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Spectral properties of a beam equation with eigenvalue parameter occurring linearly in the boundary conditions

Ziyatkhan S. Aliyev
Baku State University, Baku AZ1148, Azerbaijan
Institute of Mathematics and Mechanics NAS of Azerbaijan, Baku AZ1141, Azerbaijan
National Aviation Academy of Azerbaijan, Baku AZ1045, Azerbaijan
(z_aliyev@mail.ru)

Gunay T. Mamedova
Ganja State University, Ganja AZ2001, Azerbaijan
(gunaymamedova614@gmail.com)

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In this paper, we consider an eigenvalue problem for ordinary differential equations of fourth order with a spectral parameter in the boundary conditions. The location of eigenvalues on real axis, the structure of root subspaces and the oscillation properties of eigenfunctions of this problem are investigated, and asymptotic formulas for the eigenvalues and eigenfunctions are found. Next, by the use of these properties, we establish sufficient conditions for subsystems of root functions of the considered problem to form a basis in the space $L_p, 1 < p < \infty$.

Keywords: Differential operator; Eigenvalue; Eigenfunction; Basis property of root functions

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1. Introduction

We consider the following eigenvalue problem

$$\ell(y)(x) \equiv y^{(4)}(x) - (q(x)y'(x))' = \lambda y(x), 0 < x < 1, \tag{1.1}$$

$$y''(0) = 0, \tag{1.2}$$

$$Ty(0) - a\lambda y(0) = 0, \tag{1.3}$$

$$y'(1)\cos\gamma + y''(1)\sin\gamma = 0, \tag{1.4}$$

$$Ty(1) - c\lambda y(1) = 0, \tag{1.5}$$

where $\lambda \in \mathbb{C}$ is a spectral parameter, $Ty \equiv y''' - qy'$, $q(x)$ is a positive absolutely continuous function on $[0, 1]$, a, c and γ are real constants such that $a < 0$, $c < 0$ and $\gamma \in [0, \pi/2]$.

The study of spectral problems for ordinary differential equations with boundary conditions depending on the spectral parameter has a long history which is well reflected in [**10**, **21**]. Problems of this type arise when solving various specific problems of mechanics, physics and mathematical physics. The eigenvalue problem (1.1)–(1.5) describes the bending vibrations of a homogeneous rod, in cross-sections of which the longitudinal force acts, on the left end of which a mass is concentrated. Moreover, the right end of the rod is fixed elastically by a spring that prevents it from turning (the case of $\gamma \in (0, \pi/2)$), and on this end a tracking force acts (see [**19**, pp. 152–154]).

The general theory of spectral problems for ordinary differential equations with polynomial occurrence of the spectral parameter in the equations and boundary conditions was constructed in [**38**] and [**39**]. In these papers, various classes of boundary value problems (normal, regular and strongly regular) were distinguished, and spaces $W_{2,U}^r \oplus \mathbb{C}^{N_r}$ were constructed in which these problems admit natural linearization. For strongly regular problems, in [**38**] the Riesz basis property (after normalization) of the system of eigenvectors and associated vectors of linearizing operators in the space $W_{2,U}^r \oplus \mathbb{C}^{N_r}$ was established, and in [**39**] a condition was found under which the system of eigen- and associated functions of the original problem form a defective Riesz basis (with a finite number of defects) in space W_2^r.

Oscillatory properties of eigenfunctions and basis properties in various functional spaces of root functions of Sturm-Liouville problems with a spectral parameter in the boundary conditions were investigated in [**1**, **3**, **4**, **11**, **15**–**18**, **21**, **24**, **26**–**28**, **31**, **34**, **36**–**38**, **40**]. These properties of the root functions of eigenvalue problems for ordinary differential equations of the fourth order, one of the boundary conditions of which depends on the spectral parameter, were studied in detail in [**2**, **12**, **13**, **22**, **23**, **29**, **30**, **32**, **33**, **39**]. In the case when two of the boundary conditions contain a spectral parameter, these problems were studied in [**5**, **6**, **9**, **10**, **32**], and when three of the boundary conditions contain a spectral parameter, they were studied in [**8**]. The problems studied in these works describe bending vibrations of a rod, the left end of which is either fixed or at this end a load is concentrated or a tracking force acts, and at the right end an inertial load is concentrated (the tracking force can also act at this end) (see. [**19**, Ch. 8, § 5]). In [**7**], the authors establish conditions under which the Fourier series expansions of continuous functions in the system of eigenfunctions of the problem converge uniformly.

The aim of this work is to study the position of the eigenvalues on the real axis, the structure of root subspaces, and the oscillatory properties of the eigenfunctions, and also to obtain asymptotic formulas for the eigenvalues and eigenfunctions of problem (1.1)–(1.5). Moreover, using these properties and the operator interpretation of this problem, we establish sufficient conditions for the subsystems of root functions to form a basis in the space L_p, $1 < p < \infty$. It should be noted that the results of this paper will allow us in the future to investigate the eigenvalue problem for the equation (1.1) with boundary conditions, three of which depend on the spectral parameter.

The structure of this paper is as follows. In § 2, we consider the initial-boundary value problem (1.1), (1.2), (1.4), (1.5), in contrast to [**14**] where initial-boundary value problem (1.1)–(1.4) is considered for $\gamma = \pi/2$. This is due to the fact that great difficulties arise in the study of the oscillatory properties of the solution to the problem (1.1)–(1.4) for $\lambda < 0$. Here we show the existence and uniqueness of a solution of problem (1.1), (1.2), (1.4), (1.5) for each $\lambda \in \mathbb{C}$, and investigate some properties of this solution, including its oscillatory properties depending on the parameter $\lambda \in \mathbb{R}$. In § 3, we study the location of the eigenvalues on the real axis, the structure of root subspaces and the oscillation properties of the eigenfunctions corresponding to both positive and negative eigenvalues of problem (1.1)–(1.5). In § 4, using the oscillatory properties of the eigenfunctions, we find asymptotic formulas for the eigenvalues and eigenfunctions of the considered problem. In § 5, problem (1.1)–(1.5) is reduced to the eigenvalue problem for a some nonself-adjoint operator in a Hilbert space $H = L_2(0, 1) \oplus \mathbb{C}^2$ with corresponding scalar product. This operator is J -self-adjoint in the Pontryagin space $\Pi_1 = L_2(0, 1) \oplus \mathbb{C}^2$ with the corresponding inner product, and the system of its root vectors forms an unconditional basis in H. We also find the system adjoint to the system of root vectors of this operator. Next, with the use of these results and oscillatory properties of eigenfunctions we establish sufficient conditions for the system of root functions of problem (1.1)–(1.5) to form a basis in the space $L_p(0, 1), 1 < p < \infty$ after removing two functions.

2. The existence and main properties of the solution of problem (1.1), (1.2), (1.4), (1.5)

We consider the boundary condition

$$y(0) \cos \beta + Ty(0) \sin \beta = 0, \beta \in [0, \pi/2]. \tag{2.1}$$

Alongside the problem (1.1)–(1.5) we also consider the eigenvalue problem (1.1), (1.2), (2.1), (1.4), (1.5). The spectral properties of this problem in a more general form of boundary conditions were investigated in [**29**, **30**].

It follows from [**30**, lemma 2.2 and theorem 2.2] that the following result holds for this problem.

Theorem 2.1. *For each β and each γ the eigenvalues of the boundary value problem (1.1), (1.2), (2.1), (1.4), (1.5) are real and simple, and form an unbounded increasing sequence $\{\lambda_k(\beta, \gamma)\}_{k=1}^{\infty}$ such that*

$$0 < \lambda_1(\beta, \gamma) < \lambda_2(\beta, \gamma) < \ldots < \lambda_k(\beta, \gamma) < \ldots \textit{for } \beta \in [0, \pi/2),$$

and

$$0 = \lambda_1(\pi/2, \gamma) < \lambda_2(\pi/2, \gamma) < \ldots < \lambda_k(\pi/2, \gamma) < \ldots .$$

Moreover, the eigenfunction $y_{k,\beta,\gamma}(x)$, $k \in \mathbb{N}$, corresponding to the eigenvalue $\lambda_k(\beta, \gamma)$ has $k - 1$ simple zeros in the interval $(0, 1)$.

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Let $\tilde{H} = L_2(0,1) \oplus \mathbb{C}$ be the Hilbert space with the scalar product

$$(\tilde{y}, \tilde{v}) = (\{y,n\},\{v,t\}) = \int_0^1 u(x)\,\overline{v(x)}\,\mathrm{d}x + |c|^{-1} n\bar{t}.$$

As is known [2] that problem (1.1), (1.2), (2.1), (1.4), (1.5) is equivalent the following eigenvalue problem

$$\tilde{L}\tilde{y} = \lambda\tilde{y}, \tilde{y} \in D(\tilde{L}),$$

where L is a self-adjoint bounded below operator in $\tilde{H}$ defined by

$$\tilde{L}\tilde{y} = L\{y,n\} = \{\ell y, Ty(0)\} \tag{2.2}$$

with the domain

$$D(\tilde{L}) = \Big\{\tilde{y} = \{y,n\} \in \tilde{H} : y \in W_2^4(0,1), \ell y \in L_2(0,1), y''(0) = 0,$$
$$y(0)\cos\beta + Ty(0)\sin\beta = y(1)\cos\gamma + y''(1)\sin\gamma = 0, n = cy(1)\Big\}.$$

It is known that the eigenvalues of problem (2.2) are given by the max-min principle [20]

$$\lambda_k(\beta,\gamma) = \max_{\tilde{V}^{(k-1)}} \quad \min_{\substack{\tilde{y}\in\mathfrak{L} \\ (\tilde{y},\tilde{V}^{(k-1)})=0}} R[\tilde{y}]$$

where $R[\tilde{y}]$ is the Rayleigh quotient

$$R[\tilde{y}] = \frac{(\tilde{L}\tilde{y},\tilde{y})}{(\tilde{y},\tilde{y})} = \frac{\int_0^1 \left(y''^2(x) + q(x)y'^2(x)\right)\mathrm{d}x + N[y]}{\int_0^1 y^2(x)\,\mathrm{d}x - cy^2(1)},$$
$$N[y] = y^2(0)\cot\beta + y'^2(1)\cot\gamma,$$

(we use the convention that if any of the parameters β or γ is zero, then the boundary value of y at 0 or y' at 1 is taken to be zero and the corresponding term in $N[y]$ does not appear), $\mathfrak{L}$ is the set of vectors $\tilde{y} = \{y,n\} \in \tilde{H}$ such that the function y satisfies the boundary conditions (1.2), (2.1), (1.4), $\tilde{V}^{(k-1)}$ is an arbitrary set of linearly independent vectors $\tilde{v}_j = \{v_j, t_j\}$, $1 \leqslant j \leqslant k-1$, such that the functions v_j, $1 \leqslant j \leqslant k-1$, satisfy the boundary conditions (1.2), (2.1), (1.4).

Using this max-min characterization, by following the argument in theorem 9 of [20, p. 419] for eigenvalues of (1.1), (1.2), (2.1), (1.4), (1.5) we have the following property.

LEMMA 2.2. *The eigenvalues of problem* (1.1), (1.2), (2.1), (1.4), (1.5) *are continuous, strictly decreasing functions of* β *and* γ *for* $\beta,\gamma \in [0,\pi/2]$.

By virtue of theorem 2.1 and lemma 2.2 for each $\gamma \in [0,\pi/2]$ we have

$$\lambda_1(\pi/2,\gamma) < \lambda_1(0,\gamma) < \lambda_2(\pi/2,\gamma) < \lambda_2(0,\gamma) < \ldots \tag{2.3}$$

THEOREM 2.3. *For each fixed* $\lambda \in \mathbb{C}$ *there exists a nontrivial solution* $y(x,\lambda)$ *of* (1.1), (1.2), (1.4), (1.5) *which is unique up to a constant coefficient.*

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Proof. Let $\psi_k(x,\lambda), k = 1, 2, 3, 4$, denote the solutions of equation (1.1) normalized for $x = 1$ by the Cauchy conditions

$$\psi_k^{(s-1)}(1,\lambda) = \delta_{ks}, s = 1, 2, 3, T\psi_k(1,\lambda) = \delta_{k4}, \tag{2.4}$$

where δ_{ks} is the Kronecker delta.

As in [**8**, **33**], we will seek the solution $y(x,\lambda)$ of (1.1), (1.2), (1.4), (1.5) in the form

$$y(x,\lambda) = \sum_{k=1}^{4} A_k\psi_k(x,\lambda), \tag{2.5}$$

where A_k, $k = 1,\ 2, 3, 4$, are some constants.

By (1.4), (1.5) and (2.4) it follows from (2.5) that $A_2 \cos\gamma + A_3 \sin\gamma = 0$, $A_4 - c\lambda A_1 = 0$. Consequently, for the function $y(x,\lambda)$ we have

$$y(x,\lambda) = \begin{cases} A_1\left\{\psi_1(x,\lambda) + c\lambda\psi_4(x,\lambda)\right\} + A_3\,\psi_3(x,\lambda) & \text{if } \gamma = 0, \\ A_1\left\{\psi_1(x,\lambda) + c\lambda\psi_4(x,\lambda)\right\} & \\ \quad +A_2\{\psi_2(x,\lambda) - \psi_3(x,\lambda)\,\cot\gamma\} & \text{if}\, \gamma \in (0, \pi/2]. \end{cases} \tag{2.6}$$

By (1.2) from (2.6) we get

$$\begin{aligned} &A_1\left(\psi_1''(0,\lambda) + c\lambda\psi_4''(0,\lambda)\right) + A_3\psi_3''(0,\lambda) = 0 \quad \text{if } \gamma = 0, \\ &A_1\left(\psi_1''(0,\lambda) + c\lambda\psi_4''(0,\lambda)\right) + A_2\left(\psi_2''(0,\lambda) - \psi_3''(0,\lambda)\cot\gamma\right) = 0 \quad \text{if } \gamma \in (0, \pi/2]. \end{aligned} \tag{2.7}$$

For brevity, we use the following notations:

$$\begin{aligned} C_1(\lambda) &= \psi_1''(0,\lambda) + c\lambda\psi_4''(0,\lambda), \\ C_2(\lambda) &= \begin{cases} \psi_3''(0,\lambda) & \text{if } \gamma = 0, \\ \psi_2''(0,\lambda) - \psi_3''\,(0,\lambda)\cot\gamma & \text{if } \gamma \in (0, \pi/2]. \end{cases} \end{aligned} \tag{2.8}$$

It can be seen from (2.7) that to complete the proof of theorem it suffices to show that for each $\lambda \in \mathbb{C}$ the relation

$$|\,C_1(\lambda)| + |\,C_2(\lambda)| > 0 \tag{2.9}$$

holds.

If $\lambda > 0$, then by the second part of [**14**, lemma 2.1] we get

$$\begin{aligned} &\psi_1(0,\lambda) > 0,\ \psi_1'(0,\lambda) < 0,\ \psi_1''(0,\lambda) > 0,\ T\psi_1(0,\lambda) < 0, \\ &\psi_2(0,\lambda) < 0,\ \psi_2'(0,\lambda) > 0,\ \psi_2''(0,\lambda) < 0,\ T\psi_2(0,\lambda) > 0, \\ &\psi_3(0,\lambda) > 0,\ \psi_3'(0,\lambda) < 0,\ \psi_3''(0,\lambda) > 0,\ T\psi_3(0,\lambda) < 0, \\ &\psi_4(0,\lambda) < 0,\ \psi_4'(0,\lambda) > 0,\ \psi_4''(0,\lambda) < 0,\ T\psi_4(0,\lambda) > 0. \end{aligned} \tag{2.10}$$

Indeed, by (1.1) and (2.4) for the function $\psi_1(x,\lambda)$ we have

$$\lim_{\substack{x\to 1\\ x<1}} (T\psi_1(x,\lambda))' = \lambda \lim_{\substack{x\to 1\\ x<1}} \psi_1(x,\lambda) = \lambda > 0,$$

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which implies that there exists $(T\psi_1)'(1,\lambda) = \lambda > 0$. Consequently, there exists $\psi_1^{(4)}(1,\lambda) = (T\psi_1)'(1,\lambda) + q(1)\psi_1''(1,\lambda) + q'(1)\psi_1'(1,\lambda) = (T\psi_1)'(1,\lambda) > 0$. Hence $T\psi_1(x,\lambda) < 0$ and $\psi_1'''(x,\lambda) < 0$ in a sufficiently small left punctured neighbourhood V_1^- of the point $x = 1$. Since $x = 1$ is a triple zero of the function $\psi_1'(x,\lambda)$ it follows that $\psi_1''(x,\lambda) > 0$, $\psi_1'(x,\lambda) < 0$ for $x \in V_1^-$. Moreover, $\psi_1(x,\lambda) > 0$ for $x \in V_1^-$. Then it follows from the second part of [**14**, lemma 2.1] that $\psi_1(0,\lambda) > 0$, $\psi_1'(0,\lambda) < 0$, $\psi_1''(0,\lambda) > 0$, $T\psi_1(0,\lambda) < 0$. The remaining relations in (2.10) for the functions $\psi_2(x,\lambda)$, $\psi_3(x,\lambda)$ and $\psi_4(x,\lambda)$ are proved similarly.

Let $\lambda > 0$. Then, in view of $c < 0$, by (2.8) we have $C_1(\lambda) = \psi_1''(0,\lambda) + c\lambda\psi_4''(0,\lambda) > 0$, and consequently, (2.9) holds.

Now let $\lambda \in \mathbb{C}\setminus(0,+\infty)$. If (2.8) fails for some such λ, then $C_1(\lambda) = C_2(\lambda) = 0$. Hence the functions $\psi_1(x,\lambda) + c\lambda\psi_4(x,\lambda)$ and $\psi_3(x,\lambda)$ for $\gamma = 0$, $\psi_2(x,\lambda) - \psi_3(x,\lambda)\cot\gamma$ for $\gamma \in (0,\pi/2]$ are solutions of problem (1.1), (1.2), (1.4), (1.5) for such λ. We consider the function $u(x,\lambda)$ which is defined as follows:

$$u(x,\lambda) = \begin{cases} \psi_3(0,\lambda)\left(\psi_1(x,\lambda) + c\lambda\psi_4(x,\lambda)\right) - \left(\psi_1(0,\lambda) + c\lambda\psi_4(0,\lambda)\right) \\ \psi_3(x,\lambda) \text{ for } \gamma = 0, \\ \left(\psi_2(0,\lambda) - \psi_3(0,\lambda)\cot\gamma\right)\left(\psi_1(x,\lambda) + c\lambda\psi_4(x,\lambda)\right) \\ \quad - \left(\psi_1(0,\lambda) + c\lambda\psi_4(0,\lambda)\right) \\ \left(\psi_2(x,\lambda) - \psi_3(x,\lambda)\cot\gamma\right) \text{ for } \gamma \in (0,\pi/2]. \end{cases}$$

Note that $u(0,\lambda) = 0$. Hence the function $u(x,\lambda)$ is an eigenfunction of the eigenvalue problem (1.1), (1.2), (2.1), (1.4), (1.5) for $\beta = 0$ and $\gamma \in [0,\pi/2]$. Then by theorem 2.1 we have $\lambda > 0$ which contradicts the condition $\lambda \in \mathbb{C}\setminus(0,+\infty)$. The proof of this theorem is complete.

Remark 2.4. By (2.6)–(2.8), for each $\lambda \in \mathbb{C}$ the nontrivial solutions $y(x,\lambda)$ of problem (1.1), (1.2), (1.4), (1.5) are nonzero multiples of

$$v(x,\lambda) = C_2(\lambda)\left\{\psi_1(x,\lambda) + c\lambda\psi_4(x,\lambda)\right\} - C_1(\lambda)\left\{\operatorname{sgn}\gamma\,\psi_2(x,\lambda) - (-1)^{1-\operatorname{sgn}\gamma}\left(1 + \operatorname{sgn}\gamma\left(\cot\gamma - 1\right)\right)\psi_3(x,\lambda)\right\}. \tag{2.11}$$

As is known (see [**35**, Ch. 1, § 2.1]) that for each fixed $x \in [0,1]$ the functions $\psi_k(x,\lambda)$, $k = 1,2,3,4$, and their derivatives are entire functions of λ, and consequently, $v(x,\lambda)$ is also an entire function of λ for each fixed $x \in [0,1]$.

Lemma 2.5. *Let $y(x,\lambda)$, $\lambda \in \mathbb{C}$, be nontrivial solutions of problem* (1.1), (1.2), (1.4), (1.5). *Then $y(1,\lambda) \neq 0$ for $\lambda > 0$ and $y(0,\lambda) \neq 0$ for $\lambda \leqslant 0$.*

Proof. If $y(1,\lambda) = 0$ for some $\lambda > 0$, then from (1.5) we get $Ty(1,\lambda) = 0$. Since $\gamma \in [0,\pi/2]$ it follows from (1.4) that $y'(1,\lambda)y''(1,\lambda) \leqslant 0$. Then by the second part of [**14**, lemma 2.1] we have $y'(0,\lambda)\,y''(0,\lambda) < 0$ which contradicts the condition (1.2).

If $y(0,\lambda) = 0$ for some $\lambda \leqslant 0$, then $y(x,\lambda)$ is an eigenfunction of problem (1.1), (1.2), (2.1), (1.4), (1.5) for $\beta = 0$ and $\gamma \in [0,\pi/2]$. Then by theorem 2.1 we have $\lambda > 0$ which contradicts the condition $\lambda \leqslant 0$. The proof of this lemma is complete.

Now, using lemma 2.5, we can normalize the function $y(x,\lambda), x\in[0,1], \lambda\in\mathbb{R}$, as follows:

$$y(1,\lambda)=1 \tag{2.12}$$

if $\lambda>0$, and

$$y(0,\lambda)=1 \tag{2.13}$$

if $\lambda\leqslant 0$.

For $\lambda\in\mathbb{R}$ we consider the following equation

$$y(x,\lambda)=0,\ x\in[\,0,1].$$

It is obvious that the zeros of this equation are functions of the parameter λ.

LEMMA 2.6. *The zeros of function $y(x,\lambda)$ contained in the half-open interval $[0,1)$ are simple and continuously differentiable functions of λ, $\lambda\in\mathbb{R}$.*

Proof. Let λ_0 be an arbitrary fixed positive number. If $y(x_0,\lambda_0)=0$ for $x_0\in(0,1)$, then it follows from [**14**, lemma 2.2] that $y'(x_0,\lambda_0)\neq 0$. If $y(0,\lambda_0)=y'(0,\lambda_0)=0$, then in view of (1.2), by the first part of [**14**, lemma 2.1] we have $y'(1,\lambda_0)y''(1,\lambda_0)>0$ in contradiction with the boundary condition (1.4).

Let $\lambda_0\leqslant 0$ and $x_0\in[0,1)$ such that $y(x_0,\lambda_0)=y'(x_0,\lambda_0)=0$. Then $y(x,\lambda_0)$ solves the eigenvalue problem defined on $[x_0,1]$ and determined by equation (1.1) with the boundary conditions $y(x_0)=y'(x_0)=0$ and (1.4), (1.5). By theorem 2.1 the eigenvalues of this problem are simple and positive which contradicts the condition $\lambda_0\leqslant 0$.

The continuous differentiability of the zeros contained in $[0,1)$ of the function $y(x,\lambda)$ follows from the well-known implicit function theorem, and the proof of this lemma is complete.

By lemma 2.5, lemma 2.6 implies the following statement.

COROLLARY 2.7. *As $\lambda>0$ $(\lambda\leqslant 0)$ varies the function $y(x,\lambda)$ can lose or gain zeros only by these zeros leaving or entering the interval $[0,1]$ through its endpoint $x=0$ $(x=1)$.*

We consider the function

$$H(x,\lambda)=\frac{y(x,\lambda)}{Ty(x,\lambda)}\,.$$

By theorem 2.3, remark 2.4 and lemma 2.6 the function $H(x,\lambda)$ is a finite order meromorphic function of λ for each fixed $x\in[\,0,1]$.

Let $\mathcal{D}_k=(\lambda_{k-1}(0,\gamma),\lambda_k(0,\gamma))$, $k\in\mathbb{N}$, where $\lambda_0(0,\gamma)=-\infty$.

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Obviously, the function

$$F(\lambda) = \frac{1}{H(0,\lambda)} = \frac{Ty(0,\lambda)}{y(0,\lambda)}$$

which is well defined for

$$\lambda \in \mathcal{D} \equiv (\mathbb{C}\backslash\mathbb{R}) \cup \left(\bigcup_{k=1}^{\infty} \mathcal{D}_k\right)$$

and is a meromorphic function of finite order. The eigenvalues $\lambda_k(0,\gamma)$ and $\lambda_k(\pi/2,\gamma), k = 1, 2, \dots,$ of problem (1.1), (1.2), (2.1), (1.4), (1.5) for $\beta = 0$ and $\beta = \pi/2$ are poles and zeros of function $F(\lambda)$, respectively.

LEMMA 2.8. *For each $\lambda \in \mathcal{D}$ the relation*

$$\frac{dF(\lambda)}{d\lambda} = -\frac{1}{y^2(0,\lambda)}\left\{\int_0^1 y^2(x,\lambda)\,dx - cy^2(1,\lambda)\right\} \tag{2.14}$$

holds.

Proof. By virtue of equation (1.1) we have

$$(Ty(x,\mu))'\,y(x,\lambda) - (Ty(x,\lambda))'\,y(x,\mu) = (\mu-\lambda)y(x,\mu)y(x,\lambda). \tag{2.15}$$

Integrating equality (2.15) from 0 to 1, using the formula for the integration by parts and taking boundary conditions (1.2), (1.4) and (1.5) into account we obtain

$$\begin{aligned} &-Ty(0,\mu)\,y(0,\lambda) + Ty(0,\lambda)\,y(0,\mu) \\ &\quad = (\mu-\lambda)\left\{\int_0^1 y(x,\mu)\,y(x,\lambda)\mathrm{d}x - cy(1,\mu)\,y(1,\lambda)\right\}. \end{aligned} \tag{2.16}$$

By (2.16) for $\mu, \lambda \in \mathcal{D}, \mu \neq \lambda$, we have

$$\begin{aligned} &\frac{Ty(0,\mu)}{y(0,\mu)} - \frac{Ty(0,\lambda)}{y(0,\lambda)} \\ &\quad = -(\mu-\lambda)\frac{1}{y(0,\mu)\,y(0,\lambda)}\left\{\int_0^1 y(x,\mu)y(x,\lambda)\mathrm{d}x - cy(1,\mu)y(1,\lambda)\right\}. \end{aligned} \tag{2.17}$$

Dividing both sides of relation (2.17) by $\mu-\lambda$ $(\mu \neq \lambda)$ and by passing to the limit as $\mu \to \lambda$ we get (2.14). The proof of this lemma is complete.

COROLLARY 2.9. *The function $F(\lambda)$ strictly decreases on each of intervals $\mathcal{D}_k,\ k = 1, 2, \dots$.*

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Lemma 2.10. *The following relation holds:*

$$\lim_{\lambda \to -\infty} F(\lambda) = +\infty. \tag{2.18}$$

Proof. In equation (1.1) we set $\lambda = \rho^4$. By theorem 1 of [**35**, Ch. II, § 4.5] in each subdomain $\mathcal{T}$ of the complex ρ-plane this equation has four linearly independent solutions $\varphi_k(x, \rho), k = 1, 2, 3, 4$, which are regular with respect to ρ (for sufficiently large $|\rho|$) and satisfying the following relations

$$\varphi_k^{(s)}(x, \rho) = (\rho\omega_k)^s e^{\rho\omega_k x} \left\{1 + O\left(\rho^{-1}\right)\right\}, \quad k = 1, 2, 3, 4, \ \ s = 0, 1, 2, 3, \tag{2.19}$$

where $\omega_k, k = 1, 2, 3, 4$, are distinct fourth roots of unity.

Let $\lambda < 0$. Then, without loss of generality, we can assume that ρ lies on the bisector of the first quadrant, and the numbers $\omega_k, k = 1, 2, 3, 4$, are numbered in the following order: $\omega_1 = -1$, $\omega_2 = i$, $\omega_3 = -i$ and $\omega_4 = 1$.

For brevity, we introduce the notation

$$[1] = 1 + O(\rho^{-1}).$$

Assuming that the initial condition $y(0, \lambda) = 1$ is imposed, the unique solution of (1.1), (1.2), (1.4), (1.5) together with the initial condition $y(0, \lambda) = 1$ can be written in the form

$$y(x, \lambda) = \sum_{k=1}^{4} B_k(\rho)\, \varphi_k(x, \rho).$$

Writing $B = (B_1, B_2, B_3, B_4)^T$, the coefficients $C_k(\rho)$ are solution of the linear algebraic system

$$M(\rho)\, B(\rho) = (0, 0, 0, 1)^T,$$

where the matrix $M(\rho)$ is given by

$$M(\rho) = \begin{pmatrix} [1] & -[1] & -[1] & [1] \\ -e^{-\rho}[1] & ie^{i\rho}[1] & -ie^{-i\rho}[1] & e^{\rho}[1] \\ e^{-\rho}[1] & e^{i\rho}[1] & e^{-i\rho}[1] & e^{\rho}[1] \\ [1] & [1] & [1] & [1] \end{pmatrix}$$

for $\gamma = 0$, and

$$M(\rho) = \begin{pmatrix} [1] & -[1] & -[1] & [1] \\ e^{-\rho}[1] & -e^{i\rho}[1] & -e^{-i\rho}[1] & e^{\rho}[1] \\ e^{-\rho}[1] & e^{i\rho}[1] & e^{-i\rho}[1] & e^{\rho}[1] \\ [1] & [1] & [1] & [1] \end{pmatrix}$$

for $\gamma \in (0, \pi/2]$.

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The solution of the system $M(\rho)\,B(\rho) = (0,0,0,1)^T$ is

$$B_1(\rho) = \frac{1}{2}[1], \quad B_2(\rho) = -\frac{(1+i)\,e^{-\,i\rho}}{2((1-i)e^{i\rho} - (1+i)e^{-i\rho})}[1],$$
$$B_3(\rho) = \frac{(1-i)\,e^{i\rho}}{2((1-i)e^{i\rho} - (1+i)e^{-i\rho})}[1], \quad B_4(\rho) = -\frac{2ie^{-\,\rho}}{2((1-i)e^{i\rho} - (1+i)e^{-i\rho})}[1],$$

if $\gamma = 0$, and

$$B_1(\rho) = \frac{1}{2}[1], \quad B_2(\rho) = \frac{-\,e^{-\,i\rho}}{2(e^{i\rho} - e^{-\,i\rho})}[1],$$
$$B_3(\rho) = \frac{e^{i\rho}}{2(e^{i\rho} - e^{-\,i\rho})}[1], B_4(\rho) = \frac{1}{2}\,e^{-2\,\rho}[1].$$

if $\gamma \in (0, \pi/2]$. Then for $F(\lambda) = Ty(0,\lambda)/\,y(0,\lambda)$ we get the following representation

$$F(\lambda) = \{((1-i)e^{i\rho} - (1+i)e^{-i\rho})[1] - (1+i)\,e^{-i\rho}[1] + (1-i)\,e^{i\rho}[1] + 2ie^{-\rho}[1]\}^{-1}$$
$$\rho^3\{-((1-i)e^{i\rho} - (1+i)e^{-i\rho})[1] - (1-i)\,e^{-i\rho}[1] + (1+i)\,e^{i\rho}[1] + 2ie^{-\rho}[1]\} \tag{2.20}$$

if $\gamma = 0$, and

$$F(\lambda) = \rho^3\{(e^{i\rho} - e^{-i\rho})[1] - \,e^{-i\rho}[1] + \,e^{i\rho}[1] - e^{-2\rho}(e^{i\rho} - e^{-i\rho})[1]\}^{-1}$$
$$\{-(e^{i\rho} - e^{-i\rho})[1] + i\,e^{-i\rho}[1] + i\,e^{i\rho}[1] - e^{-2\rho}(e^{i\rho} - e^{-i\rho})[1]\} \tag{2.21}$$

if $\gamma \in (0, \pi/2]$.

Since ρ lies on the bisector of the first quadrant it follows that $\rho = (1+i)u$, where $u > 0$, and consequently, $|\rho| = \sqrt{2}\,u$. Then, from (2.20) and (2.21) by a straightforward computation, we obtain

$$F(\lambda) = -(1-i)^{-1}\rho^3[1] = -(1-i)^{-1}(1+i)^3u^3[1] = (\sqrt{2})^{-1}|\rho|^3(1 + O(|\rho|^{-1})$$
$$= (\sqrt{2})^{-1}\,\sqrt[4]{|\lambda|^3}\left(1 + O\left(\left(\sqrt[4]{|\lambda|}\right)^{-1}\right)\right) \text{ as } \lambda \to -\infty. \tag{2.22}$$

The proof of this lemma is complete.

Lemma 2.11. *Let $x \in [0,1)$ and $\lambda > 0$ such that $y(x,\lambda) = 0$. Then*

$$\frac{\partial H(x,\lambda)}{\partial x} < 0. \tag{2.23}$$

Proof. Let $y(x,\lambda) = 0$ for some $x \in [0,1)$ and $\lambda > 0$. If $x \in (0,1)$, then it follows from [**14**, lemma 2.2] that $y'(x,\lambda)\,Ty(x,\lambda) < 0$. If $y(0,\lambda) = 0$, then in view of (1.2),

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by the first part of [**14**, lemma 2.1] we have $y'(0,\lambda)Ty(0,\lambda)<0$. Therefore, by virtue of (1.1), we get

$$\frac{\partial H(x,\lambda)}{\partial x}=\frac{\partial}{\partial x}\left(\frac{y(x,\lambda)}{Ty(x,\lambda)}\right)=\frac{y'(x,\lambda)\,Ty(x,\lambda)-\lambda y^2(x,\lambda)}{(Ty(x,\lambda))^2}=\frac{y'(x,\lambda)}{Ty(x,\lambda)}<0.$$

The proof of lemma 2.11 is complete.

By $\tau(\lambda)$ we denote the number of zeros of the function $y(x,\lambda)$ contained in $(0,1)$.

Lemma 2.12. *Let $0<\mu<\nu$. Then $\tau(\mu)\leqslant\tau(\nu)$.*

Proof. By corollary 2.7 as $\lambda>0$ varies the zeros of the function $y(x,\lambda)$ can enter or leave the interval $(0,1)$ only through the endpoint $x=0$. Moreover, by lemma 2.6 and the implicit function theorem for every zero $x(\lambda)$ of the function $y(x,\lambda)$ the following relation holds:

$$x'(\lambda)=-\frac{H'_\lambda(x,\lambda)}{H'_x(x,\lambda)}.$$

If $x(\lambda)=0$, then it follows from this and relations (2.14), (2.23) that

$$x'(\lambda)>0.$$

Therefore, as $\lambda>0$ increases the zeros of the function $y(x,\lambda)$ cannot leave the interval $(0,1)$ through the point $x=0$. Hence, as λ, $\mu<\lambda<\nu$, increases the number of zeros of the function $y(x,\lambda)$ cannot decrease, i.e. $\tau(\mu)\leqslant\tau(\nu)$. The proof of this lemma is complete.

Theorem 2.13. *If $\lambda\in[0,+\infty)\cap(\lambda_{k-1}(0,\gamma),\lambda_{k-1}(0,\gamma)$, then $\tau(\lambda)=k-1$.*

Proof. It is obvious that $\psi_1(x,0)\equiv1$. Then by (2.8), (2.13) it follows from (2.11) that $y(x,0)\equiv1$. Hence, for all $\lambda\in\mathbb{R}$ sufficiently close to zero, the function $y(x,\lambda)$ has no zeros in $(0,1)$. Moreover, by theorem 2.1 we have $\tau(\lambda_k(0,\gamma)=k-1$, $k=1,2,\ldots$. Therefore, by lemma 2.12 it follows that if $\lambda>0$ and $\lambda\in(\lambda_{k-1}(\gamma,0),\lambda_{k-1}(\gamma,0)$, then $\tau(\lambda)=k-1$. The proof of theorem 2.13 is complete.

It follows from lemma 2.6 that as $\lambda<0$ varies then the zeros of the function $y(x,\lambda)$ can enter or leave the interval $(0,1)$ only through the endpoint $x=1$. To find the number of zeros contained in the interval $(0,1)$ of the function $y(x,\lambda)$ for $\lambda<0$ consider the following spectral problem

$$\begin{aligned}&\ell(y)(x)=\lambda y(x),0<x<1,\\&\quad y''(0)=y'(1)\cos\gamma+y''(1)\sin\gamma=y(1)=Ty(1)=0.\end{aligned}\tag{2.24}$$

It follows from the second part of [**14**, lemma 2.1] that the eigenvalues of problem (2.24) cannot be positive. Let η be a real eigenvalue of this problem and $\epsilon>0$ be the fixed sufficiently small number. The oscillation index of the eigenvalue η which denotes by $i(\eta)$ is the difference between the number of zeros of the function $y(x,\lambda)$ for $\lambda\in(\eta-\epsilon,\eta)$ contained in the interval $(0,1)$ and the number of the same zeros for $\lambda\in(\eta,\eta+\epsilon)$. This definition directly implies that the number of zeros of the function $y(x,\lambda)$ for $\lambda<0$ contained in the interval $(0,1)$ is equal to the sum of the oscillation indices of all eigenvalues of problem (2.24) contained in the interval $(\lambda, 0)$ (see [**13**, § 4]).

By following the arguments in theorem 4.1 of [**13**] one can justify the following statement.

THEOREM 2.14. *There exists $\xi < 0$ such that the eigenvalues η_k, $k = 1, 2, \ldots,$ of problem (2.24) are simple, lying on the interval $(-\infty, \xi)$, form an unbounded decreasing sequence $\{\eta_k\}_{k=1}^{\infty}$ such that $i(\eta_k) = 1$, $k \in \mathbb{N}$, and*

$$\eta_k = -4\pi^4 k^4 + o(k^4).$$

Now, based on the above reasoning, we obtain the following formula for the number of zeros contained in $(0, 1)$ of the function $y(x, \lambda)$ for $\lambda < 0$:

$$\tau(\lambda) = \sum_{\eta_k \in (\lambda, 0)} i(\eta_k). \tag{2.25}$$

3. The location of eigenvalues and the oscillatory properties of eigenfunctions of problem (1.1)–(1.5)

LEMMA 3.1. *The eigenvalues of the boundary value problem (1.1)–(1.5) are real and form an at most countable set without finite limit point.*

Proof. Note that the eigenvalues of problem (1.1)–(1.5) are the roots of the equation

$$Ty(0, \lambda) - a\lambda\, y(0, \lambda) = 0. \tag{3.1}$$

Let λ be the nonreal eigenvalue of this problem. Then $\bar{\lambda}$ is also eigenvalue of (1.1)–(1.5) because the coefficients $q(x)$, a, c and γ are real. Moreover, in this case $y(x, \bar{\lambda}) = \overline{y(x, \lambda)}$, and consequently, equality (3.1) holds for $\bar{\lambda}$.

Setting $\mu = \bar{\lambda}$ in relation (2.16) and taking (1.4) into account we get

$$-a\,(\overline{\lambda} - \lambda)\,|y(0, \lambda)|^2 = (\overline{\lambda} - \lambda)\left\{\int_0^1 |y(x, \lambda)|^2 \mathrm{d}x - c|y(1, \lambda)|^2\right\}, \tag{3.2}$$

whence, by $\overline{\lambda} \neq \lambda$, implies that

$$\int_0^1 |y(x, \lambda)|^2 \mathrm{d}x + a\,|y(0, \lambda)|^2 - c|y(1, \lambda)|^2 = 0. \tag{3.3}$$

Putting $y(x, \lambda)$ in (1.1)–(1.5), then multiplying both sides of (1.1) by $\overline{y(x, \lambda)}$, integrating this relation from 0 to 1, using the formula for the integration by parts,

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and taking into account conditions (1.2)–(1.5), we obtain

$$\int_0^1 |y''(x,\lambda)|^2 \mathrm{d}x + \int_0^1 q(x)\,|y'(x,\lambda)|^2 \mathrm{d}x + \mathcal{N}[y(x,\lambda)]$$
$$= \lambda \left\{ \int_0^1 |y(x,\lambda)|^2 \mathrm{d}x + a\,|y(0,\lambda)|^2 - c|y(1,\lambda)|^2 \right\}, \tag{3.4}$$

where $\mathcal{N}[y(x,\lambda)] = 0$ for $\gamma = 0$, $\mathcal{N}[y(x,\lambda)] = |y'(1,\lambda)|^2 \cot\gamma$ for $\gamma \in (0, \pi/2]$. Hence it follows from (3.3) and (3.4) that

$$\textstyle\int_0^1 |y''(x,\lambda)|^2 \mathrm{d}x + \int_0^1 q(x)\,|y'(x,\lambda)|^2 \mathrm{d}x + \mathcal{N}[y(x,\lambda)] = 0. \tag{3.5}$$

By the boundary condition (1.3), relation (3.5) implies that $y(x,\lambda) \equiv 0$, a contradiction.

By the above arguments the entire function on the left-hand side of (3.1) does not vanish for non-real λ. Consequently, it does not vanish identically. Therefore, its zeros form an at most countable set without finite limit point. The proof of this lemma is complete.

Remark 3.2. If λ is an eigenvalue of (1.1)–(1.5), then $y(0,\lambda) \neq 0$. Indeed, if $y(0,\lambda) = 0$, then it follows from (1.2) that $Ty(0,\lambda) = 0$. Consequently, λ is an eigenvalue of problem (1.1), (1.2), (2.1), (1.4), (1.5) for $\beta = 0$ and $\beta = \pi/2$ in contradiction with relation (2.3).

Lemma 3.3. *The nonzero eigenvalues of problem* (1.1)–(1.5) *are simple.*

Proof. Let λ be an eigenvalue of (1.1)–(1.5). Then by remark 3.2 we have $y(0,\lambda) \neq 0$. Therefore each root (with regard of multiplicities) of equation (3.1) is a root of the equation

$$F(\lambda) = a\lambda. \tag{3.6}$$

Let $\lambda = \lambda^*$ be a multiple root of (3.6). Then the following relations hold:

$$F(\lambda^*) = a\lambda^*, F'(\lambda^*) = a. \tag{3.7}$$

By remark 3.2 and formula (2.14), the second relation of (3.7) implies that

$$\int_0^1 y^2(x,\lambda^*)\,\mathrm{d}x + a\,y^2(0,\lambda^*) - c\,y^2(1,\lambda^*) = 0. \tag{3.8}$$

Since $\lambda^* \in \mathbb{R}$ it follows from (3.4) that

$$\int_0^1 y''^2(x,\lambda^*)\mathrm{d}x + \int_0^1 q(x)\,y'^2(x,\lambda^*)\mathrm{d}x + \mathcal{N}[y(x,\lambda^*)]$$
$$= \lambda \left\{ \int_0^1 y^2(x,\lambda^*)\mathrm{d}x + a\,y^2(0,\lambda^*) - c\,y^2(1,\lambda^*) \right\}, \tag{3.9}$$

whence, by virtue of (3.8), we get

$$\int_0^1 y''^2(x,\lambda^*)\mathrm{d}x + \int_0^1 q(x)\,y'^2(x,\lambda^*)\mathrm{d}x + \mathcal{N}[y(x,\lambda^*)] = 0. \tag{3.10}$$

Consequently, by condition (1.3), from (3.10) we obtain $y(x,\lambda^*) \equiv 0$ which contradicts the condition $y(x,\lambda^*) \not\equiv 0$. The proof of this lemma is complete.

Following the reasoning in [**6**, lemma 3.3], we can justify the following result.

Lemma 3.4. *One has the following representation:*

$$F(\lambda) = \sum_{k=1}^{\infty} \frac{\lambda c_k}{\lambda_k(0,\gamma)\,(\lambda - \lambda_k(0,\gamma))}\,, \tag{3.11}$$

where $c_k = \underset{\lambda=\lambda_k(0,\gamma)}{res} F(\lambda)$ *and* $c_k > 0$, $k = 1, 2, \ldots.$

We have the following oscillation theorem for problem (1.1)–(1.5).

Theorem 3.5. *For each* $\gamma \in [0, \pi/2]$ *the eigenvalues of problem* (1.1)–(1.5) *form an unbounded nondecreasing sequence* $\{\lambda_k(\gamma)\}_{k=1}^{\infty}$ *such that*

$$\lambda_1(\gamma) < 0 = \lambda_2(\gamma) < \lambda_3(\gamma) < \ldots < \lambda_k(\gamma) < \ldots \ \textit{if} \ a > c-1,$$
$$\lambda_1(\gamma) = \lambda_2(\gamma) = 0 < \lambda_3(\gamma) < \ldots < \lambda_k(\gamma) < \ldots \ \textit{if} \ a = c-1,$$
$$\lambda_1(\gamma) = 0 < \lambda_2(\gamma) < \lambda_3(\gamma) < \ldots < \lambda_k(\gamma) < \ldots \ \textit{if} \ a < c-1,$$

(*in the case* $c = a+1$ *the eigenvalue* $\lambda_1(\gamma) = 0$ *is double, and it corresponds to the chain consisting of the eigenfunction* $y_{1,\gamma}(x)$ *and the associated function* $y_{2,\gamma}(x)$). *The eigenfunction* $y_k(x)$*, corresponding to the eigenvalue* λ_k*, for* $k \geqslant 3$ *has exactly* $k-2$ *simple zeros in* $(0,1)$*; moreover, if* $a < c-1$*, then the eigenfunctions* $y_{1,\gamma}(x)$ *and* $y_{2,\gamma}(x)$ *have no zeros in* $(0,1)$*, if* $a = c-1$*, then the eigenfunction* $y_{1,\gamma}(x)$ *has no zeros in* $(0,1)$*, if* $a > c-1$*, then* $y_{2,\gamma}(x)$ *has no zeros in* $(0,1)$ *and the number of zeros of the eigenfunction* $y_{1,\gamma}(x)$ *in* $(0,1)$ *is equal* $\sum_{\eta_k \in (\lambda_1(\gamma),0)} i(\eta_k)$.

Proof. Recall that the eigenvalues of problem (1.1)–(1.5), taking into account their multiplicities, are the roots of equation (3.6). It follows from (3.11) that

$$F''(\lambda) = 2\sum_{k=1}^{\infty} \frac{c_k}{(\lambda - \lambda_k(0,\gamma))^3}, \lambda \in \mathcal{D}.$$

From this we obtain the relation

$$F''(\lambda) < 0 \text{ for } \lambda \in \mathcal{D}_1, \tag{3.12}$$

i.e. the function $F(\lambda)$ is concave in $\mathcal{D}_1$. Moreover, by (2.14), (2.18) and (3.11) we have

$$F(0) = 0, F'(0) = c-1, \tag{3.13}$$

$$\lim_{\lambda\to-\infty} F(\lambda) = +\infty, \lim_{\lambda\to\lambda_1(0,\gamma)-0} F(\lambda) = -\infty. \tag{3.14}$$

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Let $f(\lambda) = F(\lambda) - a\lambda$. Then it follows from relations (3.12)–(3.14) that

$$f''(\lambda) < 0 \text{ for } \lambda \in \mathcal{D}_1,$$
$$f(0) = 0 \text{ and } f'(0) = c - 1 - a.$$

Moreover, by (2.22) and (3.14) we have

$$\lim_{\lambda \to -\infty} f(\lambda) = -\infty, \lim_{\lambda \to \lambda_1(0,\gamma)-0} f(\lambda) = -\infty. \tag{3.15}$$

If $a > c - 1$, then $f'(0) < 0$. Since $f''(\lambda) < 0$ in $\mathcal{D}_1$ it follows that $f'(\lambda) < f'(0) < 0$ for $\lambda \in (0, \lambda_1(0,\gamma))$. Then $f(\lambda) < 0$ for $\lambda \in (0, \lambda_1(0,\gamma))$. By the relations $f(0) = 0$ and $f'(0) < 0$ we have $f(\lambda) > 0$ for all $0 > \lambda$ small. Then it follows from the first relation of (3.15) that there exists $\lambda_*(\gamma) \in (-\infty, 0)$ such that $f(\lambda_*(\gamma)) = 0$. Hence there exists $\lambda_{**}(\gamma) \in (\lambda_*(\gamma), 0)$ such that $f'(\lambda_{**}(\gamma)) = 0$. Consequently, $f'(\lambda) > 0$ for $\lambda \in (-\infty, \lambda_{**}(\gamma))$ and $f'(\lambda) < 0$ for $\lambda \in (\lambda_{**}(\gamma), \lambda_1(0,\gamma))$. Therefore, in this case equation (3.6) in the interval $\mathcal{D}_1$ has two simple roots $\lambda_1(\gamma) < \lambda_2(\gamma)$, where $\lambda_1(\gamma) = \lambda_*(\gamma) < 0$ and $\lambda_2(\gamma) = 0$.

Let $a = c - 1$. Then we have $f'(0) = 0$. Since $f''(\lambda) < 0$ in $\mathcal{D}_1$ it follows that $f'(\lambda) > 0$ for $\lambda \in (-\infty, 0)$ and $f'(\lambda) < 0$ for $\lambda \in (0, \lambda_1(0,\gamma))$. Hence by relations (3.15) $f(\lambda) < 0$ for $\lambda \in (-\infty, 0) \cup (0, \lambda_1(0,\gamma))$. Therefore, in this case $f(0) = f'(0) = 0$, $f''(0) < 0$ and $f(\lambda) \neq 0$ for $\lambda \in D_1 \backslash \{0\}$, i.e. equation (3.6) has one double root $\lambda_1(\gamma) = \lambda_2(\gamma) = 0$ for $a = c - 1$.

If $a < c - 1$, then $f'(0) > 0$. Hence $f'(\lambda) > f'(0) > 0$ for $\lambda \in (-\infty, 0)$. Then $f(\lambda) < 0$ for $\lambda < 0$. Since $f(0) = 0$ and $f'(0) > 0$ it follows that $f(\lambda) > 0$ for all $0 < \lambda$ small. Then by virtue of second relation of (3.15) there exists $\lambda^*(\gamma) \in (0, \lambda_1(0,\gamma)$ such that $f(\lambda^*(\gamma)) = 0$. Hence there exists $\lambda^{**}(\gamma) \in (0, \lambda^*(\gamma))$ such that $f'(\lambda^{**}(\gamma)) = 0$. Then $f'(\lambda) > 0$ for $\lambda \in (-\infty, \lambda^{**}(\gamma))$ and $f'(\lambda) < 0$ for $\lambda \in (\lambda^{**}(\gamma), \lambda_1(0,\gamma))$. Thus, in this case equation (3.6) in the interval $\mathcal{D}_1$ has two simple root $\lambda_1(\gamma) < \lambda_2(\gamma)$, where $\lambda_1(\gamma) = 0$ and $\lambda_2(\gamma) = \lambda^*(\gamma) > 0$.

By theorem 2.13 and formula (2.25) it follows from the above reasoning that $\tau(\lambda_1(\gamma)) = \sum_{\eta_k \in (\lambda_1(\gamma),0)} i(\eta_k)$ and $\tau(\lambda_2(\gamma)) = 0$ for $a > c - 1$, $\tau(\lambda_1(\gamma)) = 0$ for $a = c - 1$, and $\tau(\lambda_1(\gamma)) = \tau(\lambda_2(\gamma)) = 0$ for $a < c - 1$.

From representation (3.11) we obtain the following relations

$$\lim_{\lambda \to \lambda_{k-1}(0,\gamma)+0} F(\lambda) = +\infty, \lim_{\lambda \to \lambda_k(0,\gamma)-0} F(\lambda) = -\infty, k = 2, 3, \ldots. \tag{3.16}$$

If equation (3.6) has a root in the interval $\mathcal{D}_k$ for $k \geqslant 2$, then by lemma 3.3 this root must be simple. Since $F(\lambda)$ is continuous in each of intervals $\mathcal{D}_k$, $k \in \mathbb{N}$, by (3.16) it follows that equation (3.6) has at least one root in each of intervals $\mathcal{D}_k$, $k \geqslant 2$. Let us show that this equation has only one simple root in $\mathcal{D}_k$ for $k \geqslant 2$. Indeed, if (3.6) has more than one root, then the two smallest roots $\lambda^*_{k1} < \lambda^*_{k2}$ satisfy

$$F'(\lambda^*_{k1}) - a < 0 \text{ and } F'(\lambda^*_{k2}) - a > 0. \tag{3.17}$$

On the other hand it follows from (3.4) that

$$\int_0^1 y''^2(x,\lambda_{k2}^*)\mathrm{d}x + \int_0^1 q(x)\, y'^2(x,\lambda_{k2}^*)\mathrm{d}x + \mathcal{N}[y(x,\lambda_{k2}^*)]$$
$$= \lambda_{k2}^* \left\{ \int_0^1 y^2(x,\lambda_{k2}^*)\mathrm{d}x + a\, y^2(0,\lambda_{k2}^*) - c\, y^2(1,\lambda_{k2}^*) \right\},$$

whence, by $\lambda_{k2}^* > 0$, we get

$$\int_0^1 y^2(x,\lambda_{k2}^*)\mathrm{d}x + a\, y^2(0,\lambda_{k2}^*) - c\, y^2(1,\lambda_{k2}^*) > 0.$$

By (2.14) we obtain from the last relation

$$F'(\lambda_{k2}^*) - a < 0,$$

which contradicts the second relation of (3.17). Therefore, problem (1.1)–(1.5) in each interval $\mathcal{D}_k$, $k \geqslant 2$, has a unique simple eigenvalue $\lambda_{k+1}(\gamma)$. Then it follows from theorem 2.13 that $\tau(\lambda_{k+1}(\gamma)) = k-1$. The proof of theorem 3.5 is complete.

4. Asymptotic behaviour of eigenvalues and eigenfunctions of problem (1.1)–(1.5)

Let

$$\nu_\gamma = (3 + \mathrm{sgn}\gamma)/4, \tilde{\nu}_\gamma = \nu_\gamma + 1,\ \gamma \in [0, \pi/2].$$

By [**30**, theorem 3.1] we have the following asymptotic formulas

$$\sqrt[4]{\lambda_k(0,\gamma)} = (k - \nu_\gamma)\,\pi + O\left(k^{-1}\right), \tag{4.1}$$

$$y_{k,0,\gamma}(x) = \sin\left(k - \nu_\gamma\right)\pi x - (1 - \mathrm{sgn}\gamma)(-1)^k(\sqrt{2}\,)^{-1} e^{(k-\nu_\gamma)\,\pi\,(x-1)} + O\left(k^{-1}\right), \tag{4.2}$$

where relation (4.2) holds uniformly for $x \in [0,1]$.

THEOREM 4.1. *The following asymptotic formulas hold:*

$$\sqrt[4]{\lambda_k(\gamma)} = (k - \tilde{\nu}_\gamma)\,\pi + O\left(k^{-1}\right), \tag{4.3}$$

$$y_{k,\gamma}(x) = \sin\left(k - \tilde{\nu}_\gamma\right)\pi x + (1 - sgn\gamma)(-1)^k \left(\sqrt{2}\right)^{-1} e^{(k-\tilde{\nu}_\gamma)\pi(x-1)} + O\left(k^{-1}\right), \tag{4.4}$$

where relation (4.4) holds uniformly for $x \in [0,1]$.

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Proof. Taking (2.19) into account in the boundary conditions (1.2)–(1.4) we get

$$\sqrt[4]{\lambda_{k+i}(\gamma)} = (k - \nu_\gamma + 1)\,\pi + O\left(k^{-1}\right), \tag{4.5}$$

where i is some fixed integer. Then using (2.19) and (4.5), and following the arguments in pp. 84–87 of [**35**] we obtain the following asymptotic formula

$$y_{k+i,\gamma}(x) = \sin\left(k - \nu_\gamma + 1\right)\pi x + (1 - \text{sgn}\gamma)(-1)^k \left(\sqrt{2}\right)^{-1} e^{(k-\nu_\gamma+1)\pi(x-1)} + O\left(k^{-1}\right), \tag{4.6}$$

which holds uniformly for $x \in [0, 1]$. Next, using the oscillation properties of the eigenfunctions of problem (1.1)–(1.5) and following the proof of [**30**, theorem 3.1] we get $i = 2$. Hence by setting $i = 2$ in (4.5) and (4.6) we obtain (4.3) and (4.4) respectively. The proof of this theorem is complete.

5. Operator interpretation and basis properties of the root functions of problem (1.1)–(1.5)

Let $H = L_2(0, 1) \oplus \mathbb{C}^2$ be the Hilbert space with the scalar product

$$(\hat{y}, \hat{v})_H = (\{y, m, n\}, \{v, s, t\})_H = \int_0^1 y(x)\,\overline{v(x)}\,\mathrm{d}x + |a|^{-1} m\bar{s} + |c|^{-1} n\bar{t}.$$

In H we define the operator

$$L\hat{y} = L\{y, m, n\} = \{\ell(y), Ty(0), Ty(1)\}$$

with the domain

$$D(L) = \{\{y\ (x), m, n\} \in H\ :\ y \in W_2^4(0, 1), \ell(y) \in L_2(0, 1), y''(0) = 0,$$
$$y'(1)\cos\gamma + y''(1)\sin\gamma = 0, m = ay(0), n = cy(1)\},$$

which dense everywhere in H. Then problem (1.1)–(1.5) is equivalent to the eigenvalue problem

$$L\hat{y} = \lambda\hat{y}, y \in D(L). \tag{5.1}$$

In this the eigenvalues $\lambda_{k,\gamma}, k \in \mathbb{N}$, of problems (1.1)–(1.5) and (5.2) coincide considering their multiplicity, and between the root functions, there is a one-to-one correspondence

$$y_{k,\gamma}(x) \leftrightarrow \hat{y}_{k,\gamma} = \{y_{k,\gamma}(x), m_{k,\gamma}, n_{k,\gamma}\}, m_{k,\gamma} = ay_{k,\gamma}(0), n_{k,\gamma} = cy_{k,\gamma}(1), k \in \mathbb{N}.$$

We define the operator $J : H \to H$ by

$$J\hat{y} = J\{y, m, n\} = \{y, -m, n\}.$$

Note that operator J generates the Pontryagin space $\Pi_1 = L_2(0, 1) \oplus \mathbb{C}^2$ equipped with an inner product

$$(\hat{y}, \hat{v})_{\Pi_1} = (J\hat{y}, \hat{v})_H = \int_0^1 y(x)\,\overline{v(x)}\,\mathrm{d}x + a^{-1} m\bar{s} - c^{-1} n\bar{t},$$

where $\hat{y} = \{y, m, n\}$, $\hat{v} = \{v, s, t\}$.

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THEOREM 5.1 (see [9, lemma 4.1, theorem 4.2]). *L is a J-self-adjoint operator in Π_1; $L^* = JLJ$, where L^* is an adjoint operator of L in H; the system of root vectors $\{\hat{y}_{k,\gamma}\}_{k=1}^{\infty}$, $\hat{y}_{k,\gamma} = \{y_{k,\gamma}(x), m_{k,\gamma}, n_{k,\gamma}\}$, $m_{k,\gamma} = ay_{k,\gamma}(0)$, $n_{k,\gamma} = cy_{k,\gamma}(1)$, of problem (5.2) forms an unconditional basis in H.*

Theorem 4.1 implies that

$$\begin{aligned} &y_{k,\gamma}(x) = y(x, \lambda_k(\gamma)) \text{ if } a \neq c-1,\ k \in \mathbb{N}, \text{ and } a = c-1,\ k \geqslant 3; \\ &y_{1,\gamma}(x) = y(x, \lambda_1(\gamma)), y_{2,\gamma}(x) = y^*_{2,\gamma}(x) + b\, y_{1,\gamma}(x) \text{ if } a = c-1, \end{aligned} \tag{5.2}$$

where $y^*_{2,\gamma}(x) = y'_{\lambda}(x, \lambda_1(\gamma))$ and b is an arbitrary constant

Let $\{\hat{v}^*_{k,\gamma}\}_{k=1}^{\infty}$, $\hat{v}^*_{k,\gamma} = \{v^*_{k,\gamma}, s^*_{k,\gamma}, t^*_{k,\gamma}\}$, is the system of root vectors of the operator L^*. Then by [**25**, formula (7)] we have

$$\begin{aligned} &L\hat{y}_{k,\gamma} = \lambda_k(\gamma)\, \hat{y}_{k,\gamma},\ L^*\hat{v}^*_{k,\gamma} = \lambda_k(\gamma)\, \hat{v}^*_{k,\gamma}, \text{ if } a \neq c-1,\ k \in \mathbb{N}, \text{ and } a = c-1,\ k \geqslant 3; \\ &L\hat{y}_{1,\gamma} = \lambda_1(\gamma)\, \hat{y}_{1,\gamma}, L\hat{y}_{2,\gamma} = \lambda_1(\gamma)\, \hat{y}_{2,\gamma} + \hat{y}_{1,\gamma}, L^*\hat{v}^*_{1,\gamma} = \lambda_1(\gamma)\, \hat{v}^*_{1,\gamma} + \hat{v}^*_{2,\gamma}, \\ &L^*\hat{v}^*_{1,\gamma} = \lambda_1(\gamma)\, \hat{v}^*_{1,\gamma}. \end{aligned} \tag{5.3}$$

In view of (5.2), by (5.3) we obtain

$$\begin{aligned} &\hat{v}^*_{k,\gamma} = J\hat{y}_{k,\gamma} \text{ if } a \neq c-1, k \in \mathbb{N}, \text{ and } a = c-1, k \geqslant 3; \\ &\hat{v}^*_{1,\gamma} = J\hat{y}^*_{2,\gamma} + \tilde{b}J\hat{y}_{1,\gamma}, \hat{v}^*_{2,\gamma} = J\hat{y}_{1,\gamma}, \text{ if } c = a+1, \end{aligned} \tag{5.4}$$

where $\tilde{b}$ is an arbitrary constant.

By following the arguments in lemma 4.1 of [**9**, pp. 15–16] we can show that the following assertion holds.

LEMMA 5.2. *Let $\{\hat{v}_{k,\gamma}\}_{k=1}^{\infty}$, $\hat{v}_{k,\gamma} = \{v_{k,\gamma}, s_{k,\gamma}, t_{k,\gamma}\}$, be the system that is adjoint to the system $\{\hat{y}_{k,\gamma}\}_{k=1}^{\infty}$. Then*

$$\hat{v}_k = \delta^{-1}_{k,\gamma}\, \hat{v}^*_{k,\gamma}, k \in \mathbb{N}, \tag{5.5}$$

where $\delta_{k,\gamma} = (y_{k,\gamma}, y_{k,\gamma})_{\Pi_1}$, if $a \neq c-1, k \in \mathbb{N}$, and $a = c-1, k \geqslant 3$; $\delta_{1,\gamma} = \delta_{2,\gamma} = (\hat{y}_{1,\gamma}, \hat{y}^*_{2,\gamma})_{\Pi_1}$ if $a = c-1$, and $\delta_{k,\gamma} \neq 0$ for $k \in \mathbb{N}$, and $\tilde{b} = -(b + \delta^{-1}_{1,\gamma}\, (\hat{y}^*_{2,\gamma}, \hat{y}^*_{2,\gamma})_{\Pi_1})$.

Let

$$\Delta_{r,l,\gamma} = \begin{vmatrix} s_{r,\gamma} & s_{l,\gamma} \\ t_{r,\gamma} & t_{l,\gamma} \end{vmatrix}. \tag{5.6}$$

where r and l $(r \neq l)$ are arbitrary fixed positive integers.

THEOREM 5.3. *If $\Delta_{r,l,\gamma} \neq 0$, then the system of root functions $\{y_{k,\gamma}(x)\}_{k=1,k\neq r,l}^{\infty}$ of problem (1.1)–(1.5) forms a basis in $L_p(0,1), 1 < p < \infty$, which is an unconditional basis for $p = 2$; if $\Delta_{r,l,\gamma} = 0$ then this system is incomplete and nonminimal in $L_p(0,1), 1 < p < \infty$.*

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Proof. The assertions of this theorem for $p = 2$ follow from [**1**, theorems 3.1, 3.2 and corollary 3.1]. The proof of theorem 5.3 for $p \in (1, +\infty), p \neq 2$, is similar to that of [**30**, theorem 5.1] by using asymptotic formulas (4.1)–(4.4). The proof of this theorem is complete.

Using the oscillatory properties of the eigenfunctions of problem (1.1)–(1.5), by theorem 5.3 we can establish sufficient conditions for the system $\{y_{k,\gamma}(x)\}_{k=1, k\neq r,l}^{\infty}$ of root functions of this problem to form a basis in $L_p(0,1)$, $1 < p < \infty$.

THEOREM 5.4. *Let r and l $(r < l)$ be arbitrary fixed natural numbers. Then in the cases (i) $r, l \geqslant 3$ and have different parities; (ii) $a < c - 1$, $r = 1$ or $r = 2$, and l is odd; (iii) $a > c - 1$, $r = 2$ and l is odd; (iv) $a > c - 1$, $r = 1$, $\tau(\lambda_1(\gamma))$ and l have different parities; (v) $a = c - 1$, $r = 2$ and l is odd; (vi) $a = c - 1$, $r = 1$ and $y^*_{2,\gamma}(0) - y^*_{2,\gamma}(1)\, y_{l,\gamma}(0) \neq 0$, the system $\{y_{k,\gamma}(x)\}_{k=1,k\neq r,l}^{\infty}$ is a basis in $L_p(0,1), 1 < p < \infty$, which is an unconditional basis in $L_2(0,1)$.*

Proof. By relations (5.4) and (5.5), it follows from (5.6) that

$$\begin{aligned}\Delta_{r,l,\gamma} &= \begin{vmatrix} -\delta_{r,\gamma}^{-1}\, m_{r,\gamma} - \delta_{l,\gamma}^{-1}\, m_{l,\gamma} \\ \delta_{r,\gamma}^{-1}\, n_{r,\gamma} \delta_{l,\gamma}^{-1}\, n_{l,\gamma} \end{vmatrix} = -\,\delta_{r,\gamma}^{-1}\,\delta_{l,\gamma}^{-1} \begin{vmatrix} m_{r,\gamma}\; m_{l,\gamma} \\ n_{r,\gamma}\; n_{l,\gamma} \end{vmatrix} \\ &= -\,\delta_{r,\gamma}^{-1}\,\delta_{l,\gamma}^{-1} \begin{vmatrix} a\, y_{r,\gamma}(0)\; a\, y_{l,\gamma}(0) \\ c\, y_{r,\gamma}(1)\;\; c\, y_{l,\gamma}(1) \end{vmatrix} = -a\, c\, \delta_{r,\gamma}^{-1}\,\delta_{l,\gamma}^{-1} \begin{vmatrix} y_{r,\gamma}(0)\;\; y_{l,\gamma}(0) \\ y_{r,\gamma}(1)\;\; y_{l,\gamma}(1) \end{vmatrix} \qquad (5.7)\end{aligned}$$

for $r, l \in \mathbb{N}$ in the case $a \neq c - 1$, and for $r, l \geqslant 3$ in the case $a = c - 1$.

By (2.12) and (2.13), relation (5.7) implies that

$$\Delta_{r,l,\gamma} = -\,a\, c\, \delta_{r,\gamma}^{-1}\, \delta_{l,\gamma}^{-1}\, \left(y_{r,\gamma}(0) - y_{l,\gamma}(0)\right) \qquad (5.8)$$

for $r, l \in \mathbb{N}$ in the case $a < c - 1$, for $r, l \geqslant 2$ in the case $a > c - 1$, and for $r, l \geqslant 3$ in the case $a = c - 1$, and

$$\Delta_{1,l,\gamma} = -\,a\, c\, \delta_{r,\gamma}^{-1}\, \delta_{l,\gamma}^{-1}\, \left(1 - y_{1,\gamma}(1)\, y_{l,\gamma}(0)\right) \qquad (5.9)$$

for $l \geqslant 2$ in the case $a > c - 1$.

Moreover, in the case $a = c - 1$ for $r = 2$, $l \geqslant 3$, and for $r = 1$ and $l \geqslant 2$ we have

$$\begin{aligned}\Delta_{2,l,\gamma} &= -\,\delta_{2,\gamma}^{-1}\delta_{l,\gamma}^{-1} \begin{vmatrix} m_{1,\gamma}\; m_{l,\gamma} \\ n_{1,\gamma}\;\; n_{l,\gamma} \end{vmatrix} = -\,\delta_{2,\gamma}^{-1}\delta_{l,\gamma}^{-1} \begin{vmatrix} a\, y_{1,\gamma}(0)\; a\, y_{l,\gamma}(0) \\ c\, y_{1,\gamma}(1)\;\; c\, y_{l,\gamma}(1) \end{vmatrix} \\ &= -a\, c\, \delta_{2,\gamma}^{-1}\delta_{l,\gamma}^{-1} \begin{vmatrix} y_{1,\gamma}(0) & y_{l,\gamma}(0) \\ 1 & 1 \end{vmatrix} = -a\, c\, \delta_{1,\gamma}^{-1}\, \delta_{l,\gamma}^{-1}\, \left(y_{1,\gamma}(0) - y_{l,\gamma}(0)\right),\end{aligned} \qquad (5.10)$$

and

$$\begin{aligned}\Delta_{1,l,\gamma} &= -\begin{vmatrix} \delta_{1,\gamma}^{-1}\, \{m^*_{2,\gamma} + \tilde{b}\, m_{1,\gamma}\}\; \delta_{l,\gamma}^{-1}\, m_{l,\gamma} \\ \delta_{1,\gamma}^{-1}\, \{n^*_{2,\gamma} + \tilde{b}\, n_{1,\gamma}\}\; \delta_{l,\gamma}^{-1}\, n_{l,\gamma} \end{vmatrix} = -\,\delta_{1,\gamma}^{-1}\,\delta_{l,\gamma}^{-1} \begin{vmatrix} m^*_{2,\gamma} + \tilde{b}\, m_{1,\gamma}\; m_{l,\gamma} \\ n^*_{2,\gamma} + \tilde{b}\, n_{1,\gamma}\; n_{l,\gamma} \end{vmatrix} \\ &= -a\, c\, \delta_{1,\gamma}^{-1}\,\delta_{l,\gamma}^{-1} \begin{vmatrix} y^*_{2,\gamma}(0) + \tilde{b}\, y_{1,\gamma}(0)\; y_{l,\gamma}(0) \\ y^*_{2,\gamma}(1) + \tilde{b}\, y_{1,\gamma}(1)\; y_{l,\gamma}(1) \end{vmatrix} = -\,a\, c\, \delta_{1,\gamma}^{-1}\,\delta_{l,\gamma}^{-1} \begin{vmatrix} y^*_{2,\gamma}(0)\; y_{l,\gamma}(0) \\ y^*_{2,\gamma}(1)\;\; 1 \end{vmatrix} \\ &= -a\, c\, \delta_{1,\gamma}^{-1}\,\delta_{l,\gamma}^{-1}\, \{\, y^*_{2,\gamma}(0) - y^*_{2,\gamma}(1)\, y_{l,\gamma}(0)\}. \qquad (5.11)\end{aligned}$$

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respectively.

Next, by (2.12) and (2.25), theorem 3.5 implies that

$$\operatorname{sgn} y_{k,\gamma}(0) = (-1)^k \text{ for } k \geqslant 3, \tag{5.12}$$

$$y_{1,\gamma}(0) = 1 \text{ for } a \leqslant c-1, y_{2,\gamma}(0) > 0 \text{ for } a < c-1,$$

$$\operatorname{sgn} y_{1,\gamma}(1) = (-1)^{\tau(\lambda_1(\gamma))}, y_{2,\gamma}(0) = 1 \text{ for } a > c-1, \tag{5.13}$$

where $\tau(\lambda_1(\gamma)) = \sum\limits_{\xi_k \in (\lambda_1(\gamma),0)} i(\xi_k)$.

Now the statements (i)–(vi) of this theorem follow from (5.8)–(5.13) in view of theorem 5.3. The proof of this theorem is complete.

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