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On Homogeneous Polynomials Determined by their Partial Derivatives

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Abstract. We prove that a generic homogeneous polynomial of degree d is determined, up to a nonzero constant multiplicative factor, by the vector space spanned by its partial derivatives of order k for $k \leqslant \frac{d}{2} - 1$.

1 Introduction

We investigate in this note the reconstructibility of a homogeneous polynomial from its partial derivatives. The study dates back to J. Carlson and Ph. Griffiths, who in [1] showed that a generic homogeneous polynomial could be reconstructed, up to a nonzero constant multiple, from its Jacobian ideal, or equivalently, from its first order partial derivatives; in that paper, they used this result to study variation of Hodge structures and proved the global Torelli theorem for hypersurfaces. For further developments of the determination of a homogeneous polynomial by its Jacobian ideal, see [3] and references therein.

In the classical theory of variation of Hodge structures for smooth hypersurfaces, as in [1], only first order derivatives of the defining homogeneous polynomials are involved. We can also construct higher order versions of this classical theory; see for instance [2]. In this higher order analogous theory, a problem arises concerning the reconstructibility of a homogeneous polynomial from its higher order partial derivatives. In this paper, we will solve this problem and prove that a generic homogeneous polynomial has the desired property.

Let $S = \mathbb{C}[x_0, x_1, \dots, x_n]$ be the graded polynomial ring in $n+1$ variables with coefficients in $\mathbb{C}$,

$$S = \bigoplus_{d=0}^{\infty} S_{n,d},$$

where $S_{n,d}$ is the vector space of homogeneous polynomials of degree d. Given $f \in S_{n,d}$ and a natural number $k \geq 0$, denote by $J_k(f)$ the graded ideal of S generated by all partial derivatives of f of order k and by $E_k(f)$ the degree $d-k$ homogeneous component of $J_k(f)$, that is, the vector space spanned by all k-th order partial derivatives of f. We will prove the following theorem.

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Theorem 1.1 *Given $n \geq 1$, $d \geq 3$, and a natural number $k \geq 1$ such that $k \leq \frac{d}{2} - 1$, suppose f is a generic homogeneous polynomial in $S_{n,d}$. Let g be another homogeneous polynomial in $S_{n,d}$ such that $E_k(f) = E_k(g)$. Then $g \in \mathbb{C}^* f$.*

The underlying idea in the proof is very simple, so we give an outline here. We will show that $E_{k-1}(g) = E_{k-1}(f)$, then apply induction on k to obtain $E_r(g) = E_r(g)$ for all $0 \leq r \leq k$. Since $E_0(f)$ is essentially nothing but $\mathbb{C}f$, the conclusion follows immediately.

Note that we already have a more precise result, Theorem 1.1 in [3], for the case $k = 1$. But we do not need to use it to prove Theorem 1.1 here; instead, we will use induction on k until the case $k = 0$ is reached. In addition, the restriction $k \leq \frac{d}{2} - 1$ is given in order to ensure that $\dim E_{k+1}(f) = \dim S_{n,k+1}$ for a generic f; see Lemma 2.5 below.

Notations

As in the introduction, $S_{n,d}$ denotes the vector space of homogeneous polynomials of degree d.

Consider the multi-index set

$$\mathbb{N}^{n+1} = \{(i_0, i_1, \dots, i_n) : i_j \geq 0 \text{ for } j = 0, 1, \dots, n\}.$$

We denote an element of $\mathbb{N}^{n+1}$ by I. We shall interpret $\mathbb{N}^{n+1}$ as a subset of the vector space $\mathbb{R}^{n+1}$; among the operations on $\mathbb{N}^{n+1}$ are addition and subtraction,

$$I \pm I' = (i_0 \pm i'_0, i_1 \pm i'_1, \dots, i_n \pm i'_n),$$

and multiplication by a positive integer,

$$mI = (mi_0, \dots, mi_n),$$

for $I = (i_0, \dots, i_n)$, $I' = (i'_0, \dots, i'_n)$, and $m \in \mathbb{N}$.

Denote by $\mathbf{e}_j$, $j = 0, \dots, n$ the canonical basis of $\mathbb{R}^{n+1}$,

$$\mathbf{e}_j = (0, \dots, 0, 1, 0, \dots, 0),$$

where 1 lies in the j-th entry. Using this basis, we may write $I = (i_0, \dots, i_n)$ as $I = \sum_{j=0}^{n} i_j \mathbf{e}_j$.

Moreover, there is an obvious partial ordering "$\geq$" on $\mathbb{N}^{n+1}$, with

$$I = (i_0, i_1, \dots, i_n) \geq I' = (i'_0, \dots, i'_n) \Leftrightarrow i_j \geq i'_j, \quad j = 0, \dots, n,$$

or more concisely,

$$I \geq I' \Leftrightarrow I - I' \in \mathbb{N}^{n+1}.$$

The *order* of $I = (i_0, \dots, i_n)$ is

$$|I| = i_0 + \cdots + i_n.$$

For $f \in S_{n,d}$, the partial derivative of f of type I is

$$D_I f = \frac{\partial^{|I|} f}{\partial x_0^{i_0} \partial x_1^{i_1} \cdots \partial x_n^{i_n}}.$$

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By definition, $E_k(f)$ is the vector subspace of $S_{n,d-k}$ spanned by $D_I f$, $|I| = k$; thus we have

$$E_k(f) = \langle D_I f : |I| = k \rangle.$$

2 Polynomials Determined by Higher Order Derivatives

In this section, we will give the proof of Theorem 1.1.

We begin our proof with the following lemma.

Lemma 2.1 *Let $f \in S_{n,d}$. If $k \geq 1$ and $\dim E_k(f) = \dim S_{n,k}$, then $\dim E_{k-1}(f) = \dim S_{n,k-1}$.*

Proof The proof is almost obvious. If we are given a linear relation

$$\sum_{|I|=k-1} a_I D_I f = 0,$$

by taking differentiation with respect to the variable x_0, it follows that

$$\sum_{|I|=k-1} a_I D_{I+\mathbf{e}_0} f = 0.$$

On the other hand, the assumption of $E_k(f)$ implies that $\{D_{I+\mathbf{e}_0} f : |I| = k-1\}$ are linearly independent, so $a_I = 0$ for all I. ■

An induction on k gives the following corollary.

Corollary 2.2 *Let $f \in S_{n,d}$. If $k \geq 1$ and $\dim E_k(f) = \dim S_{n,k}$, then $\dim E_r(f) = \dim S_{n,r}$ for all $0 \leq r \leq k$.*

As a second step in the proof of Theorem 1.1, we show the following proposition.

Proposition 2.3 *Given $n \geq 1$, $d \geq 3$, and $k \geq 1$, let $f, g \in S_{n,d}$ be such that $E_k(g) = E_k(f)$ and $\dim E_{k+1}(f) = \dim S_{n,k+1}$. Then $E_{k-1}(g) = E_{k-1}(f)$.*

Proof We will show $E_{k-1}(g) \subseteq E_{k-1}(f)$. This is sufficient for our purpose because, the two vector spaces have the same dimension, by Corollary 2.2.

From $E_k(g) = E_k(f)$, we consider the following system of linear relations: for all $I \in \mathbb{N}^{n+1}$ such that $|I| = k$, we have

$$D_I g = \sum_{|I'|=k} a_{I,I'} D_{I'} f \tag{1}$$

for some $a_{I,I'} \in \mathbb{C}$.

Our discussions in the sequel will be divided into two steps.

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Step 1: Differentiating equations

Fix I and $0 \le p \le n$ such that $I \ge \mathbf{e}_p$. For any $0 \le q \le n$, we will apply the equality $D_{\mathbf{e}_q} D_I g = D_{\mathbf{e}_p} D_{(I-\mathbf{e}_p)+\mathbf{e}_q} g$ to equation (1). To this end, we obtain first

$$\begin{aligned} D_{\mathbf{e}_q} D_I g &= D_{\mathbf{e}_q}\Big(\sum_{|I'|=k} a_{I,I'} D_{I'} f \Big) \\ &= \sum_{|I'|=k} a_{I,I'} D_{I'+\mathbf{e}_q} f, \end{aligned}$$

and second,

$$\begin{aligned} D_{\mathbf{e}_p} D_{(I-\mathbf{e}_p)+\mathbf{e}_q} g &= D_{\mathbf{e}_p}\Big(\sum_{|I'|=k} a_{(I-\mathbf{e}_p)+\mathbf{e}_q,I'} D_{I'} f \Big) \\ &= \sum_{|I'|=k} a_{I-\mathbf{e}_p+\mathbf{e}_q,I'} D_{I'+\mathbf{e}_p} f. \end{aligned}$$

From our assumption $\dim E_{k+1}(f) = \dim S_{n,k+1}$, it follows that $\{D_J f : |J| = k+1\}$ are linearly independent. Therefore, using $D_{\mathbf{e}_q} D_I g = D_{\mathbf{e}_p} D_{(I-\mathbf{e}_p)+\mathbf{e}_q} g$ and comparing the coefficients of each term $D_J f$, we obtain that

$$a_{I,J-\mathbf{e}_q} = a_{I-\mathbf{e}_p+\mathbf{e}_q,J-\mathbf{e}_p}$$

for all $|J| = k+1$. Here we used the convention that $a_{I,J-\mathbf{e}_q} = 0$ if $J \ngeq \mathbf{e}_q$.

Since the above conclusion holds for all I, J, p, q satisfying $I \ge \mathbf{e}_p$, it follows that for all I, I', p, q such that $|I| = |I'| = k$ and $I \ge \mathbf{e}_p$,

$$a_{I,I'} = a_{I-\mathbf{e}_p+\mathbf{e}_q,I'-\mathbf{e}_p+\mathbf{e}_q}. \tag{2}$$

Step 2: Considering $(k-1)$-th order partial derivatives

Let $K \in \mathbb{N}^{n+1}$ be such that $|K| = k-1$, then the Euler formula for $D_K g$ gives

$$(d-k+1) D_K g = \sum_{p=0}^{n} x_p D_{K+\mathbf{e}_p} g. \tag{3}$$

Substituting (1) into (3), we have

$$(d-k+1) D_K g = \sum_{p=0}^{n} \sum_{|I'|=k} x_p a_{K+\mathbf{e}_p,I'} D_{I'} f.$$

By (2), we deduce first of all that $a_{K+\mathbf{e}_p,I'} = 0$ if $I' \ngeq \mathbf{e}_p$, and thus

$$\begin{aligned} (d-k+1) D_K g &= \sum_{p=0}^{n} \sum_{I' \ge \mathbf{e}_p} x_p a_{K+\mathbf{e}_p,I'} D_{I'} f \\ &= \sum_{p=0}^{n} \sum_{I' \ge \mathbf{e}_p} x_p a_{K+\mathbf{e}_p,(I'-\mathbf{e}_p)+\mathbf{e}_p} D_{I'} f, \end{aligned}$$

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or written in a more convenient way,

$$(d-k+1)D_K g = \sum_{p=0}^{n} \sum_{|K'|=k-1} x_p a_{K+\mathbf{e}_p, K'+\mathbf{e}_p} D_{K'+e_p} f$$
$$= \sum_{|K'|=k-1} \Big(\sum_{p=0}^{n} x_p a_{K+\mathbf{e}_p, K'+\mathbf{e}_p} D_{K'+e_p} f \Big).$$

Now the relations (2) imply that $a_{K+\mathbf{e}_p,K'+\mathbf{e}_p} = a_{K+\mathbf{e}_q,K'+\mathbf{e}_q}$ for any $p, q = 0, \dots, n$; therefore, we obtain

$$(d-k+1)D_K g = \sum_{|K'|=k-1} a_{K+\mathbf{e}_0, K'+\mathbf{e}_0} \Big(\sum_{p=0}^{n} x_p D_{K'+e_p} f \Big).$$

By the Euler formula for $D_{K'} f$, we have that

$$\sum_{p=0}^{n} x_p D_{K'+e_p} f = (d-k+1) D_{K'} f,$$

so

$$D_K g = \sum_{|K'|=k-1} a_{K+\mathbf{e}_0, K'+\mathbf{e}_0} D_{K'} f.$$

Since this holds for all K satisfying $|K| = k-1$, it follows that $E_{k-1}(g) \subseteq E_{k-1}(f)$. ∎

2.4 Linear Independence of Partial Derivatives

As a final step to our proof of Theorem 1.1, we need the following lemma, which is interesting in its own right; see also [2, Proposition 3.4].

Lemma 2.5 *Given $n \geq 1$ and $d \geq 3$, suppose $0 \leq k \leq \frac{d}{2}$. Then for a generic $f \in S_{n,d}$, we have*

$$\dim E_k(f) = \dim S_{n,k}.$$

Proof Suppose given a linear relation

$$\sum_{|I|=k} a_I D_I f = 0. \tag{4}$$

The remaining proof will be divided into two steps. We first show that our proof can be reduced to the case where $d = 2k$. Indeed, the cases $k = 0, 1$ are rather trivial.

Step 1: Reduction

If $k > 1$ and $d > 2k$, take derivative $D_{(d-2k)\mathbf{e}_0}$ in (4), we obtain

$$\sum_{|I|=k} a_I D_I (D_{(d-2k)\mathbf{e}_0} f) = 0.$$

Note that $D_{(d-2k)\mathbf{e}_0} : S_{n,d} \to S_{n,2k}$ is a linear surjective morphism, hence for a generic $f \in S_{n,d}$, the polynomial $D_{(d-2k)\mathbf{e}_0} f \in S_{n,2k}$ is also generic.

Step 2: The case $d = 2k$

From the linear relation (4), we obtain for any $I' \in \mathbb{N}^{n+1}$ with $|I'| = k$,

$$\sum_{|I|=k} a_I D_{I+I'} f = 0.$$

Hence $\{a_I : |I| = k\}$ satisfies a system of linear equations with coefficients given by $\{D_{I+I'} f : |I| = k, |I'| = k\}$. Note that $D_{I+I'} f$ is a constant since $|I + I'| = 2k = \deg f$.

Recall that the lexicographic order on the set $\{I = (i_0, \dots, i_n) : |I| = k\}$ is given as follows:

$$I = (i_0, \dots, i_n) \prec I' = (i'_0, \dots, i'_n)$$

if and only if there exists $0 \le j \le n$ such that

$$i_0 = i'_0, \dots, i_j = i'_j, i_j < i'_j.$$

One can use this order to write the sequence $\{D_{I+I'} f : |I| = k, |I'| = k\}$ into a square matrix, denoted by $\mathbf{S}(f)$, whose rows and columns are both indexed by the set $\{I \in \mathbb{N}^{n+1} : |I| = k\}$ and whose (I, I')-entry is given by $D_{I+I'} f$.

To finish the proof of Lemma 2.5, we need to show the matrix $\mathbf{S}(f)$ is nonsingular for a generic f. To this end, it suffices to find one f for which $\mathbf{S}(f)$ is nonsingular, because the subset of $f \in S_{n,2k}$ with nonsingular $\mathbf{S}(f)$ is clearly a Zariski open subset of $S_{n,2k}$. Just take the polynomial $f = \sum_{|I|=k} x^{2I}$; then the matrix $\mathbf{S}(f)$ is a diagonal matrix whose (I, I)-entry is the nonzero number $(2I)!$, hence it is nonsingular. ■

Remark 2.6 In view of the obvious bound for $\dim E_k(f)$ given by

$$\dim E_k(f) \le \min\{\dim S_{n,k}, \dim S_{n,d-k}\},$$

the condition on k in Lemma 2.5 is optimal.

2.7 Proof of Theorem 1.1

Let f be a generic polynomial in $S_{n,d}$ and $E_k(g) = E_k(f)$. Under the assumption $k \le \frac{d}{2} - 1$, it follows that $k + 1 \le \frac{d}{2}$, hence, by Lemma 2.5, we have $\dim E_{k+1}(f) = \dim S_{n,k+1}$; therefore the requirements in Proposition 2.3 are satisfied. By Proposition 2.3, it follows that $E_{k-1}(g) = E_{k-1}(f)$. Note that by Corollary 2.2, we have $\dim E_k(f) = \dim S_{n,k}$, so the requirements in Proposition 2.3 are satisfied with k replaced by $k - 1$ and we obtain $E_{k-2}(g) = E_{k-2}(f)$. These arguments can be repeated until we obtain $E_0(g) = E_0(f)$. By definition, we have $E_0(g) = \mathbb{C}g$ and $E_0(f) = \mathbb{C}f$, therefore g is a constant multiple of f.

3 Applications

As pointed out in the introduction, the most remarkable application of the results in this paper lies in the study of a higher order analogue of variation of Hodge structures for hypersurfaces; see [2]. In this section, we give some other applications in the study of deformations of homogeneous polynomials.

For $k \ge 0$, denote by $\mathcal{U}_{n,d}(k)$ the set

$$\mathcal{U}_{n,d}(k) = \{f \in S_{n,d} : \dim E_k(f) = \dim S_{n,k}\}.$$

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From semi-continuity of $\dim E_k(f)$ with respect to f, we see that $\mathcal{U}_{n,d}(k)$ is a Zariski open subset of $S_{n,d}$. Obviously, we have $\mathcal{U}_{n,d}(k) = \varnothing$ if $k > \frac{d}{2}$. From Lemma 2.5, we have the following result.

Corollary 3.1 *Given $n \geq 1$ and $d \geq 3$, for $k \leq \frac{d}{2}$, the set $\mathcal{U}_{n,d}(k)$ is a Zariski open dense subset of $S_{n,d}$.*

In addition, for any $f \in \mathcal{U}_{n,d}(k)$, we have by definition that $\dim E_k(f) = \dim S_{n,k}$; by Lemma 2.1, we deduce that $\dim E_{k-1}(f) = \dim S_{n,k-1}$, that is $f \in \mathcal{U}_{n,d}(k-1)$. In other words, for fixed n and d, the sequence of sets $\{\mathcal{U}_{n,d}(k)\}$ satisfies the relations

$$\mathcal{U}_{n,d}(0) \supseteq \mathcal{U}_{n,d}(1) \supseteq \cdots \supseteq \mathcal{U}_{n,d}(k) \supseteq \mathcal{U}_{n,d}(k+1) \supseteq \cdots .$$

Note that $\mathcal{U}_{n,d}(k)$ is a cone in $S_{n,d}$, hence we can consider its projectivization, denoted by $\mathbb{P}\big(\mathcal{U}_{n,d}(k)\big)$, in $\mathbb{P}(S_{n,d})$. Similar to the construction in [3], the assignment

$$[f] \mapsto \mathbb{P}\big(E_k(f)\big)$$

gives a well-defined map, denoted by φ_k, from $\mathbb{P}\big(\mathcal{U}_{n,d}(k)\big)$ to an obvious Grassmannian for $k \leq \frac{d}{2}$.

Using Proposition 2.3 and Lemma 2.1, we prove the following result, which gives an extension of Corollary 7.7 in [3].

Corollary 3.2 *For $k \leq \frac{d}{2}-1$, the map $\varphi_k\colon \mathbb{P}\big(\mathcal{U}_{n,d}(k)\big) \ni [f] \mapsto \mathbb{P}\big(E_k(f)\big)$ is injective when restricted to $\mathbb{P}\big(\mathcal{U}_{n,d}(k+1)\big)$. In particular, it is generically injective.*

Proof To begin the proof, suppose $[f]$ and $[g]$ are two elements of $\mathbb{P}\big(\mathcal{U}_{n,d}(k+1)\big)$ such that $\varphi_k([f]) = \varphi_k([g])$. By the definition of φ_k, this means that $E_k(f) = E_k(g)$. Now the assumption $[f] \in \mathbb{P}\big(\mathcal{U}_{n,d}(k+1)\big)$ implies that $\dim E_{k+1}(f) = \dim S_{n,k+1}$, hence by Proposition 2.3, we obtain $E_{k-1}(f) = E_{k-1}(g)$. An induction argument on k gives $[f] = [g]$, which goes exactly the same as the proof of Theorem 1.1, where only the properties $\dim E_{k+1}(f) = \dim S_{n,k+1}$ and $E_k(f) = E_k(g)$ are essentially used. Thus, φ_k is injective on $\mathbb{P}(\mathcal{U}_{n,d}(k+1))$. ∎

Remark 3.3 We do not know whether φ_k is injective on $\mathbb{P}\big(\mathcal{U}_{n,d}(k)\big)$ or not, except the case $k = \frac{d}{2}$, where $\varphi_{\frac{d}{2}}$ is a constant map, because in this case $E_k(f) = S_{n,k}$ for any $f \in \mathcal{U}_{n,d}(k)$.

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