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PAPER

# A representation of proper BC domains based on conjunctive sequent calculi†

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**Abstract**
We build a logical system named a conjunctive sequent calculus which is a conjunctive fragment of the classical propositional sequent calculus in the sense of proof theory. We prove that a special class of formulae of a consistent conjunctive sequent calculus forms a bounded complete continuous domain without greatest element (for short, a proper BC domain), and each proper BC domain can be obtained in this way. More generally, we present conjunctive consequence relations as morphisms between consistent conjunctive sequent calculi and build a category which is equivalent to that of proper BC domains with Scott-continuous functions. A logical characterization of purely syntactic form for proper BC domains is obtained.

**Keywords:** Conjunctive sequent calculus; proper BC domain; categorical equivalence

## 1. Introduction

Domain theory used to specify denotational semantics of programming languages has many close connections with logic. Such connections are often established by constructing appropriate logical systems to represent certain domains. At present, there are mainly two types of frameworks on applying logic to represent various domains.

One type is the information system which has some features of a logic calculus viewed from proof theoretical point. An information system is a triple $\langle X, Con, \vdash\rangle$, in which $X$ is a set of atomic formulae, $Con$ is a consistency predicate, and entailment relation $\vdash$ satisfies some axioms. Most of these axioms are closely related to the consistency predicate, so the consistency predicate is essential. The historical roots of information systems must go back to Scott (1982) in which Scott gave a kind of information systems as a logical representation of algebraic bounded complete domains. And soon after, Larsen and Winskel (1984) proved that there is an equivalence between categories of such information systems with approximable mappings as morphisms and algebraic bounded complete domains with Scott-continuous functions as morphisms. As to representations of various domains, many scholars have devised several kinds of information systems and similar structures (Hoofman (1993); Huang et al. (2015); Spreen et al. (2008); Vickers (1993); Wu et al. (2016)).

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The other type is based on Abramsky's work. In Abramsky (1991), he constructed a logical system for sequences of finite posets (SFP) domains and yielded new insights for domain theory. In Abramsky's opinion, the Lindenbaum algebra of a domain logic must be categorically equivalent to certain domains, and the categorical equivalence is a strengthened form of logician's completeness. Then Abramsky's presentation of domains is deliberately suggestive of semantics theory. Following Abramsky's idea, Chen and Jung (2006) built a disjunctive propositional logic to capture algebraic L-domains. A logic approach to algebraic L-domains was also introduced by Zhang (1992). Jung et al. (1999) presented coherent sequent calculi which is a logic corresponding to strong proximity lattices and extended much of Abramsky's work. For a variety of results, see Jung (2013). In a coherent sequent calculus, the formulae are built by binary conjunction and disjunction. Chen and Jung (2006) substitute the binary disjunctive by an arbitrary disjunctive, while in Zhang (1992), there is even no connective and constant.

Our work focuses on a syntactic representation of domains. As a collection of special domains, bounded complete continuous domains are of great importance in domain theory. Specifically, the category of bounded complete domains with Scott-continuous functions is Cartesian closed. The commonness between Scott and Abramsky's approaches is to extract an appropriate syntax from a category of domains. Motivated by this observation, we use connectives $\wedge$ and $T$ to build formulae, and exactly continue to use derivation rules of the SFP domains logic (Abramsky 1991) and the disjunctive propositional logic (Chen and Jung 2006) corresponding to connectives $\wedge$ and $T$. Then we define a logical system for every proper BC domain, which is called a conjunctive sequent calculus. Similar to the conjunctive proposition logic proposed in Hitzler et al. (2006), our sequent calculus is a conjunctive fragment of the classical Gentzen propositional sequent calculus in the sense of proof theory.

The concept of filters of a consistent conjunctive sequent calculus can serve as a bridge between consistent conjunctive sequent calculi and proper BC domains. A filter of a consistent conjunctive sequent calculus corresponds to a special theory in logic. The collection of all filters of a consistent conjunctive sequent calculus forms a proper BC domain ordered by set inclusion, and each proper BC domain can be obtained in this way, up to isomorphism. We present a notion of conjunctive consequent relation between consistent conjunctive sequent calculi. This is a generalization of sequents of a consistent conjunctive sequent calculus in the sense of multilingual sequent calculus (Jung et al. 1999). With conjunctive consequent relations as morphisms, we prove that the category of consistent conjunctive sequent calculi is equivalent to that of proper BC domains with Scott-continuous functions. This result enables us to express proper BC domains in a purely syntactic form.

The difference of our approach from Abramsky's is that we use the proof system itself to capture domains rather than its Lindenbaum algebra. Since the Lindenbaum algebra of a logic is only corresponding to its tautologies, the logic can not be recovered from a given domain. Although our approach is motivated by Scott's information systems, it is not a Scott-type information system. Our system is a formal propositional logic which deviates from Gentzen propositional sequent calculus in some way. A conjunctive sequent calculus neither relies on consistency predicate to make inferences, nor revolves around only the atomic formulae.

An outline of this paper is as follows. Section 2 contains necessary definitions and results for domain theory and categories. Section 3 introduces the notion of conjunctive sequent calculus and shows that a conjunctive sequent calculus is the conjunctive fragment of classical propositional sequent calculus in the sense of proof theory. In Section 4, we define the notions of a consistent conjunctive sequent calculus and a filter of the consistent conjunctive sequent. We obtain the result that consistent conjunctive sequent calculi and proper BC domains can be represented with each other. The equivalence between the categories of consistent conjunctive sequent calculi with conjunctive sequence relations and proper BC domains with Scott-continuous functions will be given in Section 5.

## 2. Preliminaries

Briefly, we first review some basic definitions and notions. Most of them come from Davey and Priestly (2002), Gierz et al. (2003), and Goubault-Larrecq (2013). For any set $X$, the symbol $A \sqsubseteq X$ indicates that $A$ is a nonempty finite subset of $X$. Let $P$ be a poset, for any nonempty subset $D$ of $P$, $D$ is directed if every pair of elements of $D$ has an upper bound in $D$. We write $x \sqcup y$ as the least upper bound of $\{x, y\}$ and write $\bigsqcup X$ as the least upper bound of $X$, respectively. The least element of a poset $P$ is denoted by $0_P$. If $P$ has a least element $0_P$, $P$ is called pointed. A directed complete poset (dcpo, for short) $P$ is a poset in which every directed subset $D$ has a least upper bound $\bigsqcup D$.

Let $P$ be a dcpo and $x, y \in P$. Then $x$ is said to be way-below $y$, in symbols $x \ll y$, if and only if for any directed subset $D$ of $P$ the relation $y \le \bigsqcup D$ always implies the existence of some $d \in D$ with $x \le d$. A subset $B \subseteq P$ is called a basis of $P$ if and only if for any $x \in P$, there exists a directed subset $D$ of the set $\{y \in B \mid y \ll x\}$ such that $x = \bigsqcup D$. An element is finite or compact if $x \ll x$. We write $K(P)$ for the set of compact elements of $P$. A pointed dcpo $P$ is called a continuous domain if it has a basis, and it is said algebraic if $K(P)$ forms a basis. The way-below relation on a continuous domain $P$ satisfies the interpolation property, that is, $x \ll y$ implies $x \ll z \ll y$ for some $z$. Moreover, for a continuous domain $P$ and $X \sqsubseteq P$ with $\bigsqcup X \in P$, we have $\bigsqcup X \ll y$ if and only if $x \ll y$ for all $x \in X$.

**Proposition 2.1.** (Wu et al. 2016) *Let $(D, \le)$ be a continuous domain and $B(D)$ the set of all elements $x \in D$ such that $x \ll p$ for some maximal element $p$ of $D$. Then $B(D)$ is a basis of $D$.*

A continuous domain is called a bounded complete continuous domain if every subset that is bounded above has a least upper bound. If a bounded complete continuous domain is algebraic, it is called Scott domain. In this paper, we choose to emphasize the logical representation of bounded complete domains without greatest element, and we call a bounded complete continuous domain which has no greatest element a proper BC domain.

A function $f : P \to Q$ between two continuous domains is Scott-continuous if and only if for all directed subset $D$ of $P$, $f(\bigsqcup D) = \bigsqcup\{f(x) \mid x \in D\}$. Let **PBCD** be the category of proper BC domains with Scott-continuous functions, and **PSD** the category of algebraic proper BC domains.

**Proposition 2.2.** (Awodey 2006) *Let* **C** *and* **D** *be two categories. Then* **C** *and* **D** *are categorically equivalent if and only if there exists a functor $\mathfrak{F} : \mathbf{C} \to \mathbf{D}$ such that $\mathfrak{F}$ is full, faithful and for every object $D$ of* **D**, *there exists some object $C$ of* **C** *such that $\mathfrak{F}(C) \cong D$.*

We refer to Gallier (2015) and Wang and Zhou (2009) for the standard definitions and notations of logical theory.

## 3. Conjunctive Sequent Calculi

Similar to (Hitzler et al., 2006, Definition 6.1), we use a nonempty set $P$ to denote atomic formulae and use connectives $\wedge$ and $T_P$ to build compound formulae for developing a calculus theory.

**Definition 3.1.** *Given a nonempty set $P$ with $T_P \in P$. Each element of $P$ is called an atomic formula. The set $\mathcal{L}(P)$ of formulae is defined inductively in an obvious way:*

(1) *Every atomic formula $p$ is in $\mathcal{L}(P)$.*
(2) *Whenever $\varphi, \psi$ are in $\mathcal{L}(P)$, $\varphi \wedge \psi$ is also in $\mathcal{L}(P)$.*
(3) *All formulae are generated by* (1) *and* (2).

Given a nonempty finite subset $\Gamma = \{\varphi_1, \varphi_2, \cdots, \varphi_n\}$ of $\mathcal{L}(P)$, we abbreviate the formula $\varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_n$ as $\bigwedge \Gamma$. For any $\varphi \in \mathcal{L}(P)$, we write $\overline{\varphi} = \{p_1, p_2, \cdots, p_n\}$ , where $p_1, p_2, \cdots, p_n$ are

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all the atomic formulae which occur in $\varphi$. Similarly, we write $\overline{X} = \{p \in \overline{\varphi} \mid \varphi \in X\}$ for any nonempty subset $X$ of $\mathcal{L}(P)$. In the sequel, if there is no ambiguity, we abbreviate a singleton $\{\varphi\}$ as $\varphi$.

**Definition 3.2.** *A conjunctive sequent calculus is a pair* $(\mathcal{L}(P), \vdash_P)$*, where* $\vdash_P$ *is a relation on nonempty finite subsets of* $\mathcal{L}(P)$ *and closed under the following derivation rules*[1]*:*

$$\frac{}{\Gamma \vdash_P T_P}\,(T_P) \qquad \frac{\Gamma \vdash_P \Delta}{\Gamma', \Gamma \vdash_P \Delta}\,(Weakening)$$

$$\frac{\varphi, \psi, \Gamma \vdash_P \Delta}{\overline{\varphi \wedge \psi, \Gamma \vdash_P \Delta}}\,(\wedge : left) \qquad \frac{\Gamma \vdash_P \Delta, \varphi \quad \Gamma \vdash_P \Delta, \psi}{\overline{\Gamma \vdash_P \Delta, \varphi \wedge \psi}}\,(\wedge : right)$$

$$\frac{\Gamma \vdash_P \varphi \quad \varphi \vdash_P \Delta}{\overline{\Gamma \vdash_P \Delta}}\,(Cut)\,.$$

*When the following additional identity rule is put on* $\vdash_P$*:*

$$\varphi \vdash_P \varphi \ (Id),$$

*we call a conjunctive sequent calculus* $(\mathcal{L}(P), \vdash_P)$ *algebraic.*

A sequent is an object of the form $\Gamma \vdash_P \Delta$, where $\Gamma$ and $\Delta$ are nonempty finite subsets of formulae.

**Example 3.1.** Let $P = [0, 1)$ and $T_P = 0$. For any $\Gamma, \Delta \sqsubseteq \mathcal{L}(P)$, define

$$\Gamma \vdash_P \Delta \text{ if and only if } \overline{\Delta} = \{0\} \text{ or } p < \bigsqcup \overline{\Gamma} \text{ for every } p \in \overline{\Delta}.$$

Then $(\mathcal{L}(P), \vdash_P)$ is a conjunctive sequent calculus.

**Example 3.2.** Let $(D, \leq)$ be a proper BC domain, and let $B(D)$ be the basis of $D$ defined in Proposition 2.1. Using the way of presenting in Definition 3.1, we obtain the set $\mathcal{L}(B(D))$ of formulae, which has $B(D)$ as the set of atomic formulae and $0_D$ as the constant $T_{B(D)}$. For $\Gamma, \Delta \sqsubseteq \mathcal{L}(B(D))$, define $\Gamma \vdash_{B(D)} \Delta$ if and only if one of the following conditions holds:

(1) $\overline{\Gamma}$ has no upper bound in $D$.
(2) $\overline{\Gamma}$ has an upper bound in $D$, and $d \ll \bigsqcup \overline{\Gamma}$ for any $d \in \overline{\Delta}$.

Then $\vdash_{B(D)}$ is closed under the rules of a conjunctive sequent calculus, and $(\mathcal{L}(B(D)), \vdash_{B(D)})$ is a conjunctive sequent calculus. We only illustrate this for the rule $(Weakening)$.

Let $\Gamma \vdash_{B(D)} \Delta$ and $\Gamma' \sqsubseteq \mathcal{L}(B(D))$. If $\overline{\Gamma \cup \Gamma'}$ has no upper bound in $D$, then by the above condition (1), $\Gamma', \Gamma \vdash_{B(D)} \Delta$ follows. If $\bigsqcup \overline{\Gamma' \cup \Gamma}$ exists in $D$, then $\bigsqcup \overline{\Gamma}$ exists in $D$ and $\bigsqcup \overline{\Gamma} \leq \bigsqcup \overline{\Gamma' \cup \Gamma}$. This implies that $d \ll \bigsqcup \overline{\Gamma} \leq \bigsqcup \overline{\Gamma' \cup \Gamma}$ for any $d \in \overline{\Delta}$. So that $\Gamma', \Gamma \vdash_{B(D)} \Delta$.

Particularly, if $(D, \leq)$ is algebraic, then $(\mathcal{L}(K(D)), \vdash_{K(D)})$ is algebraic.

**Example 3.3.** Let $P = \{T_P, p_1, p_2, p_3, \cdots\}$. For $\Gamma, \Delta \sqsubseteq \mathcal{L}(P)$, define

$$\Gamma \vdash_P \Delta \text{ if and only if } \Delta = \{T_P\} \text{ or } \overline{\Delta} \subseteq \overline{\Gamma}, \text{ or } \{p_1, p_2\} \subseteq \overline{\Gamma}.$$

Then $(\mathcal{L}(P), \vdash_P)$ is an algebraic conjunctive sequent calculus.

**Remark 3.1.** Consider a conjunctive sequent calculus $(\mathcal{L}(P), \vdash_P)$. It is easy to see that if $\overline{\psi} \subseteq \overline{\varphi}$ and $\psi \vdash_P \rho$, then $\varphi \vdash_P \rho$. But the converse is not true, see Example 3.3.

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The next proposition tells us that for any algebraic conjunctive sequent calculus, the double lines in the rules $(\wedge : \mathit{left})$, $(\wedge : \mathit{right})$, and $(\mathit{Cut})$ can be replaced by horizontal lines. That is to say, if $(\mathcal{L}(P), \vdash_P)$ is an algebraic conjunctive sequent calculus, then the rules $(\wedge : \mathit{left})$, $(\wedge : \mathit{right})$, and $(\mathit{Cut})$ can be replaced by the following forms, respectively:

$$\frac{\varphi, \psi, \Gamma \vdash_P \Delta}{\varphi \wedge \psi, \Gamma \vdash_P \Delta} (\wedge^* : \mathit{left})$$

$$\frac{\Gamma \vdash_P \Delta, \varphi \quad \Gamma \vdash_P \Delta, \psi}{\Gamma \vdash_P \Delta, \varphi \wedge \psi} (\wedge^* : \mathit{right})$$

$$\frac{\Gamma \vdash_P \varphi \quad \varphi \vdash_P \Delta}{\Gamma \vdash_P \Delta} (\mathit{Cut}^*) .$$

**Proposition 3.1.** *Let $\mathcal{L}(P)$ be the set of formulae and $\vdash_P$ a relation on the collection of nonempty finite subsets of $\mathcal{L}(P)$. If $\vdash_P$ is closed under the derivation rules: $(Id)$, $(Weakening)$, $(\wedge^* : left)$, $(\wedge^* : left)$, and $(Cut^*)$. Then the following statements hold.*

(1) *If $\varphi \wedge \psi \vdash_P \Delta$, then $\varphi, \psi \vdash_P \Delta$.*
(2) *If $\Gamma \vdash_P \varphi \wedge \psi$, then $\Gamma \vdash_P \varphi$ and $\Gamma \vdash_P \psi$.*
(3) *If $\Gamma \vdash_P \Delta$, then $\Gamma \vdash_P \varphi$ and $\varphi \vdash_P \Delta$ for some $\varphi \in \mathcal{L}(P)$.*

*Proof.* (1) Suppose that $\varphi \wedge \psi \vdash_P \Delta$, then

$$\frac{\varphi \wedge \psi \vdash_P \Delta \qquad \dfrac{\dfrac{\varphi \vdash_P \varphi \ (Id)}{\varphi, \psi \vdash_P \varphi} (\mathit{Weakening}) \quad \dfrac{\psi \vdash_P \psi \ (Id)}{\varphi, \psi \vdash_P \psi} (\mathit{Weakening})}{\varphi, \psi \vdash_P \varphi \wedge \psi} (\wedge^* : \mathit{right})}{\varphi, \psi \vdash_P \Delta} (\mathit{Cut}^*)$$

(2) Suppose that $\Gamma \vdash_P \varphi \wedge \psi$, then

$$\frac{\dfrac{\Gamma \vdash_P \varphi \wedge \psi \qquad \dfrac{\varphi, \psi \vdash_P \varphi, \psi \ (Id)}{\varphi \wedge \psi \vdash_P \varphi, \psi} (\wedge^* : \mathit{left})}{\Gamma \vdash_P \varphi, \psi} (\mathit{Cut}^*) \quad \dfrac{\varphi \vdash_P \varphi \ (Id)}{\varphi, \psi \vdash_P \varphi} (\mathit{Weakening})}{\Gamma \vdash_P \varphi} (\mathit{Cut}^*).$$

Similarly, we have $\Gamma \vdash_P \psi$.

(3) Let $\varphi = \bigwedge \Gamma$. It is clear that $\Gamma \vdash_P \varphi$ and $\varphi \vdash_P \Delta$ by induction. □

The above studies show that a conjunctive sequent calculus is a conjunctive fragment of the classical Gentzen propositional sequent calculus in the sense of proof theory. Naturally, one can include disjunctive or other connectives into a conjunctive sequent calculus. It is worth noting that there have been around two remarkable conservation extensions of our conjunctive sequent calculus: one is of the SFP domains logic (Abramsky 1991) and the other is of the disjunctive proposition logic for algebraic L-domains (Chen and Jung 2006).

## 4. Representation Theorem

Let $(\mathcal{L}(P), \vdash_P)$ be a conjunctive sequent calculus, we call $\varphi \in \mathcal{L}(P)$ satisfiable if there exists some $\psi \in \mathcal{L}(P)$ with $\varphi \nvdash_P \psi$. The set of all satisfiable formulae is denoted by $\mathcal{S}_P$.

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**Definition 4.1.** *A conjunctive sequent calculus* $(\mathcal{L}(P), \vdash_P)$ *is consistent if* $P \subseteq \mathcal{S}_P \neq \mathcal{L}(P)$*, and for every* $\psi \in \mathcal{S}_P$ *there exists some* $\varphi \in \mathcal{S}_P$ *such that* $\varphi \vdash_P \psi$.

Roughly speaking, a formula is satisfiable if and only if it is not a contradiction. In usual logic, the assumptions of Definition 4.1 naturally hold. Set

$$\mathbb{S}(P) = \{\Gamma \sqsubseteq \mathcal{L}(P) \mid \bigwedge \Gamma \in \mathcal{S}_P\} \tag{1}$$

The relation $\vdash_P$ is restricted to the set $\mathbb{S}(P)$ which gives rise to a new relation $\Vdash_P$:

$$\Gamma \Vdash_P \Delta \text{ if and only if } \Gamma \vdash_P \Delta \text{ with } \Gamma \in \mathbb{S}(P). \tag{2}$$

We claim that whenever $\varphi \Vdash_P \psi$ then $\varphi \in \mathcal{S}_P$, and thus $\psi \in \mathcal{S}_P$. Indeed, if not, then $\varphi$ is also unsatisfiable by the rules (*Cut*), ($\wedge$ : *left*), and ($\wedge$ : *right*). Hence, $\Vdash_P \subseteq \mathbb{S}(P) \times \mathbb{S}(P)$ is closed under all the derivation rules of a conjunctive sequent calculus. In such a case, the predecessor $\Gamma$ of $\Gamma \Vdash_P \Delta$ must belong to $\mathbb{S}(P)$. Moreover, by equation (2), it is straightforward to see that $\vdash_P$ is uniquely determined by its restriction $\Vdash_P$, and vice versa.

Given a consistent conjunctive sequent calculus $(\mathcal{L}(P), \vdash_P)$, for any $X \subseteq \mathcal{L}(P)$, we write

$$X[\Vdash_P] = \{\psi \in \mathcal{L}(P) \mid \rho \Vdash_P \psi \text{ with } \overline{\rho} \subseteq \overline{\Gamma} \text{ for some } \Gamma \sqsubseteq X\}. \tag{3}$$

**Proposition 4.1.** *Let* $(\mathcal{L}(P), \vdash_P)$ *be a consistent conjunctive sequent calculus. Then the following statements hold.*

1. *If* $X \subseteq Y \subseteq \mathcal{L}(P)$*, then* $X[\Vdash_P] \subseteq Y[\Vdash_P] \subseteq \mathcal{S}_P$.
2. *If* $\varphi \in X[\Vdash_P]$ *and* $\varphi \Vdash_P \psi$*, then* $\psi \in X[\Vdash_P]$.
3. *For any* $\Gamma \sqsubseteq \mathcal{L}(P)$*,* $\Gamma[\Vdash_P] = (\bigwedge \Gamma)[\Vdash_P]$.
4. *For any* $\varphi \in \mathcal{S}_P$*,* $\psi \in \varphi[\Vdash_P]$ *if and only if* $\varphi \Vdash_P \psi$.

*Proof.* (1) $X[\Vdash_P] \subseteq Y[\Vdash_P]$ is given from equation (3) and $Y[\Vdash_P] \subseteq \mathcal{S}_P$ follows from the fact that $\varphi \Vdash_P \psi$ implies $\psi \in \mathcal{S}_P$. (2) This is true by the rule (*Cut*). (3) It is immediate from equation (3). (4) If $\psi \in \varphi[\Vdash_P]$, then there exists $\rho \Vdash_P \psi$ with $\overline{\rho} \subseteq \overline{\varphi}$. Because $\overline{\rho} \subseteq \overline{\varphi}$ and $\varphi \in \mathcal{S}_P$, by the rule ($\wedge$ : *right*), it follows that $\varphi \Vdash_P \rho$. Therefore, $\varphi \Vdash_P \psi$ using the rule (*Cut*). The reverse implication is trivial by equation (3). □

**Definition 4.2.** *Let* $(\mathcal{L}(P), \vdash_P)$ *be a consistent conjunctive sequent calculus. A filter of* $\mathcal{L}(P)$ *is a nonempty subset* $\mathcal{F}$ *of* $\mathcal{S}_P$ *such that* $\mathcal{F} = \mathcal{F}[\Vdash_P]$ *and* $\bigwedge \Gamma \in \mathcal{F}$ *for any* $\Gamma \sqsubseteq \mathcal{F}$.

Next, we give two equivalent characterizations for the above concept. The first can be seen as a usual logic theoretical version and the second is an order theoretical one.

**Proposition 4.2.** *A subset* $\mathcal{F}$ *of* $\mathcal{S}_P$ *is a filter of* $\mathcal{L}(P)$ *if and only if*

1. $T_P \in \mathcal{F}$;
2. *If* $\varphi, \psi \in \mathcal{F}$*, then* $\varphi \wedge \psi \in \mathcal{F}$;
3. $\psi \in \mathcal{F}$ *if and only if* $\varphi \Vdash_P \psi$ *for some* $\varphi \in \mathcal{F}$.

*Proof.* If $\mathcal{F}$ is a filter, then it is straightforward to check that the three conditions hold.

For the converse implication, suppose that $\Gamma \sqsubseteq \mathcal{F}$. With condition (2), it is easy to see that $\bigwedge \Gamma \in \mathcal{F}$. For any $\varphi \in \mathcal{F}$, by condition (3), there exists $\psi \in \mathcal{F}$ such that $\psi \Vdash_P \varphi$. This implies that $\varphi \in \mathcal{F}[\Vdash_P]$, and hence $\mathcal{F} \subseteq \mathcal{F}[\Vdash_P]$. Conversely, for any $\varphi \in \mathcal{F}[\Vdash_P]$, by equation (3), there exists some $\Gamma \sqsubseteq \mathcal{F}$ such that $\overline{\rho} \subseteq \overline{\Gamma}$ with $\rho \Vdash_P \varphi$. Since $\bigwedge \Gamma \in \mathcal{F}$, we have $\bigwedge \Gamma \Vdash \rho$. Using the rule (*Cut*) twice, we have $\bigwedge \Gamma \Vdash_P \varphi$. From condition (3), it follows that $\varphi \in \mathcal{F}$, and then $\mathcal{F}[\Vdash_P] \subseteq \mathcal{F}$. As a result, $\mathcal{F} = \mathcal{F}[\Vdash_P]$. □

**Proposition 4.3.** *A subset $\mathcal{F}$ of $\mathcal{S}_P$ is a filter of $\mathcal{L}(P)$ if and only if the set $\{\varphi[\Vdash_P] \mid \varphi \in \mathcal{F}\}$ is directed, and $\mathcal{F}$ is its union.*

*Proof.* Let $\mathcal{F}$ be a filter of $\mathcal{L}(P)$. The set $\{\varphi[\Vdash_P] \mid \varphi \in \mathcal{F}\}$ is not empty since $T_P \in \mathcal{F}$. For any $\varphi_1, \varphi_2 \in \mathcal{F}$, we have $\varphi_1 \wedge \varphi_2 \in \mathcal{F}$, and then $\varphi_1[\Vdash_P], \varphi_2[\Vdash_P], (\varphi_1 \wedge \varphi_2)[\Vdash_P] \subseteq \mathcal{F}$. By Proposition 4.1(1,3), it follows that $\varphi_1[\Vdash_P], \varphi_2[\Vdash_P] \subseteq (\varphi_1 \wedge \varphi_2)[\Vdash_P]$, and hence the set $\{\varphi[\Vdash_P] \mid \varphi \in \mathcal{F}\}$ is directed. For any $\psi \in \mathcal{F}$, by condition (3) of Proposition 4.2, there exists some $\varphi \in \mathcal{F}$ such that $\psi \in \varphi[\Vdash_P]$. Therefore, $\mathcal{F} \subseteq \bigcup\{\varphi[\Vdash_P] \mid \varphi \in \mathcal{F}\}$. Since $\varphi[\Vdash_P] \subseteq \mathcal{F}[\Vdash_P] = \mathcal{F}$ for any $\varphi \in \mathcal{F}$, it follows that $\bigcup\{\varphi[\Vdash_P] \mid \varphi \in \mathcal{F}\} \subseteq \mathcal{F}$. Therefore, $\mathcal{F} = \bigcup\{\varphi[\Vdash_P] \mid \varphi \in \mathcal{F}\}$.

For the reverse implication, suppose that $\mathcal{F} = \bigcup\{\varphi[\Vdash_P] \mid \varphi \in \mathcal{F}\}$ and $\{\varphi[\Vdash_P] \mid \varphi \in \mathcal{F}\}$ is a directed set. We show that $\mathcal{F}$ is a filter by checking the conditions of Definition 4.2. Firstly, since the set $\{\varphi[\Vdash_P] \mid \varphi \in \mathcal{F}\}$ is directed, we know that $\mathcal{F}$ is not empty. Secondly, for any $\Gamma \sqsubseteq \mathcal{F}$, there exists a satisfiable formula $\psi \in \mathcal{F}$ such that $\Gamma \sqsubseteq \psi[\Vdash_P]$. Then using Proposition 4.1(4) and the rule ($\wedge_P$ : *right*), we see that $\bigwedge \Gamma \in \psi[\Vdash_P] \subseteq \mathcal{F}$. Finally, since $\mathcal{F} = \bigcup\{\varphi[\Vdash_P] \mid \varphi \in \mathcal{F}\}$ and $\varphi[\Vdash_P] \subseteq \mathcal{F}[\Vdash_P]$ for every $\varphi \in \mathcal{F}$, it follows that $\mathcal{F} \subseteq \mathcal{F}[\Vdash_P]$. Conversely, let $\psi \in \mathcal{F}[\Vdash_P]$. Then there exists $\Delta \sqsubseteq \mathcal{F} = \bigcup\{\varphi[\Vdash_P] \mid \varphi \in \mathcal{F}\}$ such that $\overline{\rho} \subseteq \overline{\Delta}$ and $\rho \Vdash_P \psi$. Note that $\bigwedge \Delta \in \mathcal{F}$ and $\bigwedge \Delta \Vdash_P \rho$, by the rule (*Cut*), we get $\bigwedge \Delta \Vdash \psi$. Thus $\psi \in (\bigwedge \Delta)[\Vdash_P] \subseteq \bigcup\{\varphi[\Vdash_P] \mid \varphi \in \mathcal{F}\} = \mathcal{F}$. □

In fact, Proposition 4.3 has a more general form: a subset $\mathcal{F}$ of $\mathcal{S}_P$ is a filter of $\mathcal{L}(P)$ if and only if there exists a directed subset $\{\varphi_i[\Vdash_P] \mid i \in I\}$ of the set $\{\varphi[\Vdash_P] \mid \varphi \in \mathcal{F}\}$ such that $\mathcal{F} = \bigcup_{i \in I} \varphi_i[\Vdash_P]$. The proof of this assertion is analogous to that in Proposition 4.3.

By Definition 4.1 and using the rule ($\wedge$, *right*), we have the following proposition, which shows that there are enough filters.

**Proposition 4.4.** *If $\varphi \in \mathcal{S}_P$, then $\varphi[\Vdash_P]$ is a filter of $\mathcal{L}(P)$.*

Given a consistent conjunctive sequent calculus $(\mathcal{L}(P), \vdash_P)$, for convenience, we use $\mathbf{Filt}(P)$ to denote the set of all filters of $\mathcal{L}(P)$.

**Lemma 4.1.** *$(\mathbf{Filt}(P), \subseteq)$ is a pointed dcpo.*

*Proof.* Since $\top_P[\Vdash] \in \mathbf{Filt}(P)$ and $\top_P[\Vdash] \subseteq \mathcal{F}$ for any $\mathcal{F} \in (\mathbf{Filt}(P), \subseteq)$, we have $(\mathbf{Filt}(P), \subseteq)$ is pointed. Let $\{\mathcal{F}_i | i \in I\}$ be a directed subset of $\mathbf{Filt}(P)$ and $\mathcal{F} = \bigcup_{i \in I} \mathcal{F}_i$. For any $\Gamma \sqsubseteq \mathcal{F}$, there exists some $i_0 \in I$ such that $\Gamma \sqsubseteq \mathcal{F}_{i_0}$. This yields that $\bigwedge \Gamma \in \mathcal{F}_{i_0} \subseteq \mathcal{F}$. Assume that $\varphi \in \mathcal{F}[\Vdash_P]$, by equation (3), we have that $\rho \Vdash_P \varphi$ with $\overline{\rho} \subseteq \overline{\Delta}$ for some $\Delta \sqsubseteq \mathcal{F}$. From $\Delta \sqsubseteq \mathcal{F}$, it follows that $\Delta \sqsubseteq \mathcal{F}_{i_1}$ for some $i_1 \in I$. Then $\varphi \in \mathcal{F}_{i_1}[\Vdash_P] = \mathcal{F}_{i_1} \subseteq \mathcal{F}$. This means that $\mathcal{F}[\Vdash_P] \subseteq \mathcal{F}$. For the other direction, let $\varphi \in \mathcal{F} = \bigcup_{i \in I} \mathcal{F}_i$. Then there exists some $i_2 \in I$ such that $\varphi \in \mathcal{F}_{i_2} = \mathcal{F}_{i_2}[\Vdash_P] \subseteq \mathcal{F}[\Vdash_P]$, and hence $\mathcal{F} \subseteq \mathcal{F}[\Vdash_P]$. As a result, $(\mathbf{Filt}(P), \subseteq)$ is a dcpo. □

**Lemma 4.2.** *For any $\mathcal{F}_1, \mathcal{F}_2 \in \mathbf{Filt}(P)$,*

$$\mathcal{F}_1 \ll \mathcal{F}_2 \text{ if and only if } \mathcal{F}_1 \subseteq \varphi[\Vdash_P] \text{ for some } \varphi \in \mathcal{F}_2.$$

*Proof.* Assume that $\mathcal{F}_1 \ll \mathcal{F}_2$. Since $\mathcal{F}_2$ is the directed union of the set $\{\varphi[\Vdash_P] \mid \varphi \in \mathcal{F}_2\}$, it follows that $\mathcal{F}_1 \subseteq \varphi[\Vdash_P]$ for some $\varphi \in \mathcal{F}_2$ by the definition of the way-below relation.

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Conversely, assume that $\mathcal{F}_1 \subseteq \varphi[\Vdash_P]$ for some $\varphi \in \mathcal{F}_2$. If $\{\mathcal{F}_i \in \mathbf{Filt}(P) \mid i \in I\}$ is directed and $\mathcal{F}_2 \subseteq \bigcup_{i\in I} \mathcal{F}_i$, then there exists some $j \in I$ such that $\varphi \in \mathcal{F}_j$. Since $\mathcal{F}_j$ is a filter, we have $\varphi[\Vdash_P] \subseteq \mathcal{F}_j$, and hence $\mathcal{F}_1 \subseteq \mathcal{F}_j$. As a result, $\mathcal{F}_1 \ll \mathcal{F}_2$. □

This result nicely reflects the intuition that $\mathcal{F}_1 \ll \mathcal{F}_2$ if $\mathcal{F}_1$ is covered by a finite part of $\mathcal{F}_2$.

**Theorem 4.1.** *If $(\mathcal{L}(P), \vdash_P)$ be a consistent conjunctive sequent calculus, then $(\mathbf{Filt}(P), \subseteq)$ is a proper BC domain with a basis $\{\varphi[\Vdash_P] \mid \varphi \in \mathcal{S}_P\}$.*

*Proof.* According to Proposition 4.3, Lemmas 4.1 and 4.2, we know that $(\mathbf{Filt}(P), \subseteq)$ is a domain which has a basis $\{\varphi[\Vdash_P] \mid \varphi \in \mathcal{S}_P\}$.

We now prove that $(\mathbf{Filt}(P), \subseteq)$ has no greatest element. Suppose for a contradiction that $\mathcal{F}_0$ is the greatest element of $(\mathbf{Filt}(P), \subseteq)$. Then for any $\varphi \in \mathcal{S}_P$, we have that $\varphi[\Vdash_P] \subseteq \mathcal{F}_0$. As $(\mathcal{L}(P), \vdash_P)$ is consistent, it follows that $p \in \mathcal{S}_P$ for any $p \in P$. Thus there exists $\varphi_p \in \mathcal{S}_P$ such that $p \in \varphi_p[\Vdash_P]$, which implies that $P \subseteq \mathcal{F}_0$. With Definitions 3.1 and 4.2, we have $\mathcal{L}(P) \subseteq \mathcal{F}_0 \subseteq \mathcal{S}_P \subseteq \mathcal{L}(P)$. This is a contradiction to that $\mathcal{S}_P \neq \mathcal{L}(P)$.

Next, we check that any two filters which are bounded above have a least upper bound. Suppose that $\mathcal{F}_1, \mathcal{F}_2 \in \mathbf{Filt}(P)$ and they have an upper bound $\mathcal{F}_3 \in \mathbf{Filt}(P)$. Then $\varphi_1 \wedge \varphi_2 \in \mathcal{F}_3$ for any $\varphi_1 \in \mathcal{F}_1$ and $\varphi_2 \in \mathcal{F}_2$. Set

$$\mathcal{F} = \{\varphi \in \mathcal{L}(P) \mid \varphi_1 \wedge \varphi_2 \Vdash_P \varphi \text{ for some } \varphi_1 \in \mathcal{F}_1, \varphi_2 \in \mathcal{F}_2\}.$$

For $\varphi_1 \in \mathcal{F}_i$, there exists $\psi_i \in \mathcal{F}_i$ such that $\psi_i \Vdash_P \varphi_i$ for $i = 1, 2$. By the rules (*Weakening*), $(\wedge : \mathit{left})$ and $(\wedge : \mathit{right})$, we have $\psi_1 \wedge \psi_2 \Vdash_P \varphi_i (i = 1, 2)$. This yields that $\mathcal{F}_1, \mathcal{F}_2 \subseteq \mathcal{F} \subseteq \mathcal{F}_3$. So that to finish the proof, it suffices to show that $\mathcal{F}$ is a filter.

For any $\varphi, \psi \in \mathcal{F}$, there exist $\varphi_1, \psi_1 \in \mathcal{F}_1 \subseteq \mathcal{F} \subseteq \mathcal{F}_3$ and $\varphi_2, \psi_2 \in \mathcal{F}_2 \subseteq \mathcal{F} \subseteq \mathcal{F}_3$ such that $\varphi_1 \wedge \varphi_2 \Vdash_P \varphi$ and $\psi_1 \wedge \psi_2 \Vdash_P \psi$. Then $\varphi_1 \wedge \varphi_2 \wedge \psi_1 \wedge \psi_2 \Vdash_P \varphi \wedge \psi$, which implies that $\varphi \wedge \psi \in \mathcal{F}$. With Proposition 4.1(1,3), we therefore have $\varphi[\Vdash_P], \psi[\Vdash_P] \subseteq (\varphi \wedge \psi)[\Vdash_P]$, and hence the set $\{\varphi[\Vdash_P] \mid \varphi \in \mathcal{F}\}$ is directed. By Proposition 4.3, we have to show that $\mathcal{F} = \bigcup\{\varphi[\Vdash_P] \mid \varphi \in \mathcal{F}\}$. First let $\psi \in \mathcal{F}$. Then $\psi \in (\psi_1 \wedge \psi_2)[\Vdash_P]$ for some $\psi_1 \in \mathcal{F}_1$ and $\psi_2 \in \mathcal{F}_2$. Note that $\psi_1 \wedge \psi_2 \in \mathcal{F}$ whenever $\psi_1 \in \mathcal{F}_1$ and $\psi_2 \in \mathcal{F}_2$, we obtain that $\mathcal{F} \subseteq \bigcup\{\varphi[\Vdash_P] \mid \varphi \in \mathcal{F}\}$.

Conversely, if $\psi \in \{\varphi[\Vdash_P] \mid \varphi \in \mathcal{F}\}$, then there exists some $\varphi \in \mathcal{F}$ such that $\psi \in \varphi[\Vdash_P]$. Since $\varphi \in \mathcal{F}$ if and only if $\varphi_1 \wedge \varphi_2 \Vdash_P \varphi$ for some $\varphi_1 \in \mathcal{F}_1, \varphi_2 \in \mathcal{F}_2$, by the rule (*Cut*), it follows that $\varphi_1 \wedge \varphi_2 \Vdash_P \psi$. Thus, $\psi \in \mathcal{F}$ and $\bigcup\{\varphi[\Vdash_P] \mid \varphi \in \mathcal{F}\} \subseteq \mathcal{F}$. □

**Corollary 4.1.** *If $(\mathcal{L}(P), \vdash_P)$ is an algebraic consistent conjunctive sequent calculus, then $(\mathbf{Filt}(P), \subseteq)$ is an algebraic proper domain.*

*Proof.* By Theorem 4.1, it suffices to prove that the compact elements are of the form $\varphi[\Vdash_P]$ with $\varphi \in \mathcal{S}_P$. To this end, let $\mathcal{F}$ be a compact element. Note that $\mathcal{F}$ is the union of the directed set $\{\varphi[\Vdash_P] \mid \varphi \in \mathcal{F}\}$, it follows that $\mathcal{F} = \varphi[\Vdash_P]$ for some $\varphi \in \mathcal{F} \subseteq \mathcal{S}_P$. Conversely, suppose that $\varphi[\Vdash_P] \subseteq \bigcup_{i\in I} \mathcal{F}_i$ for some directed subset $\{\mathcal{F}_i \mid i \in I\}$ of $\mathbf{Filt}$(P), then there exists $i_0 \in I$ such that $\varphi \in \mathcal{F}_{i_0}$, Since $\mathcal{F}_{i_0}$ is a filter, we have $\varphi[\Vdash_P] \subseteq \mathcal{F}_{i_0}[\Vdash_P]$. Therefore, $\varphi[\Vdash_P]$ is a compact element. □

Theorem 4.1 tells us that each consistent conjunctive sequent calculus can induce a proper BC domain. Conversely, we will show that every proper BC domain can also be generated by a consistent conjunctive sequent calculus, up to isomorphism.

Recall that Example 3.2, we have known that each proper BC domain $(D, \leq)$ associates with a conjunctive sequent calculus $(\mathcal{L}(B(D)), \vdash_{B(D)})$. We claim that it is consistent. In fact, note that $(D, \leq)$ has no greatest element, it is obvious that

$$\varphi \text{ is satisfiable if and only if } \overline{\varphi} \text{ has an upper bound in } D, \tag{4}$$

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and then $B(D) \subseteq \mathcal{S}_{B(D)} \neq \mathcal{L}(B(D))$. We therefore obtain that

$$\Gamma \Vdash_{B(D)} \Delta \text{ if and only if } \bigsqcup \overline{\Delta} \ll \bigsqcup \overline{\Gamma}. \tag{5}$$

For every satisfiable formula $\psi$ of $(\mathcal{L}(B(D)), \vdash_{B(D)})$, there exists a satisfiable formula $\varphi$ such that $\varphi \Vdash_{B(D)} \psi$ because of the interpolation property of $\ll$ in $(D, \leq)$.

**Lemma 4.3.** *Let $(D, \leq)$ be a proper BC domain.*

(1) *If $d \in D$, then $\mathcal{F}_d = \{\varphi \in \mathcal{L}(B(D)) \mid p \in \overline{\varphi}, p \ll d\} \in \mathbf{Filt}(B(D))$.*
(2) *If $\mathcal{F} \in \mathbf{Filt}(B(D))$, then $\bigsqcup \overline{\mathcal{F}} \in D$ and $\mathcal{F} = \mathcal{F}_{\bigsqcup \overline{\mathcal{F}}}$.*

*Proof.* (1) For each $d \in D$, we prove $\mathcal{F}_d$ is a filter by checking the conditions of Proposition 4.2. The first two conditions follow from $T_D = 0_D \ll d$ and $\overline{\varphi \wedge \psi} = \overline{\varphi} \cup \overline{\psi}$, respectively. By the definitions of $\mathcal{F}_d$ and the interpolation property of $\ll$ in $(D, \leq)$, we see that $\psi \in \mathcal{F}_d$ if and only if $\bigsqcup \overline{\psi} \ll d$ if and only if $\bigsqcup \overline{\psi} \ll p \ll d$ for some $p \in B(D)$. Let $\varphi = p$, then $\psi \in \mathcal{F}_d$ if and only if $\varphi \Vdash_P \psi$. Thus, the third condition of Proposition 4.2 hold.

(2) We first claim that $\bigsqcup \overline{\varphi[\Vdash_{B(D)}]} = \bigsqcup \overline{\varphi}$. Indeed, for every $\varphi \in \mathcal{S}_{B(D)}$, by equation (4), we have $\bigsqcup \overline{\varphi} \in D$. Since

$$\varphi[\Vdash_{B(D)}] = \{\psi \in \mathcal{S}_{B(D)} \mid \varphi \Vdash_{B(D)} \psi\} = \{\psi \in \mathcal{S}_{B(D)} \mid (\forall p \in \overline{\psi}) p \ll \bigsqcup \overline{\varphi}\},$$

it follows that $\overline{\varphi[\Vdash_{B(D)}]} = \{p \in B(D) \mid p \ll \bigsqcup \overline{\varphi}\}$. Note that $(D, \leq)$ is a domain, we obtain that $\bigsqcup \overline{\varphi[\Vdash_{B(D)}]} = \bigsqcup \overline{\varphi}$. For any $\mathcal{F} \in \mathbf{Filt}(B(D))$, let $A_{\mathcal{F}} = \{\varphi[\Vdash_{B(D)}] \mid \varphi \in \mathcal{F}\}$ and $\overline{A}_{\mathcal{F}} = \{\bigsqcup \overline{\varphi[\Vdash_{B(D)}]} \mid \varphi \in \mathcal{F}\}$. As $A_{\mathcal{F}}$ is directed with respect to set inclusion, it follows that $\overline{A}_{\mathcal{F}}$ is directed in $(D, \leq)$, which implies that $\bigsqcup \overline{A}_{\mathcal{F}} \in D$. Because $\bigcup A_{\mathcal{F}} = \mathcal{F}$ and $\bigsqcup \overline{\varphi[\Vdash_{B(D)}]} = \bigsqcup \overline{\varphi}$, we see that $\bigsqcup \overline{A}_{\mathcal{F}} = \bigsqcup \overline{\mathcal{F}}$. Then combining equation (5) with the definition of $\mathcal{F}_d$, it is easy to detect that $\mathcal{F} = \{\varphi \in \mathcal{L}(B(D)) \mid \bigsqcup \overline{\varphi} \ll \bigsqcup \overline{\mathcal{F}}\}$. □

After these preparations, we can prove the main result of this section.

**Theorem 4.2.** **(Representation theorem).** *Let $(D, \leq)$ be a proper BC domain. Then it is order isomorphic to $(\mathbf{Filt}(B(D)), \subseteq)$.*

*Proof.* Define

$$f : D \to \mathbf{Filt}(B(D)) \text{ given by } f : d \mapsto \mathcal{F}_d,$$
$$g : \mathbf{Filt}(B(D)) \to D \text{ given by } g : \mathcal{F} \mapsto \bigsqcup \overline{\mathcal{F}}.$$

The above definitions are well defined thanks to Lemma 4.3. It is obvious that $f$ and $g$ are order-preserving mappings and mutually inverse. Consequently, $(D, \leq)$ is order isomorphic to $(\mathbf{Filt}(B(D)), \subseteq)$. □

## 5. Categorical Equivalence

We are interested in constructing a category of consistent conjunctive sequent calculi and exploring the equivalence relation between this category and that of proper BC domains. We first state proper morphisms between consistent conjunctive sequent calculi which correspond to Scott-continuous functions.

**Definition 5.1.** *Let $(\mathcal{L}(P), \vdash_P)$ and $(\mathcal{L}(Q), \vdash_Q)$ be two consistent conjunctive sequent calculi. A conjunctive consequence relation from $\mathcal{L}(P)$ to $\mathcal{L}(Q)$ is a binary relation $\Vdash \subseteq \mathbb{S}(P) \times \mathbb{S}(Q)$ closed under the following derivation rules:*

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$$\frac{}{\Gamma \Vdash T_Q}\,(T) \qquad \frac{\Gamma \Vdash \Delta}{\Gamma', \Gamma \Vdash \Delta}\,(W)$$

$$\frac{\Gamma \Vdash \Delta, \varphi \quad \Gamma \Vdash \Delta, \psi}{\Gamma \Vdash \Delta, \varphi \wedge \psi}\,(\wedge : R) \qquad \frac{\varphi, \psi, \Gamma \Vdash \Delta}{\varphi \wedge \psi, \Gamma \Vdash \Delta}\,(\wedge : L)$$

$$\frac{\Gamma \Vdash_P \varphi \quad \varphi \Vdash \Delta}{\Gamma \Vdash \Delta}\,(Cut : L) \qquad \frac{\Gamma \Vdash \varphi \quad \varphi \Vdash_Q \Delta}{\Gamma \Vdash \Delta}\,(Cut : R)\,.$$

**Remark 5.1.** By letting $\Gamma \vdash \Delta$ for any $\Delta \sqsubseteq \mathcal{L}(Q)$ whenever $\Gamma \notin \mathbb{S}(P)$, a conjunctive consequence relation $\Vdash$ from $\mathcal{L}(P)$ to $\mathcal{L}(Q)$ can extend to a binary relation $\vdash$ between nonempty finite sets from $\mathcal{L}(P)$ and $\mathcal{L}(Q)$. As the relationship of $\vdash_P$ and $\Vdash_P$, a conjunctive consequence relation $\Vdash$ and its extension $\vdash$ are also uniquely determined by each other.

Considering $\Gamma \vdash \Delta$ as a sequent between two consistent conjunctive sequent calculi, Definition 5.1 actually presents a new so-called multilingual sequent calculus (Jung et al. 1999). And each consistent conjunctive sequent calculus defined in Definition 4.1 is such a special one.

In the following proposition, we will see that there is a canonical way to pass from a Scott-continuous function between two proper BC domains to a conjunctive consequence relations between the corresponding consistent conjunctive sequent calculi.

**Proposition 5.1.** *Let $(D, \leq)$ and $(D', \leq')$ be two proper BC domains, and $f$ a Scott-continuous function from $D$ to $D'$. For any $\Gamma \in \mathbb{S}(B(D))$ and $\Delta \in \mathbb{S}(B(D'))$, define*

$$\Gamma \Vdash \Delta \textit{ if and only if } \bigsqcup \overline{\Delta} \ll' f(\bigsqcup \overline{\Gamma}).$$

*Then the relation $\Vdash$ is a conjunctive consequence relation from $\mathcal{L}(B(D))$ to $\mathcal{L}(B(D'))$.*

*Proof.* Note that a Scott-continuous function is order-preserving. The rule $(T)$ holds since $T_{B(D')} = 0_{D'}$. The rules $(\wedge : L)$ and $(\wedge : R)$ follow from the equalities $\overline{\varphi, \psi, \Gamma} = \overline{\varphi \wedge \psi, \Gamma}$ and $\overline{\Delta, \varphi \wedge \psi} = \overline{\Delta, \varphi} \bigcup \overline{\Delta, \psi}$, respectively. While the rule $(W)$ is due to the inequality $\bigsqcup \overline{\Gamma} \leq \bigsqcup \overline{\Gamma', \Gamma}$.

It remains to show the rules $(Cut : L)$ and $(Cut : R)$. We only need to check the rule $(Cut : L)$ because the rule $(Cut : R)$ follows in a similar manner. Assume that there exists some $\varphi \in \mathcal{S}_{B(D)}$ such that $\Gamma \Vdash_{B(D)} \varphi$ and $\varphi \Vdash \Delta$. Since $\varphi \in \mathcal{S}_{B(D)}$, we have that $\bigsqcup \overline{\varphi} \in D$. From $\Gamma \Vdash_{B(D)} \varphi$, it follows that $\bigsqcup \overline{\varphi} \ll \bigsqcup \overline{\Gamma}$, and hence $f(\bigsqcup \overline{\varphi}) \leq f(\bigsqcup \overline{\Gamma})$. From $\varphi \Vdash \Delta$, we refer that $\bigsqcup \overline{\Delta} \ll f(\bigsqcup \overline{\varphi})$. Therefore, $\bigsqcup \overline{\Delta} \ll f(\bigsqcup \overline{\Gamma})$. That is, $\Gamma \Vdash \Delta$. Conversely, suppose that $\Gamma \Vdash \Delta$, then $\bigsqcup \overline{\Delta} \ll \bigsqcup \overline{\Gamma} = \bigsqcup (\twoheaddownarrow(\bigsqcup \overline{\Gamma}) \cap B(D))$. By the definition of way-below and the interpolation property, there exists some $d \in B(D)$ such that $\bigsqcup \overline{\Delta} \ll d \ll \bigsqcup \overline{\Gamma}$. Thus $\Gamma \Vdash_{B(D)} d$ and $d \Vdash \Delta$. □

Given a conjunctive sequence relation $\Vdash$ from $\mathcal{L}(P)$ to $\mathcal{L}(Q)$, for any $X \subseteq \mathcal{L}(P)$, set

$$X[\Vdash] = \{\varphi \in \mathcal{L}(Q) \mid \rho \Vdash \varphi \text{ with } \overline{\rho} \subseteq \overline{\Gamma} \text{ for some } \Gamma \sqsubseteq X\}. \tag{6}$$

By an argument similar to that given in Proposition 4.1, the next proposition follows.

**Proposition 5.2.** *Let $\Vdash$ be a conjunctive consequence relation from $\mathcal{L}(P)$ to $\mathcal{L}(Q)$.*

(1) *If $X \subseteq Y \subseteq \mathcal{L}(P)$, then $X[\Vdash] \subseteq Y[\Vdash] \subseteq \mathcal{S}_Q$.*
(2) *$\Gamma[\Vdash] = (\bigwedge \Gamma)[\Vdash]$ for any $\Gamma \sqsubseteq \mathbb{S}(P)$.*
(3) *For any $\varphi \in \mathcal{S}_P$, $\psi \in \varphi[\Vdash]$ if and only if $\varphi \Vdash \psi$.*

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**Proposition 5.3.** *Let* $\Vdash$ *be a conjunctive consequence relation from* $\mathcal{L}(P)$ *to* $\mathcal{L}(Q)$. *Then* $(\varphi[\Vdash_P])[\Vdash] = \varphi[\Vdash]$ *whenever* $\varphi \in \mathcal{S}_P$.

*Proof.* $\psi \in (\varphi[\Vdash_P])[\Vdash] \Leftrightarrow \rho \Vdash \psi$ for some $\Gamma \sqsubseteq \varphi[\Vdash_P]$ with $\overline{\rho} \subseteq \overline{\Gamma}$ (by equation (6))
$\Leftrightarrow$ there exist some $\rho \Vdash \psi$ and $\varphi \Vdash_P \rho$ (by Proposition 4.1(4))
$\Leftrightarrow \varphi \Vdash \psi$ (by the rule $(Cut : L)$ )
$\Leftrightarrow \psi \in \varphi[\Vdash]$ (by Proposition 5.2(3)). □

**Proposition 5.4.** *Let* $\Vdash$ *be a conjunctive consequence relation from* $\mathcal{L}(P)$ *to* $\mathcal{L}(Q)$, *and* $\mathcal{F}$ *a filter of* $\mathcal{L}(P)$. *Then the following statements hold.*

(1) $\Delta \sqsubseteq \mathcal{F}[\Vdash]$ *if and only if* $\varphi \Vdash \Delta$ *for some* $\varphi \in \mathcal{F}$.
(2) $\mathcal{F}[\Vdash]$ *is a filter of* $\mathcal{L}(Q)$.

*Proof.* (1) Since the if part is obvious, we need only to prove the converse implication. Assume that $\Delta \sqsubseteq \mathcal{F}[\Vdash]$. Then for any $\varphi \in \Delta$, by equation (6), there exist some $\rho \Vdash \varphi$ and $\Gamma_\varphi \sqsubseteq \mathcal{F}$ with $\overline{\rho} \subseteq \overline{\Gamma_\varphi}$. Note that $\mathcal{F}$ is a filter of $\mathcal{L}(P)$, it follows that $\Gamma_\varphi \Vdash \varphi$, and hence $\bigwedge_{\varphi\in\Delta} \Gamma_\varphi \Vdash \Delta$.

(2) It is clear that $\mathcal{F}[\Vdash]$ is a nonempty subset of $\mathcal{S}_P$ since $T_Q \in \mathcal{F}[\Vdash]$. By the rule $(\wedge : R)$, we know that $\mathcal{F}[\Vdash]$ is closed under $\wedge$. Next, we have to prove that $\mathcal{F}[\Vdash] = (\mathcal{F}[\Vdash])[\Vdash_Q]$. For any $\psi \in \mathcal{F}[\Vdash]$, there exists some $\varphi \in \mathcal{F}$ such that $\varphi \Vdash \psi$. By the rule $(Cut : R)$, it follows that $\varphi \Vdash \rho$ and $\rho \Vdash_Q \psi$ for some $\rho \in \mathcal{S}_Q$. Thus $\psi \in (\varphi[\Vdash])[\Vdash_Q] \subseteq (\mathcal{F}[\Vdash])[\Vdash_Q]$, that is $\mathcal{F}[\Vdash] \subseteq (\mathcal{F}[\Vdash])[\Vdash_Q]$. Let conversely $\psi \in (\mathcal{F}[\Vdash])[\Vdash_Q]$. Then there exist some $\rho \Vdash_Q \psi$ and $\Gamma \sqsubseteq \mathcal{F}[\Vdash]$ with $\overline{\rho} \subseteq \overline{\Gamma}$. Since $\Gamma \sqsubseteq \mathcal{F}[\Vdash]$ and $\mathcal{F}$ is a filter of $\mathcal{L}(P)$, there exists $\varphi \in \mathcal{F}$ such that $\varphi \Vdash_P \bigwedge \Gamma$. From $\overline{\rho} \subseteq \overline{\Gamma}$, it follows that $\bigwedge \Gamma \Vdash_P \rho$. Using the rules $(Cut : L)$ and $(Cut : R)$ successively, we have $\varphi \Vdash_Q \psi$. Consequently, $\psi \in \varphi[\Vdash] \subseteq \mathcal{F}[\Vdash]$, and hence $(\mathcal{F}[\Vdash])[\Vdash_Q] \subseteq \mathcal{F}[\Vdash]$. □

**Proposition 5.5.** *Let* $\Vdash$ *be a conjunctive consequence relation from* $\mathcal{L}(P)$ *to* $\mathcal{L}(Q)$ *and* $\varphi \in \mathcal{S}_P$. *Then the following are equivalent.*

(1) $\varphi \Vdash \psi$.
(2) $\rho \Vdash \psi$ *for some* $\rho \in \varphi[\Vdash_P]$.
(3) $\rho \Vdash_Q \psi$ *for some* $\rho \in \varphi[\Vdash]$.

*Proof.* (1)$\Leftrightarrow$(2): According to Proposition 5.2(3), $\varphi \in \mathcal{S}_P$ and $\rho \in \varphi[\Vdash_P]$ if and only if $\varphi \Vdash_P \rho$. By the rule $(Cut : L)$, $\varphi \Vdash_P \rho$ and $\rho \Vdash \psi$ if and only if $\varphi \Vdash \psi$. So $\varphi \Vdash \psi$ if and only if $\rho \Vdash \psi$ for some $\rho \in \varphi[\Vdash_P]$. (1)$\Leftrightarrow$(3): Similarly. □

Proposition 5.5 explores the relationship among $\Vdash_P$, $\Vdash_Q$, and $\Vdash$. Now we put forward this in a more general form. Let $\Vdash$ be a conjunctive consequence relation from $\mathcal{L}(P)$ to $\mathcal{L}(Q)$ and $\vdash'$ a conjunctive consequence relation from $\mathcal{L}(Q)$ to $\mathcal{L}(R)$. A logical composition of $\Vdash$ and $\Vdash'$ is given by the following rule:

$$\frac{\Gamma \Vdash \varphi \quad \varphi \Vdash' \Delta}{\Gamma(\Vdash \circ \Vdash')\Delta}(Comp).$$

It is easy to check that the relation $\Vdash \circ \Vdash'$ that arises from the rule $(Comp)$ is also a conjunctive consequence relation from $\mathcal{L}(P)$ to $\mathcal{L}(R)$, and the composition of conjunctive consequence relations is associative. Proposition 5.5 shows that the rules $(Cut : L)$ and $(Cut : R)$ can be equivalently described by $\Vdash_P \circ \Vdash = \Vdash$, and $\Vdash = \Vdash \circ \Vdash_Q$, respectively.

In conclusion, consistent conjunctive sequent calculi with conjunctive consequence relations form a category **CCSC**. The identity morphism on a consistent conjunctive sequent calculus

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$(\mathcal{L}(P), \vdash_P)$ is the relation $\Vdash_P$, and the composition just is the rule (*Comp*). Analogously, algebraic consistent conjunctive sequent calculi with conjunctive consequence relations form a category **ASC**.

**Lemma 5.1.** *Let* $\Vdash$ *be a conjunctive consequence relation from* $\mathcal{L}(P)$ *to* $\mathcal{L}(Q)$. *Then* $\mathfrak{F}(\Vdash): \mathbf{Filt}(P) \to \mathbf{Filt}(Q)$, *given by* $\mathfrak{F}(\Vdash)(\mathcal{F}) = \mathcal{F}[\Vdash]$ *is a Scott-continuous function.*

*Proof.* By Proposition 5.4(2), the assignment $\mathfrak{F}(\Vdash)$ is well defined. Let $\{F_i \mid i \in I\}$ be a directed subset of $\mathbf{Filt}(P)$. Since $(\mathbf{Filt}(P), \subseteq)$ is a dcpo, we have $\bigcup_{i\in I} \mathcal{F}_i \in \mathbf{Filt}(P)$. So that to show $\mathfrak{F}(\Vdash)$ is Scott-continuous, it suffices to prove that $\bigcup_{i\in I} (\mathfrak{F}(\Vdash)(\mathcal{F}_i)) = \mathfrak{F}(\Vdash)(\bigcup_{i\in I} \mathcal{F}_i)$.

For every $\psi \in \mathfrak{F}(\Vdash)(\bigcup_{i\in I} \mathcal{F}_i) = (\bigcup_{i\in I} \mathcal{F}_i)[\Vdash]$, there exists some $\varphi \Vdash \psi$ with $\overline{\varphi} \subseteq \overline{\Gamma}$ and $\Gamma \sqsubseteq \bigcup_{i\in I} \mathcal{F}_i$. Because the set $\{\mathcal{F}_i \mid i \in I\}$ is directed and $\Gamma$ is finite, it follows that $\Gamma \sqsubseteq \mathcal{F}_{i_0}$ for some $i_0 \in I$. Therefore, $\psi \in \mathcal{F}_{i_0}[\Vdash] = \mathfrak{F}(\Vdash)(\mathcal{F}_{i_0}) \subseteq \bigcup_{i\in I} (\mathfrak{F}(\Vdash)(\mathcal{F}_i))$, and hence $\mathfrak{F}(\Vdash)(\bigcup_{i\in I} \mathcal{F}_i) \subseteq \bigcup_{i\in I} (\mathfrak{F}(\Vdash)(\mathcal{F}_i))$. The other direction is immediate by Proposition 5.2(1). □

**Lemma 5.2.** $\mathfrak{F}: \mathbf{CCSC} \to \mathbf{PBCD}$ *is a functor which maps every consistent conjunctive sequent calculus* $(\mathcal{L}(P), \vdash_P)$ *to* $(\mathbf{Filt}(P), \subseteq)$ *and conjunctive consequence relation* $\Vdash$ *to* $\mathfrak{F}(\Vdash)$.

*Proof.* Let $(\mathcal{L}(P), \vdash_P)$ and $(\mathcal{L}(Q), \vdash_Q)$ be two consistent conjunctive sequent calculi. Recall that Theorem 4.1 and Lemma 5.1, we have that $(\mathbf{Filt}(P), \subseteq)$ is a proper BC domain and $\mathfrak{F}(\Vdash): \mathfrak{F}(P) \to \mathbf{Filt}(Q)$ is Scott-continuous.

For each $\mathcal{F} \in \mathbf{Filt}(P)$, since $\mathfrak{F}(\Vdash_P)(\mathcal{F}) = \mathcal{F}[\Vdash_P] = \mathcal{F}$, we see that the identity is preserved. Next, we show that the composition is also preserved. Suppose that $\Vdash: \mathcal{L}(P) \to \mathcal{L}(Q)$ and $\Vdash'$: $\mathcal{L}(Q) \to \mathcal{L}(R)$ are two conjunctive consequence relations. We claim that $\mathfrak{F}(\Vdash \circ \Vdash') = \mathfrak{F}(\Vdash) \circ \mathfrak{F}(\Vdash')$. Indeed, we have

$\psi \in \mathfrak{F}(\Vdash \circ \Vdash')(\mathcal{F}) = \mathcal{F}[\Vdash \circ \Vdash']$ (by Lemma 5.1)
$\Leftrightarrow$ there exists $\rho \in \mathcal{F}$ such that $\rho \Vdash \circ \Vdash' \psi$ (by Proposition 5.4(1))
$\Leftrightarrow$ there exist $\rho \in \mathcal{F}$ and $\varphi \in \mathcal{S}_P$ such that $\rho \Vdash \varphi$ and $\varphi \Vdash' \psi$ (by the rule (*Comp*))
$\Leftrightarrow$ $\varphi \in \mathcal{F}[\Vdash]$ and $\varphi \Vdash' \psi$ ((by Proposition 5.4(1))
$\Leftrightarrow$ $\psi \in (\mathcal{F}(\Vdash))[\Vdash'] = (\mathfrak{F}(\Vdash) \circ \mathfrak{F}(\Vdash'))(\mathcal{F})$ (by Lemma 5.1).

As a result, $\mathfrak{F}: \mathbf{CCSC} \to \mathbf{PBCD}$ is a functor. □

**Theorem 5.1.** *The category* **CCSC** *is equivalent to* **PBCD**.

*Proof.* According to Proposition 2.2 and Theorem 4.2, it suffices to show that the functor $\mathfrak{F}$ defined in Lemma 5.2 is full and faithful.

(1) $\mathfrak{F}$ is full.

Let $(\mathcal{L}(P), \vdash_P)$ and $(\mathcal{L}(Q), \vdash_Q)$ be two consistent conjunctive sequent calculi, and $f: \mathbf{Filt}(P) \to \mathbf{Filt}(Q)$ a Scott-continuous function. For any $\Gamma \in \mathbb{S}(P)$ and $\Delta \in \mathbb{S}(Q)$, define

$$\Gamma \Vdash \Delta \text{ if and only if } \Delta \subseteq f((\bigwedge \Gamma)[\Vdash_P]). \tag{7}$$

An argument similar to the one used in the proof of Proposition 5.1 shows that $\Vdash$ is a conjunctive consequence relation. Next, we show that $\mathfrak{F}(\Vdash) = f$. For any $\mathcal{F} \in \mathbf{Filt}(P)$,

$\mathfrak{F}[\Vdash](\mathcal{F}) = \mathcal{F}[\Vdash]$ (by Lemma 5.1)
$= \{\psi \in \mathcal{L}(Q) \mid \varphi \Vdash \psi \text{ for some } \varphi \in \mathcal{F}\}$ (by Proposition 5.4(1))
$= \{\psi \in \mathcal{L}(Q) \mid \psi \in f(\varphi[\Vdash_P]) \text{ for some } \varphi \in \mathcal{F}\}$ (by equation (7))
$= \bigcup\{f(\varphi[\Vdash_P]) \mid \varphi \in \mathcal{F}\}$ (with a standard set calculus)
$= f(\bigcup\{\varphi[\Vdash_P] \mid \varphi \in \mathcal{F}\})$ (since $f$ is Scott-continuous)
$= f(\mathcal{F})$ (by Proposition 4.3).

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(2) $\mathfrak{F}$ is faithful.

Let $\Vdash_1, \Vdash_2$ be two conjunctive sequence relations from $\mathcal{L}(P)$ to $\mathcal{L}(Q)$ with $\mathfrak{F}(\Vdash_1) = \mathfrak{F}(\Vdash_2)$. Then for any $\Gamma \in \mathbb{S}(P)$, we see that $((\bigwedge \Gamma)[\Vdash_P])[\Vdash_1] = ((\bigwedge \Gamma)[\Vdash_P])[\Vdash_2]$ since $(\bigwedge \Gamma)[\Vdash_P]$ is a filter of $\mathcal{L}(P)$. An application of Proposition 5.3 gives that $(\bigwedge \Gamma)[\Vdash_1] = (\bigwedge \Gamma)[\Vdash_2]$. Combining equation (6) and Proposition 5.2(2), we therefore have $\Vdash_1 = \Vdash_2$. □

By Corollary 4.1 and Theorem 5.1, the following observation is immediate.

**Corollary 5.1.** *The category* **ASC** *is equivalent to* **PSD**.

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