Efficient DOP Calculation for GPS with and without Altimeter Aiding

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Geometric Dilution Of Precision (GDOP) is a factor that describes the effect of geometry on the relationship between measurement error and position determination error. It is used to provide an indication of the quality of the solution. Conventional closed-form GDOP calculation formula applied to all possible combinations of visible satellites is rather time consuming, especially as the number of satellites grows. Approximations, such as the maximum volume method, are faster but optimum selection is not guaranteed. In this paper, a more concise but efficient solution for the calculation of GDOP value in the case of four Global Positioning System (GPS) satellites is firstly reviewed and then extended to cover the other forms of dilution of precision (DOP) values, including vertical DOP (VDOP) and horizontal DOP (HDOP). Secondly, a review and extension of the conventional solution is performed in the case of three GPS satellites aided by an altimeter. Based on the ideas gained from these two approaches, a simpler closed-form DOP formula for three GPS satellites aided by an altimeter is derived. The advantage of the proposed formulation is that it is simpler and thus reduces the computational load in comparison to the conventional one.

KEY WORDS

1. GPS. 2. GDOP. 3. Altimetry.

1. INTRODUCTION. Geometric dilution of precision, usually referred to as the GDOP, is a factor that describes the effect of geometry on the relationship between measurement error and position determination error. It is used to provide an indication of the quality of the solution. The smaller the GDOP factor, the more accurate the navigation fix. The most straightforward approach is to use the closed-form solution to all combinations and select the minimum one. However, this is very time-consuming when the number of satellites is large. It has been shown (Kihara and Okada, 1984) that GDOP is approximately inversely proportional to the volume of the tetrahedron formed by four satellites. Therefore, the optimum solution is to select satellites such that the volume is as large as possible, which is sometimes called the maximum volume method. The disadvantage of this method is that it does not guarantee an optimum selection of satellites. To avoid the above disadvantages, a modified closed-form solution with simpler calculation was later proposed (Zhu, 1992), which worked only for a four-satellite case.

2. DILUTION OF PRECISION. Consider the vectors depicted in Figure 1 relating to the Earth's centre, satellites and user position. The vector **s** represents the vector from the Earth's centre to a satellite, **u** represents the vector from the Earth's centre to the user's position, and **r** represents the vector from the user to satellite. Thus, we can write the vector relation

$$\mathbf{r} = \mathbf{s} - \mathbf{u}. \tag{1}$$

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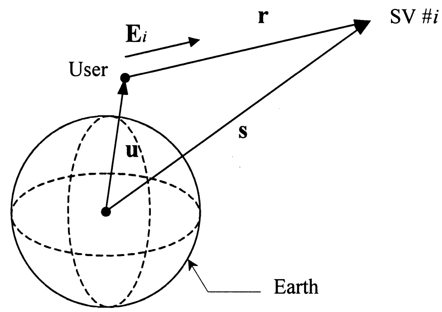

Figure 1. Definition of vectors.

The distance $\|\mathbf{r}\|$ is computed by measuring the propagation time from the transmitting satellite to the user/receiver. The pseudo-range ρ_i is defined for the i-th satellite by:

$$\rho_i = \|\mathbf{s}_i - \mathbf{u}\| + ct_b + v_{\rho_i}, \tag{2}$$

where: c is the speed of light, t_b is the receiver clock offset from system time, and v_{ρ_i} is the pseudo-range measurement noise. Considering the user position in three dimensions, denoted by (x_u, y_u, z_u), the GPS pseudo-range measurements made to the n satellites can then be written as:

$$\rho_i = \sqrt{(x_i - x_u)^2 + (y_i - y_u)^2 + (z_i - z_u)^2} + ct_b + v_{\rho_i}, i = 1, \ldots, n, \tag{3}$$

where: (x_i, y_i, z_i) denotes the i-th satellite's position in three dimensions.

Equation (3) is linearised by expanding Taylor's series around the approximate (or nominal) user position $(\hat{x}_n, \hat{y}_n, \hat{z}_n)$ and neglecting the higher terms. Defining $\hat{\rho}_i$ as ρ_i at $(\hat{x}_n, \hat{y}_n, \hat{z}_n)$ gives:

$$\Delta\rho_i = \rho_i - \hat{\rho}_i = e_{i1}\,\Delta x_u + e_{i2}\,\Delta y_u + e_{i3}\,\Delta z_u + ct_b + v_{\rho_i}. \tag{4}$$

where:

$$e_{i1} = \frac{\hat{x}_n - x_i}{\hat{r}_i}; \quad e_{i2} = \frac{\hat{y}_n - y_i}{\hat{r}_i}; \quad e_{i3} = \frac{\hat{z}_n - z_i}{\hat{r}_i} \tag{5}$$

$$\hat{r}_i = \sqrt{(\hat{x}_n - x_i)^2 + (\hat{y}_n - y_i)^2 + (\hat{z}_n - z_i)^2}.$$

The vector $(e_{i1}, e_{i2}, e_{i3}) \equiv \mathbf{E}_i$, $i = 1, \ldots, n$, denotes the line-of-sight vector from the user to the satellites. Equation (4) can be written in a matrix formulation

$$\Delta\rho_i = [\Delta\rho_i \quad \Delta\rho_2 \quad \Delta\rho_3 \quad \ldots \quad \Delta\rho_n]^T$$

$$= \begin{bmatrix} e_{11} & e_{12} & e_{13} & 1 \\ e_{21} & e_{22} & e_{23} & 1 \\ e_{31} & e_{32} & e_{33} & 1 \\ \vdots & \vdots & \vdots & \vdots \\ e_{n1} & e_{n2} & e_{n3} & 1 \end{bmatrix} \begin{bmatrix} \Delta x_u \\ \Delta y_u \\ \Delta z_u \\ c\Delta t_b \end{bmatrix} + v_{\rho_i} \tag{6}$$

which can be represented as:

$$\mathbf{z} = \mathbf{H}\mathbf{x} + \mathbf{v}. \tag{7}$$

The dimension of matrix $\mathbf{H}$ is $n \times 4$ (bold, uppercase letters are used to denote matrices) with $n \geqslant 4$, and $\mathbf{H}$ is usually referred to as the 'geometry matrix' or 'visibility matrix'. The least squares solution to Equation (7) is given by:

$$\hat{\mathbf{x}} = (\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T\mathbf{z},$$

and the quality of navigation solution for a linearised pseudo-range equation is represented by taking the difference between the estimated and true positions:

$$\begin{aligned}\tilde{\mathbf{x}} &= \hat{\mathbf{x}} - \mathbf{x} \\ &= (\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T\mathbf{H}\mathbf{x} + (\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T\mathbf{v} - \mathbf{x} \\ &= (\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T\mathbf{v},\end{aligned} \tag{8}$$

where: $\mathbf{v}$ has zero mean, and so does $\tilde{\mathbf{x}}$. The covariance between the errors in the components of the estimated position is:

$$E\{\tilde{\mathbf{x}}\tilde{\mathbf{x}}^T\} = (\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T E\{\mathbf{v}\mathbf{v}^T\}\mathbf{H}(\mathbf{H}^T\mathbf{H})^{-1}, \tag{9}$$

where: $E\{\cdot\}$ is the expected value operator. If all components of $\mathbf{v}$ are pairwise uncorrelated and have variance σ^2, then:

$$E\{\mathbf{v}\mathbf{v}^T\} = \sigma^2\mathbf{I},$$

and Equation (9) becomes:

$$E\{\tilde{\mathbf{x}}\tilde{\mathbf{x}}^T\} = \sigma^2(\mathbf{H}^T\mathbf{H})^{-1}. \tag{10}$$

The GDOP factor is defined as:

$$\text{GDOP} = \sqrt{\text{tr}(\mathbf{H}^T\mathbf{H})^{-1}} = \sqrt{\frac{E\{\tilde{\mathbf{x}}\tilde{\mathbf{x}}^T\}}{\sigma^2}} = \frac{\sqrt{\sigma_{xx}^2 + \sigma_{yy}^2 + \sigma_{zz}^2 + \sigma_{tt}^2}}{\sigma}, \tag{11}$$

where:

$\sigma_{xx}, \sigma_{yy}, \sigma_{zz}$ = root-mean-squared (RMS) errors in x, y, and z positions, respectively;

σ_{tt} = RMS error in time;

σ = RMS GPS pseudo-range error.

It can be seen from Equation (11) that the GDOP factor gives a simple interpretation of how much one unit of measurement error contributes to the derived position solution error for a given situation. It determines the magnification factor of the measurement noise that is translated into the derived solution. Clearly, the GDOP expression can be relatively simple if all the measurements exhibit the same RMS errors. The modified version of GDOP, hereinafter called the weighted GDOP (WGDOP), is utilized for measurements with different error variances. These measurements could be from two different systems, such as the GPS with GLONASS or an altimeter. It is known that the GPS navigation algorithm using weighted least-squares estimation provides a better solution than the ordinary least-squares estimation, which is especially evident when the random errors in pseudo-range measurements are significantly different from each other. The quality of the navigation solution regarding measurements with different random errors can be modified as follows:

$$\tilde{\mathbf{x}} = (\mathbf{H}^T\mathbf{W}\mathbf{H})^{-1}\mathbf{H}^T\mathbf{W}\mathbf{v}. \tag{12}$$

Assuming the covariance matrix of $\mathbf{v}$ is:

$$E\{\mathbf{v}\mathbf{v}^T\} = \begin{bmatrix} \sigma_n^2 & 0 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & \sigma_3^2 & 0 & 0 \\ 0 & 0 & 0 & \sigma_2^2 & 0 \\ 0 & 0 & 0 & 0 & \sigma_1^2 \end{bmatrix}, \tag{13}$$

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and the weighting function is defined as:

$$\mathbf{W} = \begin{bmatrix} \sigma_1^2/\sigma_n^2 & 0 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & \sigma_1^2/\sigma_3^2 & 0 & 0 \\ 0 & 0 & 0 & \sigma_1^2/\sigma_2^2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{14}$$

where: $\sigma_i, i = 1, \ldots, n$ represents the RMS error of the i-th measurement, consequently, the estimation covariance becomes:

$$\begin{aligned} E\{\tilde{\mathbf{x}}\tilde{\mathbf{x}}^T\} &= (\mathbf{H}^T\mathbf{W}\mathbf{H})^{-1}\mathbf{H}^T\mathbf{W}E\{\mathbf{v}\mathbf{v}^T\}\mathbf{W}^T\mathbf{H}(\mathbf{H}^T\mathbf{W}\mathbf{H})^{-1} \\ &= \sigma_1^2(\mathbf{H}^T\mathbf{W}\mathbf{H})^{-1}, \end{aligned} \tag{15}$$

and the WGDOP is recognized as:

$$\text{WGDOP} = \sqrt{\text{tr}(\mathbf{H}^T\mathbf{W}\mathbf{H})^{-1}} \tag{16}$$

3. CALCULATION OF DOP FOR FOUR VISIBLE SATELLITES. Four satellites will generally be required to provide sufficient information for an acceptable three-dimensional position fix. When less than four satellites are available, other sensor data may be fused with available pseudo-range data such as altimeter information (Stein, 1985) or accurate clock information (Sturza, 1985). The aiding information is simply used as a source of additional data that will make the navigation solution mathematically possible. The selection of four visible satellites for obtaining suitable navigation accuracy has appeared in several literatures. From Equation (7), the geometry/visibility matrix for four satellites is:

$$\mathbf{H} = \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \\ \mathbf{E}_4 \end{bmatrix} = \begin{bmatrix} e_{11} & e_{12} & e_{13} & 1 \\ e_{21} & e_{22} & e_{23} & 1 \\ e_{31} & e_{32} & e_{33} & 1 \\ e_{41} & e_{42} & e_{43} & 1 \end{bmatrix}. \tag{17}$$

From basic matrix algebra, since:

$$(\mathbf{H}^T\mathbf{H})^{-1} = \frac{\text{adj}(\mathbf{H}^T\mathbf{H})}{\det(\mathbf{H}^T\mathbf{H})} = \frac{\text{adj}(\mathbf{H}^T\mathbf{H})}{\det^2(\mathbf{H})}, \tag{18}$$

the GDOP factor can be written as:

$$\text{GDOP} = \sqrt{\text{tr}(\mathbf{H}^T\mathbf{H})^{-1}} = \frac{\sqrt{\text{tr}[\text{adj}(\mathbf{H}^T\mathbf{H})]}}{|\det(\mathbf{H})|}. \tag{19}$$

The determinant of $\mathbf{H}$ is not changed through the elementary row operations, consequently,

$$\begin{aligned} \det(\mathbf{H}) &= \begin{vmatrix} e_{11} & e_{12} & e_{13} & 1 \\ e_{21} & e_{22} & e_{23} & 1 \\ e_{31} & e_{32} & e_{33} & 1 \\ e_{41} & e_{42} & e_{43} & 1 \end{vmatrix} \\ &= - \begin{vmatrix} e_{41}-e_{11} & e_{42}-e_{12} & e_{43}-e_{13} \\ e_{41}-e_{21} & e_{42}-e_{22} & e_{43}-e_{23} \\ e_{41}-e_{31} & e_{42}-e_{32} & e_{43}-e_{33} \end{vmatrix}. \end{aligned} \tag{20}$$

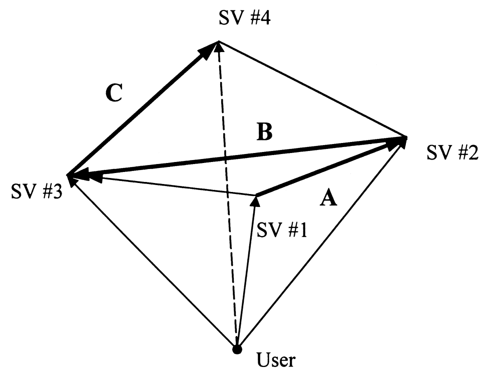

Figure 2. The tetrahedron formed by four satellites.

In Equation (18), the following two properties have been employed:

$$\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B}),$$

and

$$\det \mathbf{A}^T = \det \mathbf{A},$$

provided **A** and **B** are both square matrices.

The existing methods for GDOP calculation include (Stein, 1985):

(a) Matrix inversion by computer;
(b) Closed form algorithm;
(c) Maximum volume of a tetrahedron.

The matrix inversion by computer presents a computational burden on the navigation computer. The idea of the maximum volume approach is based on the property from Equation (19), where it can be seen that the GDOP is approximately proportional to $1/\det \mathbf{H}$. It can also be seen from Equation (20) that the volume of the tetrahedron formed by four satellites is $\det(\mathbf{H})/6$,

$$\text{Volume} = \frac{1}{6}(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C},$$

where: the vectors **A**, **B**, and **C** are defined as in Figure 2. However, the maximum volume method does not guarantee the selection of the four satellites with minimum GDOP.

By use of the following properties:

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1},$$

and

$$\operatorname{tr}(\mathbf{AB}) = \operatorname{tr}(\mathbf{BA}),$$

we have:

$$\operatorname{tr}(\mathbf{AB})^{-1} = \operatorname{tr}(\mathbf{B}^{-1}\mathbf{A}^{-1}) = \operatorname{tr}(\mathbf{A}^{-1}\mathbf{B}^{-1}) = \operatorname{tr}(\mathbf{BA})^{-1}.$$

Therefore, the GDOP of Equation (19) can be alternatively written as:

$$\text{GDOP} = \sqrt{\operatorname{tr}(\mathbf{H}^T\mathbf{H})^{-1}} = \sqrt{\operatorname{tr}(\mathbf{HH}^T)^{-1}}. \tag{21}$$

Denoting

$$E_{ij} = e_{i1}e_{j1} + e_{i2}e_{j2} + e_{i3}e_{j3} + 1, \quad 1 \leqslant i < j \leqslant 4, \tag{22}$$

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and using the fact that:

$$e_{i1}^2 + e_{i2}^2 + e_{i3}^2 = 1, \quad i = 1, \ldots, 4,$$

we have:

$$\mathbf{HH}^T = \begin{bmatrix} 2 & E_{12} & E_{13} & E_{14} \\ E_{12} & 2 & E_{23} & E_{24} \\ E_{13} & E_{23} & 2 & E_{34} \\ E_{14} & E_{24} & E_{34} & 2 \end{bmatrix}$$

Based on Equation (21), several forms of DOPs are defined as:

(a) Geometric Dilution of Precision (GDOP):

$$\text{GDOP} = \sqrt{(\mathbf{HH}^T)^{-1}_{1,1} + (\mathbf{HH}^T)^{-1}_{2,2} + (\mathbf{HH}^T)^{-1}_{3,3} + (\mathbf{HH}^T)^{-1}_{4,4}}. \tag{23}$$

(b) Horizontal Dilution of Precision (HDOP):

$$\text{HDOP} = \sqrt{(\mathbf{HH}^T)^{-1}_{1,1} + (\mathbf{HH}^T)^{-1}_{2,2}}. \tag{24}$$

(c) Vertical Dilution of Precision (VDOP):

$$\text{VDOP} = \sqrt{(\mathbf{HH}^T)^{-1}_{3,3}}. \tag{25}$$

(d) Position Dilution of Precision (PDOP):

$$\text{PDOP} = \sqrt{(\mathbf{HH}^T)^{-1}_{1,1} + (\mathbf{HH}^T)^{-1}_{2,2} + (\mathbf{HH}^T)^{-1}_{3,3}}. \tag{26}$$

Here the notation $(\mathbf{HH}^T)^{-1}_{i,i}$ is defined as the i-th element on the main diagonal of $(\mathbf{HH}^T)^{-1}$:

$$(\mathbf{HH}^T)^{-1}_{i,i} = \frac{\text{adj}_{i,i}(\mathbf{HH}^T)}{\det(\mathbf{HH}^T)} = \frac{\text{cof}_{i,i}(\mathbf{HH}^T)}{\det(\mathbf{HH}^T)}$$

and the cofactor, $\text{cof}_{i,i}(\mathbf{A})$, is the determinant of the sub-matrix of $\mathbf{A}$ formed by deleting the i-th row and the i-th column. The cofactors can be derived to be

$$\text{cof}_{1,1}(\mathbf{HH}^T) = 8 + 2[E_{23}E_{24}E_{34} - (E_{23}^2 + E_{24}^2 + E_{34}^2)], \tag{27a}$$

$$\text{cof}_{2,2}(\mathbf{HH}^T) = 8 + 2[E_{13}E_{14}E_{34} - (E_{13}^2 + E_{14}^2 + E_{34}^2)], \tag{27b}$$

$$\text{cof}_{3,3}(\mathbf{HH}^T) = 8 + 2[E_{12}E_{14}E_{24} - (E_{12}^2 + E_{14}^2 + E_{24}^2)], \tag{27c}$$

$$\text{cof}_{4,4}(\mathbf{HH}^T) = 8 + 2[E_{12}E_{13}E_{23} - (E_{12}^2 + E_{13}^2 + E_{23}^2)], \tag{27d}$$

and the trace of the matrix $(\mathbf{HH}^T)^{-1}$ is:

$$\text{tr}[\text{adj}(\mathbf{HH}^T)] = \sum_{i=1}^{4} \text{cof}_{i,i}(\mathbf{HH}^T). \tag{28}$$

Defining the following variables (Zhu, 1992):

$$a = (E_{12}E_{34} + E_{13}E_{24} - E_{14}E_{23})^2 - 4(E_{12}E_{34}E_{13}E_{24}), \tag{29a}$$

$$b = 16 - 4(E_{12}^2 + E_{13}^2 + E_{14}^2 + E_{23}^2 + E_{24}^2 + E_{34}^2), \tag{29b}$$

$$c = 2[E_{12}(E_{13}E_{23} + E_{14}E_{24}) + E_{34}(E_{13}E_{14} + E_{23}E_{24})], \tag{29c}$$

the GDOP can be written as:

$$\text{GDOP} = \sqrt{\frac{16 + b + c}{a + b + 2c}}$$

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This closed-form equation needs only 39 multiplications, 34 additions, 1 division, and 1 square root. Since both $E_{12}E_{34}$ and $E_{13}E_{24}$ appears twice, two multiplications can be eliminated.

4. CALCULATION OF DOP USING ALTIMETER AIDING.

Conceptually, the altimeter can be viewed as a pseudo-satellite located at the centre of the Earth. See Figure 3. Therefore, one of the satellite observables can be replaced by

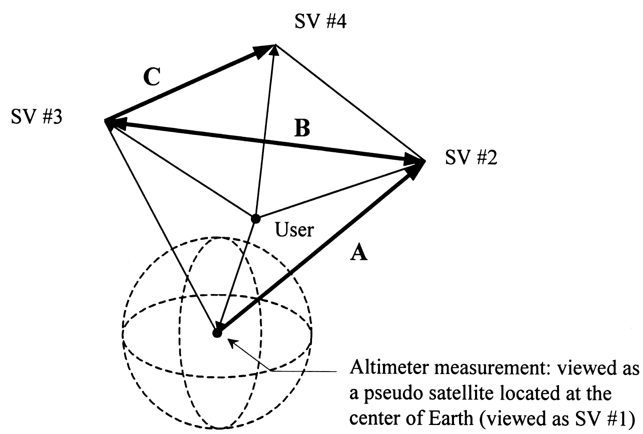

Figure 3. The tetrahedron formed by three satellites and an altimeter.

altitude information provided by the altimeter. In this case, three visible satellites will be sufficient for a three-dimensional navigation fix. The **H** matrix then can be expressed as:

$$\mathbf{H} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ e_{21} & e_{22} & e_{23} & 1 \\ e_{31} & e_{32} & e_{33} & 1 \\ e_{41} & e_{42} & e_{43} & 1 \end{bmatrix}. \tag{30}$$

The weighting matrix regarding the two types of measurements is defined as:

$$\mathbf{W} = \begin{bmatrix} 1/r & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{31}$$

where: r represents the ratio of the altimeter error variance to the GPS pseudo-range error variance:

$$r = \frac{\sigma_{alt}^2}{\sigma_{gps}^2}. \tag{32}$$

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As mentioned in Section 2, DOP performance measures indicate the error as an estimated navigation quantity 'per unit of measurement noise'. The use of weighted DOP is more appropriate when measurements from two different systems are fused. The WGDOP in Equation (16) is as follows

$$\begin{aligned} \text{WGDOP} &= \sqrt{\text{tr}(\mathbf{H}^T\mathbf{W}\mathbf{H})^{-1}} \\ &= \sqrt{\frac{\text{tr}[\text{adj}(\mathbf{H}^T\mathbf{W}\mathbf{H})]}{\det(\mathbf{H}^T\mathbf{W}\mathbf{H})}} \\ &= \sqrt{\frac{\text{tr}[\text{adj}(\mathbf{H}^T\mathbf{W}\mathbf{H})]}{\det(\mathbf{H}^T)\det(\mathbf{W})\det(\mathbf{H})}}. \end{aligned} \tag{33}$$

Defining the following variable

$$E_{ii} = e_{2i}^2 + e_{3i}^2 + e_{4i}^2, \quad 1 \leqslant i \leqslant 2 \tag{34a}$$

$$E_{i4} = e_{2i} + e_{3i} + e_{4i}, \quad 1 \leqslant i \leqslant 3 \tag{34b}$$

$$E_{ij} = e_{2i}\,e_{2j} + e_{3i}\,e_{3j} + e_{4i}\,e_{4j}, \quad 1 \leqslant i < j \leqslant 3 \tag{34c}$$

$$E_{33} = e_{23}^2 + e_{33}^2 + e_{43}^2 + 1/r \tag{34d}$$

$$E_{44} = 3 \tag{34e}$$

we have:

$$\mathbf{H}^T\mathbf{W}\mathbf{H} = \begin{bmatrix} E_{11} & E_{12} & E_{13} & E_{14} \\ E_{12} & E_{22} & E_{23} & E_{24} \\ E_{13} & E_{23} & E_{33} & E_{34} \\ E_{14} & E_{24} & E_{34} & E_{44} \end{bmatrix}. \tag{35}$$

After some algebraic manipulation, the cofactors on the main diagonal are found to be:

$$\text{cof}_{1,1}(\mathbf{H}^T\mathbf{W}\mathbf{H}) = E_{22}\,E_{33}\,E_{44} + 2E_{23}\,E_{24}\,E_{34} - (E_{34}^2\,E_{22} + E_{24}^2\,E_{33} + E_{23}^2\,E_{44}) \tag{36a}$$

$$\text{cof}_{2,2}(\mathbf{H}^T\mathbf{W}\mathbf{H}) = E_{11}\,E_{33}\,E_{44} + 2E_{13}\,E_{14}\,E_{34} - (E_{34}^2\,E_{11} + E_{14}^2\,E_{33} + E_{13}^2\,E_{44}) \tag{36b}$$

$$\text{cof}_{3,3}(\mathbf{H}^T\mathbf{W}\mathbf{H}) = E_{11}\,E_{22}\,E_{44} + 2E_{12}\,E_{14}\,E_{24} - (E_{24}^2\,E_{11} + E_{14}^2\,E_{22} + E_{12}^2\,E_{44}) \tag{36c}$$

$$\text{cof}_{4,4}(\mathbf{H}^T\mathbf{W}\mathbf{H}) = E_{11}\,E_{22}\,E_{33} + 2E_{12}\,E_{13}\,E_{23} - (E_{23}^2\,E_{11} + E_{13}^2\,E_{22} + E_{12}^2\,E_{33}) \tag{36d}$$

and the trace is:

$$\begin{aligned} \text{tr}[\text{adj}(\mathbf{H}^T\mathbf{W}\mathbf{H})] &= \sum_{i=1}^{4} \text{cof}_{i,i}(\mathbf{H}^T\mathbf{W}\mathbf{H}) \\ &= E_{33}[(E_{11}+3)\,E_{22} - E_{24}^2 - E_{12}^2 + 3E_{11} - E_{14}^2] \\ &\quad - E_{34}^2(E_{11} + E_{22}) + 2E_{34}(E_{13}\,E_{14} + E_{23}\,E_{24}) \\ &\quad - E_{23}^2(E_{11} + 3) + 2E_{12}\,E_{13}\,E_{23} \\ &\quad - E_{22}(E_{13}^2 - 3E_{11} + E_{14}^2) \\ &\quad - E_{11}\,E_{24}^2 + 2E_{12}\,E_{14}\,E_{24} - 3(E_{12}^2 + E_{13}^2). \end{aligned} \tag{37}$$

The determinant of **H** and **W** matrices are:

$$\det(\mathbf{H}) = (e_{31} - e_{21})\,(e_{42} - e_{32}) - (e_{41} - e_{31})\,(e_{32} - e_{22}) = \det \mathbf{H}^T, \tag{38}$$

and

$$\det(\mathbf{W}) = 1/r, \tag{39}$$

respectively. Substituting Equations (37) to (39) into Equation (33) yields the GDOP value. This closed-form formula has a relatively simple form in the denominator, but is more complicated in the numerator. It needs 47 multiplications, 42 additions, 1 division, and 1 square root. Notice that E_{24}^2, E_{12}^2, E_{14}^2, $3E_{11}$ and $(E_{11}+3)$ all appear twice in the expression. Besides, the value of $1/r$ is assumed to be already known outside the GDOP calculation loop, and thus can be treated as a constant during the calculation. The weighted HDOP (WHDOP) and weighted VDOP (WVDOP) are obtained by using Equations (24) and (25) – except replacing $(\mathbf{H}^T\mathbf{W}\mathbf{H})$ by $(\mathbf{H}^T\mathbf{H})$:

$$\mathrm{WHDOP} = \sqrt{\frac{\mathrm{cof}_{1,1}(\mathbf{H}^T\mathbf{W}\mathbf{H}) + \mathrm{cof}_{2,2}(\mathbf{H}^T\mathbf{W}\mathbf{H})}{\det(\mathbf{H}^T)\det(\mathbf{W})\det(\mathbf{H})}},$$

$$\mathrm{WVDOP} = \sqrt{\frac{\mathrm{cof}_{3,3}(\mathbf{H}^T\mathbf{W}\mathbf{H})}{\det(\mathbf{H}^T)\det(\mathbf{W})\det(\mathbf{H})}}$$

respectively, where the cofactors are given by Equation (36).

5. DEVELOPMENT OF THE ALTERNATIVE FORMULA.

A more concise and efficient formulation that reduces the computational load is derived based on Sections 3 and 4. The following relation will be employed in the approach.

$$\mathrm{tr}(\mathbf{H}^T\mathbf{W}\mathbf{H})^{-1} = \mathrm{tr}(\mathbf{H}\mathbf{H}^T\mathbf{W})^{-1} = \mathrm{tr}(\mathbf{W}\mathbf{H}\mathbf{H}^T)^{-1} \tag{40}$$

A brief proof is given now.

Proof. By use of the equality in basic matrix algebra:

$$(\mathbf{A}\mathbf{B})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1},$$
$$\mathrm{tr}(\mathbf{A}\mathbf{B}) = \mathrm{tr}(\mathbf{B}\mathbf{A}),$$

it is shown that:

$$\mathrm{tr}(\mathbf{A}\mathbf{B})^{-1} = \mathrm{tr}(\mathbf{B}^{-1}\mathbf{A}^{-1}) = \mathrm{tr}(\mathbf{A}^{-1}\mathbf{B}^{-1}) = \mathrm{tr}(\mathbf{B}\mathbf{A})^{-1},$$

hence:

$$\mathrm{tr}(\mathbf{H}^T\mathbf{W}\mathbf{H})^{-1} = \mathrm{tr}[\mathbf{H}^{-1}(\mathbf{H}^T\mathbf{W})^{-1}] = \mathrm{tr}[(\mathbf{H}^T\mathbf{W})^{-1}\mathbf{H}^{-1}] = \mathrm{tr}(\mathbf{H}\mathbf{H}^T\mathbf{W})^{-1}.$$

Manipulation yields:

$$\mathrm{tr}(\mathbf{H}\mathbf{H}^T\mathbf{W})^{-1} = \mathrm{tr}[\mathbf{W}^{-1}(\mathbf{H}\mathbf{H}^T)^{-1}] = \mathrm{tr}[(\mathbf{H}\mathbf{H}^T)^{-1}\mathbf{W}^{-1}] = \mathrm{tr}(\mathbf{W}\mathbf{H}\mathbf{H}^T)^{-1}.$$

By denoting:

$$E_{1j} = -e_{j3}, \quad 2 \leqslant j \leqslant 4, \quad \text{and} \tag{41a}$$

$$E_{ij} = e_{i1}e_{j1} + e_{i2}e_{j2} + e_{i3}e_{j3} + 1, \quad 2 \leqslant i < j \leqslant 4 \tag{41b}$$

we have:

$$\mathbf{H}\mathbf{H}^T\mathbf{W} = \begin{bmatrix} 1/r & E_{12} & E_{13} & E_{14} \\ E_{12}/r & 2 & E_{23} & E_{24} \\ E_{13}/r & E_{23} & 2 & E_{34} \\ E_{14}/r & E_{24} & E_{34} & 2 \end{bmatrix},$$

where the following relation has been used:

$$e_{i1}^2 + e_{i2}^2 + e_{i3}^2 = 1, \quad i = 2, \ldots, 4.$$

The cofactors on the main diagonal can be obtained as:

$$\text{cof}_{1,1}(\mathbf{HH}^T\mathbf{W}) = 8 + 2[E_{23}E_{24}E_{34} - (E_{23}^2 + E_{24}^2 + E_{34}^2)] \tag{42a}$$

$$\text{cof}_{2,2}(\mathbf{HH}^T\mathbf{W}) = [4 + 2E_{13}E_{14}E_{34} - (2E_{13}^2 + 2E_{14}^2 + E_{34}^2)]/r \tag{42b}$$

$$\text{cof}_{3,3}(\mathbf{HH}^T\mathbf{W}) = [4 + 2E_{12}E_{14}E_{24} - (2E_{12}^2 + 2E_{14}^2 + E_{24}^2)]/r \tag{42c}$$

$$\text{cof}_{4,4}(\mathbf{HH}^T\mathbf{W}) = [4 + 2E_{12}E_{13}E_{23} - (2E_{12}^2 + 2E_{13}^2 + E_{23}^2)]/r \tag{42d}$$

Compared with Equations (36), Equations (42) are improved. The trace of $(\mathbf{HH}^T\mathbf{W})^{-1}$ is:

$$\begin{aligned} \text{tr}[\text{adj}(\mathbf{HH}^T\mathbf{W})] &= \sum_{i=1}^{4} \text{cof}_{i,i}(\mathbf{HH}^T\mathbf{W}) \\ &= 8 + 2[E_{23}E_{24}E_{34} - (E_{23}^2 + E_{24}^2 + E_{34}^2)] \\ &\quad + \{12 + 2(E_{13}E_{14}E_{34} + E_{12}E_{14}E_{24} + E_{12}E_{13}E_{23}) \\ &\quad - 4(E_{12}^2 + E_{13}^2 + E_{14}^2) - (E_{23}^2 + E_{24}^2 + E_{34}^2)\}/r. \end{aligned} \tag{43}$$

and

$$\begin{aligned} \det(\mathbf{HH}^T\mathbf{W}) = \{&8 + 2[E_{23}E_{24}E_{34} - (E_{23}^2 + E_{24}^2 + E_{34}^2)] \\ &+ 4[E_{13}E_{14}E_{34} + E_{12}E_{13}E_{24} + E_{12}E_{13}E_{23} - (E_{12}^2 + E_{13}^2 + E_{14}^2)] \\ &+ (E_{12}E_{34} + E_{13}E_{24} - E_{14}E_{23})^2 - 4(E_{12}E_{34}E_{13}E_{24})\}/r. \end{aligned} \tag{44}$$

In Equation (43), E_{23}^2, E_{24}^2, E_{34}^2 and $E_{23}^2 + E_{24}^2 + E_{34}^2$ appear twice in the expression. The determinant by Equation (44) is complicated and is not a good choice. Instead, the much simpler formula previously obtained by multiplying three determinants, $\det(\mathbf{H}^T)\det(\mathbf{W})\det(\mathbf{H})$ can be used. The negative sign in Equation (41a) is treated as an addition. The alternative closed-form formula only needs 31 multiplications, 29 additions, 1 division, and 1 square root.
Again, the value of $1/r$ is treated as a constant in GDOP calculation loop.

6. CONCLUSIONS.

Issues related to GDOP calculation have been reviewed. Closed-form formulae for the calculation of GDOP factors for the cases of four satellites and three satellites plus an altimeter, have been discussed. Extended results on other forms of DOPs, including HDOP and VDOP, have also been presented. A derivation to obtain an alternative closed-form formulation in the case of three satellites plus an altimeter has been presented, in which the random errors of measurements from two systems are significantly different and, consequently, a weighting function appears in the calculation of DOPs (which yields the WGDOP, WHDOP and WVDOP, etc.). Related proofs for deriving the alternative formulae are provided. The computational load based on the proposed algorithm is reduced.

REFERENCES

Kihara, M. and Okada, T. (1984). A satellite selection method and accuracy for the Global Positioning System. *Navigation*: Journal of the US Institute of Navigation, 8–20, Vol. 31, No. 1.

Stein, B. A. (1985). Satellite selection criteria during altimeter aiding of GPS. *Navigation*: Journal of the US Institute of Navigation, 149–157, Vol. 32, No. 2.

Sturza, M. A. (1985). GPS navigation using three satellites and a precise clock. *Global Positioning System*, Volume 2, US Institute of Navigation, 122–132.

Zhu, J. (1992). Calculation of Geometric Dilution of Precision. *IEEE Transaction on Aerospace and Electronic Systems*, Vol. 28, No. 3.

Nielsen, R. (1997). Relationship between dilution of precision for point positioning and for relative positioning with GPS. *IEEE Transaction on Aerospace and Electronic Systems*, Vol. 33, No. 1.

Phatak, M. (1999). Position fix from three GPS satellites and altitude: a direct method. *IEEE Transaction on Aerospace and Electronic Systems*, Vol. 35, No. 1.