}_1}} < \infty,$$ □

Now we are ready for the proof of Theorem 1. Since T_1 is a stopping time, Z_{T_1} is a well-defined random variable. Now consider the case $Z_{T_1} = k \geq K + 1$. As $(Z_{T_1+n})_n$ is then a subcritical or critical branching process, Z_{T_1+n} will reach the set of the states $\{0, 1, 2, \dots, K\}$ with probability 1. Hence, we can define

$$D_1 \equiv \begin{cases} \min\{n\colon n \geq 1,\ Z_{T_1+n} \leq K\} & \text{when } Z_{T_1} \geq K+1, \\ 0 & \text{when } Z_{T_1} = 0. \end{cases}$$

At $T_1 + D_1$, $(Z_n)_{n\in\mathbb{N}_0}$ has completed its *first cycle*. If $Z_{T_1+D_1} \neq 0$ then the process starts with $Z_{T_1+D_1} (\in \{1, \dots, K\})$ its *second cycle*, which can be treated the same way as the first one, and so on.

Hence, we can now define the random times $T_2, D_2, \dots, T_n, D_n, \dots$ accordingly. In particular, if $Z_{T_1+D_1} = 0$ then set $(T_i, D_i) = (0, 0)$ for all $i \geq 2$, and so on. The extinction time T defined in Theorem 1 can now be written as $T = T_1 + D_1 + \dots + T_{n^*} + D_{n^*}$, where n^* is the smallest n with $Z_{T_1+D_1+\dots+T_n+D_n} = 0$.

Proof of Theorem 1. For $1 \leq i, j \leq K < k$ define

$$\begin{aligned} \lambda_{i,j}(k) &\equiv \mathbb{P}(Z_{T_1} = k \mid Z_{T_1-1} = i,\ Z_0 = j) \\ &= \frac{\mathbb{P}(Z_{T_1} = k,\ Z_{T_1-1} = i,\ Z_0 = j)}{\mathbb{P}(Z_{T_1-1} = i,\ Z_0 = j)} \\ &= \pi_k^{*i} \frac{\sum_{l=1}^{\infty} \mathbb{P}(Z_{l-1} = i,\ Z_{l-2} \leq K, \dots, Z_1 \leq K,\ Z_0 = j)}{\sum_{l=1}^{\infty} \mathbb{P}(Z_l \geq K+1,\ Z_{l-1} = i,\ Z_{l-2} \leq K, \dots, Z_1 \leq K,\ Z_0 = j)}, \end{aligned}$$

where the ratio is positive. Here, $(\pi_k^{*i})_{k\in\mathbb{N}_0}$ is the i-fold convolution of the supercritical offspring distribution $(\pi_k)_{k\in\mathbb{N}_0}$.

Set $g_{i,j}(s) \equiv \sum_{h=K+1}^{\infty} \lambda_{i,j}(h) s^h$. Then $\mathbb{P}(Z_{T_1+1} = 0 \mid Z_{T_1} = 0) = 1$, and $\mathbb{P}(Z_{T_1+1} = 0 \mid Z_{T_1} \geq K+1) \geq \min\{g_{i,j}(p_0)\colon 1 \leq i, j \leq K\} \equiv \lambda$, say. Now $p_0 > 0$ implies that $\lambda > 0$. Hence, $\mathbb{P}(Z_{T_1+D_1} = 0) \geq \lambda$ and $\mathbb{P}(Z_{T_1+D_1} \in \{1, \dots, K\}) \leq 1 - \lambda$. By the conditional independence of the cycles as defined above, for each $n \geq 1$, $\mathbb{P}(Z_{T_1+D_1+T_2+D_2+\dots+T_n+D_n} \in \{1, 2, \dots, K\}) \leq (1-\lambda)^n$. Since $Q_j = \lim_{n\to\infty} \mathbb{P}(T = T_1 + D_1 + T_2 + D_2 + \dots + T_n + D_n \mid Z_0 = j)$, it follows that

$$Q_j = 1 - \lim_{n\to\infty} \mathbb{P}(Z_{T_1+D_1+\dots+T_n+D_n} \in \{1, \dots, K\} \mid Z_0 = j) \geq 1 - \lim_{n\to\infty} (1-\lambda)^n = 1$$

as $\lambda > 0$. □

Proof of Theorem 2. We know by Lemma 2 and Lemma 3 that the conditions of Theorem 2 imply that the extinction time τ of the Galton–Watson process with offspring distribution $(p_j)_{j\in\mathbb{N}_0}$ has a finite expectation, and in case this process is subcritical, $\mathbb{E}(\tau) \leq 1/(1-m)$ if $\zeta_0 = 1$.

Now we have to estimate $\mathbb{E}(D_1)$. Suppose that $Z_{T_1} = k \geq K + 1$. If we remove the threshold K, $(Z_{T_1+n})_{n\in\mathbb{N}_0} = (\zeta_{n,1} + \dots + \zeta_{n,k})_{n\in\mathbb{N}_0}$, where the $(\zeta_{n,i})_{n\in\mathbb{N}_0}$ are i.i.d. subcritical or critical Galton–Watson processes with offspring distribution $(p_j)_{j\in\mathbb{N}_0}$. Now the process $(\zeta_{n,1} + \dots + \zeta_{n,k})_{n\in\mathbb{N}_0}$ dies out, when all the $(\zeta_{n,i})_n$ have died out, i.e. at $\max\{\tau_1, \tau_2, \dots, \tau_k\}$. This is clearly less than or equal to $\tau_1 + \tau_2 + \dots + \tau_k$. As $(Z_{T_1+n})_{n\in\mathbb{N}_0}$ reaches the states $\{0, 1, 2, \dots, K\}$ not later than when $(\zeta_{n,1} + \dots + \zeta_{n,k})_{n\in\mathbb{N}_0}$ dies out, we obtain

$$\mathbb{E}(D_1 \mid Z_{T_1} = k) \leq \mathbb{E}(\max\{\tau_1, \dots, \tau_k\}) \leq k\mathbb{E}(\tau), \qquad \mathbb{E}(D_1 \mid Z_{T_1} = 0) = 0.$$

Hence,

$$\mathbb{E}(D_1 \mid Z_{T_1} \geq K+1) \leq \max\left\{ \sum_{k=K+1}^{\infty} \lambda_{i,j}(k) k \mathbb{E}(\tau) \colon 1 \leq i, j \leq K \right\}$$
$$= \max\{g'_{i,j}(1-) \colon 1 \leq i, j \leq K\} \mathbb{E}(\tau).$$

This is finite as $g'(1-) = M < \infty$.

Next, $\mathbb{P}(T > T_1 + D_1) = \mathbb{P}(Z_{T_1+D_1} \in \{1, 2, \dots, K\}) \leq 1 - \lambda$. Then, using the cycle argument as in the discussion prior to Theorem 1, and since $T \leq \sum_{j=1}^{n^*}(T_j + D_j)$, using Wald's identity we see that $\mathbb{E}(T) \leq \mathbb{E}(T_1 + D_1)\mathbb{E}(n^*)$, (where n^* as defined before the proof of Theorem 1) implying

$$\mathbb{E}(T) \leq \mathbb{E}(T_1 + D_1) \sum_{k=1}^{\infty} (1-\lambda)^k$$
$$\leq (C_1 + \mathbb{E}(D_1))(1-\lambda)\frac{1}{\lambda}$$
$$= (C_1 + \mathbb{E}(D_1))\left(\frac{1}{\lambda} - 1\right)$$
$$< \infty \quad \text{as } \lambda > 0. \qquad \square$$

Remark 5. In the proof of Theorem 2 we bound above the maximum of i.i.d. random variables by their sum. Hence, the conditions given in Theorem 2 are only sufficient, and by far not necessary, to prove the finiteness of $\mathbb{E}(D_1)$, as can be seen in Theorem 3.

Lemma 6. *Let* $(\zeta_{n,1}+\cdots+\zeta_{n,k})_{n\in\mathbb{N}_0}$ *be the sum of k i.i.d. subcritical Galton–Watson processes with an offspring distribution such that* $p_0 + p_1 = 1$, $p_1 < 1$ *and such that* $\zeta_{0,i} = 1$. *Let* $\tau_i \equiv \min\{n \colon n \geq 1,\ \zeta_{n,i} = 0\}$ *be the extinction time of* $(\zeta_{n,i})_n$, $1 \leq i \leq k$.

Let $\tilde{\tau} = \max\{\tau_1, \tau_2, \dots, \tau_k\}$ and $\mathbb{E}_k := E(\tilde{\tau})$. Then, for all $k \in \mathbb{N}$,

$$-\frac{1}{\log(1-p_0)} H_k \leq \mathbb{E}_k \leq 1 - \frac{1}{\log(1-p_0)} H_k,$$

where $H_k := (1 + \frac{1}{2} + \cdots + 1/k)$.

Proof. Here the extinction times τ_i of these subcritical processes are geometrically distributed: $\mathbb{P}(\tau_i = k) = p_o(1-p_0)^{k-1}$, $k \in \mathbb{N}_0$. Set $q_0 := 1 - p_0$. Then

$$\mathbb{P}(\tau_i \leq l) = \sum_{k=1}^{l} p_0 q_0^{k-1} = p_0 \frac{1-q_0^l}{1-q_0} = 1 - q_0^l.$$

Hence, $\mathbb{P}(\tilde{\tau} \leq l) = \mathbb{P}(\max\{\tau_1, \tau_2, \dots, \tau_k\} \leq l) = (1-q_0^l)^k$, $\mathbb{P}(\tilde{\tau} > l) = 1 - (1-q_0^l)^k$. This implies that $\mathbb{E}_k = \mathbb{E}(\tilde{\tau}) = \sum_{l=0}^{\infty} 1 - (1-q_0^l)^k$.

Comparing this infinite sum with the corresponding integral (see [2]), we obtain

$$\int_0^{\infty} 1 - (1-q_0^x)^k \,\mathrm{d}x \leq \mathbb{E}_k \leq 1 + \int_0^{\infty} 1 - (1-q_0^x)^k \,\mathrm{d}x.$$

Substituting $y = 1 - q_0^x$ leads to

$$\int_0^{\infty} 1 - (1-q_0^x)^k \,\mathrm{d}x = -\frac{1}{\log q_0} \int_0^1 \frac{1-y^k}{1-y} \,\mathrm{d}y$$

$$= -\frac{1}{\log q_0}\int_0^1 (1+y+\cdots+y^{k-1})\,\mathrm{d}y$$
$$= -\frac{1}{\log q_0} H_k. \qquad \square$$

Remark 6. The integral in the proof of Lemma 6 is the value for $\mathbb{E}(\tilde{\tau})$, if the τ_i were exponentially distributed with parameter $-\log q_0$.

Lemma 7. (i) *We have* $L_M = \sum_{j=2}^{\infty}(\log j)\pi_j < \infty$ *if and only if* $\sum_{j=2}^{\infty}(\log j)\pi_j^{*i} < \infty$ *for all* $i \geq 2$.

(ii) *Let* $(\zeta_{n,i})_{n\in\mathbb{N}_0}$, $1 \leq i \leq K$, *be i.i.d. subcritical Galton–Watson processes with offspring distribution* $(p_j)_{j\in\mathbb{N}_0}$ *and extinction times* τ_i, $\zeta_{0,i} = 1$. *Then the mean values of the extinction times* $\tilde{\tau}_j$ *of* $(\zeta_{n,1} + \cdots + \zeta_{n,j})_{n\in\mathbb{N}_0}$, $1 \leq j \leq K$, *have a finite common upper bound.*

Proof. (i) Let $Y_1, \ldots, Y_i \geq 0$ be i.i.d. random variables with distribution $(\pi_j)_{j\in\mathbb{N}_0}$. Then

(a) We have
$$\begin{aligned}\sum_{j=2}^{\infty}(\log(1+j))\pi_j^{*i} &= \mathbb{E}(\log(1+Y_1+\cdots+Y_i))\\ &\leq \mathbb{E}(\log(1+Y_1)+\cdots+\log(1+Y_i))\\ &= i\,\mathbb{E}(\log(1+Y_1)) < \infty \quad \text{as } L_M < \infty\end{aligned}$$

and $\log j \sim \log(1+j)$, i.e. $\log j/\log(1+j) \to 1$ as $j \to \infty$.

(b) We have $\mathbb{E}(\log(1+Y_1)) \leq \mathbb{E}(\log(1+Y_1+\cdots+Y_i)) < \infty$ implies that $L_M < \infty$.

(ii) We have $\mathbb{E}(\tilde{\tau}) \leq K\mathbb{E}(\tau_1) < \infty$. $\square$

Proof of Theorem 3. We have $\mathbb{E}_k \sim -(1/\log(1-p_0))H_k$ by Lemma 6. Furthermore, $H_k \sim \log k$. Hence, there exist constants $0 < C_3 < C_2 < \infty$ with $C_3 \log k \leq E_k \leq C_2 \log k$ for all $k \geq K+1$.

(i) As $\mathbb{E}(D_1 \mid Z_{T_1} = k) \leq C_2 \log k$, $k \geq K+1$, and $\mathbb{E}(D_1 \mid Z_{T_1} = 0) = 0$, we obtain

$$\mathbb{E}(D_1 \mid Z_{T_1} \geq K+1) \leq C_2 \max_{1\leq i\leq K,\,1\leq j\leq K} \sum_{k=K+1}^{\infty} \frac{\lambda_{i,j}(k)\log k}{\mathbb{P}(Z_{T_1} \geq K+1)} < \infty \quad \text{as } L_M < \infty$$

(implies that $\sum_{k=2}^{\infty}(\log k)\pi_k^{*i} < \infty$ by Lemma 7(i)). Hence, $\mathbb{E}(D_1) < \infty$ and, therefore,

$$\mathbb{E}(T) \leq \mathbb{E}(T_1 + D_1)\bigg(\frac{1}{\lambda} - 1\bigg) < \infty.$$

(ii) Let $L_M = \infty$. Then

$$\mathbb{E}(D_1 \mid Z_{T_1} \geq K+1) \geq C_3 \min_{1\leq i\leq K,\,1\leq j\leq K} \sum_{k=K+1}^{\infty} \lambda_{i,j}(k)\log k - \mathbb{E}(\tilde{\tau}) = \infty,$$

by Lemma 7(ii) and as $L_M = \infty$. As $T \geq D_1$, we obtain $\mathbb{E}(T) = \infty$.

This completes the proof. $\square$

Proof of Theorem 4. By Theorem 1 we know that the branching process $(Z_n)_{n\in\mathbb{N}_0}$ dies out eventually, i.e. $\mathbb{P}(T < \infty) = 1$. The condition $\pi_0 = 0$ ensures that this can happen only by a

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jump from some $k \geq K+1$ to 0 (with probability p_0^k). Using the tree structure of the paths of the process $(Z_n)_{n\in\mathbb{N}_0}$, we embed a critical Galton–Watson process $(\varrho_n)_{n\in\mathbb{N}_0}$ governed by the offspring PGF $f(s)$ with $\varrho_0 \equiv 1$. The assumptions of Theorem 4 imply that $\mathbb{E}(\tau_\varrho) = \infty$, where τ_ϱ is the extinction time of $(\varrho_n)_{n\in\mathbb{N}_0}$. Construction of $(\varrho_n)_n$: $(Z_{T_1+n})_n$ behaves like a sum of i.i.d. critical Galton–Watson processes governed by $f(s)$ for $n = 0, 1, \dots, D_1$.

Step 1. Choose at random one of the $Z_{T_1} (\geq K+1)$ individuals and define $(\varrho_n)_{n=0,\dots,D_1}$ as the values of the process generated by this individual. Obviously, $\varrho_n \leq Z_{T_1+n}$, $n = 0, \dots, D_1$.

Step 2. Since $Z_{T_1+D_1} \leq K$, either $\varrho_{D_1} = 0$ and, therefore, $\varrho_n = 0$ for all $n \geq D_1$, or $\varrho_{D_1} = j \in \{1, \dots, K\}$. In the latter case choose j individuals at random from the $Z_{T_1+D_1+T_2} (\geq K+1)$ individuals. Define $(\varrho_{D_1+n})_{n=0,\dots,D_2}$ as the values of the sum of the processes generated by these j individuals, i.e. $(\varrho_n)_n$ is now defined for $0 \leq n \leq D_1 + D_2$ and so on. So $(\varrho_n)_{n\in\mathbb{N}_0}$ will eventually die out, at the latest at time T, when $(Z_n)_{n\in\mathbb{N}_0}$ dies out. Hence, $\tau_\varrho \leq T$. Since $\mathbb{E}(\tau_\varrho) = \infty$, it follows that $\mathbb{E}(\tau) = \infty$. □

4. Final remarks

Let us modify the process $(Z_n)_{n\in\mathbb{N}_0}$ in such a way that on reaching the state 0 there is an immigration of exactly one individual, i.e. if $Z_r = 0$ then $Z_{r+1} = 1$, $r \in \mathbb{N}_0$. The state 1 is now a point of renewal. The renewal time is distributed like $T + 1$, where T is the extinction time of the original process. The modified process is a recurrent Markov chain as $\mathbb{P}(\tau + 1 < \infty) = 1$ by Theorem 1. If $\mathbb{E}(T + 1) < \infty$ then this chain is positively recurrent and nullrecurrent otherwise. An interesting problem is to investigate the stationary distribution in the positive recurrent case.

One might introduce a finite number of thresholds $1 \leq K_1 < K_2 < \cdots < K_r$. As long as the different Galton–Watson processes are supercritical below K_1 and subcritical or critical, respectively, between K_j and K_{j+1} and above K_r, the behaviour of the new branching process should not differ much from the one studied in this paper.

All processes looked at so far belong to the class of size-dependent Galton–Watson processes.

The extension of the results of this paper to the multitype case will appear elsewhere.

An interesting problem is to investigate the asymptotics of the distribution of Z_n conditioned on the event $\{Z_n > 0\}$ as $n \to \infty$.

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